Heat kernel estimates for elliptic operators with Robin boundary conditions

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Abstract

Consider the elliptic operator

$$A = -\sum_{k,l=1}^{d} \partial_l c_{kl} \partial_k - \sum_{k=1}^{d} \partial_k b_k + \sum_{k=1}^{d} a_k \partial_k + a_0$$

on a bounded connected open set $\Omega \subset \mathbb{R}^d$ where $d \geq 2$, subject to Robin boundary conditions $\partial_\nu u + \beta u = 0$. We show that the kernel for the semigroup generated by $-A$ satisfies Gaussian and Hölder Gaussian bounds given domain and coefficients regularities.

In particular we show that when the domain is Lipschitz and the principal coefficients are real, then the kernel is ν-Hölder continuous for some $\nu \in (0,1)$. We also show that if the domain is $C^{1+\kappa}$, where $\kappa \in (0,1)$, and the coefficients are κ-Hölder continuous, then the kernel is differentiable and the derivative is κ-Hölder continuous.

We use these kernel estimates to prove other properties of the semigroup, including holomorphy and irreducibility. Moreover, we prove lower bounds for the kernel if the domain is Lipschitz, all coefficients are real and A is self-adjoint.

As an application we also associate the elliptic operator with the Dirichlet-to-Neumann operator $\mathcal{N}$. We show that if Ω is $C^{1+\kappa}$, where $\kappa \in (0,1)$, $c_{kl} = c_{lk}$ are real κ-Hölder continuous, $a_k = b_k = 0$ and a_0 is real, then the kernel of the semigroup generated by $-\mathcal{N}$ has a Hölder Poisson bounds.

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Chapter 1

Introduction

1.1 Background

Let $d \geq 2$ and consider the Laplacian

$$\Delta = \partial_1^2 + \partial_2^2 + \ldots + \partial_d^2.$$

Over $\mathbb{R}^d$, the famous heat equation $u_t = \Delta u$ is known to have a **heat kernel**

$$K_t^\Delta(x,y) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x-y|^2}{4t}}$$

for all $t > 0$. Given any $u \in L_2(\Omega)$, the function

$$v(t,x) = \int_{\mathbb{R}^d} K_t^\Delta(x,y) u(y)\, dy$$

solves the heat equation $v_t = \Delta_x v$ for all $t > 0$ and it satisfies the initial condition $\lim_{t \to 0^+} \|v_t - u\|_2 = 0$.

We note a few properties of interest. First since the kernel is C^∞, we have that $v_t \in C^\infty(\mathbb{R}^d)$ for all $t > 0$. Secondly kernel bounds in the form of

$$|K_t(x,y)| \leq c\, t^{-d/2}\, e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t}$$

for some $b, c, \omega > 0$ are known as **Gaussian kernel bounds**. The heat kernel is precisely the archetype of kernels with such bounds. The heat kernel is also Hölder continuous. More precisely, it possesses **Hölder Gaussian kernel bounds**: for all $\nu \in (0,1)$ there exist $b, c, \omega > 0$ such that

$$|K_t^\Delta(x,y) - K_t^\Delta(x',y)| \leq c\, t^{-d/2}\, e^{-b\frac{|x-y|^2}{t}} \Big(\frac{|x-x'|}{\sqrt{t}}\Big)^\nu\, e^{\omega t}$$

for all $x, x', y \in \mathbb{R}^d$ with $|x - x'| \leq \sqrt{t} + \frac{1}{2}|x - y|$. It is also differentiable with another Gaussian estimate: there exist $b', c' > 0$ such that for all $i \in \{1, \dots, d\}$,

$$|\partial_{x_i} K_t^{\Delta}(x, y)| \leq c'\, t^{-d/2}\, t^{-1/2} e^{-b' \frac{|x-y|^2}{t}}.$$

We can generalize the above results in two ways. First the Laplacian can be replaced by a strongly elliptic operator (in divergence form):

$$A = -\sum_{k,l=1}^{d} \partial_l\, c_{kl}\, \partial_k - \sum_{k=1}^{d} \partial_k\, b_k + \sum_{k=1}^{d} a_k\, \partial_k + a_0,$$

where the coefficients are bounded measurable and the principal coefficients (c_{kl}) satisfy an ellipticity condition, i.e., there exists a $\mu > 0$ such that

$$\operatorname{Re} \sum_{k,l}^{d} c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geq \mu |\xi|^2$$

for all x. Note that (c_{kl}) is not necessarily symmetric.

If we go back to the the heat equation $u_t = \Delta_x u$, one wonders if an operator S_t would exist, such that $v(t, x) = (S_t u)(x)$ solves the heat equation whenever $u \in L_2(\Omega)$. That connects to the study of semigroups, in particular C_0-semigroups. There has been extensive studies on C_0-semigroups, or more generally on one-parameter semigroups, [Dav80], [Paz83] and [EN00] for example. The answer to the above question is affirmative — by the Hille-Yosida theorem such operator exists, denoted by $e^{t\Delta}$ nowadays ([HP57], [Yos80]).

The starting point of this thesis is to show that the elliptic operator $-A$ also generates a C_0-semigroup (e^{-tA}) with appropriate properties that we can make use of. For example, we want the semigroup to be defined not just only on $(0, \infty)$, but one that can be analytically extended (in the sense of vector-valued functions, see [ABHN01]) to a sector on $\mathbb{C}$. This can be done by investigating the sesquilinear form that is associated with the operator. Such method is called the form method following from the work of Kato ([Kat80]) and Lions ([DL92]). Simply speaking, we find a sesquilinear form $\mathfrak{a}$ such that $\mathfrak{a}(u, v) = (Au, v)$ for all u, v in appropriate spaces.

For example let Δ_D be the Dirichlet Laplacian in $W_0^{1,2}(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is open.

Using the theory of semigroups, we want to replicate properties of the heat kernel on kernels of semigroups generated by elliptic operators. One of the most important property is the Gaussian upper bounds for the kernels.

Gaussian upper bounds for a kernel is a powerful tool that relates to other properties of the operator and the semigroup associated to that kernel. Arendt listed some consequences in [Are04] if a kernel has Gaussian upper bounds. These include holomorphy, maximal regularity and p-independence of the spectrum of the semigroup in L_p for $p \in (1, \infty)$. We also need the Gaussian upper bound in order to obtain the Gaussian lower bound ([Dav99]).

In this thesis, we will present holomorphy and irreducibility results based on the kernel estimates.

Upper bounds for heat kernels in $\mathbb{R}^d$ were first obtained by Aronsen for real measurable coefficients ([Aro67]), other works include [Dav89] or on manifolds in [Gri09]. Auscher, McIntosh and Tchamitchian proved in [AMT98] Gaussian upper bounds for second-order elliptic operators with complex Hölder continuous principal coefficients. Auscher also proved in [Aus96] Hölder Gaussian bounds for second-order elliptic operators with complex uniformly continuous principal coefficients. Note that for merely complex measurable principal coefficients the kernel fails to exist in general if $d \geq 3$ ([HMM11]).

The second generalization is to investigate the problem in a domain $\Omega \neq \mathbb{R}^d$ with various boundary regularity and boundary conditions. Other than the usual Dirichlet or Neumann boundary conditions, we are also interested in the Robin boundary condition $\partial_\nu u + \beta \operatorname{Tr} u = 0$ where ∂_ν reduces to the normal derivative for the Laplacian. This can be viewed as a generalization of the Neumann boundary condition, which is the Robin boundary condition with $\beta = 0$.

Heat kernels associated to operators with Dirichlet boundary conditions with real coefficients can be obtained from the $\mathbb{R}^d$ case ([Dav89])). However there are technical problems once we take other boundary conditions (see discussions in [AE97] Section 4). In [AE97], an extension property was assumed, together with real coefficients and differentiable first-order coefficients, in order to prove Gaussian upper bounds for the kernel in case of Neumann boundary condition. Daners then removed the differentiability requirement in [Dan00]. Auscher and Tchamitchian proved in [AT01] Hölder Gaussian estimates for second-order elliptic operators with real principal coefficients for the kernel over a Lipschitz domain subjected to Dirichlet or Neumann boundary condition. More recently, ter Elst and Rehberg proved in [ER15] Gaussian upper bounds and Hölder Gaussian bounds of kernels for second-order elliptic operators with real principal coefficients on a Lipschitz domain in case of a mix between Dirichlet and Neumann boundary condition.

Now we turn to results with Robin boundary conditions. Arendt and ter Elst gave in [AE97] Gaussian upper bounds for the kernel if the coefficients are real, the first-order terms are differentiable, $\beta \geq 0$ and the domain is Lipschitz. Daners removed the restrictions on the first-order coefficients ([Dan00]) and then relaxed the requirement on β to be real measurable in [Dan09]. Note that there were no Hölder Gaussian estimates under Robin boundary conditions — which we will discuss in this thesis, even with complex lower-order coefficients and complex β.

Another question is whether one obtains estimates for derivatives of the kernel if the domain is of higher regularity as well. In [EO19] ter Elst and Ouhabaz proved that the kernel for second-order elliptic operators, in a $C^{1+\kappa}$-domain subject to Dirichlet boundary conditions, is $C^{1+\kappa}$ with corresponding Gaussian and Hölder Gaussian estimates. We will extend this result to Robin boundary conditions in the thesis.

It is also possible to obtain lower bounds for these kernels if all the coefficients are real. Nash gave a lower bound for the heat kernels in the early days ([Nas58]). Gaussian lower bound on $\mathbb{R}^d$ were proved by Aronsen ([Aro67]). More recently ter Elst and Robinson gave in [ER98] local lower bounds for second-order elliptic operators with real or uniformly continuous coefficients. Davies in [Dav99] gave local lower bounds for second-order elliptic operators via a different method.

Choulli and Kayser in [CK15] gave lower bounds for elliptic operators in non-divergence form in $C^{1,1}$-domains with Neumann boundary conditions. There are also results for lower bounds in a more general setting, including [Cou03] and [GHL03] for metric measure spaces. In this thesis we follow these results to give a Gaussian lower bound for the heat kernel.

Lastly one may also ask for, instead of regularity of the kernel, the regularity of the solutions for the elliptic and parabolic equations. There are early works from the Russian school, in particular [LU68] and [LSU68] from Ladyzhenskaya, Solonnikov and Ural'tseva on elliptic operators with Neumann and Robin boundary conditions. There are also results on oblique derivative problems which can be seen as a generalization of the Robin boundary condition, especially from Lieberman, for example in [Lie89] and [Lie96]. These results however only hold for operators with real coefficients so their techniques do not fully apply here, since all our results allow the operators to have lower order terms with complex coefficients (except the Gaussian lower bounds).

The tools that we use to prove Hölder bounds for the kernel associated to the Robin operator can also be used for the Dirichlet-to-Neumann operator, which we describe as follows. Consider $\Omega \subset \mathbb{R}^d$ to be bounded Lipschitz with boundary Γ. One wants to investigate functions $\varphi \in L_2(\Gamma)$ such that there is a $u \in W^{1,2}(\Omega)$ with $\Delta u = 0$ weakly on Ω with $\operatorname{Tr} u = \varphi$ and $\partial_\nu \varphi \in L_2(\Gamma)$. We can then define the Dirichlet-to-Neumann operator $\mathcal{N}$ by $\mathcal{N}\varphi = \partial_\nu u$. The Dirichlet-to-Neumann operator has the physical significance of applying a voltage to Γ and measuring the induced current. It has applications in electrical impedance tomography. We can also define the Dirichlet-to-Neumann operator with respect to a second-order elliptic operator instead of the Laplacian. In the thesis we will give estimates for the semigroup generated by such an operator.

1.2 Outline of the thesis

In this thesis we investigate elliptic operators with Robin boundary condition under one of the following conditions:

1. The $C^{1+\kappa}$ case: the domain is $C^{1+\kappa}$, and all the coefficients are complex and κ-Hölder continuous.

2. The Lipschitz case: the domain is Lipschitz, the principal coefficients are real measurable and the lower-order coefficients are complex measurable.

3. The C^1 case: the domain is C^1, the principal coefficients are complex and uniformly continuous and the lower-order coefficients are complex measurable.

The main content of this thesis is based on two articles, the first of which is [EW19], and the other is in preparation. The first article, [EW19] focused on the Lipschitz case, and the other will focus on the $C^{1+\kappa}$ case. Instead of presenting the proofs separately, they are merged in the way that similar results can be shown in neighbouring sections.

Chapter 2 covers the preliminaries of the thesis. In Section 2.1, we introduce the basic terminology and results for semigroups, the associated operators and the sesquilinear forms. We also define the elliptic operators that we are interested in.

Section 2.2 covers the classical Morrey and Campanato spaces. We give the basic properties of the these spaces and give intuitions on why we would like to use these. We will focus on Morrey and Campanato space with $p = 2$, since we will mainly work with the L_2 or the $W^{1,2}$-space.

The classical Morrey and Campanato space is however problematic if the domain is not smooth enough or is unbounded. It was a breakthrough in [ER15] that the pointwise Morrey and Campanato seminorms was used instead. As such we can do estimation near the boundary as long as it is smooth enough locally. In this thesis the domain is at least Lipschitz which is smooth enough for all estimates in Section 2.2. The advantage of using the pointwise seminorm is that one can separate the arguments close to and far from the boundary. In Section 2.3, we define the pointwise Morrey and Campanato seminorms and give a few results that follow almost directly from the properties introduced in Section 2.2. These will be used frequently throughout the rest of the thesis.

Chapter 3 sets up the estimates required on the boundary. Since we only work with domains that are at least Lipschitz, there exists bi-Lipschitz maps from a given part of the boundary to a flat plane. So we first give the estimates for functions in the half space. After that we give the regularity results in various spaces.

The main component of the chapter are Sections 3.2 and 3.3 where the domain is $C^{1+\kappa}$, using the estimates obtained in Section 3.1. The corresponding results for Lipschitz domains and C^1 domains can then be shown in a similar fashion. We will next describe the main results in the separate sections.

Section 3.1 contains estimates on functions that satisfy $Au = 0$ weakly where A is an elliptic operator with constant coefficients. We start from various Cacciopoli inequalities, then we show that u satisfies higher order elliptic bounds. Finally we show that u and ∇u satisfy De Giorgi estimates over the half space.

In Section 3.2 we prove regularity results for ∇u of order less than d (on the Morrey-Campanato scale). The main result is Proposition 3.20 where we show that we can improve the order of the Morrey norm of ∇u slightly every time. In Chapter 4 we will perform an induction showing that ∇u satisfies estimates on Morrey norm of order $\gamma \in (d-2, d)$, and that allows us to prove Hölder continuity.

In Section 3.3, we prove regularity results of order above d for $C^{1+\kappa}$-domains. That allows us to obtain Hölder continuity of the derivatives directly. This is a modification of regularity estimates for Dirichlet boundary conditions from [EO19], but requires much more hard work.

Sections 3.4 and 3.5 contain regularity results on Lipschitz and C^1 domains respectively. The De Giorgi estimates for the Lipschitz case follow from [ER15] and we can use Section 3.1 for the C^1 case.

Chapter 4 gives the heat kernel estimates for the semigroup. Again we follow the same structure as in Chapter 3. The aim is to obtain the $L_1 \to C^\nu$ estimate for the semigroup. That gives the existence of the kernel and the Hölder continuity for the kernel. We cover the Davies perturbation in Section 4.1. This is the main tool for us to obtain the desired Gaussian bounds. We give formulae and estimates with respect to the perturbed operator in this section. For $C^{1+\kappa}$ domains we prove $L_1 \to C^\kappa$ bounds for the derivatives of the semigroup, which then imply the differentiability of the kernel and Hölder continuity of the derivative.

Section 4.2 gives the kernel estimates for the $C^{1+\kappa}$ case. We use that the semigroup maps into $C^\alpha(\Omega)$ for all $\alpha \in (0,1)$ to show that the kernel exists, and satisfies Gaussian bounds and Hölder Gaussian bounds for any Hölder exponent $\alpha \in (0,1)$.

We show in Section 4.3 for $C^{1+\kappa}$-domains that the kernel obtained in Section 4.2 is differentiable and the derivative is κ-Hölder continuous, which is the best possible.

Sections 4.4 and 4.5 give kernel estimates for the Lipschitz and C^1 cases respectively.

Chapter 5 covers some consequences of the kernel estimate. With the heat kernel estimates there are a number of properties that we can give without too much effort.

In Section 5.1, we first prove that the semigroup admits $L_p \to L_q$ bounds for all $1 \le p \le q \le \infty$. They also satisfy $L_p \to C^\nu$ bounds for appropriate Hölder exponents. For the $C^{1+\kappa}$ case, the derivative of the semigroup also satisfies similar bounds. Moreover, we show that the semigroup is holomorphic in $L_p(\Omega)$ for any $p \in [1,\infty)$ with similar kernel bounds that extend to the sector.

We then turn into irreducibility. We first show that the semigroup restricted to $C(\overline{\Omega})$ is a holomorphic C_0-semigroup. This can be used to show that the semigroup is irreducible.

Section 5.2 gives Gaussian lower bounds for the kernel of the semigroup on a Lipschitz domain, where all coefficients are real and the operator is self-adjoint. These assumptions allow the use of [ER98] to obtain a local lower bound over small time. In order to extend that into a global bound we prove that Lipschitz domains satisfy a so-called chain condition, then the bound can be extended to a Gaussian lower bound following arguments in [Ouh05] or [Cou03].

In **Chapter 6** we investigate the semigroup generated by the Dirichlet-to-Neumann operator. This is part of [EW19] as an application of the regularity estimates obtained in Section 3.4. The link is to estimate functions on the boundary by their harmonic liftings. At the end of this chapter we give L_p to L_q bounds and Hölder bounds for the semigroup generated by minus the Dirichlet-to-Neumann operator.

If in addition the principal coefficients are symmetric and Hölder continuous, the first-order terms vanish, the constant term is real and the domain is of class $C^{1+\kappa}$, then we combine with the Poisson bounds from [EO19] to give Hölder Poisson bounds for the kernel of the semigroup generated by minus the Dirichlet-to-Neumann operator.

1.3 Methods and contribution

In this section, we give a brief summary on the the techniques we use and the references to these techniques.

The elliptic operators and semigroups are studies by the form method following the work of Kato ([Kat80]) and Lions ([DL92]). We show that the elliptic operators are m-sectorial such that they generates C_0-semigroups.

The technique of using Morrey and Campanato norms to deduce Hölder Gaussian kernel bounds originated from [Aus96]. We apply the technique of separating estimates close to the boundary and the interior by using pointwise Morrey and Campanato seminorms as in [ER15].

The main difference between [ER15] and the thesis is the boundary condition assumed. In [ER15] the operator was subject to mixed (Dirichlet and Neumann) boundary conditions. We use the same De Giorgi estimates of [ER15] on Lipschitz domains and operators with real measurable principal coefficients subject to Neumann boundary conditions.

We utilize the same pointwise Morrey and Campanato seminorm technique for the $C^{1+\kappa}$ case. The regularity results for the derivatives of the semigroup are similar to [EO19] where Dirichlet boundary condition was assumed. In [EO19], the De Giorgi estimates were known from [Cam65]. Instead we take the method of difference quotient to prove the required De Giorgi estimates. In our case with Neumann and Robin boundary conditions, the difference in boundary condition also caused much more delicate work in order to obtain the regularity estimates for the derivative of the semigroup of order above d. This resulted in the long proof of Proposition 3.25 (cf. [EO19] Proposition 3.8).

The C^1 case is much easier because no extra De Giorgi estimates are needed. It follows as in the $C^{1+\kappa}$ case by replacing the Hölder estimates by the modulus of continuity. The technique is also observed in [Aus96].

We refer to various sources for the results in Chapter 5. The semigroup in other L_p spaces and the kernel bounds on the sector follow from [ER98]. The irreducibility follows from [Are06] and [Ouh05]. Finally, we use results from [ER98] to obtain a local lower bound, then it can be extended to a global lower bound following the proof of [CK15].

