SYMMETRIC CUBIC GRAPHS VIA RIGID CELLS

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Abstract

Properties of symmetric cubic graphs are described via their *rigid cells*, which are maximal connected subgraphs fixed pointwise by some involutory automorphism of the graph. This paper completes the description of rigid cells and the corresponding involutions for each of the 17 'action types' of symmetric cubic graphs.

Keywords: automorphism group, arc-transitive, symmetric cubic graph, rigid cell.

1 Introduction

When attempting to gain a thorough understanding of symmetric graphs, it is often helpful to know the structure of vertex-stabilisers and how they act, both locally and globally. In particular, for symmetric cubic graphs the structure of vertex-stabilisers is well known. By the work of Tutte [15, 16], every finite symmetric cubic graph is s-arc-regular for some $s \leq 5$, and then each vertex-stabiliser is isomorphic to C_3, S_3, $S_3 \times C_2$, S_4, $S_4 \times C_2$, according to whether the graph is 1-, 2-, 3-, 4- or 5-arc-regular, respectively (see [6]). It follows that these vertex-stabilisers contain automorphisms of orders only 2, 3, 4 or 6.

Also it can be an important question to ask about the subgraph induced on the set of vertices fixed by an automorphism belonging to a vertex-stabiliser. This question was addressed in [11], in the context of the 'even/odd automorphism' dichotomy, as a tool in determining which symmetric cubic graphs have (or do not have) an odd automorphism — namely an automorphism that induces an odd permutation on the vertices; see also [10].

We recall the definition of a *rigid cell*. Given a graph X and an automorphism α of X, let $\mathrm{Fix}(\alpha)$ denote the set of all vertices of X fixed by α. With the assumption that $\mathrm{Fix}(\alpha)$ is non-empty, we call the subgraph $X[\mathrm{Fix}(\alpha)]$ induced on $\mathrm{Fix}(\alpha)$ the *rigid subgraph of* α, or in short, the α*-rigid subgraph*, and then call each component of $X[\mathrm{Fix}(\alpha)]$ an α*-rigid cell.*

In a finite symmetric cubic graph, the length of any path that can be fixed by a non-trivial automorphism of the graph is 4 (by Tutte's theorem), and consequently, a rigid cell must be a

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single vertex, or one of the graphs shown in Figure 1. For these types of rigid cells we will use the terms *I-tree*, *H-tree*, *Y-tree*, *A-tree* and *B-tree*, according to the labels in the figure.

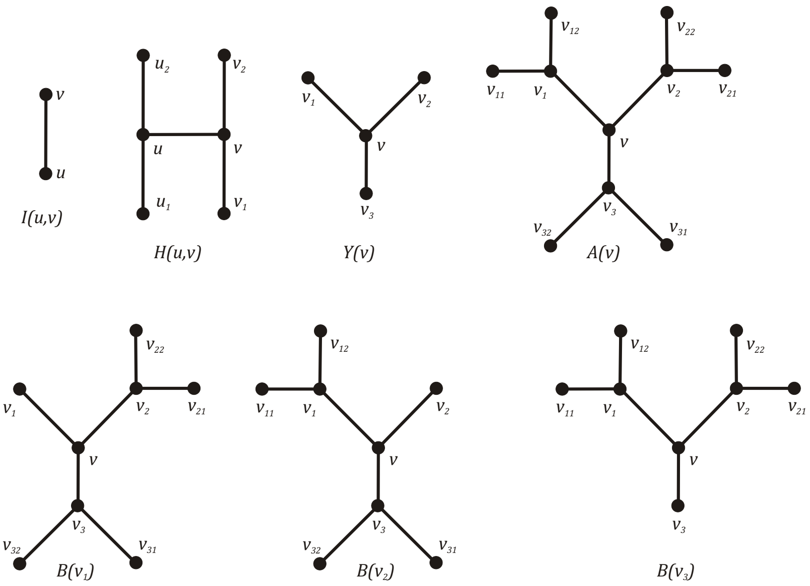

Figure 1: I-tree, H-tree, Y-tree, A-tree and three B-trees

It is easy to see that in a symmetric cubic graph, the single-vertex graph K_1 can be a rigid cell for automorphisms of order 3 and 6, and the 2-vertex complete graph K_2 (which is the I-tree) can be a rigid cell for automorphisms of order 2 or 4. Furthermore, because the stabiliser of any 2-arc in a finite symmetric cubic graph is an elementary abelian 2-group, all the other graphs given in Figure 1 can occur only as rigid cells of involutory automorphisms, and consequently the question about rigid cells is essentially a question about the orbit structure of these involutions.

This matter was first considered by Djoković and Miller in [6], where it was shown that the three graphs given in the second line of Figure 1, the so-called B-trees, cannot occur as rigid cells in symmetric cubic graphs. See also [11] for an alternative proof.

Tutte proved in two seminal papers [15, 16] that every finite symmetric cubic graph is s-arc-regular for some $s \leq 5$. Some further insight into the structure of symmetric cubic graphs was given by Djoković and Miller, who proved in [6] that each vertex-stabiliser in an s-arc-regular subgroup of automorphisms of a symmetric cubic graph is isomorphic to C_3, S_3, $S_3 \times C_2$, S_4 or $S_4 \times C_2$, when $s = 1, 2, 3, 4$ or 5, respectively. Accordingly, the automorphism group of a cubic s-arc-regular graph of order n has order $3 \cdot 2^{s-1} \cdot n$ (for $1 \leq s \leq 5$). Djoković and Miller [6] also proved that for $s \in \{1, 3, 5\}$ there is just one possibility for edge-stabilisers, while there exist up to two possibilities for $s \in \{2, 4\}$, and existence of graphs of both kinds (for for $s \in \{2, 4\}$) was later confirmed by Conder and Lorimer [4]. The resulting seven classes of arc-transitive finite group actions on cubic graphs are summarised in Table 1, where D_n denotes the dihedral group of degree n and order $2n$ (see also [7]). In particular, for $s = 2$ the edge-stabiliser (of order 4) is isomorphic to either $V_4 \cong C_2 \times C_2$ or C_4, while for $s = 4$ the edge-stabiliser (of order 16) is isomorphic to either D_8 or the quasi-dihedral group $C_8 \rtimes_3 C_2$.

Table 1: Vertex and edge-stabilisers of an s-arc-regular group G of automorphisms of a cubic graph

Class	s	G_v	G_e
1	1	C_3	C_2
2^1	2	S_3	V_4
2^2	2	S_3	C_4
3	3	$S_3 \times C_2$	D_4
4^1	4	S_4	D_8
4^2	4	S_4	$C_8 \rtimes_3 C_2$
5	5	$S_4 \times C_2$	$(D_4 \times C_2) \rtimes C_2$

More recently, Conder and Nedela [5] proved that there are exactly 17 combinations of the above seven classes realisable by the arc-transitive subgroups of the full automorphism group of a finite symmetric cubic graph X; see Table 2. We call these 17 combinations the 'action types'.

Table 2: The 17 possible action types for symmetric cubic graphs

s	Action type	Bipartite?	s	Action type	Bipartite?	s	Action type	Bipartite?
1	$\{1\}$	Sometimes	3	$\{2^1, 3\}$	Never	5	$\{1, 4^1, 4^2, 5\}$	Always
2	$\{1, 2^1\}$	Sometimes	3	$\{2^2, 3\}$	Never	5	$\{4^1, 4^2, 5\}$	Always
2	$\{2^1\}$	Sometimes	3	$\{3\}$	Sometimes	5	$\{4^1, 5\}$	Never
2	$\{2^2\}$	Sometimes	4	$\{1, 4^1\}$	Always	5	$\{4^2, 5\}$	Never
3	$\{1, 2^1, 2^2, 3\}$	Always	4	$\{4^1\}$	Sometimes	5	$\{5\}$	Sometimes
3	$\{2^1, 2^2, 3\}$	Always	4	$\{4^2\}$	Sometimes			

In [11], complete information on the existence of odd automorphisms was given for each of the action types of finite symmetric cubic graphs.

It was proved in [5] that with the exception of action types $\{3\}$ and $\{5\}$, every involution belonging to a vertex-stabiliser in the automorphism group of a finite symmetric cubic graph has only one type of rigid cell. More details are given in Section 3.

For convenience we let $\mathcal{I}(X)$ denote the set of all involutory automorphisms of a symmetric cubic graph X that fix some vertex of X, and we let $\mathcal{S}(X)$ denote the set of all involutions in $\mathrm{Aut}(X)$ that are semiregular (that is, with all of its cycles having the same length; see [12, 13]).

We will say that an involution in $\mathrm{Aut}(X)$ is

- an I-, Y-, H- or A-*involution* if all of its rigid cells are isomorphic to the I-, Y-, H- or A-tree, respectively,
- an M-*involution* if it admits non-isomorphic rigid cells (so that its rigid cells are of *mixed* structure), and/or
- an S-*involution* if it is semiregular (that is, all of its cycles have the same length; see [12, 13]).

Here also we mention that although semiregular automorphisms of order 2 usually exist in bipartite symmetric graphs (such as involutory 'edge-flippers'), there are non-bipartite symmetric cubic graphs with semiregular involutions as well, such as the 1-skeleton of the dodecahedron.

To obtain a complete characterisation of involutions in symmetric cubic graphs with respect to their conjugacy classes and the corresponding rigid cells, we need to analyse symmetric cubic graphs with action types $\{3\}$ and $\{5\}$. This is the main goal of this paper. We will prove the following theorem (in which $X+Y$ denotes the disjoint union of graphs X and Y, and kX denotes the disjoint union of k copies of X).

Theorem 1.1 *Let X be a connected symmetric cubic graph of order $2n$, where n is odd, and with action type $\{s\}$, where $s \in \{3, 5\}$. Then all involutions in $Aut(X)$ that fix some vertex of X are mutually conjugate. Moreover, every involution $\alpha \in \mathcal{I}(X)$ is an M-involution, and if $s = 3$ then $X[\mathrm{Fix}(\alpha)] \cong 3kI + 2kY$, while if $s = 5$ then $X[\mathrm{Fix}(\alpha)] \cong 3kH + 2kA$, for some positive integer k.*

For symmetric cubic graphs with action type $\{3\}$ or $\{5\}$ and of order divisible by 4, the situation is somewhat different. Some of these graphs have mixed type involutions (as in Example 2.8), while for others $\mathcal{I}(X)$ contains more than one conjugacy class of involutions, with none being an M-involution (as in Examples 2.6 and 2.7).

Theorem 1.1 will be proved in Section 5. One of the tools in the proof involves the group-theoretic concept of transfer, often used in character theory, and explained in Section 4. Before that, in Section 2 we give a number of examples illustrating the above discussion, and in Section 3 we give some further background about rigid cells.

2 Examples

In this section we give examples of symmetric cubic graphs with different action types, together with information on the conjugacy classes of their non-semiregular involutory automorphisms.