The key of Chapter 6 is to relate functions on $\partial\Omega$ with functions over Ω. We define the Dirichlet-to-Neumann operator with respect to a general elliptic second-order operator on Ω with real principal coefficients. We first prove that every function in $\operatorname{Tr} W^{1,2}(\Omega)$ has a harmonic lifting in $W^{1,2}(\Omega)$. Using [AE12], we can prove that the Dirichlet-to-Neumann operator is m-sectorial. It also gives estimates between the boundary functions and their harmonic liftings. Regularity results in Chapter 3 can then be applied on the representatives to obtain the desired estimates.

Chapter 2

Preliminaries

2.1 Elliptic operators

We introduce in this section the basic notations we need. We define elliptic operators with complex measurable coefficients in this section, but depending on the context in later chapters, some coefficients may only be real instead. Firstly we give a brief definition on semigroups as in [ABHN01]. For the rest of the thesis, we always assume $d \geq 2$.

Definition 2.1. Let X be a Banach space. A **semigroup** $(S_t)_{t>0}$ is a linear operator mapping from $(0, \infty)$ into $B(X)$ such that $S_{t+s} = S_t S_s$ for all $t, s > 0$. If in addition that $\lim_{t \to 0^+} \|S_t x - x\|_X = 0$ for all $x \in X$, then it is called a C_0**-semigroup** .

Let $\theta \in (0, \pi/2]$, define the (closed) sector $\Sigma_\theta = \{r\, e^{i\alpha} \mid r \geq 0 \text{ and } |\alpha| \leq \theta\}$. If X is a complex Banach space, we call $(S_t)_{t>0}$ a **holomorphic semigroup** of angle θ if it admits a holomorphic extension to Σ_θ, which we also identify by $(S_z)_{z \in \Sigma_\theta}$. It is a holomorphic C_0-semigroup if additionally that for any $\theta' \in (0, \theta)$, $\lim_{z \to 0, z \in \Sigma_{\theta'}} \|S_z x - x\|_X = 0$ for all $x \in X$.

For any set $U \subset \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $r > 0$, define $U(x, r) = U \cap B(x, r)$. Now let $\Omega \subset \mathbb{R}^d$ be open and define $\Gamma = \partial\Omega$. Let $\mu, M > 0$. We let $\mathcal{E}_p(\Omega, \mu, M)$ be the set of all complex measurable $C \colon \Omega \to \mathbb{C}^{d \times d}$ such that

$$\operatorname{Re} \sum_{k,l=1}^{d} c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geq \mu\, |\xi|^2$$

and

$$\|C(x)\| \leq M$$

for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$, where $\|C(x)\|$ is the ℓ_2-norm of $C(x)$ in $\mathbb{C}^d$. The subscript p is to denote the principal part of the coefficients of the elliptic operator which we will define soon. Let $\mathcal{E}_p(\Omega) = \bigcup_{\mu, M > 0} \mathcal{E}_p(\Omega, \mu, M)$. If $C \in \mathcal{E}_p(\Omega)$, then define the form $\mathfrak{a}_p \colon W^{1,2}(\Omega) \times$

$W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}_p(u,v) = \int_\Omega \sum_{k,l=1}^d c_{kl}\,(\partial_k u)\,\overline{\partial_l v},$$

where $c_{kl}(x)$ is the appropriate matrix coefficient of $C(x)$.

Next, let $\kappa \in (0,1)$. Define $|||\cdot|||_{C^\kappa(\Omega)} \colon C(\Omega) \to [0,\infty]$ by

$$|||u|||_{C^\kappa(\Omega)} = \sup_{\substack{x,y\in\Omega \\ 0<|x-y|\le 1}} \frac{|u(x)-u(y)|}{|x-y|^\kappa}.$$

Let $C^\kappa(\Omega) = \{u \in C(\Omega) : |||u|||_{C^\kappa(\Omega)} < \infty\}$ be the space of Hölder continuous functions of order κ. Similarly we define $|||\cdot|||_{C^\kappa(\Gamma)} \colon C(\Gamma) \to [0,\infty]$ by

$$|||\varphi|||_{C^\kappa(\Gamma)} = \sup_{\substack{x,y\in\Gamma \\ 0<|x-y|\le 1}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^\kappa}.$$

and let $C^\kappa(\Gamma) = \{\varphi \in C(\Gamma) : |||\varphi|||_{C^\kappa(\Gamma)} < \infty\}$. Let $\mathcal{E}_p^\kappa(\Omega,\mu,M)$ be the set of all continuous $C \colon \Omega \to \mathbb{C}^{d\times d}$ such that $C \in \mathcal{E}_p(\Omega,\mu,M)$ and $|||c_{kl}|||_{C^\kappa(\Omega)} \le M$ for all $k,l \in \{1,\dots,d\}$.

Now we add lower-order terms and the boundary term. Let $\mathcal{E}(\Omega,\mu,M)$ to be the set of all quintuples (C,a,b,a_0,β), where $C \in \mathcal{E}_p(\Omega,\mu,M)$, $a_k, b_k \colon \Omega \to \mathbb{C}^d$, $a_0 \colon \Omega \to \mathbb{C}$ and $\beta \colon \Gamma \to \mathbb{C}$, all measurable, with $\|a(x)\|, \|b(x)\|, |a_0(x)| \le M$ for all $x \in \Omega$ and $|\beta(z)| \le M$ for all $z \in \Gamma$. Similarly we define $\mathcal{E}(\Omega) = \bigcup_{\mu,M>0} \mathcal{E}(\Omega,\mu,M)$.

Let $\mathcal{E}^\kappa(\Omega,\mu,M)$ be quintuples $(C,a,b,a_0,\beta) \in \mathcal{E}(\Omega,\mu,M)$ that satisfies the Hölder bounds $|||c_{kl}|||_{C^\kappa(\Omega)}, |||a_k|||_{C^\kappa(\Omega)}, |||b_k|||_{C^\kappa(\Omega)}, |||\beta|||_{C^\kappa(\Gamma)} \le M$ for all $k,l \in \{1,\dots,d\}$.

Now let Ω be open with Lipschitz boundary Γ. For all $(C,a,b,a_0,\beta) \in \mathcal{E}(\Omega)$ define the forms $\mathfrak{a}, \mathfrak{a}_\beta \colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = \int_\Omega \sum_{k,l=1}^d c_{kl}\,(\partial_k u)\,\overline{\partial_l v} + \int_\Omega \sum_{k=1}^d a_k\,(\partial_k u)\,\overline{v} + \int_\Omega \sum_{l=1}^d b_l\,u\,\overline{\partial_l v} + \int_\Omega a_0\,u\,\overline{v}, \tag{2.1}$$

$$\mathfrak{a}_\beta(u,v) = \mathfrak{a}(u,v) + \int_\Gamma \beta\,(\operatorname{Tr} u)\,\overline{\operatorname{Tr} v}. \tag{2.2}$$

Let A be the operator associated to $\mathfrak{a}_\beta$. That is, $u \in \operatorname{dom}(A) \subset W^{1,2}(\Omega)$ if there exists an $f \in L_2(\Omega)$ such that $\mathfrak{a}_\beta(u,v) = (f,v)_{L_2(\Omega)}$ for all $v \in W^{1,2}(\Omega)$. Define then $Au = f$. We also define A_p to be the operator associated with $\mathfrak{a}_p$.

Define the operator $\mathcal{A} \colon W^{1,2}(\Omega) \to (W_0^{1,2}(\Omega))^*$ by

$$\langle \mathcal{A}u, v\rangle_{(W_0^{1,2}(\Omega))^* \times W_0^{1,2}(\Omega)} = \mathfrak{a}(u,v).$$

Let $u \in W^{1,2}(\Omega)$ and suppose that $\mathcal{A}u \in L_2(\Omega)$. Let $\psi \in L_2(\Gamma)$. Then we say that ψ is a **weak co-normal derivative of** u if

$$\mathfrak{a}(u,v) - (\mathcal{A}u, v)_{L_2(\Omega)} = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in W^{1,2}(\Omega)$. Then ψ is unique by the Stone–Weierstraß theorem and we write $\partial_\nu u = \psi$. If u and Ω are smooth enough, then $\partial_\nu u = \sum_{k,l=1}^d \nu_k \, c_{lk} \, \partial_l u + \sum_{k=1}^d \nu_k \, b_k \, u$, where ν is the outward normal vector. We now show that A realizes the elliptic operator with the Robin boundary condition $\partial_\nu u + \beta \operatorname{Tr} u = 0$.

Lemma 2.2. *$D(A) = \{u \in W^{1,2}(\Omega) : \mathcal{A}u \in L_2(\Omega)$ and $\partial_\nu u + \beta \operatorname{Tr} u = 0\}$. If $u \in D(A)$, then $Au = \mathcal{A}u$.*

Proof. ($\Rightarrow$) Suppose $u \in D(A)$. Let $f \in L_2(\Omega)$ be such that $Au = f$. Now for all $v \in W_0^{1,2}(\Omega)$, since the boundary term vanishes, we have

$$(f, v)_{L_2(\Omega)} = \mathfrak{a}_\beta(u, v) = \mathfrak{a}(u, v) = \langle \mathcal{A}u, v \rangle_{(W_0^{1,2}(\Omega))^* \times W_0^{1,2}(\Omega)}.$$

Therefore $\mathcal{A}u = f \in L_2(\Omega)$. We calculate the weak co-normal derivative

$$\begin{aligned} \mathfrak{a}(u, v) - (\mathcal{A}u, v)_{L_2(\Omega)} &= \mathfrak{a}_\beta(u, v) - (\mathcal{A}u, v)_{L_2(\Omega)} - \int_\Gamma \beta \operatorname{Tr} u \, \overline{\operatorname{Tr} v} \\ &= (f, v)_{L_2(\Omega)} - (f, v)_{L_2(\Omega)} - \int_\Gamma \beta \operatorname{Tr} u \, \overline{\operatorname{Tr} v} \\ &= (-\beta \operatorname{Tr} u, \operatorname{Tr} v)_{L_2(\Gamma)} \end{aligned}$$

for all $v \in W^{1,2}(\Omega)$. Therefore the weak co-normal derivative of u exists with $\partial_\nu u = -\beta \operatorname{Tr} u$, that gives $\partial_\nu u + \beta \operatorname{Tr} u = 0$.

($\Leftarrow$) Suppose $\mathcal{A}u = f \in L_2(\Omega)$ and $\partial_\nu u = -\beta \operatorname{Tr} u$. Then $\langle \mathcal{A}u, v \rangle_{(W_0^{1,2}(\Omega))^* \times W_0^{1,2}(\Omega)} = (f, v)_{L_2(\Omega)}$ for all $v \in W_0^{1,2}(\Omega)$. Equating the weak co-normal derivative we have

$$\mathfrak{a}(u, v) - (\mathcal{A}u, v)_{L_2(\Omega)} = (\partial_\nu u, \operatorname{Tr} v)_{L_2(\Gamma)} = \int_\Gamma -\beta \operatorname{Tr} u \, \overline{\operatorname{Tr} v}$$

for all $v \in W^{1,2}(\Omega)$. That gives $(\mathcal{A}u, v) = \mathfrak{a}_\beta(u, v)$ for all $v \in W^{1,2}(\Omega)$, so $u \in D(A)$ with $Au = \mathcal{A}u$. □

Now we show that A is m-sectorial and that $-A$ generates a holomorphic C_0-semigroup by classic results. First we need a few definitions. We say a sesquilinear form $\mathfrak{a} : V \times V \to \mathbb{C}$ is **bounded** if there exists a $M > 0$ such that $|\mathfrak{a}(u, v)| \leq M \|u\|_V \|v\|_V$ for all $u, v \in V$.

Definition 2.3. Let V, H be Hilbert spaces. Let $j \colon V \to H$ be a bounded linear operator such that $j(V)$ is dense in H. Let $\mathfrak{a} \colon V \times V \to \mathbb{C}$ be a bounded sesquilinear form. We say $\mathfrak{a}$ is **j-elliptic** if there exist $\omega \in \mathbb{R}$ and $c > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u) + \omega \|j(u)\|_H^2 \geq c \|u\|_V^2$$

for all $u \in V$. If j is the natural embedding then we may drop the j and call the form **elliptic**.

The definition is taken from [AE12], that j does not have to be the inclusion map. In Chapter 6, we consider the embedding $\mathrm{Tr} : W^{1,2}(\Omega) \to L_2(\Gamma)$ in order to investigate semigroups acting on functions over Γ.

Definition 2.4. Let H be a Hilbert space and $\theta \in [0, \pi/2)$. An operator $A\colon D(A) \to H$, where $D(A) \subset H$, is called **sectorial** (with angle θ) if there exists a vertex $\omega \in \mathbb{R}$ such that $(Au, u)_H - \omega\|u\|_H^2 \in \Sigma_\theta$ for all $u \in D(A)$. Furthermore, A is called **m-sectorial** if in addition $A - (\omega - 1)I$ is surjective.

We now prove ellipticity.

Proposition 2.5. *The forms $\mathfrak{a}_p, \mathfrak{a}$ and $\mathfrak{a}_\beta$ are bounded and elliptic.*

Proof. Take the natural embedding $j : W^{1,2}(\Omega) \hookrightarrow L_2(\Omega)$ here. The boundedness of the form is clear. It suffices to prove the ellipticity of $\mathfrak{a}_\beta$. The second order terms can easily be estimated by ellipticity of the coefficients:

$$\mathrm{Re} \int_\Omega \sum_{k,l=1}^{d} c_{kl}\,(\partial_k u)\,\overline{\partial_l u} \geq \mu \int_\Omega |\nabla u|^2.$$

The first order terms can be estimated by

$$\begin{aligned}\left|\mathrm{Re} \int_\Omega \sum_{k=1}^{d} b_k\, u\, \overline{\partial_k u}\right| &\leq \sum_{k=1}^{d} \int_\Omega |b_k|\,|u|\,|\partial_k u| \\ &\leq d\,M \|u\|_{L_2(\Omega)} \|\nabla u\|_{L_2(\Omega)} \\ &\leq d\,M \left(\varepsilon \|\nabla u\|^2_{L_2(\Omega)} + \frac{1}{4\varepsilon}\|u\|^2_{L_2(\Omega)}\right)\end{aligned}$$

for all $\varepsilon > 0$. Putting $\varepsilon = \mu(4d\,M)^{-1}$ gives

$$\left|\mathrm{Re} \int_\Omega \sum_{k=1}^{d} b_k\, u\, \overline{\partial_k u}\right| \leq \frac{\mu}{4}\|\nabla u\|^2_{L_2(\Omega)} + \frac{d^2 M^2}{\mu}\|u\|^2_{L_2(\Omega)}.$$

The estimate the other first order term is identical. The zeroth order term is clear since a_0 is bounded. Finally, we use Lemma 2.3 of [AM07] that there exists a $c > 0$ such that

$$\|\mathrm{Tr}\, u\|^2_{L_2(\Gamma)} \leq \frac{\mu}{4M}\|\nabla u\|^2_{L_2(\Omega)} + c\|u\|^2_{L_2(\Omega)} \tag{2.3}$$

for all $u \in W^{1,2}(\Omega)$. Therefore

$$\left|\int_\Gamma \beta |\mathrm{Tr}\, u|^2\right| \leq M \int_\Gamma |\mathrm{Tr}\, u|^2 \leq \frac{\mu}{4}\|\nabla u\|^2_{L_2(\Omega)} + c\,M\|u\|^2_{L_2(\Omega)}$$

for all $u \in W^{1,2}(\Omega)$. Combining we have $\mathrm{Re}\, \mathfrak{a}_\beta(u) \geq \frac{\mu}{4}\|\nabla u\|^2_{L_2(\Omega)} - \omega\|u\|^2_{L_2(\Omega)}$, where $\omega = 2d^2M^2\mu^{-1} + (c+1)M$, which gives ellipticity of $\mathfrak{a}_\beta$. The ellipticity for $\mathfrak{a}_p$ and $\mathfrak{a}$ follows by setting the lower-order terms and β to zero. □

The operator $-A$ generates a holomorphic C_0-semigroup e^{-tA}, which we usually denote by S.

Proposition 2.6. *The operator A is m-sectorial with angle $\theta = \arctan \frac{M}{\mu}$. Moreover, $-A$ generates a C_0-semigroup which is holomorphic in the sector $\Sigma^\circ_{\frac{\pi}{2}-\theta}$.*

Proof. The operator A_p and A are m-sectorial by Theorem VI.2.1 of [Kat80]. The operator A_p is sectorial with angle $\theta = \arctan \frac{M}{\mu}$. To see this note that for all $u \in D(A)$:

$$\operatorname{Re}(\mathfrak{a}_p(u)) \geq \mu \,\|\nabla u\|^2_{L_2(\Omega)} = \frac{\mu}{M} M \,\|\nabla u\|^2_{L_2(\Omega)} \geq \frac{\mu}{M} |\operatorname{Im}(\mathfrak{a}_p(u))|.$$

Therefore $(A_p u, u)_{L_2(\Omega)} = \mathfrak{a}_p(u) \in \Sigma_\theta$ for all $u \in D(A)$.

Define the form $\mathfrak{b} = \mathfrak{a}_\beta - \mathfrak{a}_p$ be the lower terms. Now the factor $\frac{\mu}{4}$ in the proof of Proposition 2.5 can be changed to $\varepsilon\mu$ for any $\varepsilon \in (0, 1)$ (including the estimate in (2.3) using Lemma 2.3 of [AM07]). Therefore there exists a $c > 0$ such that

$$|\mathfrak{b}(u)| \leq \varepsilon \|\nabla u\|^2_{L_2(\Omega)} + c\,\varepsilon^{-1} \|u\|^2_{L_2(\Omega)}$$

for all $\varepsilon > 0$.

Let $\alpha \in (-(\frac{\pi}{2} - \theta), \frac{\pi}{2} - \theta)$, then $e^{i\alpha}\mathfrak{a}_p(u) \in \Sigma_{\theta+|\alpha|}$, and hence $|\operatorname{Im}(e^{i\alpha}\mathfrak{a}_p(u))| \leq \tan(\theta + |\alpha|) \operatorname{Re}(e^{i\alpha}\mathfrak{a}_p(u))$. Therefore we have

$$\mu \|\nabla u\|^2_{L_2(\Omega)} \leq \operatorname{Re} \mathfrak{a}_p(u) \leq |e^{i\alpha}\mathfrak{a}_p(u)| \leq (1 + \tan(\theta + |\alpha|)) \operatorname{Re}(e^{i\alpha}\mathfrak{a}_p(u)).$$

Combining the above we get

$$\begin{aligned} \frac{\mu}{1 + \tan(\theta + |\alpha|)} \|\nabla u\|^2_{L_2(\Omega)} &\leq \operatorname{Re}(e^{i\alpha}\mathfrak{a}_p(u)) \\ &\leq \operatorname{Re}(e^{i\alpha}\mathfrak{a}(u)) + |\mathfrak{b}(u)| \\ &\leq \operatorname{Re}(e^{i\alpha}\mathfrak{a}(u)) + \varepsilon \|\nabla u\|^2_{L_2(\Omega)} + c\,\varepsilon^{-1} \|u\|^2_{L_2(\Omega)} \end{aligned}$$

for all $\varepsilon > 0$. Choosing $\varepsilon = \mu(2(1 + \tan(\theta + |\alpha|)))^{-1}$ gives ellipticity of $e^{i\alpha}\mathfrak{a}$ which is associated with $e^{i\alpha}A$, so $-e^{i\alpha}A$ generates a C_0-semigroup. It follows from [Kat80] Theorem IX.1.24 that $-A$ generates a holomorphic C_0-semigroup of angle $\frac{\pi}{2} - \theta$. ☐

Lastly we also need the dual of the semigroup and its generator. There is a simple expression for the dual semigroup.

Lemma 2.7. *Let A be a m-sectorial operator associated with the form $\mathfrak{a} : V \times V \to \mathbb{C}$. Then the dual A^* is associated with the form $\mathfrak{a}^* : V \times V \to \mathbb{C}$, where $\mathfrak{a}^*(u, v) = \overline{\mathfrak{a}(v, u)}$.*

Proof. Replacing A by $A - \omega I$ for some $\omega > 0$ if necessary, we may assume that A is surjective. Let B be the operator associated with $\mathfrak{a}^*$, then without loss of generality B is also surjective. For all $u \in D(A)$ and $v \in D(B)$ we have

$$(Au, v)_H = \mathfrak{a}(u, v) = \overline{\mathfrak{a}^*(v, u)} = \overline{(Bv, u)_H} = (u, Bv)_H,$$

hence $B \subset A^*$.