Example 2.1 Let X be the Petersen graph, with vertices labeled as in Figure 2. Then X has action type $\{2^1, 3\}$, and there exist two conjugacy classes of involutions in $\mathrm{Aut}(X)$, with one consisting of Y-involutions and the other consisting of I-involutions. Representatives of these two classes are illustrated in Figure 2. In particular, every involution in $\mathrm{Aut}(X)$ is conjugate to either the Y-involution $(0)(0')(1)(4)(1'\ 2)(3\ 4')(2'\ 3')$ with one rigid cell, or the I-involution $(0)(0')(1\ 4)(2'\ 3')(1'\ 4')(2\ 3)$ with one rigid cell. Note also that by Proposition 3.3, every 3-arc-regular cubic graph admitting a 2-arc-regular group of automorphisms has this property, with two classes of non-semiregular involutions (namely I-involutions and Y-involutions, separately).

Example 2.2 Let X be the Heawood graph, with vertices labeled as in the left-hand side of Figure 3. Then X has action type $\{1, 4^1\}$, and there exist two conjugacy classes of involutions in $\mathrm{Aut}(X)$, one consisting of S-involutions, and the other consisting of H-involutions. A representative of the class of H-involutions is illustrated in the left-hand-side of Figure 3. In particular, every non-semiregular involution in $\mathrm{Aut}(X)$ is conjugate to the H-involution

$$(0)(1)(2)(6)(9)(13)(3\ 11)(4\ 12)(5\ 7)(8\ 10),$$

with one rigid cell. Note also that by Proposition 3.4, every non-semiregular involutory automorphism of a 4-arc-regular cubic graph is an H-involution.

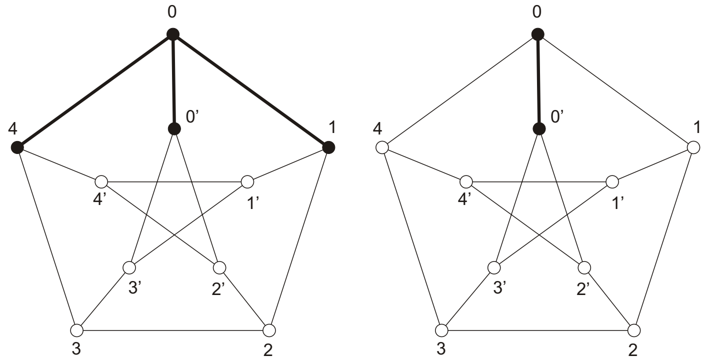

Figure 2: Illustration of rigid cells of representatives of the two classes of involutions for the Petersen graph

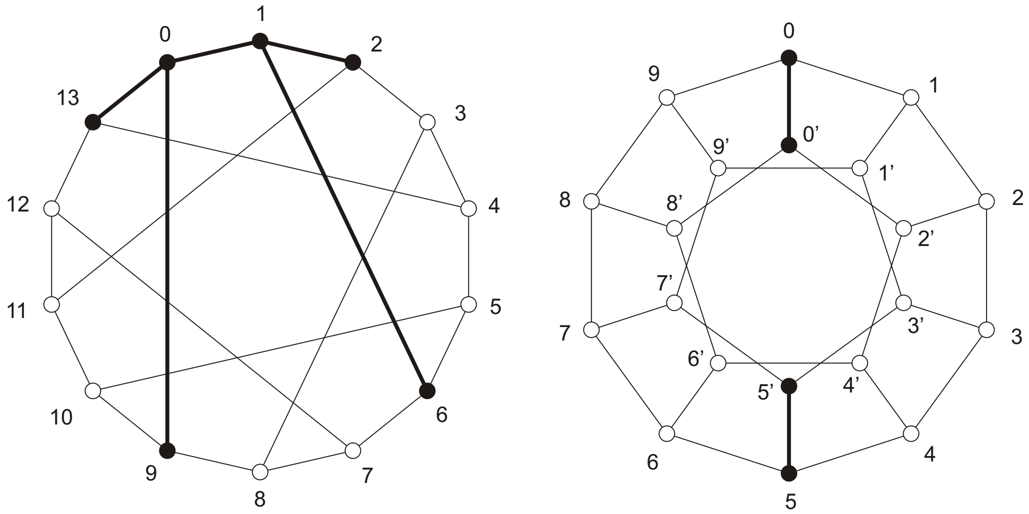

Figure 3: Illustration of the rigid cells of a representative of the class of involutions fixing a vertex for the Heawood graph (on the left-hand side) and the dodecahedral graph (on the right-hand side)

Example 2.3 Let X be the dodecahedral graph, with vertices labeled as in the right-hand side of Figure 3. Then X has action type $\{1, 2^1\}$, and there exist three conjugacy classes of involutions in $\mathrm{Aut}(X)$, with two consisting of S-involutions, and one of I-involutions. A representative of the class consisting of I-involutions in X is illustrated in the right-hand side of Figure 3. In particular, any non-semiregular involution in X is conjugate to the I-involution

$$(0)(5)(0')(5')(1\ 9)(2\ 8)(3\ 7)(4\ 6)(1'\ 9')(2'\ 8')(3'\ 7')(4'\ 6'),$$

which has two rigid cells (both of type I).

Example 2.4 Let X be Tutte's 8-cage, with vertices labeled as in Figure 4. Then X has action type $\{4^1, 4^2, 5\}$, and there exist three conjugacy classes of involutions in $\mathrm{Aut}(X)$, with one consisting of S-involutions, one consisting of H-involutions, and the third consisting of A-involutions. Representatives of the two classes of non-semiregular involutions in X are illustrated in Figure 4.

In particular, any non-semiregular involution in $\mathrm{Aut}(X)$ is conjugate to either the A-involution

$$(0)(1)(2)(6)(10)(22)(23)(24)(28)(29)(3\ 15)(5\ 7)(9\ 11)(13\ 21)(17\ 25)(19\ 27)(4\ 16)(8\ 12)(14\ 20)(18\ 26)$$

with one rigid cell, or the H-involution

$$(0)(1)(2)(10)(23)(29)(3\ 15)(6\ 28)(9\ 11)(22\ 24)(4\ 14)(5\ 27)(7\ 19)(8\ 18)(12\ 26)(13\ 25)(16\ 20)(17\ 21)$$

with one rigid cell. Note also that by Proposition 3.5, every 5-arc-regular cubic graph admitting a 4-arc-regular subgroup of automorphisms has two conjugacy classes of non-semiregular involutions, one consisting of H-involutions and the other of A-involutions.

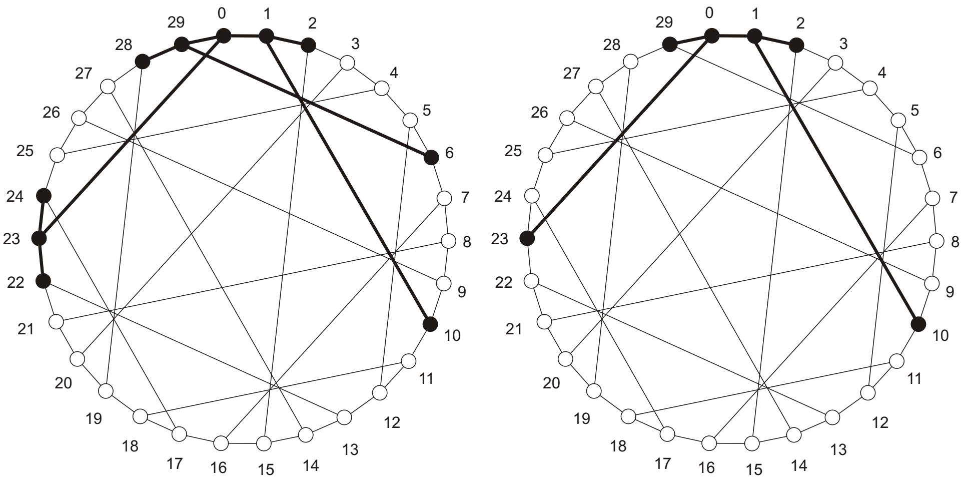

Figure 4: Illustration of the rigid cells of representatives of two classes of involutions for Tutte's 8-cage

Example 2.5 Let X be the unique symmetric cubic graph of order 110 (see [3]). This graph has action type $\{3\}$ and is shown in Figure 5, using the so-called Frucht notation [8] with respect to a $(10,11)$-semiregular automorphism ρ of X. The 110 vertices of X can be labeled as u_i^j for $i \in \mathbb{Z}_{10}$ and $j \in \mathbb{Z}_{11}$ such that the orbits of ρ are the sets $O_i = \{u_i^j \mid j \in \mathbb{Z}_{11}\}$ for $i \in \mathbb{Z}_{10}$. There exist two conjugacy classes of involutions in $\mathrm{Aut}(X)$, one of S-involutions, and the other of M-involutions having some rigid cells isomorphic to the I-tree and others isomorphic to the Y-tree. One can check that there exists a non-semiregular involution $\alpha \in \mathrm{Aut}(X)$ fixing the vertices

$$u_0^0,\ u_0^4,\ u_1^0,\ u_1^7,\ u_3^4,\ u_3^9,\ u_3^{10},\ u_4^1,\ u_4^2,\ u_4^7,\ u_7^4,\ u_7^{10},\ u_8^1 \text{ and } u_8^7,$$

and it follows that α has two rigid cells isomorphic to the Y-tree, and three rigid cells isomorphic to the I-tree. Note also that by our Theorem 1.1, every symmetric cubic graph with action type $\{3\}$ and of twice odd order has only one conjugacy class of non-semiregular involutory automorphisms, as in this example. For symmetric cubic graphs with action type $\{3\}$ which have order divisible by 4, however, the situation is quite different: see Example 2.6.

Example 2.6 A computation in Magma [1] similar to the ones that produced the graphs listed in [2, 3, 5] shows there exists a symmetric cubic graph with action type $\{3\}$ and order 39 916 800,

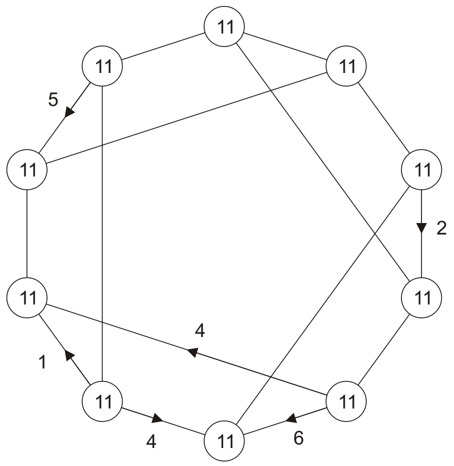

Figure 5: The symmetric cubic graph of order 110, given in Frucht's notation with respect to a $(10, 11)$-semiregular automorphism

and with automorphism group isomorphic to the symmetric group S_{12}. This group contains two conjugacy classes of involutions with fixed points (on the vertex-set). In one of these two classes, each involution fixes 15360 vertices that can be partitioned into 3840 rigid cells, with every rigid cell isomorphic to the Y-tree, while in the other class, each involution fixes 2304 vertices that can be partitioned into 1152 rigid cells, with every rigid cell isomorphic to the I-tree.

Example 2.7 Another computation in MAGMA [1] produces a symmetric cubic graph with action type $\{5\}$ and order 50 685 458 503 680 000, and automorphism group isomorphic to the symmetric group S_{20}. For this graph, there are two conjugacy classes of non-semiregular involutions, one consisting of A-involutions, and the other consisting of H-involutions.