On the other hand note that since A is surjective, A^* is injective. For each $u \in D(A^*)$ there exists $v \in D(B)$ such that $Bv = A^*u$, but since $B \subset A^*$ we have $A^*v = Bv = A^*u$. Therefore $u = v \in D(B)$ by injectivity, which proves $A^* = B$. □

Using the forms it is clear that if A is elliptic with parameters $(C, a, b, a_0, \beta) \in \mathcal{E}(\Omega, \mu, M)$, then A^* is also elliptic with parameters $(C^*, \overline{b}, \overline{a}, \overline{a_0}, \overline{\beta}) \in \mathcal{E}(\Omega, \mu, M)$. That allows us to obtain identical estimates for S_t^*.

In order to estimate the second-order terms we need the following definitions in order to investigate A_p in the weak sense we shall define below. If $N \subset \partial\Omega$ is relatively open in $\partial\Omega$, define

$$C_N^\infty(\Omega) = \{w|_\Omega : w \in C_c^\infty(\mathbb{R}^d) \text{ and } (\operatorname{supp} w) \cap (\partial\Omega \setminus N) = \varnothing\}.$$

We denote the closure of $C_N^\infty(\Omega)$ in $W^{1,2}(\Omega)$ by $W_N^{1,2}(\Omega)$. Roughly speaking, we impose Dirichlet boundary conditions on $\partial\Omega \setminus N$ and Neumann boundary conditions on N.

Lemma 2.8. *Let $V, \Omega \subset \mathbb{R}^d$ be open. Then $\{v|_{\Omega\cap V} : v \in C_c^\infty(V)\}$ is dense in $W^{1,2}_{V\cap\partial\Omega}(\Omega\cap V)$.*

Proof. By density it suffices to show that for all $w \in C^\infty_{\partial\Omega\cap V}(\Omega\cap V)$ there exists a $v \in C_c^\infty(V)$ such that $v|_{\Omega\cap V} = w|_{\Omega\cap V}$. Let $w \in C_c^\infty(\mathbb{R}^d)$ and suppose that

$$(\operatorname{supp} w) \cap (\partial(\Omega \cap V)) \setminus (\partial\Omega \cap V) = \varnothing. \tag{2.4}$$

The condition is to ensure that w is well-behaved at the intersection between the 'Dirichlet part' and the 'Neumann part' of the boundary $\partial\Omega$.

We first want to show that

$$V^c \cap (\operatorname{supp} w) \cap \overline{\Omega \cap V} = \varnothing. \tag{2.5}$$

Note that $\overline{\Omega \cap V} \cap V^c \subset \overline{\Omega \cap V} \cap (\Omega \cap V)^c = \partial(\Omega \cap V)$. Also since $V^c \subset (\partial\Omega \cap V)^c$ we have

$$V^c \cap \overline{\Omega \cap V} \subset (\partial(\Omega \cap V)) \setminus (\partial\Omega \cap V) \subset (\operatorname{supp} w)^c$$

by (2.4). This proves (2.5).

Since $(\operatorname{supp} w) \cap \overline{\Omega \cap V}$ is compact, there exists a $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi(x) = 1$ for all $x \in (\operatorname{supp} w) \cap \overline{\Omega \cap V}$ and $\operatorname{supp} \chi \subset V$. Then $w|_{\Omega\cap V} = (\chi w)|_{\Omega\cap V}$. Moreover, we have $\operatorname{supp}(\chi w) \subset V$, so $\chi w \in C_c^\infty(V)$. □

Let $\Omega \subset \mathbb{R}^d$ be open and let $C \in \mathcal{E}_p(\Omega)$. Let $U \subset \Omega$ be open and $u \in W^{1,2}(U)$. Furthermore let $V \subset \mathbb{R}^d$ be open. Then we say that $A_p u = 0$ **weakly on** V if

$$\int_{U\cap V} \sum_{k,l=1}^d c_{kl}\,(\partial_k u)\,\overline{\partial_l v} = 0 \tag{2.6}$$

for all $v \in W^{1,2}_{\partial\Omega\cap V}(\Omega \cap V)$. It follows from Lemma 2.8 that $A_p u = 0$ weakly on V if and only if (2.6) is valid for all $v \in C_c^\infty(V)$. Hence by density $A_p u = 0$ weakly on V if and only if (2.6) is valid for all $v \in W_0^{1,2}(V)$.

2.2 Morrey and Campanato spaces

In this section, we introduce the (classical) Morrey and Campanato spaces and their basic properties. The Morrey and Campanato spaces can be defined over all L_p-spaces where $p \in [1, \infty)$, but we will focus on the case $p = 2$.

Definition 2.9. Let $u \colon \Omega \to \mathbb{C}$ be a measurable function and $\gamma \geq 0$. Define the **Morrey norm** by

$$\|u\|_{M_\gamma(\Omega)}^2 = \sup_{x_0 \in \Omega, r \in (0,1]} r^{-\gamma} \int_{\Omega(x_0,r)} |u|^2 \in [0, \infty].$$

We then define the **Morrey space** by $M_\gamma(\Omega) = \big\{u \colon \Omega \to \mathbb{C} \text{ measurable} \mid \|u\|_{M_\gamma(\Omega)} < \infty\big\}$. Call γ the **order** of the Morrey norm/space.

By a similar proof as for the completeness of the $L_2(\Omega)$ we see that $M_\gamma(\Omega)$ is a Banach space. By considering the case $r = 1$ and by boundedness, it is also clear that $M_\gamma(\Omega) \subset L_2(\Omega)$. Note that it is not a subspace of $L_2(\Omega)$ when Ω would be unbounded. One can even show that $M_\gamma(\Omega) \cap L_2(\Omega)$ is not complete in such case.

Example 2.10. Consider $\Omega = (1, \infty)$ and the sequence of functions $u_n(x) = x^{-1/2}\chi_{(1,n+1)}(x)$. Then $u_n \in M_0(\Omega)$ for all $n \in \mathbb{N}$. For any $m, n \in \mathbb{N}$ with $m \geq n$, we have $\|u_m - u_n\|_{M_0(\Omega)} \leq \int_n^{n+2} \frac{1}{x}\,dx$ so that (u_n) is Cauchy. However it converges pointwisely to $x \mapsto x^{-1/2}$ which is not in $L_2(\Omega)$. Note that this example applies to all $\gamma \in [0, 1]$.

We give a few immediate properties.

Proposition 2.11. *Let $\Omega \subset \mathbb{R}^d$ be open bounded. Then the following are true:*

1. *$M_0(\Omega) \approx L_2(\Omega)$.*
2. *$M_d(\Omega) \approx L_\infty(\Omega)$.*
3. *$M_\gamma(\Omega)$ is trivial for all $\gamma > d$.*

Proof. Denote $\omega_d = |B_1(0)|$ the volume of the unit ball in $\mathbb{R}^d$. Let R be the diameter of Ω.

1. Suppose $u \in M_0(\Omega)$. Clearly $\|u\|_{L_2(\Omega)} \leq R^d \|u\|_{M_0(\Omega)}$. The opposite inclusion is obvious.

2. Suppose $u \in L_\infty(\Omega)$. Let $r > 0$ and $x_0 \in \Omega$. We have $r^{-d} \int_{\Omega(x_0,r)} |u|^2 \leq \omega_d \|u\|_\infty^2$ so $\|u\|_{M_d(\Omega)} \leq \omega_d^{1/2} \|u\|_\infty$.

   We use Hölder inequality for the opposite inclusion. Let $u \in M_d(\Omega)$. For each Lebesgue point $x_0 \in \Omega$ for u in Ω by the Hölder inequality

$$|u(x_0)| = \lim_{r \to 0} \frac{1}{|\Omega(x_0,r)|} \int_{\Omega(x_0,r)} |u| \leq \lim_{r \to 0} \frac{1}{|\Omega(x_0,r)|} \Big(\int_{\Omega(x_0,r)} |u|^2\Big)^{1/2} \Big(\int_{\Omega(x_0,r)} 1\Big)^{1/2}$$

$$= \lim_{r \to 0} \frac{r^{d/2}}{|\Omega(x_0,r)|^{1/2}} \Big(r^{-d} \int_{\Omega(x_0,r)} |u|^2\Big)^{1/2} = \omega_d^{-1/2} \Big(r^{-d} \int_{\Omega(x_0,r)} |u|^2\Big)^{1/2}. \quad (2.7)$$

Since the above holds a.e., we have $\|u\|_\infty \leq \omega_d^{-1/2}\|u\|_{M_d(\Omega)}$. Note that we have also get the equality $\omega_d^{1/2}\|u\|_\infty = \|u\|_{M_d(\Omega)}$.

3. Suppose $u \neq 0$. Choose a Lebesgue point $x_0 \in \Omega$ such that $|u(x_0)| > 0$. Let $R_0 \in (0, 1]$ be such that $B_{R_0}(x_0) \subset \Omega$. For all $r \in (0, R_0)$ we have

$$\begin{aligned}\left(r^{-\gamma}\int_{B_r(x_0)}|u|^2\right)^{1/2} &= \frac{r^{-(d+\gamma)/2}}{\omega_d^{1/2}}\left(\int_{B_r(x_0)}|u|^2\right)^{1/2}\left(\int_{B_r(x_0)}1\right)^{1/2} \\ &\geq \frac{r^{-(d+\gamma)/2}}{\omega_d^{1/2}}\int_{B_r(x_0)}|u| \\ &= \omega_d^{1/2}r^{(d-\gamma)/2}\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|u|\end{aligned}$$

which diverges since $|B_r(x_0)|^{-1}\int_{B_r(x_0)}|u|$ is bounded below as $r \to 0$.

□

In a similar fashion we define the Campanato space. We need to first define the function average.

Definition 2.12. Let X be a measurable set with $0 < |X| < \infty$ and $u \in L_2(X)$. Define the **average** of u over X by

$$\langle u\rangle_X = \frac{1}{|X|}\int_X u.$$

Definition 2.13. Let $u \in L_2(\Omega)$ and $\gamma \geq 0$. Define the **Campanato seminorm** by

$$|||u|||^2_{\mathcal{M}_\gamma(\Omega)} = \sup_{x_0\in\Omega, r\in(0,1]} r^{-\gamma}\int_{\Omega(x_0,r)}|u - \langle u\rangle_{\Omega(x_0,r)}|^2 \in [0, \infty].$$

We then define the **Campanato space** by $\mathcal{M}_\gamma(\Omega) = \{u \in L_2(\Omega) \mid |||u|||_{\mathcal{M}_\gamma(\Omega)} < \infty\}$, where $\|\cdot\|_{\mathcal{M}_\gamma(\Omega)} = |||\cdot|||_{\mathcal{M}_\gamma(\Omega)} + \|\cdot\|_{L_2(\Omega)}$. Call γ again the **order** of the Campanato norm/space.

By the boundedness of Ω, if $|||u|||_{\mathcal{M}_\gamma(\Omega)} < \infty$ then $u \in L_2(\Omega)$, so the above definition is well-defined. On the other hand, note that $|||u|||_{\mathcal{M}_\gamma(\Omega)} = 0$ for any constant function u, so $|||\cdot|||_{\mathcal{M}_\gamma(\Omega)}$ is not a norm, but $\|\cdot\|_{\mathcal{M}_\gamma(\Omega)}$ is. It can also be shown that this is a Banach space.

Note that for any measurable set $X \subset \mathbb{R}^d$ with finite non-zero measure and $u \in L^2(X)$, we have

$$\int_X |u - \lambda|^2 \geq \int_X |u - \langle u\rangle_X|^2$$

for any $\lambda \in \mathbb{C}$. Therefore $M_\gamma(\Omega) \subset \mathcal{M}_\gamma(\Omega)$. To obtain the reverse relation, we need an extra condition called the inner volume condition so that the average would behave properly near the boundary.

Definition 2.14. Let $\Omega \subset \mathbb{R}^d$ be open. The set Ω is said to satisfy the **inner volume condition** if there exists $c, R_0 > 0$ so that $|\Omega(x,r)| \geq c\, r^d$ for all $x \in \Omega$ and $r \in (0, R_0]$.

For example, an outward cusp fails the inner volume condition.

We can now characterize Campanato spaces. These results can be found similarly in Theorem 2.1.2 of [Gia83] or Proposition 5.4 of [GM05].

Proposition 2.15. *Suppose $\Omega \subset \mathbb{R}^d$ is open bounded and satisfies the inner volume condition. Then $M_\gamma(\Omega) \cap L_2(\Omega) \approx \mathcal{M}_\gamma(\Omega)$ for all $\gamma \in [0, d)$.*

Proof. It suffices to show that $\mathcal{M}_\gamma(\Omega) \subset M_\gamma(\Omega) \cap L_2(\Omega)$. Let $c_0 > 0$ and $R_0 \in (0, 1]$ be such that $|\Omega(x,r)| \geq c_0 r^d$ for all $x \in \Omega$ and $r \in (0, R_0]$. Now we expand the Morrey norm:

$$\begin{aligned} r^{-\gamma} \int_{\Omega(x_0,r)} |u|^2 &\leq r^{-\gamma} \int_{\Omega(x_0,r)} (2|u - \langle u \rangle_{\Omega(x_0,r)}|^2 + 2|\langle u \rangle_{\Omega(x_0,r)}|^2) \\ &\leq 2|||u|||^2_{\mathcal{M}_\gamma(\Omega)} + 2\omega_d\, r^{d-\gamma} |\langle u \rangle_{\Omega(x_0,r)}|^2. \end{aligned} \tag{2.8}$$

To estimate uniformly for all x_0 and r, note that for all $0 < r \leq R \leq R_0$ we have

$$|\langle u \rangle_{\Omega(x_0,R)} - \langle u \rangle_{\Omega(x_0,r)}|^2 \leq 2(|u - \langle u \rangle_{\Omega(x_0,r)}|^2 + |u - \langle u \rangle_{\Omega(x_0,R)}|^2).$$

Integrating over $\Omega(x_0, r)$ gives

$$\begin{aligned} |\langle u \rangle_{\Omega(x_0,R)} - \langle u \rangle_{\Omega(x_0,r)}|^2 &\leq 2|\Omega(x_0,r)|^{-1} \Big(\int_{\Omega(x_0,r)} |u - \langle u \rangle_{\Omega(x_0,r)}|^2 + \int_{\Omega(x_0,r)} |u - \langle u \rangle_{\Omega(x_0,R)}|^2 \Big) \\ &\leq 2|\Omega(x_0,r)|^{-1} \Big(\int_{\Omega(x_0,r)} |u - \langle u \rangle_{\Omega(x_0,r)}|^2 + \int_{\Omega(x_0,R)} |u - \langle u \rangle_{\Omega(x_0,R)}|^2 \Big) \\ &\leq 4\, c_0^{-1}\, r^{-d}\, R^\gamma |||u|||^2_{\mathcal{M}_\gamma(\Omega)}. \end{aligned} \tag{2.9}$$

Therefore $|\langle u \rangle_{\Omega(x_0,R)} - \langle u \rangle_{\Omega(x_0,r)}| \leq c_1 r^{-d/2} R^{\gamma/2} |||u|||_{\mathcal{M}_\gamma(\Omega)}$, where $c_1 = 2c_0^{-1/2}$. For all $i \in \mathbb{N}_0$ put $R_i = 2^{-i} R$. Then we obtain

$$|\langle u \rangle_{\Omega(x_0,R_i)} - \langle u \rangle_{\Omega(x_0,R_{i+1})}| \leq c_1 R^{-\frac{d-\gamma}{2}} 2^{-i\frac{d-\gamma}{2} + \frac{d}{2}} |||u|||_{\mathcal{M}_\gamma(\Omega)}. \tag{2.10}$$

By taking the sum over i, we conclude that

$$|\langle u \rangle_{\Omega(x_0,R)} - \langle u \rangle_{\Omega(x_0,R_i)}| \leq c_2\, R_i^{-\frac{d-\gamma}{2}} |||u|||_{\mathcal{M}_\gamma(\Omega)}, \tag{2.11}$$

where $c_2 = c_1 2^{d/2} (1 - 2^{-(d-\gamma)/2})^{-1}$.

Finally, given any $r > 0$, choose appropriate $i \in \mathbb{N}$ and $R \in (\frac{R_0}{2}, R_0]$ so that $2^{-i} R = r$. Using Jensen's (or Hölder's) inequality we obtain

$$|\langle u \rangle_{\Omega(x_0,R)}|^2 \leq \frac{1}{|\Omega(x_0,R)|} \int_{\Omega(x_0,R)} |u|^2 \leq c_0^{-1} R^{-d} \|u\|_{L_2(\Omega)} \leq 2^d R_0^{-d} c_0^{-1} \|u\|^2_{L_2(\Omega)} \tag{2.12}$$

Using (2.11) and (2.12), the second term of (2.8) can be estimated by

$$\begin{aligned} r^{d-\gamma}|\langle u\rangle_{\Omega(x_0,r)}|^2 &\le r^{d-\gamma}|\langle u\rangle_{\Omega(x_0,R)}|^2 + r^{d-\gamma}|\langle u\rangle_{\Omega(x_0,R)} - \langle u\rangle_{\Omega(x_0,r)}|^2 \\ &\le 2^d R_0^{-d} c_0^{-1}\|u\|^2_{L_2(\Omega)} + c_2^2|||u|||^2_{\mathcal{M}_\gamma(\Omega)} \\ &\le (2^d R_0^{-d} c_0^{-1} + c_2^2)\|u\|^2_{\mathcal{M}_\gamma(\Omega)} \qquad (2.13) \end{aligned}$$

for all $r \in (0, R_0]$. Finally if $r \in [R_0, 1]$ then $r^{-\gamma}\int_{\Omega(x_0,r)}|u|^2 \le R_0^{-\gamma}\|u\|_{L_2(\Omega)} \le R_0^{-\gamma}\|u\|_{\mathcal{M}_\gamma(\Omega)}$ as desired. □

On the other hand, the Campanato space behaves differently when $\gamma > d$.

Proposition 2.16. *Suppose $\Omega \subset \mathbb{R}^d$ is open and satisfies the inner volume condition. If $\gamma \in (d, \infty)$ then $\mathcal{M}_\gamma(\Omega) \approx C^\alpha(\Omega)$, the space of α-Hölder continuous function, where $\alpha = \frac{\gamma-d}{2}$. In particular, when $\gamma > d+2$ then $\mathcal{M}_\gamma(\Omega)$ is the space of constant functions.*

Proof. Let $c_0, R_0 > 0$ be such that $|\Omega(x,r)| \ge c_0\, r^d$ for all $x \in \Omega$ and $r \in (0, R_0]$.

Suppose $u \in C^\alpha(\Omega)$, $x_0 \in \Omega$ and $r > 0$. For $x \in \Omega(x_0, r)$ we have $|u(x) - \langle u\rangle_{x,r}| \le (2r)^\alpha|||u|||_{C^\alpha(\Omega)}$, therefore by integration over $\Omega(x_0, r)$ we obtain

$$r^{-\gamma}\int_{\Omega(x_0,r)} |u - \langle u\rangle_{\Omega(x_0,r)}|^2 \le 2^\alpha\,\omega_d\, r^{d+2\alpha-\gamma}|||u|||_{C^\alpha(\Omega)} = 2^\alpha\,\omega_d\,|||u|||_{C^\alpha(\Omega)}.$$

So $u \in \mathcal{M}_\gamma(\Omega)$.