Example 2.8 The cubic graph C1012.2 of order 1012 listed at [2] has action type $\{3\}$, and automorphism group isomorphic to $\mathrm{PSL}(2, 23) \times C_2$. This group contains three conjugacy classes of involutions, with two consisting of S-involutions, and one consisting of M-involutions with rigid subgraphs isomorphic to $6I + 4Y$. Similarly, there exists a symmetric cubic graph with action type $\{5\}$ and automorphism group isomorphic to A_{48}, with non-semiregular involutions being M-involutions, and the associated rigid cells being isomorphic to the H-tree and the A-tree.

3 Known information about rigid cells

The structure of vertex-stabilisers in symmetric cubic graphs ensures that only automorphisms of order 1, 2, 3, 4 and 6 can fix a vertex. In the following propositions, we recall certain facts proved in [11] about possible rigid cells for such automorphisms.

Proposition 3.1 [11] *Let X be a symmetric cubic graph, and let α be an automorphism of X fixing a vertex. Then*

(i) *if α has order* 3 *or* 6*, then every α-rigid cell is an isolated vertex, while*

(ii) *if α has order* 4*, then every α-rigid cell is an I-tree.*

Proposition 3.2 [11] *Let X be an s-arc-regular cubic graph, where $s \in \{1, 2\}$ and let α be an involution in $Aut(X)$. Then*

(i) *if $s = 1$, then* $\mathrm{Fix}(\alpha) = \emptyset$*, so that α is semiregular, while*

(ii) *if $s = 2$, then every α-rigid cell is an I-tree.*

Proposition 3.3 [11] *Let X be a* 3*-arc-regular cubic graph, and let α be a non-semiregular involution in $\mathcal{I}(X)$. Then every α-rigid cell is an I-tree or a Y-tree, and if both kinds occur* (*for α*) *then X has action type* $\{3\}$.

Our Theorem 1.1 gives information about rigid cells in 3-arc-regular cubic graphs, more detailed than in Proposition 3.3. In particular, it shows that in a symmetric cubic graph with action type $\{3\}$ and twice odd order, any involutory automorphism fixing a vertex has rigid cells isomorphic to the I-tree, as well as rigid cells isomorphic to the Y-tree. This is not always the case in symmetric cubic graphs with action type $\{3\}$ and order divisible by 4, however, as shown by Example 2.6.

Proposition 3.4 [11] *Let X be a* 4*-arc-regular cubic graph and let α be a non-semiregular involution in $\mathcal{I}(X)$. Then every α-rigid cell is an H-tree.*

Proposition 3.5 [11] *Let X be a* 5*-arc-regular cubic graph and let α be a non-semiregular involution in $\mathcal{I}(X)$. Then every α-rigid cell is an H-tree or an A-tree, and if both kinds occur* (*for α*) *then X has action type* $\{5\}$.

Just as in the 3-arc-regular case, our Theorem 1.1 gives information about the rigid cells for 5-arc-regular cubic graphs, more detailed than in Proposition 3.5. In particular, it shows that in a symmetric cubic graph with action type $\{5\}$ and twice odd order, any involutory automorphism fixing a vertex has rigid cells isomorphic to the H-tree, as well as rigid cells isomorphic to the A-tree. This is not always the case in symmetric cubic graphs with action type $\{5\}$ and order divisible by 4, however, as shown by Example 2.7.

4 The transfer

The following piece of group theory is used in the next section. We shall include some background information about it for the sake of completeness.

For a group G and a subgroup H of G, a *right transversal $\mathcal{T}$ for H in G* is a complete set of representatives for the set of right cosets Hx of H in G. The group G acts naturally and transitively by right multiplication on $\mathcal{T}$, and for $x \in \mathcal{T}$ and $g \in G$ we define $\overline{xg}$ to be the unique element of $\mathcal{T}$ that lies in the right coset Hxg.

Now suppose that H has finite index in G, and that H has a normal subgroup M such that H/M is abelian. Then the *transfer from G to H/M* is the function $\nu : G \to H/M$ given by $\nu\colon g \mapsto M\pi(g)$, where

$$\pi(g) = \prod_{x\in\mathcal{T}} xg\,\overline{xg}^{\,-1}.$$

It is not difficult to show that ν is a group homomorphism, and that its values do not depend on the choice of the transversal $\mathcal{T}$; see [14, Section 10.1] or [9, Ch. 7, Theorem 3.2].

What we need is the following application of the transfer:

Proposition 4.1 [9, Ch. 7, Ex. 3(i)] *Let G be a finite group with no normal subgroup of index 2. If P is any Sylow 2-subgroup of G, and R is any subgroup of index 2 in P, then every involution in G is conjugate to an involution in R.*

Proof. Let $\nu\colon G \to P/R$ be the transfer homomorphism from G to P/R ($\cong C_2$). Then since G has no subgroup of index 2, this homomorphism cannot be surjective, and hence is trivial.

Now let g be any involution in G, and suppose that right multiplication by g induces the permutation $(Px_1, Px_2)(Px_3, Px_4)\dots(Px_{2r-1}, Px_{2r})(Px_{2r+1})(Px_{2r+2})\dots(Px_m)$ on the m right cosets of P in G (where $m = |G:P|$). Let $\mathcal{T} = \{x_1, x_2, \dots, x_m\}$. Observe that $\overline{x_{2i-1}g} = x_{2i}$ and $\overline{x_{2i}g} = x_{2i-1}$ for $i \le r$, and $\overline{x_j g} = x_j = x_j g$ for $j \ge 2r+1$. It follows that $\nu\colon g \mapsto R\pi(g)$, where

$$\pi(g) = \prod_{x\in\mathcal{T}} xg\,\overline{xg}^{\,-1} = \prod_{1\le i\le r} x_{2i-1}gx_{2i}^{-1}\,x_{2i}gx_{2i-1}^{-1} \prod_{j\ge 2r+1} x_jgx_j^{-1} = \prod_{1\le i\le r} x_{2i-1}g^2x_{2i-1}^{-1} \prod_{j\ge 2r+1} x_jgx_j^{-1}.$$

In this product, the terms $x_{2i-1}g^2x_{2i-1}^{-1}$ are all trivial since g is an involution, and so $\pi(g)$ is the product of the terms $x_jgx_j^{-1}$ for $2r+1 \le j \le m$, each of which lies in P. If all $m-2r$ of these terms lie in $P \setminus R$, then their product $\pi(g)$ also lies in $P \setminus R$ (since $|P/R| = 2$ while $m = |G:P|$ is odd), but then $R\pi(g)$ is non-trivial, contradicting the fact that ν is trivial. Hence at least one of the terms $x_jgx_j^{-1}$ lies in R, and it follows that g is conjugate to an element of R, as required. ∎

5 Proof of Theorem 1.1

In view of the information about rigids cells for involutory automorphisms of symmetric cubic graphs given in [11] and summarised in Section 3, our Theorem 1.1 is restricted to graphs with action types $\{3\}$ and $\{5\}$. Proposition 4.1 plays an essential role in the proof, and for bipartite graphs the following lemma is also helpful.

Lemma 5.1 *Let X be a connected symmetric cubic graph of order $2n$, where n is odd, and having action type $\{3\}$ or $\{5\}$. If X is bipartite, then the index 2 subgroup G of $Aut(X)$ that preserves each of the two parts of the bipartition of X has no subgroup of index 2.*

Proof. Assume to the contrary that G has an index 2 subgroup, say H, which then has index 4 in $\mathrm{Aut}(X)$. Consider the natural action of $\mathrm{Aut}(X)$ on the right cosets of H by right multiplication. This action is transitive but imprimitive (because $H < G < \mathrm{Aut}(X)$), and hence if K is its kernel, then $\mathrm{Aut}(X)/K$ is isomorphic to one of the three imprimitive permutation groups of degree 4, namely V_4, C_4 and D_4. In the first two of these three cases, K has index 4 in $\mathrm{Aut}(X)$ and so $H = K \lhd \mathrm{Aut}(X)$, but then if a is any involutory edge-flipper (an automorphism that reverses some edge), then $\langle H, a\rangle$ is an arc-transitive proper subgroup of $\mathrm{Aut}(X)$, contradicting the assumption that X has action type $\{3\}$ or $\{5\}$. Similarly, in the third case (where $\mathrm{Aut}(X)/K \cong D_4$), the normal subgroup K contains every element of $\mathrm{Aut}(X)$ of order 3, and hence contains all such elements in the stabiliser of any vertex. It follows (by connectedness) that K is transitive on each of the two parts of X, and again if a is any involutory edge-flipper, then $\langle K, a\rangle$ is an arc-transitive proper subgroup of $\mathrm{Aut}(X)$, contradiction. ∎

Remark 5.2 *In Example 2.5 we considered the smallest cubic symmetric graph with action type* $\{3\}$ *and order* 2 *mod* 4*, which has order* 110 *and is bipartite. The smallest non-bipartite cubic symmetric graph with action type* $\{3\}$ *and order* 2 *mod* 4 *has order* 506*. This graph has one conjugacy class of non-semiregular involutions, and each such involution fixes* 14 *vertices, which induce two rigid cells isomorphic to the* Y*-tree and three rigid cells isomorphic to the* I*-tree.*

The situation for finite cubic symmetric graphs with action type $\{5\}$ *is much less straightforward. The first author has found the smallest such graph (namely one of order* $83\,966\,400$*, with the 'Held' sporadic simple group He as its automorphism group), and many other examples, some with alternating groups as their automorphism groups. At the time of writing this paper, however, no examples with action type* $\{5\}$ *and order* 2 *mod* 4 *have been found.*

We are now ready to prove our main theorem. We divide the proof into two cases, depending on whether $s = 3$ or $s = 5$. Also for ease of notation, we define an I-cell of an involutory automorphism $\alpha \in \mathrm{Aut}(X)$ to be a rigid cell of α that is isomorphic to the I-tree, and we define Y-cell, H-cell and A-cell in the analogous fashion.

Proof of Theorem 1.1.

Case 1. Action type $\{3\}$.

Let $e = \{u, v\}$ be any edge of X. Then by the work of Djoković and Miller [6] summarised in Table 1, the stabilisers $\mathrm{Aut}(X)_u$ and $\mathrm{Aut}(X)_e$ of the vertex u and edge e are isomorphic to $S_3 \times C_2$ (of order 12) and D_4 (of order 8). Since X has twice odd order, it follows that the number of edges in X is odd, and hence that $\mathrm{Aut}(X)_e$ is a Sylow 2-subgroup of $\mathrm{Aut}(X)$. We begin by distinguishing two cases, depending on whether or not X is bipartite.

If X is bipartite, then involutory edge-flippers are necessarily semiregular. In this case, let S be the stabiliser of the arc (u, v). Then S is isomorphic to V_4, and is a Sylow 2-subgroup of the index 2 subgroup G of $\mathrm{Aut}(X)$ preserving the two parts of the bipartition of X. Also by Lemma 5.1, the subgroup G has no subgroup of index 2, and hence by Proposition 4.1, we find that every involution in G is conjugate to an element of every subgroup of index 2 in S. Then since $S \cong V_4 \cong C_2 \times C_2$, it follows that all involutions in G are conjugate to each other, as required.