Assume $u \in \mathcal{M}_\gamma(\Omega)$. Taking summation on (2.10) again, we have for all $j > i$ that

$$|\langle u\rangle_{\Omega(x_0,2^{-i}R)} - \langle u\rangle_{\Omega(x_0,2^{-j}R)}| \le c_2\, 2^{-i\frac{\gamma-d}{2}}\, R^{\frac{\gamma-d}{2}}|||u|||_{\mathcal{M}_\gamma(\Omega)}, \qquad (2.14)$$

where $c_2 = c_1 2^{d/2}(1 - 2^{(\gamma-d)/2})^{-1}$. That says for each $x_0 \in \Omega$, $(\langle u\rangle_{\Omega(x_0,2^{-i}R)})_{i\in\mathbb{N}}$ is Cauchy. Denote $\tilde{u}(x_0)$ be its limit. Now if $r \in (2^{-(i+1)}R, 2^{-i}R]$ for some $i \in \mathbb{N}_0$, then

$$|\langle u\rangle_{\Omega(x_0,2^{-i}R)} - \langle u\rangle_{\Omega(x_0,r)}| \le 2^{\frac{d}{2}}\, c_1\, 2^{-i\frac{\gamma-d}{2}}\, R^{\frac{\gamma-d}{2}}|||u|||_{\mathcal{M}_\gamma(\Omega)}$$

by (2.9). Therefore $\lim_{r\to 0}|\langle u\rangle_{\Omega(x_0,r)}| = \tilde{u}(x)$. On the other hand, $\langle u\rangle_{\Omega(x,r)} \to u(x)$ in L^1, hence we have $u = \tilde{u}$ a.e.. By setting $i = 0$ and $j \to \infty$ on (2.14), we have

$$|\langle u\rangle_{\Omega(x,R)} - \tilde{u}(x)| \le c_2\, R^{\frac{\gamma-d}{2}}|||u|||_{\mathcal{M}_\gamma(\Omega)} = c_2\, R^\alpha|||u|||_{\mathcal{M}_\gamma(\Omega)} \qquad (2.15)$$

for all $x \in \Omega$ and $R \in (0, R_0]$. Therefore $\langle u\rangle_{\Omega(x,R)} \to \tilde{u}$ uniformly as $R \to 0$. Since $u \in L_2(\Omega)$, the function $x \mapsto \langle u\rangle_{\Omega(x,R)}$ is continuous, hence $\tilde{u}$ is continuous.

We now show that $\tilde{u}$ is α-Hölder continuous. Let $x, y \in \Omega$ and $R = d(x,y)$. Write

$$|\tilde{u}(x) - \tilde{u}(y)| \le |\tilde{u}(x) - \langle u\rangle_{\Omega(x,2R)}| + |\langle u\rangle_{\Omega(y,2R)} - \langle u\rangle_{\Omega(x,2R)}| + |\tilde{u}(y) - \langle u\rangle_{\Omega(y,2R)}|.$$

The estimates for the first and the third term follow immediately from (2.15). For the middle term write $|\langle u\rangle_{\Omega(y,2R)} - \langle u\rangle_{\Omega(x,2R)}| \le |\langle u\rangle_{\Omega(y,2R)} - u| + |u - \langle u\rangle_{\Omega(x,2R)}|$ and integrate over $U = \Omega(x, 2R) \cap \Omega(y, 2R)$. Since $\Omega(x, R) \subset U$, we have by inner volume condition that $|U| \geq c\, R^d$ for some $c > 0$. Therefore

$$|\langle u\rangle_{\Omega(y,2R)} - \langle u\rangle_{\Omega(x,2R)}| \leq \frac{1}{|U|}\Big(\int_U |\langle u\rangle_{\Omega(y,2R)} - u| + \int_U |u - \langle u\rangle_{\Omega(x,2R)}|\Big). \tag{2.16}$$

By Hölder inequality, we estimate the integral as follows:

$$\begin{aligned}\frac{1}{|U|}\int_U |\langle u\rangle_{\Omega(x,2R)} - u| &\leq c^{-1}\, R^{-d}\Big(\int_U |\langle u\rangle_{\Omega(x,2R)} - u|^2\Big)^{1/2} |\Omega(x, 2R)|^{1/2}\\ &\leq c^{-1}\, R^{-d}\, R^{\gamma/2} |||u|||_{\mathcal{M}_\gamma(\Omega)} \omega_d^{1/2} (2R)^{d/2}\\ &\leq 2^{d/2}\, c^{-1}\, \omega_d^{1/2} R^{\alpha} |||u|||_{\mathcal{M}_\gamma(\Omega)}.\end{aligned}$$

The same applies to the other term so the claim follows.

Note that following from the above proof, if $u \in \mathcal{M}_\gamma(\Omega)$ with $\gamma > d + 2$ then $\tilde{u}$ is α-Hölder continuous where $\alpha > 1$, so $\tilde{u}$ must be constant. □

The Campanato space $\mathcal{M}_d(\Omega)$ is different from the Morrey space $M_d(\Omega)$. For example, it was mentioned in [GM05] that $\mathcal{M}_d(\Omega)$ is the space of bounded mean oscillation (BMO) when Ω is a cube.

It follows that the Morrey and the Campanato spaces distinguish themselves for order $\gamma \in [d, d + 2]$. In Chapters 3 and 4, we will first focus on obtaining Morrey estimates of order below d. After that we will obtain Campanato estimates of order $d + 2\kappa \in (d, d + 2)$ for some $\kappa \in (0, 1)$, which gives the desired Hölder continuity.

There are problems when one tries to apply the above theory when Ω is unbounded, since $\mathcal{M}_\gamma$ cannot be treated as a subspace of $L_2(\Omega)$ any more. On the other hand, the uniformity of estimates are often hard to obtain near the boundary. Also the inner volume condition is not guaranteed by the given smoothness of the boundary (it may not even by Lipschitz). In Section 2.3, we would like to define a pointwise version of the Morrey and Campanato norms with similar properties.

2.3 Pointwise Morrey and Campanato seminorm

As we have seen in the last section, the notion of Morrey and Campanato space is problematic when Ω is unbounded or the boundary is not smooth enough. Here we would like to define a pointwise version of the Morrey and Campanato norm. Using pointwise Morrey and Campanato seminorms we can handle estimates for points in the interior and points near to the boundary separately.

Although the domains are at least Lipschitz in this thesis where inner volume condition holds, the use of pointwise Morrey and Campanato seminorms allows estimation for some unbounded Lipschitz domains (see Remark 4.12).

Definition 2.17. Let $\Omega \subset \mathbb{R}^d$ be open and $R_e \in (0,1]$. Let $u \in L_{2,loc}(\Omega)$ and $\gamma \geq 0$. For each $x \in \Omega$ we define the **pointwise Morrey seminorm** by

$$\|u\|_{M,\gamma,x,\Omega,R_e} = \sup_{r\in(0,R_e]} r^{-\gamma} \int_{\Omega(x,r)} |u|^2 \in [0,\infty].$$

Similarly, we define the **pointwise Campanato seminorm** by

$$\|u\|_{\mathcal{M},\gamma,x,\Omega,R_e} = \sup_{r\in(0,R_e]} r^{-\gamma} \int_{\Omega(x,r)} |u - \langle u\rangle_{\Omega(x,r)}|^2 \in [0,\infty].$$

Clearly if Ω is bounded, the ordinary Morrey norm is equivalent to the supremum of the pointwise norms with $R_e = 1$.

The motivation of using pointwise Morrey and Campanato seminorm is to obtain properties similar to Proposition 2.15 and Proposition 2.16 without the assumption of boundedness and inner volume condition uniformly over Ω. This method is used in [ER15] and also in [EW19].

We illustrate below properties similar to the ordinary Morrey and Campanato norms. First we give the equivalence between the two pointwise seminorms when $\gamma \in [0,d)$.

Proposition 2.18. *For all open $\Omega \subset \mathbb{R}^d$, $\gamma \in [0,d)$, $c > 0$ and $R_e \in (0,1]$ there exist $c_1, c_2 > 0$ such that*

$$\|u\|^2_{\mathcal{M},\gamma,x,\Omega,R_e} \leq \|u\|^2_{M,\gamma,x,\Omega,R_e} \leq c_1 \|u\|^2_{\mathcal{M},\gamma,x,\Omega,R_e} + c_2 \int_{\Omega(x,R_e)} |u|^2$$

for all $u \in L_2(\Omega)$ and $x \in \Omega$ such that $|\Omega(x,r)| \geq c\, r^d$ for all $r \in (0,R_e]$.

Proof. The first inequality is obvious. The second is the pointwise version of (2.8) and (2.13). □

Next we give local Hölder continuity with pointwise Campanato estimates of order $\gamma \in (d, d+2]$.

Proposition 2.19. *Let $\Omega \subset \mathbb{R}^d$ be open and $u \in L_2(\Omega)$. Let $\alpha \in (0,1)$, $c > 0$ and $x \in \Omega$. Suppose that $|\Omega(x,r)| \geq cr^d$ for all $r \in (0,R_e]$. If $\|u\|_{\mathcal{M},d+2\alpha,x,\Omega,R_e} < \infty$, then $\lim_{r\to 0}\langle u\rangle_{\Omega(x,r)}$ exists. Denote the limit by $\tilde{u}(x)$. Then there exists a $c_1 > 0$, depending only on c and α such that*

$$|\langle u\rangle_{\Omega(x,R)} - \tilde{u}(x)| \leq c_1\, R^\alpha\, \|u\|_{\mathcal{M},d+2\alpha,x,\Omega,R_e}$$

for all $R \in (0,R_e]$.

Proof. This is basically Proposition 2.16 by considering a single point $x \in \Omega$. The estimate is the same as (2.15). □

In particular if $\|u\|_{\mathcal{M},d+2\alpha,x,\Omega,R_e} < \infty$ then we can estimate $|\tilde{u}(x)|$ by

$$|\tilde{u}(x)| \leq |\langle u\rangle_{\Omega(x,R)}| + c_1\, R^\alpha\, \|u\|_{\mathcal{M},d+2\alpha,x,\Omega,R_e}.$$

On the other hand, if $\|u\|_{\mathcal{M},d+2\alpha,x,\Omega,R_e} < \infty$ for all $x \in \Omega$ then $\tilde{u}$ is well defined and equal to u a.e., so u has a continuous representative. It is even Hölder continuous, as stated below.

Proposition 2.20. *Let $\alpha \in (0,1)$ and $c > 0$. Then there exists a $c_1 > 0$ such that*

$$|\tilde{u}(x) - \tilde{u}(y)| \leq c_1(\|u\|_{\mathcal{M},d+2\alpha,x,\Omega,R_e} + \|u\|_{\mathcal{M},d+2\alpha,y,\Omega,R_e})|x-y|^\alpha$$

for all open $\Omega \subset \mathbb{R}^d$, $x, y \in \Omega$, $R_e \in (0,1]$ and $u \in L_2(\Omega)$ such that $\|u\|_{\mathcal{M},d+2\alpha,x,\Omega,R_e} < \infty$, $\|u\|_{\mathcal{M},d+2\alpha,y,\Omega,R_e} < \infty$, $|x-y| < R_e/2$ with $|\Omega(x,r)|, |\Omega(y,r)| \geq c\,r^d$ for all $r \in (0,R_e]$, where $\tilde{u}$ is as defined in Proposition 2.19.

Proof. This also follows from Proposition 2.16, where we estimate the right hand side of (2.16) by $(\|u\|_{\mathcal{M},d+2\alpha,x,\Omega,R_e} + \|u\|_{\mathcal{M},d+2\alpha,y,\Omega,R_e})$ instead of $\|u\|_{\mathcal{M}_{d+2\alpha}(\Omega)}$. □

With the freedom of R_e, the use of pointwise seminorms allow us to separate the boundary region and the interior region. If the domain is at least Lipschitz then we can transform the boundary locally to a plane such that the inner volume condition is automatically satisfied. The method of transformation will be described in Section 3.2.

Chapter 3

Estimates on the boundary

It was a breakthrough in [ER15] where the interior estimates and estimates on the boundary were treated separately using pointwise Morrey and Campanato seminorms. The interior estimate are well known so we focus on the boundary estimates instead. Since we are with domains that are at least Lipschitz, there exists locally a homeomorphism to the half-space where the boundary is flat. Furthermore, if the second-order coefficients are uniformly continuous, we can estimate them by constant coefficients locally plus an error term.

In this chapter, we will first give De Giorgi estimates for elliptic operators with constant coefficients on a half-space. Then we give the regularity results for elliptic operators in Lipschitz domains and in $C^{1+\kappa}$-domains separately.

3.1 Estimates on flat space

The aim of in this section is to prove estimates for pure second-order operators with constant coefficients on half space. We first establish that for elliptic operators A_p with constant coefficients, if $A_p u = 0$ weakly, then u admits higher regularity. We then give estimates for derivatives of u.

Throughout this section consider the half space $\Omega = \{x \in \mathbb{R}^d \mid x_d < 0\}$ and $\Gamma = \partial\Omega$. For $u \in L_{1,\mathrm{loc}}(\Omega)$ and $k \in \{1, \ldots, d\}$, we denote by $D_j u$ the distributional derivative.

In order to get estimates near the boundary we have two versions for some estimates here. For $u \in W^{1,2}(\Omega(x,R))$, we give estimates for when $A_p u = 0$ weakly on $B(x,R)$, as well as when $A_p u = 0$ weakly on $\Omega(x,R)$ with $\operatorname{Tr} u = 0$ on $\Gamma(x,R)$. We emphasize that these two versions are independent and both of them are used to obtain De Giorgi estimates over $\Omega(x,R)$.

We start with well known **Caccioppoli inequalities**.

Lemma 3.1. *Let $\mu, M > 0$ and $C \in \mathcal{E}_p(\Omega, \mu, M)$. Let $x \in \mathbb{R}^d$, $R \in (0,\infty)$ and $u \in$*

$W^{1,2}(\Omega(x,R))$. *Suppose that* $A_p u = 0$ *weakly on* $B(x,R)$. *Then*

$$\int_{\Omega(x,r)} |\nabla u|^2 \leq \frac{16M^2}{\mu^2} \frac{1}{(R-r)^2} \int_{\Omega(x,R)} |u-\lambda|^2$$

for all $r \in (0,R)$ *and* $\lambda \in \mathbb{C}$.

Proof. We include the well-known proof to obtain the explicit constant. There exists an $\eta \in C_c^\infty(B(x,R))$ such that $0 \leq \eta \leq \mathbb{1}$, $\eta(y) = 1$ for all $y \in B(x,r)$ and $\|\nabla \eta\|_\infty \leq \frac{2}{R-r}$. Then $v = \eta^2 u \in W^{1,2}(\Omega(x,R))$ and $\operatorname{supp} v \subset B(x,R)$. Since $\Omega(x,R)$ has the extension property there exists a $\tilde{v} \in W_0^{1,2}(B(x,R))$ such that $\tilde{v}|_{\Omega(x,R)} = v$. Then

$$0 = \int_{\Omega(x,R)} \sum c_{kl} \, (\partial_k u) \, \overline{\partial_l \tilde{v}} = \int_{\Omega(x,R)} \sum c_{kl} \, (\partial_k u) \, \eta^2 \, \overline{\partial_l u} + 2 \int_{\Omega(x,R)} \sum c_{kl} \, (\eta \, \partial_k u) \, \overline{u \, \partial_l \eta}.$$

So

$$\begin{aligned} \mu \int_{\Omega(x,R)} |\eta \, \nabla u|^2 &\leq \operatorname{Re} \int_{\Omega(x,R)} \sum c_{kl} \, (\partial_k u) \, \eta^2 \, \overline{\partial_l u} \\ &\leq 2 \Big| \int_{\Omega(x,R)} \sum c_{kl} \, (\eta \, \partial_k u) \, \overline{u \, \partial_l \eta} \Big| \\ &\leq 2M \Big(\int_{\Omega(x,R)} |\eta \, \nabla u|^2 \Big)^{1/2} \Big(\int_{\Omega(x,R)} |u \, \nabla \eta|^2 \Big)^{1/2}. \end{aligned}$$

Therefore

$$\int_{\Omega(x,r)} |\nabla u|^2 \leq \frac{4M^2}{\mu^2} \int_{\Omega(x,R)} |u \, \nabla \eta|^2 \leq \frac{16M^2}{\mu^2} \frac{1}{(R-r)^2} \int_{\Omega(x,R)} |u|^2.$$

Finally, replace u by $u - \lambda$. □

Nearly the same way one also proves the following lemma.

Lemma 3.2. *Let* $\mu, M > 0$ *and* $C \in \mathcal{E}_p(\Omega, \mu, M)$. *Let* $x \in \mathbb{R}^d$, $R \in (0,\infty)$ *and* $u \in W^{1,2}(\Omega(x,R))$. *Suppose that* $A_p u = 0$ *weakly on* $\Omega(x,R)$ *and* $\operatorname{Tr} u = 0$ *a.e. on* $\Gamma(x,R)$. *Then*

$$\int_{\Omega(x,r)} |\nabla u|^2 \leq \frac{16M^2}{\mu^2} \frac{1}{(R-r)^2} \int_{\Omega(x,R)} |u|^2$$

for all $r \in (0,R)$.

Proof. Now $\operatorname{Tr} v = 0$ in the proof of Lemma 3.1. Therefore $v \in W_0^{1,2}(\Omega(x,R))$ by [Alt85] Lemma A 6.10. The rest is the same. □

Before we can prove De Giorgi estimates, we need a few lemmas regarding derivatives.

Let $0 < r < R < \infty$ and $j \in \{1, \ldots, d-1\}$. For all $h \in \mathbb{R}$ with $|h| \leq R - r$ define $R_{j,h} \colon L_2(\Omega(0,R)) \to L_2(\Omega(0,r))$ by

$$(R_{j,h} u)(x) = u(x + h\, e_j).$$

Moreover, if $h \neq 0$, then define $\Delta_{j,h} = \frac{1}{h}(R_{j,h} - I)$.

There is a uniform bound for $\Delta_{j,h} u$ if a distributional derivative is in L_2.

Lemma 3.3. *Let $0 < r < R < \infty$ and $j \in \{1, \ldots, d-1\}$. Let $u \in L_2(\Omega(0,R))$ and suppose that $D_j u \in L_2(\Omega(0,R))$, where D_j is the distributional derivative. Then*

$$\|\Delta_{j,h} u\|_{L_2(\Omega(0,r))} \leq \|D_j u\|_{L_2(\Omega(0,R))}$$

for all $h \in \mathbb{R}$ with $0 < |h| \leq R - r$.

Proof. First suppose that $u \in C^\infty(\overline{\Omega(0,R)})$. Let $h \in \mathbb{R}$ with $0 < |h| \leq R - r$. Without loss of generality we may assume that $h > 0$. Then

$$\Delta_{j,h} u = \frac{1}{h} \int_0^h R_{j,t}\, D_j u \, dt.$$

So

$$\begin{aligned} \|\Delta_{j,h} u\|_{L_2(\Omega(0,r))} &\leq \frac{1}{h} \int_0^h \|R_{j,t}\, D_j u\|_{L_2(\Omega(0,r))} \, dt \\ &\leq \frac{1}{h} \int_0^h \|D_j u\|_{L_2(\Omega(0,R))} \, dt = \|D_j u\|_{L_2(\Omega(0,R))}. \end{aligned}$$

For general u there exists a sequence $(u_n)_{n \in \mathbb{N}}$ in $C^\infty(\overline{\Omega(0,R)})$ such that $u_n \to u$ and $D_j u_n \to D_j u$ in $L_2(\Omega(0,R))$. The the lemma follows by approximation. □

There is also a converse.

Lemma 3.4. *Let $0 < r < R < \infty$ and $j \in \{1, \ldots, d-1\}$. Let $u \in L_2(\Omega(0,R))$. Let $M > 0$ and $\varepsilon \in (0, R-r]$. Suppose that $\|\Delta_{j,h} u\|_{L_2(\Omega(0,r))} \leq M$ for all $h \in \mathbb{R}$ with $0 < |h| < \varepsilon$. Then $D_j u \in L_2(\Omega(0,r))$ and $\|D_j u\|_{L_2(\Omega(0,r))} \leq M$. Moreover, $\lim_{h \to 0} \Delta_{j,h} u = D_j u$ in $L_2(\Omega(0,r))$.*

Proof. The proof is similar to the proof of Lemma 8.2 in [Giu03]. □

Let $0 < r < R < \infty$ and $j \in \{1, \ldots, d-1\}$. Obviously $\Delta_{j,h} u \in W^{1,2}(\Omega(0,r))$ if $u \in W^{1,2}(\Omega(0,R))$ and $0 < |h| < R - r$. If $C \in \mathcal{E}_p(\mathbb{R}^d)$ is constant and $A_p u = 0$ weakly on $B(0,R)$, then $u \in C^\infty(\Omega(0,R))$ and

$$- \sum_{k,l=1}^{d} c_{kl}\, \partial_k \partial_l u = 0 \tag{3.1}$$

by elliptic (inner) regularity.