On the other hand, suppose that X is not bipartite. Then $\mathrm{Aut}(X)$ has no index 2 subgroup at all (since any index 2 subgroup would then be arc-transitive). Let R be the index 2 cyclic subgroup of order 4 in $\mathrm{Aut}(X)_e \cong D_4$. Then by Proposition 4.1, every involution of $\mathrm{Aut}(X)$ is conjugate to an involution in R, but R contains only one involution, and it follows that there is only one class of involutions in $\mathrm{Aut}(X)$.

Hence in both cases, all elements of $\mathcal{I}(X)$ are mutually conjugate in $\mathrm{Aut}(X)$.

Next, let Ω_I be the set of pairs (α, I) such that $\alpha \in \mathcal{I}(X)$ and I is a rigid I-cell of α, and let Ω_Y be the set of pairs (α, Y) such that $\alpha \in \mathcal{I}(X)$ and Y is a rigid Y-cell of α. Now since any two elements of $\mathcal{I}(X)$ are conjugate, we know there exist constants $a, b \in \mathbb{N}$ such that every $\alpha \in \mathcal{I}(X)$ has a rigid I-cells and b rigid Y-cells, and it follows that $X[\mathrm{Fix}(\alpha)] \cong aI + bY$, for every such α. In particular, $|\Omega_I| = |\mathcal{I}(X)| \cdot a$ and $|\Omega_Y| = |\mathcal{I}(X)| \cdot b$.

Also because X is 3-arc-regular, the stabiliser of an arc (u, v) of X contains three involutions, one of which fixes u and v but none of their other neighbours, while another fixes all the neighbours of u but not the other two neighbours of v, and the third fixes all the neighbours of v but not the other two neighbours of u. It follows that number of elements of $\mathcal{I}(X)$ having a particular I-tree in X as a rigid I-cell is 1, and similarly, the number of elements of $\mathcal{I}(X)$ having a particular Y-tree in X as a rigid Y-cell is also 1. These observations imply that $|\mathcal{I}(X)| \cdot a = |\Omega_I| = |E(X)|$

while $|\mathcal{I}(X)| \cdot b = |\Omega_Y| = |V(X)| = 2|E(X)|/3$, and it follows that $a = |E(X)|/|\mathcal{I}(X)| = 3b/2$. Hence in particular, a is divisible by 3, and b is even, and if we let $k = b/2$, then we find $a = 3k$ while $b = 2k$, and $X[\mathrm{Fix}(\alpha)] \cong 3kI + 2kY$ for every $\alpha \in \mathcal{I}(X)$.

Case 2. Action type $\{5\}$.

Again let $e = \{u, v\}$ be any edge of X. This time the work of Djoković and Miller [6], summarised in Table 1, shows that the stabilisers $\mathrm{Aut}(X)_u$ and $\mathrm{Aut}(X)_e$ are isomorphic to $S_4 \times C_2$ (of order 48) and $(D_4 \times C_2) \rtimes C_2$ (of order 32), and again since X has twice odd order, $\mathrm{Aut}(X)_e$ is a Sylow 2-subgroup of $\mathrm{Aut}(X)$.

Before proceeding to consider the bipartite and non-bipartite cases, we note that the vertices at distance at most 2 from u or v in X may be labelled 1, 2, 3, 4, $4'$, $5'$, $5'$, $6'$, $7'$ and $8'$, in such a way that the neighbourhoods of the six vertices u, v, 1, 2, 3 and 4 are given respectively by

$$\begin{aligned} X(u) &= \{v, 1, 3\}, & X(v) &= \{u, 2, 4\}, \\ X(1) &= \{u, 1', 5'\}, & X(2) &= \{v, 2', 6'\}, \\ X(3) &= \{u, 3', 7'\}, & X(4) &= \{v, 4', 8'\}. \end{aligned}$$

Moreover, this labelling can be arranged such that there exist automorphisms α, β and ρ of X whose restrictions to these vertices are given by

$$\begin{aligned} \bar{\alpha} &= (u)(v)(1)(3)(1')(5')(2\ 4)(2'\ 4')(3'\ 7')(6'\ 8'), \\ \bar{\beta} &= (u)(v)(2)(4)(2')(6')(1\ 3)(1'\ 3')(4'\ 8')(5'\ 7'), \\ \bar{\rho} &= (u\ v)(1\ 2\ 3\ 4)(1'\ 2'\ 3'\ 4'\ 5'\ 6'\ 7'\ 8'). \end{aligned}$$

Subcase 2.1. X is non-bipartite.

In this case, because X has action type $\{5\}$ there is no subgroup of index 2 in $\mathrm{Aut}(X)$, and so Proposition 4.1 can be applied. We will do that for a particular subgroup of index 2 in $\mathrm{Aut}(X)_e$.

With multiplication of permutations performed from left to right, an easy calculation shows that $\bar{\alpha}$ and $\bar{\rho}$ satisfy the relations $\bar{\alpha}^2 = \bar{\rho}^8 = 1$ and $\bar{\alpha}\bar{\rho}\bar{\alpha} = \bar{\rho}^3$, and hence α and ρ generate a subgroup H of $\mathrm{Aut}(X)$ whose restriction $\bar{H}$ to the ball $B_2(e)$ is isomorphic to the quasidihedral group $C_8 \rtimes_3 C_2$ of order 16. The elements of $\bar{H}$ are all expressible in the form $\bar{\alpha}^i\bar{\rho}^j$ with $i \in \mathbb{Z}_2$ and $j \in \mathbb{Z}_8$, and just five of these are involutions, namely $\bar{\rho}^4$ and the elements $\bar{\alpha}\bar{\rho}^j$ where $j \in \{0, 2, 4, 6\}$. Moreover, the rigid cells of these involutions in $\bar{H}$ are the induced subgraphs on the sets $\{u, v, 1, 2, 3, 4\}$, $\{u, v, 1, 3, 1', 5'\}$, $\{u, v, 2, 4, 2', 6'\}$, $\{u, v, 1, 3, 3', 7'\}$ and $\{u, v, 2, 4, 4', 8'\}$, respectively, and in particular, all of these are isomorphic to the H-tree.

Next, since $|\mathrm{Aut}(X)_e| = 32$ and $|\bar{H}| = 16$, we find that either H is an index 2 subgroup of $\mathrm{Aut}(X)_e$, or $H = \mathrm{Aut}(X)_e$. In the first case, Proposition 4.1 implies that any involution in $\mathrm{Aut}(X)$ is conjugate to an involution in H, and then since it is known that some involutions in $\mathrm{Aut}(X)$ have a rigid cell isomorphic to the A-tree (see Proposition 3.5), we conclude that every involution in $\mathrm{Aut}(X)$ has a rigid cell isomorphic to the H-tree as well as a rigid cell isomorphic to the A-tree. In the second case, the existence of an involution in $\mathrm{Aut}(X)$ with a rigid cell isomorphic to the A-tree gives the same conclusion, because vertex-stabilisers are conjugate subgroups of $\mathrm{Aut}(X)$.

Subcase 2.2. X is bipartite.

In this case, the part-preserving subgroup G of index 2 in $\mathrm{Aut}(X)$ contains the stabiliser S of the arc (u, v), which is isomorphic to $D_4 \times C_2$ of order 16, and hence a Sylow 2-subgroup of G. Also by Lemma 5.1, there is no subgroup of index 2 in G.

Here an easy calculation shows that $\bar{\alpha}$ and $\bar{\beta}$ satisfy the relations $\bar{\alpha}^2 = \bar{\beta}^2 = (\bar{\alpha}\bar{\beta})^4 = 1$, and hence α and β generate a subgroup H of $\mathrm{Aut}(X)$ whose restriction $\bar{H}$ to the ball $B_2(e)$ is isomorphic to the dihedral group D_4 of order 8. Moreover, the involutions in $\bar{H}$ are the five elements $\bar{\alpha}$, $\bar{\beta}$, $\bar{\alpha}\bar{\beta}\bar{\alpha}$, $\bar{\beta}\bar{\alpha}\bar{\beta}$ and $(\bar{\alpha}\bar{\beta})^2$, and another easy calculation shows that each of these elements has an H-tree as a rigid cell. (In fact those five H-trees are the same as for the elements $\bar{\alpha}$, $\bar{\alpha}\bar{\rho}^2$, $\bar{\alpha}\bar{\rho}^6$, $\bar{\alpha}\bar{\rho}^4$ and $\bar{\rho}^4$ respectively.)

Next, since $|S| = 16$ and $|\bar{H}| = 8$, we find that either H is an index 2 subgroup of S, or $H = S$. In the first case, since G has no subgroup of index 2 we find by Proposition 4.1 that any involution in G is conjugate to an involution in H, and then since we know that some involutions in $\mathrm{Aut}(X)$ have a rigid cell isomorphic to the A-tree, it follows that every involution in $\mathrm{Aut}(X)$ has a rigid cell isomorphic to the H-tree as well a rigid cell isomorphic to the A-tree. In the second case, the existence of an involution in $\mathrm{Aut}(X)$ with a rigid cell isomorphic to the A-tree gives the same conclusion, because vertex-stabilisers are conjugate subgroups of $\mathrm{Aut}(X)$.

Hence in both sub-cases, all elements of $\mathcal{I}(X)$ are mutually conjugate in $\mathrm{Aut}(X)$.

To conclude the proof of the theorem it remains to consider the numbers of rigid cells isomorphic to the H-tree and the A-tree, respectively. Let Ω_H be the set of pairs (α, H) such that $\alpha \in \mathcal{I}(X)$ and H is a rigid H-cell of α, and let Ω_A be the set of pairs (α, A) such that $\alpha \in \mathcal{I}(X)$ and A is a rigid A-cell of α. Then just as in Case 1 of this proof, we find that if each $\alpha \in \mathcal{I}(X)$ has a rigid H-cells and b rigid A-cells, then $X[\mathrm{Fix}(\alpha)] \cong aI + bY$ for every such α. This time, because X is 5-arc-regular, the stabiliser of an H-tree contains three involutions, one of which fixes no other neighbour of any vertex of that H-tree, while each of the remaining two involutions fixes an A-tree extending it, but not the same A-tree as the one fixed by the other one of those two involutions. It follows that number of elements of $\mathcal{I}(X)$ having a particular H-tree in X as a rigid I-cell is 1, and similarly, the number of elements of $\mathcal{I}(X)$ having a particular A-tree in X as a rigid Y-cell is also 1. Again these imply that $|\mathcal{I}(X)| \cdot a = |\Omega_I| = |E(X)|$ and $|\mathcal{I}(X)| \cdot b = |\Omega_Y| = |V(X)| = 2|E(X)|/3$, and hence that $a = 3k$ and $b = 2k$ for some $k \in \mathbb{N}$, and therefore $X[\mathrm{Fix}(\alpha)] \cong 3kI + 2kY$ for every $\alpha \in \mathcal{I}(X)$. ■

The main result of this paper is that for every cubic symmetric graph of type $\{3\}$ or $\{5\}$ with twice an odd number of vertices, every non-semiregular involution is an M-involution, and so has non-isomorphic rigid cells. The situation for cubic symmetric graphs with order divisible by 4 is more complex, as we showed in Examples 2.6, 2.7 and 2.8. In further research, we would like to find necessary and sufficient conditions for all non-semiregular involutory automorphisms of a given cubic symmetric graph with action type $\{3\}$ or $\{5\}$ and order divisible by 4 to be of M-type.

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