We next wish to prove higher order Caccioppoli inequalities. The starting point is that A_p and $\Delta_{j,h}$ commute if the coefficients are constant.

Lemma 3.5. *Let $0 < r < R < \infty$ and $j \in \{1, \ldots, d-1\}$. Let $C \in \mathcal{E}_p(\mathbb{R}^d)$ be constant. Let $u \in W^{1,2}(\Omega(0, R))$ and suppose that $A_p u = 0$ weakly on $B(0, R)$. Then $A_p \Delta_{j,h} u = 0$ weakly on $B(0, r)$ for all $h \in \mathbb{R}$ with $0 < |h| < R - r$.*

Proof. Let $v \in C_c^\infty(B(0, r))$. Then

$$\begin{aligned}
\int_{\Omega(0,r)} \sum c_{kl} \, (\partial_k \Delta_{j,h} u) \, \overline{\partial_l v} &= \frac{1}{h} \int_{\Omega(0,r)} \sum c_{kl} \, (\partial_k R_{j,h} u - \partial_k u) \, \overline{\partial_l v} \\
&= \frac{1}{h} \int_{\Omega(0,r)} \sum c_{kl} \, (\partial_k R_{j,h} u) \, \overline{\partial_l v} \\
&= \frac{1}{h} \int_{(\Omega(0,r)) + h \, e_j} \sum c_{kl} \, (\partial_k u) \, \overline{\partial_l R_{j,-h} v} \\
&= \frac{1}{h} \int_{\Omega(0,R)} \sum c_{kl} \, (\partial_k u) \, \overline{\partial_l R_{j,-h} v} = 0,
\end{aligned}$$

since $\partial_l R_{j,-h} v \in C_c^\infty(B(0, R))$. □

Similarly one proves the next lemma.

Lemma 3.6. *Let $0 < r < R < \infty$ and $j \in \{1, \ldots, d-1\}$. Let $C \in \mathcal{E}_p(\mathbb{R}^d)$ be constant. Let $u \in W^{1,2}(\Omega(0, R))$. Suppose that $A_p u = 0$ weakly on $\Omega(0, R)$ and $\operatorname{Tr} u = 0$ a.e. on $\Gamma(x, R)$. Then $A_p \Delta_{j,h} u = 0$ weakly on $B(0, r)$ and $\operatorname{Tr} \Delta_{j,h} u = 0$ a.e. on $\Gamma(x, r)$ for all $h \in \mathbb{R}$ with $0 < |h| < R - r$.*

Next we turn to second-order derivatives.

Lemma 3.7. *Let $0 < r < R < \infty$ and $j \in \{1, \ldots, d-1\}$. Let $\mu, M > 0$ and $C \in \mathcal{E}_p(\mathbb{R}^d, \mu, M)$ be constant. Let $u \in W^{1,2}(\Omega(0, R))$ and suppose that $A_p u = 0$ weakly on $B(0, R)$. Then $D_j u \in W^{1,2}(\Omega(0, r))$ and*

$$\int_{\Omega(0,r)} |\nabla D_j u|^2 \le 16 \Big(\frac{16 M^2}{\mu^2} \Big)^2 \frac{1}{(R - r)^4} \int_{\Omega(0,R)} |u|^2 .$$

Moreover, $A_p D_j u = 0$ weakly on $B(0, r)$.

Proof. Let $\rho \in (r, R)$. Then for all $h \in \mathbb{R}$ with $0 < |h| < R - \rho$ one deduces from Lemma 3.5 and the Caccioppoli inequality of Lemma 3.1 that

$$\begin{aligned}
\int_{\Omega(0,r)} |\nabla \Delta_{j,h} u|^2 &\le \frac{16 M^2}{\mu^2} \, \frac{1}{(\rho - r)^2} \int_{\Omega(0,\rho)} |\Delta_{j,h} u|^2 \qquad (3.2) \\
&\le \frac{16 M^2}{\mu^2} \, \frac{1}{(\rho - r)^2} \int_{\Omega(0,R)} |D_j u|^2 ,
\end{aligned}$$

where we use Lemma 3.3 in the last step. In particular, for all $k \in \{1, \ldots, d\}$ one has

$$\int_{\Omega(0,r)} |\Delta_{j,h}(\partial_k u)|^2 \le \frac{16M^2}{\mu^2} \, \frac{1}{(\rho - r)^2} \int_{\Omega(0,R)} |D_j u|^2$$

for all $h \in \mathbb{R}$ with $0 < |h| < R - \rho$. So $D_j(\partial_k u) \in L_2(\Omega(0,r))$ by Lemma 3.4. Therefore $D_j u \in W^{1,2}(\Omega(0,r))$. Taking the limit $h \to 0$ in (3.2) and using again Lemma 3.4 gives

$$\int_{\Omega(0,r)} |\nabla D_j u|^2 \le \frac{16M^2}{\mu^2} \, \frac{1}{(\rho - r)^2} \int_{\Omega(0,R)} |D_j u|^2.$$

Hence

$$\int_{\Omega(0,r)} |\nabla D_j u|^2 \le \frac{16M^2}{\mu^2} \, \frac{1}{(R - r)^2} \int_{\Omega(0,R)} |D_j u|^2. \tag{3.3}$$

by taking the limit $\rho \uparrow R$.

Next, choose $\rho = \frac{R+r}{2}$. Then (3.3) and the Caccioppoli inequality of Lemma 3.1 give

$$\begin{aligned}\int_{\Omega(0,r)} |\nabla D_j u|^2 &\le \frac{16M^2}{\mu^2} \, \frac{1}{(\rho - r)^2} \int_{\Omega(0,\rho)} |D_j u|^2 \\ &\le \Big(\frac{16M^2}{\mu^2}\Big)^2 \frac{1}{(\rho - r)^2} \, \frac{1}{(R-\rho)^2} \int_{\Omega(0,R)} |u|^2\end{aligned}$$

and the proof of the inequality in the lemma is complete.

Finally, $A_p \Delta_{j,h} u = 0$ weakly on $B(0,r)$ for all $h \in \mathbb{R}$ with $0 < |h| < R - r$ by Lemma 3.5. Since $\partial_k D_j u = D_j \partial_k u = \lim_{h \to 0} \Delta_{j,h} \partial_k u = \lim_{h \to 0} \partial_k \Delta_{j,h} u$ in $L_2(\Omega(0,r))$ for all $k \in \{1, \ldots, d\}$ one deduces that $A_p D_k u = 0$ weakly on $B(0,r)$. (Alternatively, this can also be proved if one uses elliptic regularity.) □

Again there is a version for functions with vanishing trace on the flat part.

Lemma 3.8. *Let $0 < r < R < \infty$ and $j \in \{1, \ldots, d-1\}$. Let $\mu, M > 0$ and $C \in \mathcal{E}_p(\mathbb{R}^d, \mu, M)$ be constant. Let $u \in W^{1,2}(\Omega(0,R))$. Suppose that $A_p u = 0$ weakly on $\Omega(0,R)$ and $\operatorname{Tr} u = 0$ a.e. on $\Gamma(x,R)$. Then $D_j u \in W^{1,2}(\Omega(0,r))$ and*

$$\int_{\Omega(0,r)} |\nabla D_j u|^2 \le 16 \Big(\frac{16M^2}{\mu^2}\Big)^2 \frac{1}{(R-r)^4} \int_{\Omega(0,R)} |u|^2.$$

Moreover, $A_p D_j u = 0$ weakly on $B(0,r)$ and $\operatorname{Tr} D_j u = 0$ a.e. on $\Gamma(x,r)$.

Proof. With exception of the last part regarding the trace, all is similar as above, using the Caccioppoli inequalities of Lemma 3.2 instead of those of Lemma 3.1.

The vanishing of the trace requires an additional argument. Fix $r \in (0,R)$ and $j \in \{1, \ldots, d-1\}$. For all large $n \in \mathbb{N}$ define $v_n = \Delta_{j,1/n} u \in W^{1,2}(\Omega(0,r))$. Then $\operatorname{Tr} v_n = 0$ a.e. on $\Gamma(x,r)$ for all large $n \in \mathbb{N}$. Then it follows from the new version of (3.2) that the sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,2}(\Omega(0,r))$. Hence passing to a subsequence if necessary, there exists a $v \in W^{1,2}(\Omega(0,r))$ such that $\lim v_n = v$ weakly in $W^{1,2}(\Omega(0,r))$. Then $\operatorname{Tr} v = \lim \operatorname{Tr} v_n = 0$ weakly in $L_2(\Gamma(x,r))$. Moreover $\lim v_n = v$ weakly in $L_2(\Omega(0,r))$. But $\lim v_n = D_j u$ strongly in $L_2(\Omega(0,r))$. So $v = D_j u$ and $\operatorname{Tr} D_j u = 0$ a.e. on $\Gamma(x,r)$. □

For the proof of higher-order derivatives it is convenient to introduce multi-index notation. Define $J(d) = \bigcup_{n=0}^\infty \{1,\ldots,d\}^n$ and $J(d-1) = \bigcup_{n=0}^\infty \{1,\ldots,d-1\}^n$.

Lemma 3.9. *For all $\mu, M > 0$, $\alpha \in J(d-1)$ and $\beta \in J(d)$ with $|\beta| \le 2$ there exists a $c > 0$ such that $D^\beta D^\alpha u \in L_2(\Omega(0,r))$ and*

$$\int_{\Omega(0,r)} |D^\beta D^\alpha u|^2 \le c\,\frac{1}{(R-r)^{2|\alpha|+2|\beta|}} \int_{\Omega(0,R)} |u|^2 \tag{3.4}$$

for all $C \in \mathcal{E}_p(\mathbb{R}^d,\mu,M)$ constant, $0 < r < R < \infty$ and $u \in W^{1,2}(\Omega(0,R))$ with $A_p u = 0$ weakly on $B(0,R)$.

Proof. If $|\alpha| + |\beta| = 0$ then (3.4) is trivial. Suppose $|\alpha| + |\beta| \ge 1$. If $|\beta| \le 1$ this follows by induction from Lemmas 3.7 and 3.1.

Suppose $|\beta| = 2$. There are $i, j \in \{1,\ldots,d\}$ such that $\beta = (i,j)$. If $(i,j) \ne (d,d)$ then we can use (3.4) again by rewriting α and β. If $(i,j) = (d,d)$ then

$$c_{dd}\,\partial_d\partial_d D^\alpha u = -\sum_{(k,l)\ne(d,d)} c_{kl}\,\partial_k\partial_l D^\alpha u \in L_2(\Omega(0,r)).$$

Moreover, there exists a $c > 0$, depending only on μ, M and α, such that

$$\int_{\Omega(0,r)} \Big| \sum_{(k,l)\ne(d,d)} c_{kl}\,\partial_k\partial_l D^\alpha u \Big|^2 \le c\,\frac{1}{(R-r)^{2|\alpha|+4}} \int_{\Omega(0,R)} |u|^2.$$

Since $\mu \le \operatorname{Re} c_{dd} \le |c_{dd}|$ the lemma follows. □

Lemma 3.10. *For all $\mu, M > 0$, $\alpha \in J(d-1)$ and $\beta \in J(d)$ with $|\beta| \le 1$ there exists a $c > 0$ such that for all $0 < r < R < \infty$, $C \in \mathcal{E}_p(\mathbb{R}^d,\mu,M)$ constant and $u \in W^{1,2}(\Omega(0,R))$ with $A_p u = 0$ weakly on $B(0,R)$ one has $D^\beta D^\alpha u \in L_\infty(\Omega(0,r))$ and*

$$|(D^\beta D^\alpha u)(x)|^2 \le c\,\frac{R^d}{(R-r)^{2d+2|\alpha|+2|\beta|}} \int_{\Omega(0,R)} |u|^2$$

for all $x \in \Omega(0,r)$.

Proof. Set $\rho = \frac{r+R}{2}$. There exists an $f \in C_c^\infty(B(0,\rho))$ such that $f|_{B(0,r)} = \mathbb{1}$ and $\|\partial^\gamma f\|_\infty \le \frac{2}{(\rho-r)^{|\gamma|}}$ for all $\gamma \in J(d)$. We consider uf as a C^∞ function on $\mathbb{R}^d$ by extending it by zero. Then

$$(D^\beta D^\alpha u)(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} (D_1 \ldots D_d\, D^\beta D^\alpha(uf))(t_1,\ldots,t_d)\,dt_d \ldots dt_1.$$

Hence by Cauchy–Schwarz one deduces that

$$|(D^\beta D^\alpha u)(x)|^2 \le (2\rho)^d \int_{\Omega(0,\rho)} |D_1 \dots D_d \, D^\beta D^\alpha (uf)|^2.$$

By the product rule, Lemmas 3.1 and 3.9 there exists a $c > 0$, depending only on μ, M and $d + |\alpha| + |\beta|$, such that

$$\int_{\Omega(0,\rho)} |D_1 \dots D_d \, D^\beta D^\alpha (uf)|^2 \le \frac{c}{(R-r)^{2d+2|\alpha|+2|\beta|}} \int_{\Omega(0,R)} |u|^2.$$

Then the lemma follows. □

Using Lemmas 3.6 and 3.8 one proves similarly the following.

Lemma 3.11. *For all $\mu, M > 0$, $\alpha \in J(d-1)$ and $\beta \in J(d)$ with $|\beta| \le 1$ there exists a $c > 0$ such that for all $0 < r < R < \infty$, $C \in \mathcal{E}_p(\mathbb{R}^d, \mu, M)$ constant and $u \in W^{1,2}(\Omega(0,R))$ with $A_p u = 0$ weakly on $\Omega(0,R)$ and $\operatorname{Tr} u = 0$ a.e. on $\Gamma(0,R)$ one has $D^\beta D^\alpha u \in L_\infty(\Omega(0,r))$ and*

$$|(D^\beta D^\alpha u)(x)|^2 \le c \, \frac{R^d}{(R-r)^{2d+2|\alpha|+2|\beta|}} \int_{\Omega(0,R)} |u|^2$$

for all $x \in \Omega(0,r)$.

Lemma 3.12. *Let $0 < r < R < \infty$, $\mu, M > 0$ and $C \in \mathcal{E}_p(\mathbb{R}^d, \mu, M)$ constant. Let $u \in W^{1,2}(\Omega(0,R))$ and suppose that $A_p u = 0$ weakly on $B(0,R)$. Then $u \in W^{2,2}(\Omega(0,r))$. Define $w = \sum_{k=1}^d c_{kd} \, \partial_k u$. Then $w \in W^{1,2}(\Omega(0,r))$ and $A_p w = 0$ weakly on $\Omega(0,r)$. Moreover, $\operatorname{Tr} w = 0$ a.e. on $\Gamma(0,r)$.*

Proof. It follows from Lemma 3.9 that $u \in W^{2,2}(\Omega(0,r))$. Since $A_p u = 0$ weakly on $\Omega(0,r)$ and C is constant one also has $A_p \partial_k u = 0$ weakly on $\Omega(0,r)$ for all $k \in \{1, \dots, d\}$. So $A_p w = 0$ weakly on $\Omega(0,r)$. Finally, let $v \in C_c^\infty(B(0,r))$. Then

$$\begin{aligned} 0 &= \sum_{k,l=1}^d \int_{\Omega(0,r)} c_{kl} \, (\partial_k u) \, \overline{\partial_l v} \\ &= -\sum_{k,l=1}^d \int_{\Omega(0,r)} c_{kl} \, (\partial_l \partial_k u) \, \overline{v} + \int_{\partial\Omega(0,r)} \Big(\operatorname{Tr} \sum_{k=1}^d c_{kd} \, \partial_k u \Big) \, \overline{v} \, d\sigma = \int_{B(0,r)\cap\Gamma} (\operatorname{Tr} w) \, \overline{v} \, d\sigma \end{aligned}$$

and $\operatorname{Tr} w = 0$ a.e. on $\Gamma(0,r)$. □

Now we are able to prove the **De Giorgi estimates**.

Lemma 3.13. *For all $\mu, M > 0$ and $\alpha \in J(d-1)$ there exists a $c > 0$ such that*

$$\int_{\Omega(0,r)} |D^\alpha u - \langle D^\alpha u \rangle_{\Omega(0,r)}|^2 \le c \Big(\frac{r}{R} \Big)^{d+2} \int_{\Omega(0,R)} |D^\alpha u - \langle D^\alpha u \rangle_{\Omega(0,R)}|^2$$

for all $0 < r < R < \infty$, $C \in \mathcal{E}_p(\mathbb{R}^d, \mu, M)$ constant and $u \in W^{1,2}(\Omega(0,R))$ with $A_p u = 0$ weakly on $B(0,R)$.

Proof. We may assume that $r \leq \frac{R}{2}$. By Lemma 3.10 there exists a $c > 0$ such that

$$\sup_{\Omega(0,\frac{R}{2})} |\nabla u|^2 \leq c \, \frac{1}{R^{d+2}} \int_{\Omega(0,R)} |u|^2.$$

Then

$$\int_{\Omega(0,r)} |u - \langle u \rangle_{\Omega(0,r)}|^2 \leq |B(0,1)| \, r^{d+2} \sup_{\Omega(0,\frac{R}{2})} |\nabla u|^2 \leq c \, |B(0,1)| \Big(\frac{r}{R}\Big)^{d+2} \int_{\Omega(0,R)} |u|^2. \quad (3.5)$$

Replacing u by $u - \langle u \rangle_{\Omega(0,R)}$ gives the lemma if $|\alpha| = 0$. Finally, replace u by $D^\alpha u$ and use the last part of Lemma 3.7 to obtain the general statement of the lemma, first with R replaced by $\frac{R}{2}$ and then by the usual extension. □

Now we prove De Giorgi estimates for half balls on the half space.

Proposition 3.14. *For all $\mu, M > 0$ there exists a $c_H > 0$ such that*

$$\int_{\Omega(0,r)} |\nabla u|^2 \leq c_H \Big(\frac{r}{R}\Big)^d \int_{\Omega(0,R)} |\nabla u|^2$$

and

$$\sum_{k=1}^d \int_{\Omega(0,r)} |D_k u - \langle D_k u \rangle_{\Omega(0,r)}|^2 \leq c_H \Big(\frac{r}{R}\Big)^{d+2} \sum_{k=1}^d \int_{\Omega(0,R)} |D_k u - \langle D_k u \rangle_{\Omega(0,R)}|^2$$

for all $0 < r \leq R < \infty$, $C \in \mathcal{E}_p(\mathbb{R}^d, \mu, M)$ constant and $u \in W^{1,2}(\Omega(0,R))$ with $A_p u = 0$ weakly on $B(0,R)$.

Proof. By the Neumann-type Poincaré inequality there exists a $c_P > 0$ such that

$$\int_{\Omega(0,1)} |v - \langle v \rangle_{\Omega(0,1)}|^2 \leq c_P \int_{\Omega(0,1)} |\nabla v|^2$$

for all $v \in W^{1,2}(\Omega(0,1))$. Then by scaling

$$\int_{\Omega(0,R)} |v - \langle v \rangle_{\Omega(0,R)}|^2 \leq c_P \, R^2 \int_{\Omega(0,R)} |\nabla v|^2$$

for all $R > 0$ and $v \in W^{1,2}(\Omega(0,R))$. Let $c > 0$ be as in Lemma 3.13 for $|\alpha| = 0$. If $r \leq \frac{R}{2}$ then it follows from the Caccioppoli inequality of Lemma 3.1 that

$$\begin{aligned}
\int_{\Omega(0,r)} |\nabla u|^2 &\leq \frac{16M^2}{\mu^2} \, \frac{1}{r^2} \int_{\Omega(0,2r)} |u - \langle u \rangle_{\Omega(0,2r)}|^2 \\
&\leq \frac{16c \, M^2}{\mu^2} \, \frac{1}{r^2} \Big(\frac{2r}{R}\Big)^{d+2} \int_{\Omega(0,R)} |u - \langle u \rangle_{\Omega(0,R)}|^2 \\
&\leq \frac{2^{d+6} c \, c_P \, M^2}{\mu^2} \, \Big(\frac{r}{R}\Big)^d \int_{\Omega(0,R)} |\nabla u|^2.
\end{aligned}$$

This proves the first estimate in the proposition.

Let $w = \sum_{k=1}^{d} c_{kd}\, \partial_k u \in W^{1,2}(\Omega(0, \frac{R}{2}))$. Then $\operatorname{Tr} w = 0$ a.e. on $\Gamma(0, \frac{R}{2}))$ and $A_p w = 0$ weakly on $\Omega(0, \frac{R}{2})$ by Lemma 3.12. We may assume that $r \leq \frac{R}{4}$. Using Lemma 3.11 instead of Lemma 3.10 it follows as in (3.5) that there exists a $c_1 > 0$, depending only on μ and M, such that

$$\int_{\Omega(0,r)} |w - \langle w \rangle_{\Omega(0,r)}|^2 \leq c_1 \Big(\frac{r}{R}\Big)^{d+2} \int_{\Omega(0,\frac{R}{4})} |w|^2.$$

Next,

$$\int_{\Omega(0,\frac{R}{4})} |w|^2 \leq R^2 \int_{\Omega(0,\frac{R}{4})} |D_d w|^2$$

by the Poincaré inequality [EE87] Theorem V.3.22. But $D_d w = -\sum_{k=1}^{d} \sum_{l=1}^{d-1} c_{kl}\, \partial_k \partial_l u$ by (3.1). Therefore

$$\int_{\Omega(0,\frac{R}{4})} |w|^2 \leq d^4\, M^2\, R^2 \sum_{k=1}^{d-1} \int_{\Omega(0,\frac{R}{4})} |\nabla D_k u|^2.$$

By Lemma 3.7 one has $D_j u \in W^{1,2}(\Omega(0, \frac{R}{2}))$ and $A_p D_j u = 0$ weakly on $B(0, \frac{R}{2})$ for all $j \in \{1, \ldots, d-1\}$. Hence by Lemma 3.1 there exists a $c_2 > 0$, depending only on μ and M, such that

$$\int_{\Omega(0,\frac{R}{4})} |\nabla D_j u|^2 \leq \frac{c_2}{R^2} \int_{\Omega(0,\frac{R}{2})} |D_j u - \langle D_j u \rangle_{\Omega(0,\frac{R}{2})}|^2.$$

Hence

$$\int_{\Omega(0,r)} |w - \langle w \rangle_{\Omega(0,r)}|^2 \leq c_3 \Big(\frac{r}{R}\Big)^{d+2} \sum_{j=1}^{d-1} \int_{\Omega(0,R)} |D_j u - \langle D_j u \rangle_{\Omega(0,R)}|^2,$$

where $c_3 = c_1\, c_2\, d^4\, M^2$. This is valid for all $r \leq \frac{R}{4}$.

By Lemma 3.13 there exists a $c_4 > 0$, depending only on μ and M, such that

$$\int_{\Omega(0,r)} |D_k u - \langle D_k u \rangle_{\Omega(0,r)}|^2 \leq c_4 \Big(\frac{r}{R}\Big)^{d+2} \int_{\Omega(0,R)} |D_k u - \langle D_k u \rangle_{\Omega(0,R)}|^2$$

for all $k \in \{1, \ldots, d-1\}$. Then

$$\begin{aligned}
|c_{dd}|^2 & \int_{\Omega(0,r)} |D_d u - \langle D_d u \rangle_{\Omega(0,r)}|^2 \\
&= \int_{\Omega(0,r)} |(w - \langle w \rangle_{\Omega(0,r)}) - \sum_{j=1}^{d-1} c_{jd} (D_j u - \langle D_j u \rangle_{\Omega(0,r)})|^2 \\
&\leq d \int_{\Omega(0,r)} |w - \langle w \rangle_{\Omega(0,r)}|^2 + d M^2 \sum_{j=1}^{d-1} \int_{\Omega(0,r)} |D_j u - \langle D_j u \rangle_{\Omega(0,r)}|^2 \\
&\leq (c_3 d + c_4\, d M^2) \Big(\frac{r}{R}\Big)^{d+2} \sum_{j=1}^{d-1} \int_{\Omega(0,R)} |D_j u - \langle D_j u \rangle_{\Omega(0,R)}|^2.
\end{aligned}$$

The rest of the proof is obvious. □

We need a well-known De Giorgi inequality for balls.

Proposition 3.15. *Let $\mu, M > 0$. Then there exists a $c_B > 0$ such that*

$$\int_{B(x,r)} |\nabla u|^2 \leq c_B \Big(\frac{r}{R}\Big)^d \int_{B(x,R)} |\nabla u|^2 \tag{3.6}$$

$$\sum_{k=1}^d \int_{B(x,r)} |D_k u - \langle D_k u\rangle_{B(x,r)}|^2 \leq c_B \Big(\frac{r}{R}\Big)^{d+2} \sum_{k=1}^d \int_{B(x,R)} |D_k u - \langle D_k u\rangle_{B(x,R)}|^2 \tag{3.7}$$

for all $C \in \mathcal{E}_p(\mathbb{R}^d, \mu, M)$ constant, $0 < r \leq R < \infty$, $x \in \mathbb{R}^d$ and $u \in W^{1,2}(B(x,R))$ with $A_p u = 0$ weakly on $B(x,R)$.

Proof. By Lemma 3.7 it follows that $D_k u \in W^{1,2}(B(0, R-\varepsilon))$ and $A_p D_k u = 0$ weakly on $B(0, R-\varepsilon)$ for all $\varepsilon \in (0, R)$ and $k \in \{1, \ldots, d\}$. Then (5.14) in [GM05] gives (3.6). Similarly (3.7) follows from (5.15) in [GM05]. □

Corollary 3.16. *For all $\mu, M > 0$ there exists a $c_{DG} > 0$ such that*

$$\int_{\Omega(x,r)} |\nabla u|^2 \leq c_{DG} \Big(\frac{r}{R}\Big)^d \int_{\Omega(x,R)} |\nabla u|^2 \tag{3.8}$$

and

$$\sum_{k=1}^d \int_{\Omega(x,r)} |D_k u - \langle D_k u\rangle_{\Omega(x,r)}|^2 \leq c_{DG} \Big(\frac{r}{R}\Big)^{d+2} \sum_{k=1}^d \int_{\Omega(x,R)} |D_k u - \langle D_k u\rangle_{\Omega(x,R)}|^2 \tag{3.9}$$

for all $j \in \{1, 2, \ldots, d\}$, $x \in \overline{\Omega}$, $0 < r \leq R < \infty$, $C \in \mathcal{E}_p(\mathbb{R}^d, \mu, M)$ constant and $u \in W^{1,2}(\Omega(x,R))$ with $A_p u = 0$ weakly on $B(x,R)$.

Proof. We only prove the second inequality, since the first one is similar. Let c_B be the constant (on the balls) as in (3.6) of Proposition 3.15 and let c_H be the constant (on the half-balls) as in Proposition 3.14. Without loss of generality we may assume that $r \leq \frac{1}{3} R$. Write $y = (x_1, \ldots, x_{d-1}, 0)$, the projection of x onto Γ. For every open non-empty $V \subset \Omega(x,R)$ set

$$\Phi(V) = \sum_{k=1}^d \int_V |D_k u - \langle D_k u\rangle_V|^2.$$

We separate three cases.
Case 1. Suppose $\frac{1}{3} R \leq |x_d|$. Then

$$\Phi(\Omega(x,r)) = \Phi(B(x,r)) \leq c_B \Big(\frac{r}{\frac{1}{3}R}\Big)^{d+2} \Phi(B(x, \tfrac{1}{3}R)) \leq 3^{d+2}\, c_B \Big(\frac{r}{R}\Big)^{d+2} \Phi(\Omega(x,R))$$

as required.
Case 2. Suppose $r \leq |x_d| \leq \frac{1}{3} R$. Note that

$$\Omega(x, r) = B(x, r) \subset B(x, |x_d|) \subset \Omega(y, 2|x_d|) \subset \Omega(y, \tfrac{2}{3}R) \subset \Omega(x, R).$$

Therefore

$$\Phi(\Omega(x, r)) = \Phi(B(x, r)) \leq c_B \Big(\frac{r}{|x_d|}\Big)^{d+2} \Phi(B(x, |x_d|)) \leq c_B \Big(\frac{r}{|x_d|}\Big)^{d+2} \Phi(\Omega(y, 2|x_d|))$$

$$\leq c_B \, c_H \Big(\frac{r}{|x_d|}\Big)^{d+2} \Big(\frac{2|x_d|}{\frac{2}{3}R}\Big)^{d+2} \Phi(\Omega(y, \tfrac{2}{3}R)) \leq 3^{d+2} \, c_B \, c_H \Big(\frac{r}{R}\Big)^{d+2} \Phi(\Omega(x, R))$$

by Propositions 3.14 and 3.15.
Case 3. Suppose $|x_d| \leq r$. Now use that $\Omega(x, r) \subset \Omega(y, 2r) \subset \Omega(y, \frac{2}{3}R) \subset \Omega(x, R)$. Then

$$\Phi(\Omega(x, r)) \leq \Phi(\Omega(y, 2r)) \leq c_H \Big(\frac{2r}{\frac{2}{3}R}\Big)^{d+2} \Phi(\Omega(y, \tfrac{2}{3}R)) \leq 3^{d+2} \, c_H \Big(\frac{r}{R}\Big)^{d+2} \Phi(\Omega(x, R))$$

by Proposition 3.14 and the proof is complete. □

We finish this section with a trace estimate.

Lemma 3.17. *If* $x \in \overline{\Omega}$ *and* $R \in (0, \infty)$, *then*

$$\int_{\Gamma(x,R)} |\mathrm{Tr}\, u|^2 \leq 2R \int_{\Omega(x,R)} |\nabla u|^2$$

for all $u \in W^{1,2}_{\Gamma(x,R)}(\Omega(x, R))$.

Proof. Let $u \in C^\infty_{\Gamma(x,R)}(\Omega(x, R))$. If $z \in \Gamma(x, R)$, then $u(z, 0) = \int_{x_d - \sqrt{R^2 - |z|^2}}^0 (\partial_d u)(z, t)\, dt$. Hence

$$|u(z, 0)|^2 \leq (\sqrt{R^2 - |z|^2} - x_d) \int_{x_d - \sqrt{R^2 - |z|^2}}^0 |(\partial_d u)(z, t)|^2 \, dt$$

$$\leq 2R \int_{x_d - \sqrt{R^2 - |z|^2}}^0 |(\partial_d u)(z, t)|^2 \, dt.$$

Integration over $\Gamma(x, R)$ gives $\int_{\Gamma(x,R)} |\mathrm{Tr}\, u|^2 \leq 2R \int_{\Omega(x,R)} |\nabla u|^2$. Since $C^\infty_{\Gamma(x,R)}(\Omega(x, R))$ is dense in $W^{1,2}_{\Gamma(x,R)}(\Omega(x, R))$, the lemma follows. □

3.2 Regularity improvement of order $< d$

In this section we focus on the regularity result that if a function under some circumstances admits the Morrey norm of order γ, then it is also admits the Morrey norm of order $\gamma + \delta$.

This will be used later in an induction argument to show that the semigroup admits Morrey norm of order γ for all $\gamma \in [0, d)$. The technique is a modification from [Aus96], later also used in [ER99a], [ER15], [EO15] and [EW19]. Here we add the extra boundary integral to cope with the Robin boundary condition.

We will first prove the regularity result without transformation in the interior of the domain. Then we will give a proof for the transformed version. Assume $C^{1+\kappa}$ domain in this section, and the regularity result for other domains will be given in the rest of this chapter. The major difference is on the De Giorgi estimate used while the proof itself is similar.

Define the reference domain

$$E = (-4, 4)^d \quad \text{and} \quad E^- = (-4, 4)^{d-1} \times (-4, 0),$$

where E is an open cube in $\mathbb{R}^d$ and E^- is its lower half. Define the midplate $P = (-4, 4)^{d-1} \times \{0\}$. We also need the half cube $\frac{1}{2}E$ and the truncated balls $E(x, r) = E \cap B(x, r)$ where $x \in \mathbb{R}^d$ and $r > 0$ are defined similarly for these spaces.

Throughout this section we assume that the domain Ω is $C^{1+\kappa}$. That is, for open $U \subset \mathbb{R}^d$ such that $U \cap \Omega \neq \varnothing$, there exists a $C^{1+\kappa}$-diffeomorphism Φ mapping from an open neighbourhood of $\overline{U}$ to an open subset of $\mathbb{R}^d$ with $\Phi(U) = E$ and $\Phi(\Omega \cap U) = E$. Note that it implies $\Phi(\partial\Omega \cap U) = P$. For the sake of convenience we say that Φ is a **standard $C^{1+\kappa}$-diffeomorphism from U onto E** if there are $\rho > 0$, an isometry $\Phi_0 \colon \mathbb{R}^d \to \mathbb{R}^d$ and a $C^{1+\kappa}$-function $\zeta \colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that $\Phi = \rho\,\Phi_1 \circ \Phi_0$, $\Phi(U) = E$ and $\Phi(U \cap \Omega) = E^-$, where $\Phi_1 \colon \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$\Phi_1(y_1, \ldots, y_{d-1}, y_d) = (y_1, \ldots, y_{d-1}, y_d - \zeta(y_1, \ldots, y_{d-1})).$$

Note that $\det(D\Phi) = \rho^d$ is constant. Then $\Phi_0(\Gamma \cap U) = \{\rho(\tilde{y}, \zeta(\tilde{y})) : \tilde{y} \in \rho^{-1}(-4, 4)^{d-1}\}$. If $\varphi \colon \Gamma \cap U \to [0, \infty)$ is a measurable function, then

$$\int_{\Gamma \cap U} \varphi = \int_P (\varphi \circ \Phi^{-1}) \cdot \Theta, \tag{3.10}$$

where $\Theta \colon \mathbb{R}^d \to \mathbb{R}$ is given by $\Theta(\tilde{y}, 0) = \rho^{-(d-1)} \sqrt{1 + |(\nabla\zeta)(\rho^{-1}\,\tilde{y})|^2}$ for all $\tilde{y} \in \mathbb{R}^{d-1}$. We emphasise that Θ is Hölder continuous of order κ.

We say that an open set Ω is Lipschitz if for all $z \in \partial\Omega$ there exist open $U \subset \mathbb{R}^d$ and a bi-Lipschitz map Φ that maps from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\Phi(U) = E$, $\Phi(\Omega \cap U) = E^-$ and $\Phi(z) = 0$. Note that the definition is more general to the one by means of Lipschitz charts, but the two versions, when the domain is C^1 or smoother, are equivalent.

We also define A_p^C to be the pure second-order elliptic operator induced by the particular coefficient matrix C as we need to transform or freeze the coefficients in the proofs below.

Before we give the estimate we need a technical lemma.

Lemma 3.18. *Let $\varepsilon, a, b, \alpha, \beta > 0$ where $\alpha > \beta$. Let $R_e \in (0,1]$. Let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing function satisfying*

$$\phi(r) \le a\Big(\Big(\frac{r}{R}\Big)^{\alpha} + \varepsilon\Big)\phi(R) + bR^{\beta}$$

for all $0 < r \le R \le R_e$. Then there exist $\varepsilon_0, c > 0$ only depending on a, α, β such that

$$\phi(r) \le c\Big(\Big(\frac{r}{R}\Big)^{\beta}\phi(R) + b\,r^{\beta}\Big)$$

for all $0 < r \le R \le R_e$ provided that $\varepsilon \le \varepsilon_0$.

Proof. The proof is as in Lemma 5.13 of [GM05]. Write $r = \tau R$ such that $\phi(\tau R) \le a\,\tau^{\alpha}(1 + \varepsilon\,\tau^{-\alpha})\phi(R) + b\,R^{\beta}$. Let $\gamma = \frac{1}{2}(\alpha + \beta)$ and assume without loss of generality that $a > \frac{1}{2}$. Now choose $\tau \in (0,1)$ such that $2a\tau^{\alpha} = \tau^{\gamma}$, and choose $\varepsilon_0 > 0$ such that $\varepsilon_0 \tau^{-\alpha} < 1$. Then the following holds:

$$\phi(\tau R) \le 2a\tau^{\alpha}\phi(R) + bR^{\beta} = \tau^{\gamma}\phi(R) + bR^{\beta}.$$

Iterating the above gives

$$\begin{aligned}
\phi(\tau^k R) &\le \tau^{\gamma}\phi(\tau^{k-1}R) + b\tau^{(k-1)\beta}\,R^{\beta} \\
&\le \ldots \le \tau^{k\gamma}\phi(R) + b\,R^{\beta}\sum_{j=0}^{k-1}\tau^{(k-j-1)\beta}\,\tau^{j\gamma} \\
&= \tau^{k\gamma}\phi(R) + b\,\tau^{(k-1)\beta}\,R^{\beta}\sum_{j=0}^{k-1}\tau^{j(\gamma-\beta)} \\
&\le \tau^{(k+1)\beta}(\phi(R) + b\,R^{\beta})(\tau^{-\beta} + \tau^{-2\beta}\sum_{j=0}^{\infty}\tau^{j(\gamma-\beta)}) \\
&= c\,\tau^{(k+1)\beta}(\phi(R) + b\,R^{\beta}),
\end{aligned}$$

where $c = \tau^{-\beta} + \tau^{-2\beta}\sum_{j=0}^{\infty}\tau^{j(\gamma-\beta)}$.

Finally given any $r \in (0,R]$ choose $k \in \mathbb{N}_0$ such that $\tau^{k+1}R \le r \le \tau^k R$. Then we have

$$\phi(r) \le \phi(\tau^k\,R) \le c\,\tau^{(k+1)\beta}(\phi(R) + b\,R^{\beta}) \le c\Big(\Big(\frac{r}{R}\Big)^{\beta}\phi(R) + b\,r^{\beta}\Big)$$

as required. □

Since we will apply Poincaré inequalities on E^- frequently throughout the thesis, we will quote two lemmas from [ER15] directly with proofs for completeness. Let $c_N > 0$ be the constant for Neumann-type Poincaré inequality for balls, i.e., it is the constant such that

$$\int_{B(x,R)} |u - \langle u\rangle_{B(x,R)}|^2 \le c_N\,R^2 \int_{B(x,R)} |\nabla u|^2 \tag{3.11}$$

for all $x \in \mathbb{R}^d$, $R > 0$ and $u \in W^{1,2}(B(x,R))$.

Lemma 3.19. *Let $c_N > 0$ be as in (3.11).*

I. *If $x_0 \in \frac{1}{2}E^-$, $R \in (0,1]$ and $u \in W^{1,2}(E^-)$, then*

$$\int_{E^-(x_0,R)} |u - \langle u \rangle_{E^-(x_0,R)}|^2 \leq 2c_N\, R^2 \int_{E^-(x_0,R)} |\nabla u|^2. \tag{3.12}$$

II. *If $x_0 \in \frac{1}{2}E^-$, $R \in (0,1]$ and $u \in W^{1,2}_{P(x,R)}(E^-(x_0,R))$, then*

$$\int_{E^-(x_0,R)} |u|^2 \leq 4R^2 \int_{E^-(x_0,R)} |\nabla u|^2.$$

Proof. I: If $E^-(x_0,R) = B(x_0,R)$ then this clear. Otherwise define $\tilde{u}\colon B(x_0,R) \to \mathbb{C}$ by

$$\tilde{u}(y) = \begin{cases} u(y) & \text{if } \pi_d(y) < 0, \\ (\operatorname{Tr} u)(y) & \text{if } \pi_d(y) = 0, \\ u(y - 2\pi_d(y)e_d) & \text{if } \pi_d(y) > 0. \end{cases}$$

Then $\tilde{u} \in W^{1,2}(B(x_0,R))$ by [ER15] Proposition 4.4(d). Also,

$$\begin{aligned} \int_{E^-(x_0,R)} |u - \langle u \rangle_{E^-(x_0,R)}|^2 &\leq \int_{E^-(x_0,R)} |u - \langle \tilde{u} \rangle_{B(x_0,R)}|^2 \\ &\leq \int_{B(x_0,R)} |\tilde{u} - \langle \tilde{u} \rangle_{B(x_0,R)}|^2 \\ &\leq c_N\, R^2 \int_{B(x_0,R)} |\nabla u|^2 \leq 2c_N\, R^2 \int_{E^-(x_0,R)} |\nabla u|^2, \end{aligned}$$

as desired.

II: This follows from an adaption of the proof of Theorem V.3.22 of [EE87]. □

The above lemma gives the following embedding between Morrey and Campanato spaces.

Lemma 3.20. *Let $c_N > 0$ be as in (3.11). Then*

$$\|u\|_{\mathcal{M},\gamma+2,x,E^-,1} \leq \sqrt{2c_N}\|\nabla u\|_{M,\gamma,x,E^-,1} \tag{3.13}$$

and

$$\|u\|_{\mathcal{M},\gamma+\delta,x,E^-,1} \leq c_1(\varepsilon^{2-\delta}\|\nabla u\|_{M,\gamma,x,E^-,1} + \varepsilon^{-(\gamma+\delta)}\|u\|_{L_2(E^-)})$$

for all $\gamma \in [0,d)$, $\delta \in (0,2]$, $\varepsilon \in (0,1]$, $u \in W^{1,2}(E^-)$ and $x \in \frac{1}{2}E^-$, where $c_1 = 2^{d+2} + 2c_N$.

Proof. The inequality (3.13) is a direct consequence of (3.12). Now if $r \in (0,\varepsilon^2]$ then

$$r^{-(\gamma+\delta)} \int_{E^-(x,r)} |u - \langle u \rangle_{E^-(x,r)}|^2 \leq 2c_N\, r^{2-\delta} r^{-\gamma} \int_{E^-(x,r)} |\nabla u|^2 \leq 2c_N \varepsilon^{2(2-\delta)} \|\nabla u\|^2_{M,\gamma,x,E^-,1}.$$

for all $x \in \frac{1}{2}E^-$ by (3.12). If $r \in [\varepsilon^2, 1]$ then

$$\int_{E^-(x,r)} |u - \langle u \rangle_{E^-(x,r)}|^2 \leq \int_{E^-(x,r)} |u|^2 \leq 2^{\gamma+\delta}\, \varepsilon^{-2(\gamma+\delta)}\, \|u\|^2_{L_2(E^-)} r^{\gamma+\delta}.$$

Combining the two gives the desired result. □

Below is the regularity estimate for the interior part of the domain.

Proposition 3.21. *Let $K \geq 1$, $\mu, M > 0$, $\kappa \in (0,1)$, $\gamma \in [0,d)$, $\delta \in (0,2]$ with $\gamma + \delta < d$ and $R_e \in (0,1]$. Then there exists a $c > 0$ so that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $C \in \mathcal{E}_p^\kappa(\Omega, \mu, M)$, $u, g \in W^{1,2}(\Omega)$, $\beta \in L_\infty(\Gamma)$ with $\|\beta\|_{L_\infty(\Gamma)} \leq M$ and $f, f_1, \ldots, f_d \in L_2(\Omega)$ with

$$\sum_{k,l=1}^{d} \int_\Omega c_{kl}\, (\partial_k u)\, \overline{\partial_l v} = \int_\Omega f\, \overline{v} - \sum_{k=1}^{d} \int_\Omega f_k\, \overline{\partial_k v} + \int_\Gamma \beta \operatorname{Tr} g\, \overline{\operatorname{Tr} v}$$

for all $v \in W^{1,2}(\Omega)$. Then

$$\|\nabla u\|_{M,\gamma+\delta,x,\Omega,R_e} \leq c \left(\varepsilon^{2-\delta} \|f\|_{M,\gamma,x,\Omega,R_e} + \sum_{k=1}^{d} \|f_k\|_{M,\gamma+\delta,x,\Omega,R_e} + \varepsilon^{-(\gamma+\delta)} \|\nabla u\|_{L_2(\Omega)} \right)$$

for all $x \in \Omega$ such that $d_\Gamma(x) > R_e$ and $\varepsilon \in (0,1)$.

Proof. The proof is a modification from Proposition 3.2 of [ER15]. Let ε_0 be as in Lemma 3.18 with respective to the parameters $(a, \alpha, \beta) = (4c_B, d, \gamma+\delta)$. Choose $R_0 \in (0,1)$ such that $R_0 \leq (c_B\, \mu^2\, M^{-2}\, (1 + 2c_B)^{-1}\, \varepsilon_0)^{-2\kappa}$.

Let $0 < r \leq R \leq R_e$. Using the Lax-Milgram theorem there exists a $\chi \in W_0^{1,2}(B(x,R))$ such that

$$\sum_{k,l=1}^{d} \int_{B(x,R)} c_{kl}(x)\, \partial_k \chi\, \overline{\partial_l w} = \sum_{k,l=1}^{d} \int_{B(x,R)} c_{kl}(x)\, \partial_k u\, \overline{\partial_l w} \tag{3.14}$$

for all $w \in W_0^{1,2}(B(x,R))$. Extend χ by zero to $\tilde{\chi} \colon \Omega \to \mathbb{C}$ which is also in $W_0^{1,2}(B(x,R))$ and define $\eta = u - \tilde{\chi} \in W^{1,2}(\Omega)$. Then we have

$$\sum_{k,l=1}^{d} \int_{B(x,R)} c_{kl}\, \partial_k \eta\, \overline{\partial_l w} = 0 \tag{3.15}$$

for all $w \in W_0^{1,2}(B(x,R))$. In other words, $A_p^{C(x)} \eta = 0$ weakly on $B(x,R)$. Applying the De Giorgi estimates of Proposition 3.15 gives

$$\int_{B(x,r)} |\nabla u|^2 \leq 2 \int_{B(x,r)} |\nabla \eta|^2 + 2 \int_{B(x,r)} |\nabla \chi|^2$$

$$\le 2c_B\Big(\frac{r}{R}\Big)^d\int_{B(x,R)}|\nabla\eta|^2+2\int_{B(x,R)}|\nabla\chi|^2$$

$$\le 4c_B\Big(\frac{r}{R}\Big)^d\int_{B(x,R)}|\nabla u|^2+(2+4c_B)\int_{B(x,R)}|\nabla\chi|^2. \tag{3.16}$$

To estimate $\int_{B(x,R)}|\nabla\chi|^2$ choose $w=\chi$ in (3.14):

$$\begin{aligned}\sum_{k,l=1}^d\int_{B(x,R)}c_{kl}(x)\,\partial_k\chi\,\overline{\partial_l\chi} &= \sum_{k,l=1}^d\int_{B(x,R)}c_{kl}(x)\,\partial_k u\,\overline{\partial_l\chi}\\ &= \sum_{k,l=1}^d\Big(\int_\Omega c_{kl}\,\partial_k u\,\overline{\partial_l\tilde\chi}+\int_\Omega(c_{kl}(x)-c_{kl})\partial_k u\,\overline{\partial_l\tilde\chi}\Big)\\ &= (f,\tilde\chi)_{L_2(\Omega)}+\sum_{i=1}^d(f_i,\partial_i\tilde\chi)_{L_2(\Omega)}+(\beta\operatorname{Tr}g,\operatorname{Tr}\tilde\chi)_{L_2(\Gamma)}\\ &\quad+\sum_{k,l=1}^d\int_\Omega(c_{kl}(x)-c_{kl})\partial_k u\,\overline{\partial_l\tilde\chi}.\end{aligned}$$

The third term is zero because $\operatorname{supp}\tilde\chi\subset\subset\Omega$. We now estimate the other two terms separately. Note the Dirichlet-type Poincaré inequality as in Theorem V 3.22 of [EE87]:

$$\int_{B(x,R)}|\chi|^2\le(2R)^2\int_{B(x,R)}|\nabla\chi|^2.$$

So Cauchy-Schwarz gives

$$\begin{aligned}&(f,\tilde\chi)_{L_2(\Omega)}+\sum_{i=1}^d(f_i,\partial_i\tilde\chi)_{L_2(\Omega)}\\ \le\ &\Big(\int_{B(x,R)}|f|^2\Big)^{\frac12}\Big(\int_{B(x,R)}|\chi|^2\Big)^{\frac12}+\sum_{i=1}^d\Big(\int_{B(x,R)}|f_i|^2\Big)^{\frac12}\Big(\int_{B(x,R)}|\partial_i\chi|^2\Big)^{\frac12}\\ \le\ &R^{\frac{\gamma+\delta}{2}}(2R^{\frac{2-\delta}{2}}\|f\|_{M,\gamma,x,\Omega,R_e}+\sum_{i=1}^d\|f_i\|_{M,\gamma+\delta,x,\Omega,R_e})\Big(\int_{B(x,R)}|\nabla\chi|^2\Big)^{\frac12}.\end{aligned}$$

Similarly the last term is given by

$$\sum_{k,l=1}^d\int_\Omega(c_{kl}(x)-c_{kl})\partial_k u\,\overline{\partial_l\tilde\chi}\le M\,R^\kappa\Big(\int_{B(x,R)}|\nabla u|^2\Big)^{\frac12}\Big(\int_{B(x,R)}|\nabla\chi|^2\Big)^{\frac12}.$$

Using the above and using ellipticity we have

$$\int_{B(x,R)}|\nabla\chi|^2\le\mu^{-2}\Big(R^{\frac{\gamma+\delta}{2}}(2R^{\frac{2-\delta}{2}}\|f\|_{M,\gamma,x,\Omega,R_e}+\sum_{i=1}^d\|f_i\|_{M,\gamma+\delta,x,\Omega,R_e})+M\,R^\kappa\Big(\int_{B(x,R)}|\nabla u|^2\Big)^{\frac12}\Big)^2$$

$$\leq 4\mu^{-2} R^{\gamma+\delta} (R^{\frac{2-\delta}{2}} \|f\|_{M,\gamma,x,\Omega,R_e} + \sum_{i=1}^{d} \|f_i\|_{M,\gamma+\delta,x,\Omega,R_e})^2 + 2\mu^{-2} M^2 R^{2\kappa} \Big(\int_{B(x,R)} |\nabla u|^2 \Big).$$

Combining the above and (3.16) we have

$$\int_{B(x,r)} |\nabla u|^2 \leq 4c_B \Big(\Big(\frac{r}{R}\Big)^d + \frac{(1+2c_B) M^2 R^{2\kappa}}{c_B \mu^2} \Big) \int_{B(x,R)} |\nabla u|^2 + c_1 (R^{\frac{2-\delta}{2}} \|f\|_{M,\gamma,x,\Omega,R_e} + \sum_{i=1}^{d} \|f_i\|_{M,\gamma+\delta,x,\Omega,R_e})^2 R^{\gamma+\delta}$$

for all $0 < r \leq R \leq R_e$, where $c_1 = 8\mu^{-2}(2 + 4c_B)$.

Since $(1 + 2c_B) M^2 R^{2\kappa} (c_B \mu^2)^{-1} \leq \varepsilon_0$ for all $R \leq R_0$, by Lemma 3.18 there exists a $c_2 > 0$, only depending on c_B, d and $\gamma + \delta$, such that

$$\int_{B(x,r)} |\nabla u|^2 \leq c_2 \Big(4c_B \Big(\frac{r}{R}\Big)^{\gamma+\delta} \int_{B(x,R)} |\nabla u|^2 + c_1 (\varepsilon^{2-\delta} \|f\|_{M,\gamma,x,\Omega,R_e} + \sum_{i=1}^{d} \|f_i\|_{M,\gamma+\delta,x,\Omega,R_e})^2 r^{\gamma+\delta} \Big)$$

for all $\varepsilon \in (0, 1]$ and $0 < r \leq R \leq R_e\, R_0\, \varepsilon^2$. If $0 < r \leq R_e\, R_0 \varepsilon^2$ then choose $R = R_e\, R_0\, \varepsilon^2$ and we get

$$\begin{aligned} \int_{B(x,r)} |\nabla u|^2 &\leq c_2 \Big(4c_B R^{-(\gamma+\delta)} \int_{B(x,R)} |\nabla u|^2 + c_1 (\varepsilon^{2-\delta} \|f\|_{M,\gamma,x,\Omega,R_e} + \sum_{i=1}^{d} \|f_i\|_{M,\gamma+\delta,x,\Omega,R_e})^2 \Big) r^{\gamma+\delta} \\ &\leq c_3 \Big((\varepsilon^{-(\gamma+\delta)} \|\nabla u\|^2_{L_2(\Omega)})^2 + (\varepsilon^{2-\delta} \|f\|_{M,\gamma,x,\Omega,R_e} + \sum_{i=1}^{d} \|f_i\|_{M,\gamma+\delta,x,\Omega,R_e})^2 \Big) r^{\gamma+\delta}, \end{aligned}$$

where $c_3 = c_2(4c_B\, R_e^{-(\gamma+\delta)}\, R_0^{-(\gamma+\delta)} + c_1)$. Alternatively if $r \geq R_e\, R_0\, \varepsilon^2$ then

$$\int_{B(x,r)} |\nabla u|^2 \leq \|\nabla u\|^2_{L_2(\Omega)} \leq R_e^{-(\gamma+\delta)}\, R_0^{-(\gamma+\delta)} (\varepsilon^{-(\gamma+\delta)} \|\nabla u\|^2_{L_2(\Omega)}) r^{\gamma+\delta}.$$

The result follows by combining the above two equations. Note that the constant c_3 only depends on $\mu, M, d, \gamma + \delta$ and R_e. □

We now give the transformed version of the above estimate near the boundary. The extra parameter $\tilde{\gamma}$ will be used in Chapter 6. We may take $\tilde{\gamma} = \gamma + \delta$ for the meantime.

Proposition 3.22. *Let $\gamma, \tilde{\gamma} \in [0, d)$, $K \geq 1$, $\delta \in (0, 2]$, $\mu, M > 0$. Suppose that $\gamma + \delta < d$ and $\gamma + \delta \leq \tilde{\gamma}$. Then there exists a $c > 0$ such that the following valid.*

Let $\Omega, U \subset \mathbb{R}^d$ be open. Let Φ be a standard $C^{1+\kappa}$-diffeomorphism from U onto E. Suppose that K is larger than the Lipschitz constant for Φ and Φ^{-1}. Moreover, suppose *that* $|||(D\Phi)_{ij}|||_{C^\kappa} \leq K$ *and* $|||(D(\Phi^{-1}))_{ij}|||_{C^\kappa} \leq K$ *for all* $i, j \in \{1, \ldots, d\}$. *Let* $C \in \mathcal{E}_p^\kappa(\Omega, \mu, M)$, $u, g \in W^{1,2}(\Omega)$ *and* $f, f_1, \ldots, f_d \in L_2(\Omega)$, *and suppose that*

$$\sum_{k,l=1}^{d} \int_\Omega c_{kl}\,(\partial_k u)\,\overline{\partial_l v} = \int_\Omega f\,\overline{v} - \sum_{k=1}^{d} \int_\Omega f_k\,\overline{\partial_k v} + \int_\Gamma \beta\,\mathrm{Tr}\,g\,\overline{\mathrm{Tr}\,v} \tag{3.17}$$

for all $v \in W^{1,2}(\Omega)$. *Define* $\tilde{u}\colon E^- \to \mathbb{C}$ *by* $\tilde{u} = u \circ \Phi^{-1}$. *Then*

$$\begin{aligned}\|\nabla \tilde{u}\|_{M,\gamma+\delta,x,E^-,1} \leq c\Big(&\varepsilon^{2-\delta}\|f \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} + \sum_{k=1}^{d} \|f_k \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} \\ &+ \varepsilon^{2-\delta}\,\|\nabla(g \circ \Phi^{-1})\|_{M,\gamma,x,E^-,1} + \varepsilon^{\tilde{\gamma}-\gamma-\delta}\|g \circ \Phi^{-1}\|_{M,\tilde{\gamma},x,E^-,1} \\ &+ \varepsilon^{-(\gamma+\delta)}\|\nabla u\|_{L_2(\Omega)}\Big).\end{aligned}$$

for all $x \in \frac{1}{2}E^-$ *and* $\varepsilon \in (0, 1]$.

Proof. The proof is a modification of the proof of [ER15] Proposition 6.5.

Define $C^\Phi : E^- \to \mathbb{C}^{d \times d}$ by:

$$C^\Phi(y) = \frac{1}{\big|\det(D\Phi)(\Phi^{-1}(y))\big|}\,(D\Phi)(\Phi^{-1}(y))\;C(\Phi^{-1}(y))\;\big(D\Phi\big)^T(\Phi^{-1}(y)). \tag{3.18}$$

By [ER15] Proposition 4.3 we know that $(C^\Phi)_{kl}$ is the corresponding transformed principal coefficients, with $C^\Phi \in \mathcal{E}_p^\kappa(E^-, (d!\,K^{d+2})^{-1}\mu, d!\,d^2\,K^{d+2}\,M)$.

Let $x \in \frac{1}{2}E^-$ and $0 < R \leq 1$. By the Dirichlet-type Poincaré inequality of Lemma 3.19(b) and the Lax–Milgram theorem there exists a unique $\tilde{\chi} \in W^{1,2}_{P(x,R)}(E^-(x, R))$ such that

$$\sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x)\,(\partial_k \tilde{\chi})\,\overline{\partial_l w} = \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x)\,(\partial_k(u \circ \Phi^{-1}))\,\overline{\partial_l w}$$

for all $w \in W^{1,2}_{P(x,R)}(E^-(x, R))$. Define $\chi\colon \Omega \to \mathbb{C}$ by

$$\chi(y) = \begin{cases} \tilde{\chi}(\Phi(y)) & \text{if } y \in \Phi^{-1}(E^-(x, R)), \\ 0 & \text{if } y \in \Omega \setminus \Phi^{-1}(E^-(x, R)). \end{cases}$$

Then $v \in W^{1,2}(\Omega)$ by in [ER15] Lemma 6.4. Moreover,

$$\begin{aligned}&\sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x)\,(\partial_k \tilde{\chi})\,\overline{\partial_l \tilde{\chi}} \\ = \;&\sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x)\,(\partial_k(u \circ \Phi^{-1}))\,\overline{\partial_l \tilde{\chi}}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x)\,(\partial_k \tilde{u})\,\overline{\partial_l \hat{\chi}} \\
&= \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}\,(\partial_k \tilde{u})\,\overline{\partial_l \tilde{\chi}} + \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big((C^\Phi)_{kl}(x) - (C^\Phi)_{kl}\Big)\,(\partial_k \tilde{u})\,\overline{\partial_l \tilde{\chi}} \\
&= \sum_{k,l=1}^{d} \int_{\Omega} c_{kl}\,(\partial_k u)\,\overline{\partial_l \chi} + \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big((C^\Phi)_{kl}(x) - (C^\Phi)_{kl}\Big)\,(\partial_k \tilde{u})\,\overline{\partial_l \tilde{\chi}} \\
&= (f,\chi)_{L_2(\Omega)} + \sum_{k=1}^{d} (f_k, \partial_k \chi)_{L_2(\Omega)} + (\beta \operatorname{Tr} g, \operatorname{Tr} \chi)_{L_2(\Gamma)} \\
&\qquad + \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big((C^\Phi)_{kl}(x) - (C^\Phi)_{kl}\Big)\,(\partial_k \tilde{u})\,\overline{\partial_l \tilde{\chi}}.
\end{aligned}
$$

By using ellipticity, the Cauchy–Schwartz inequality and the Dirichlet-type Poincaré inequality on E^- as in Lemma 3.19(b), one obtains

$$
\begin{aligned}
&(d! K^{d+2})^{-1}\,\mu \int_{E^-(x,R)} |\nabla \tilde{\chi}|^2 \\
&\quad \le d!\,K^d \Big(\int_{E^-(x,R)} |f \circ \Phi^{-1}|^2\Big)^{1/2} \Big(\int_{E^-(x,R)} |\tilde{\chi}|^2\Big)^{1/2} \\
&\qquad + d!\,K^{d+1} \sum_{i=1}^{d} \Big(\int_{E^-(x,R)} |f_i \circ \Phi^{-1}|^2\Big)^{1/2} \Big(\int_{E^-(x,R)} |\partial_i \tilde{\chi}|^2\Big)^{1/2} + \Big|\int_\Gamma \beta \operatorname{Tr} g\, \overline{\operatorname{Tr} \chi}\Big| \\
&\quad \le 2d!\,K^d\,\|f \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1}\, R^{(\gamma+2)/2} \Big(\int_{E^-(x,R)} |\nabla \tilde{\chi}|^2\Big)^{1/2} \\
&\qquad + d!\,K^{d+1} \sum_{i=1}^{d} \|f_i \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1}\, R^{(\gamma+\delta)/2} \Big(\int_{E^-(x,R)} |\nabla \tilde{\chi}|^2\Big)^{1/2} + \Big|\int_\Gamma \beta \operatorname{Tr} g\, \overline{\operatorname{Tr} \chi}\Big|,
\end{aligned}
$$

where we used the Dirichlet-type Poincaré inequality of Lemma 3.19(b) in the last step. We next estimate the boundary integral. Note that by [Neč12] Theorem 2.4.2 there exists $c_1 \ge 1$ such that $\|\operatorname{Tr} w\|_{L_1(\Gamma)} \le c_1 \|w\|_{W^{1,1}(\Omega}$ for all $w \in W^{1,1}(\Omega)$. Using the Dirichlet-type Poincaré inequality on $E^-(x,R)$ again, we have

$$
\begin{aligned}
&\Big|\int_\Gamma \beta \operatorname{Tr} g\, \overline{\operatorname{Tr} \chi}\Big| \\
&\quad \le c_1\,\|\beta\|_{L_\infty(\Gamma)} \|g\,\overline{\chi}\|_{W^{1,1}(\Omega)} \\
&\quad \le c_1\,M \int_\Omega \Big(|g|\,|\chi| + |\nabla g|\,|\chi| + |g|\,|\nabla \chi|\Big) \\
&\quad \le c_1\,d!\,d\,M\,K^{d+1} \int_{E^-(x,R)} \Big(|g \circ \Phi^{-1}|\,|\tilde{\chi}| + |\nabla (g \circ \Phi^{-1})|\,|\tilde{\chi}| + |g \circ \Phi^{-1}|\,|\nabla \tilde{\chi}|\Big)
\end{aligned}
$$

$$\leq c_1\, d!\, d\, M\, K^{d+1}\Big(R^{\tilde\gamma/2}\, \|g\circ\Phi^{-1}\|_{M,\tilde\gamma,x,E^-,1}\cdot 2R + R^{\gamma/2}\, \|\nabla(g\circ\Phi^{-1})\|_{M,\gamma,x,E^-,1}\cdot 2R + R^{\tilde\gamma/2}\, \|g\circ\Phi^{-1}\|_{M,\tilde\gamma,x,E^-,1}\Big)\Big(\int_{E^-(x,R)} |\nabla\tilde\chi|^2\Big)^{1/2}$$

$$\leq 3c_1\, d!\, d\, M\, K^{d+1}\Big(R^{(\tilde\gamma-\gamma-\delta)/2}\, \|g\circ\Phi^{-1}\|_{M,\tilde\gamma,x,E^-,1} + R^{(2-\delta)/2}\, \|\nabla(g\circ\Phi^{-1})\|_{M,\gamma,x,E^-,1}\Big)\cdot R^{(\gamma+\delta)/2}\Big(\int_{E^-(x,R)} |\nabla\tilde\chi|^2\Big)^{1/2}.$$

Lastly, since $|(C^\Phi)_{kl}(x)-(C^\Phi)_{kl}(y)| \leq |||(C^\Phi)_{kl}|||_{C^\kappa}\, |x-y|^\kappa \leq d!\, d^2\, K^{d+2}\, M\, R^\kappa$ for all $k,l\in\{1,\ldots,d\}$ and $y\in E^-(x,R)$, one deduces that

$$\sum_{k,l=1}^{d}\int_{E^-(x,R)} \Big((C^\Phi)_{kl}(x)-(C^\Phi)_{kl}\Big)\,(\partial_k\tilde u)\,\overline{\partial_l\tilde\chi} \leq d!\, d^2\, K^{d+2}\, M\, R^\kappa\Big(\int_{E^-(x,R)} |\nabla(u\circ\Phi^{-1})|^2\Big)^{\frac12}\Big(\int_{B(x,R)} |\nabla\tilde\chi|^2\Big)^{\frac12}.$$

Combining the above we have

$$\int_{E^-(x,R)} |\nabla\tilde\chi|^2 \leq c_2^2\,\mu^{-2}\Big(R^{(2-\delta)/2}\, \|f\circ\Phi^{-1}\|_{M,\gamma,x,E^-,1} + \sum_{i=1}^{d}\|f_i\circ\Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} + R^{(\tilde\gamma-\gamma-\delta)/2}\, \|g\circ\Phi^{-1}\|_{M,\tilde\gamma,x,E^-,1} + R^{(2-\delta)/2}\, \|\nabla(g\circ\Phi^{-1})\|_{M,\gamma,x,E^-,1}\Big)^2 R^{\gamma+\delta} + 2d!^4\, d^4\, K^{4d+8}\, M^2\, R^{2\kappa}\int_{E^-(x,R)} |\nabla(u\circ\Phi^{-1})|^2$$

where $c_2 = 6c_1\, d!^2\, d\, M\, K^{2d+3}$. Next let $r\in(0,R]$. Let $c_{DG}>0$ be as in Corollary 3.16 corresponding to $C^\Phi(x)$. Let ε_0 be as in Lemma 3.18 with respective to the parameters $(a,\alpha,\beta)=(4c_{DG},d,\gamma+\delta)$. Apply De Giorgi estimate to the function $\eta=u-\chi\in W^{1,2}(\Omega)$ like Proposition 3.21 gives

$$\int_{E^-(x,r)} |\nabla(u\circ\Phi^{-1})|^2 \leq 2\int_{E^-(x,r)} |\nabla(\eta\circ\Phi^{-1})|^2 + 2\int_{E^-(x,r)} |\nabla\tilde\chi|^2$$
$$\leq 2c_{DG}\Big(\frac{r}{R}\Big)^d\int_{E^-(x,R)} |\nabla(\eta\circ\Phi^{-1})|^2 + 2\int_{E^-(x,r)} |\nabla\tilde\chi|^2$$
$$\leq 4c_{DG}\Big(\frac{r}{R}\Big)^d\int_{E^-(x,R)} |\nabla(u\circ\Phi^{-1})|^2 + (2+4c_{DG})\int_{E^-(x,R)} |\nabla\tilde\chi|^2$$

$$
\begin{aligned}
&\leq 4c_{DG}\Big(\Big(\frac{r}{R}\Big)^{d} + \frac{(1+2c_{DG})d!^4\, d^4\, K^{4d+8}\, M^2\, R^{2\kappa}}{\mu^2\, c_{DG}}\Big)\int_{E^-(x,R)} |\nabla(u\circ\Phi^{-1})|^2 \\
&\quad + c_3\Big(R^{(2-\delta)/2}\, \|f\circ\Phi^{-1}\|_{M,\gamma,x,E^-,\frac{1}{2}} + \sum_{i=1}^{d} \|f_i\circ\Phi^{-1}\|_{M,\gamma+\delta,x,E^-,\frac{1}{2}} \\
&\qquad + R^{(\tilde\gamma-\gamma-\delta)/2}\, \|g\circ\Phi^{-1}\|_{M,\tilde\gamma,x,E^-,\frac{1}{2}} \\
&\qquad + R^{(2-\delta)/2}\, \|\nabla(g\circ\Phi^{-1})\|_{M,\gamma,x,E^-,\frac{1}{2}}\Big)^2 R^{\gamma+\delta},
\end{aligned}
$$

for all $x\in\frac{1}{2}E^-$ and $0<r\leq R\leq 1$, where $c_3=(2+4c_{DG})c_2^2\,\mu^{-2}$. Choose $R_0\in(0,1)$ such that $(1+2c_{DG})d!^4\, d^4\, K^{4d+8}\, M^2\, R^{2\kappa}(\mu^2\, c_{DG})^{-1}\leq\varepsilon_0$. The above can then be improved by Lemma 3.18 and one deduces that there exists a $c_4>0$, depending only on c_{DG}, $\gamma+\delta$ and d, such that

$$
\begin{aligned}
\int_{E^-(x,r)} |\nabla(u\circ\Phi^{-1})|^2 &\leq c_4\Big(\frac{r}{R}\Big)^{\gamma+\delta}\int_{E^-(x,R)} |\nabla(u\circ\Phi^{-1})|^2 \\
&\quad + c_3\, c_4\Big(\varepsilon^{2-\delta}\, \|f\circ\Phi^{-1}\|_{M,\gamma,x,E^-,1} + \sum_{i=1}^{d} \|f_i\circ\Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} \\
&\qquad + \varepsilon^{\tilde\gamma-\gamma-\delta}\, \|\beta\|_{L_\infty(\Gamma)}\, \|h\circ\Phi^{-1}\|_{M,\tilde\gamma,x,E^-,1} \\
&\qquad + \varepsilon^{2-\delta}\, \|\beta\|_{L_\infty(\Gamma)}\, \|\nabla(h\circ\Phi^{-1})\|_{M,\gamma,x,E^-,1}\Big)^2 r^{\gamma+\delta},
\end{aligned}
$$

uniformly for all $x\in\frac{1}{2}E^-$, $\varepsilon\in(0,1]$ and $0<r\leq R\leq R_0\,\varepsilon^2$. Choosing $R=R_0\,\varepsilon^2$ gives

$$
\begin{aligned}
\int_{E^-(x,r)} |\nabla(u\circ\Phi^{-1})|^2 &\leq 2^{\gamma+\delta}c_4\, (\varepsilon^{-(\gamma+\delta)}\, \|\nabla(u\circ\Phi^{-1})\|_{L_2(E^-)})^2\, r^{\gamma+\delta} \\
&\quad + c_3\, c_4\Big(\varepsilon^{2-\delta}\, \|f\circ\Phi^{-1}\|_{M,\gamma,x,E^-,1} + \sum_{i=1}^{d} \|f_i\circ\Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} \\
&\qquad + \varepsilon^{\tilde\gamma-\gamma-\delta}\|\beta\|_{L_\infty(\Gamma)}\, \|h\circ\Phi^{-1}\|_{M,\tilde\gamma,x,E^-,1} \\
&\qquad + \varepsilon^{2-\delta}\|\beta\|_{L_\infty(\Gamma)}\, \|\nabla(h\circ\Phi^{-1})\|_{M,\gamma,x,E^-,1}\Big)^2 r^{\gamma+\delta},
\end{aligned}
$$

for all $x\in\frac{1}{2}E^-$ and $0<r\leq R_0\,\varepsilon^2$.

The rest of the proof is similarly to the proof of Proposition 3.21, which is a modification of the proof of Proposition 3.2 in [ER15]. □

The above and a Poincaré inequality give conditions which imply that u is κ-Hölder continuous for all $\kappa\in(0,1)$. In the next section, we give estimates for ∇u of order above d such that ∇u is also Hölder continuous.

3.3 Regularity improvement: the $C^{1+\kappa}$ case

In order to obtain κ-Hölder continuity of ∇u we need Campanato estimates of order $d+2\kappa$, so the above propositions are not enough. To cope with this we need Proposition 3.25 which is a modification of Lemma 3.6 and Proposition 3.7 of [EO19]. The difficult part is to adopt the trace term into the estimate.

We will need two more estimates involving Campanato norms.

Lemma 3.23. *Let $\Omega \subset \mathbb{R}^d$ be open and $\gamma \in (d, d+2)$. Let $u, v \in L_2(\Omega) \cap \mathcal{L}_\infty(\Omega)$ and $x \in \Omega$. Then*

$$\begin{aligned}|||u\,v|||^2_{\mathcal{M},\gamma,x,\Omega,1} \leq\ & 2\|u\|^2_{L_\infty(\Omega(x,1))}\,|||v|||^2_{\mathcal{M},\gamma,x,\Omega,1} \\ &+ 2^{d+\gamma}\,|||u|||^2_{C^{(\gamma-d)/2}(\Omega(x,1))}\,\|v\|^2_{L_\infty(\Omega(x,1))} \\ &+ 2^{\gamma}\,\|u\|^2_{L_2(\Omega(x,1))}\,\|v\|^2_{L_\infty(\Omega(x,1))}.\end{aligned}$$

Proof. Let $r \in (0, \frac{1}{2}]$. Then

$$\begin{aligned}&\int_{\Omega(x,r)} |u\,v - \langle u\,v\rangle_{\Omega(x,r)}|^2 \\ &\quad= \int_{\Omega(x,r)} \Big|\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} \Big(u(y)\,(v(y)-v(z)) + v(z)\,(u(y)-u(z))\Big)\,dz\Big|^2\,dy \\ &\quad\leq 2\int_{\Omega(x,r)} \Big|\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} u(y)\,(v(y)-v(z))\,dz\Big|^2\,dy \\ &\qquad+ 2\int_{\Omega(x,r)} \Big|\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} v(z)\,(u(y)-u(z))\,dz\Big|^2\,dy.\end{aligned}$$

The first term can be estimated by

$$\begin{aligned}2\int_{\Omega(x,r)} \Big|\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} u(y)\,(v(y)-v(z))\,dz\Big|^2\,dy &= 2\int_{\Omega(x,r)} \Big||u(y)|\,\Big(v(y)-\langle v\rangle_{\Omega(x,r)}\Big)\Big|^2\,dy \\ &\leq 2\,\|u\|^2_{L_\infty(\Omega(x,1))}\,r^\gamma\,|||v|||^2_{\mathcal{M},\gamma,x,\Omega,1}\end{aligned}$$

and the second by

$$\begin{aligned}&2\int_{\Omega(x,r)} \Big|\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} v(z)\,(u(y)-u(z))\,dz\Big|^2\,dy \\ &\quad\leq 2\int_{\Omega(x,r)} \Big|\frac{1}{|\Omega(x,r)|}\int_{\Omega(x,r)} \|v\|_{L_\infty(\Omega(x,1))}\,|||u|||_{C^{(\gamma-d)/2}(\Omega(x,1))}\,(2r)^{(\gamma-d)/2}\,dz\Big|^2\,dy \\ &\quad= 2\,|\Omega(x,r)|\,(\|v\|_{L_\infty(\Omega(x,1))}\,|||u|||_{C^{(\gamma-d)/2}(\Omega(x,1))}\,(2r)^{(\gamma-d)/2})^2 \\ &\quad\leq 2^{\gamma+1}\,r^\gamma\,\|v\|^2_{L_\infty(\Omega(x,1))}\,|||u|||^2_{C^{(\gamma-d)/2}}.\end{aligned}$$

Alternatively, if $r \in [\frac{1}{2}, 1]$, then

$$\int_{\Omega(x,r)} |u\, v - \langle u\, v\rangle_{\Omega(x,r)}|^2 \leq \int_{\Omega(x,r)} |u\, v|^2$$
$$\leq \|u\|^2_{L_2(\Omega(x,1))} \|v\|^2_{L_\infty(\Omega(x,1))} \leq 2^\gamma\, r^\gamma\, \|u\|^2_{L_2(\Omega(x,1))} \|v\|^2_{L_\infty(\Omega(x,1))}$$

and the statement follows. □

We also need to bound the Campanato seminorm by the Hölder norm on part of the boundary. If $Z \subset \mathbb{R}^{d-1}$ is a set which contains at least two elements $z, w \in Z$ with $0 < |z - w| \leq 1$, and if $\nu \in (0,1)$, then define $|||\cdot|||_{C^\nu(Z)} \colon C(Z) \to [0, \infty]$ by

$$|||\varphi|||_{C^\nu(Z)} = \sup_{\substack{z,w \in Z \\ 0<|z-w|\leq 1}} \frac{|\varphi(z) - \varphi(w)|}{|z-w|^\nu}.$$

Let $C^\nu(Z) = \{\varphi \in C(Z) \mid |||\varphi|||_{C^\nu(Z)} < \infty\}$ be the space of Hölder continuous functions of order ν.

Lemma 3.24. *Suppose $\Omega = \{x \in \mathbb{R}^d \mid x_d < 0\}$ is the half-space. Let $x \in \Omega$ and suppose that $x_d > -1$. Let $\nu \in (0,1)$ and let $\varphi \in C^\nu(\Gamma(x,1))$. Then*

$$\int_{\Gamma(x,R)} |\varphi - \langle\varphi\rangle_{\Gamma(x,R)}|^2 \leq 2^{d-1+2\nu}\, R^{d-1+2\nu}\, |||\varphi|||^2_{C^\nu(\Gamma(x,1))} + 2^{d-1+2\nu}\, R^{d-1+2\nu}\, \|\varphi\|^2_{L_\infty(\Gamma(x,1))}$$

for all $R \in (0,1]$.

Proof. Consider separately the cases $R \in (0, \frac{1}{2})$ and $R \in [\frac{1}{2}, 1]$ and argue as in the proof of Lemma 3.23. □

We will now prove regularity estimates on the transformed space.

Proposition 3.25. *Let $\kappa \in (0,1)$, $K \geq 1$, $\delta \in [0, \kappa]$ and $\mu, M > 0$. Then there exists a $c > 0$ such that the following is valid.*

Let $\Omega, U \subset \mathbb{R}^d$ be open. Let Φ be a standard $C^{1+\kappa}$-diffeomorphism from U onto E. Suppose that K is larger than the Lipschitz constant for $\Phi|_{\Omega\cap U}$ and $\Phi^{-1}|_{E^-}$. Moreover, suppose that $|||(D\Phi)_{ij}|||_{C^\kappa} \leq K$ and $|||(D(\Phi^{-1}))_{ij}|||_{C^\kappa} \leq K$ for all $i, j \in \{1, \ldots, d\}$. Let $C \in \mathcal{E}^\kappa_p(\Omega, \mu, M)$, $u \in W^{1,2}(\Omega)$, $f, f_1, \ldots, f_d \in L_2(\Omega)$ and $h \in C(\Gamma) \cap L_2(\Gamma)$ and suppose that

$$\sum_{k,l=1}^d \int_\Omega c_{kl}\, (\partial_k u)\, \overline{\partial_l v} = \int_\Omega f\, \overline{v} - \sum_{k=1}^d \int_\Omega f_k\, \overline{\partial_k v} + \int_\Gamma h\, \overline{\operatorname{Tr} v} \tag{3.19}$$

for all $v \in W^{1,2}(\Omega)$. *Define* $\tilde{u} \colon E^- \to \mathbb{C}$ *by* $\tilde{u} = u \circ \Phi^{-1}|_{E^-}$. *Then*

$$|||\nabla \tilde{u}|||_{\mathcal{M},\gamma,x,E^-,1} \leq c \Big(\|f \circ \Phi^{-1}\|_{M,\gamma-2,x,E^-,1} + \sum_{k=1}^{d} (|||f_k \circ \Phi^{-1}|||_{\mathcal{M},\gamma,x,E^-,1} + \|f_k \circ \Phi^{-1}\|_{L_\infty(E^-)}) + |||h|||_{C^{(\gamma-d)/2}(\Gamma)} + \|h\|_{L_\infty(\Gamma)} + \|\nabla \tilde{u}\|_{M,d-\delta,x,E^-,1} \Big)$$

for all $x \in \frac{1}{2} E^-$, *where* $\gamma = d + 2\kappa - \delta$.

Proof. For all $r \in (0,1]$ define

$$\Psi(r) = \sum_{k=1}^{d} \int_{E^-(x,r)} |\partial_k \tilde{u} - \langle \partial_k \tilde{u} \rangle_{E^-(x,r)}|^2.$$

Let $R \in (0,1]$. Define $C^\Phi \colon E^- \to \mathbb{C}^{d \times d}$ by:

$$C^\Phi(y) = \frac{1}{\big|\det(D\Phi)(\Phi^{-1}(y))\big|} (D\Phi)(\Phi^{-1}(y))\, C(\Phi^{-1}(y))\, \big(D\Phi\big)^T(\Phi^{-1}(y)). \tag{3.20}$$

Let $c_{DG} > 0$ be as in the De Giorgi estimates (3.9) of Corollary 3.16 respective to C^Φ. We will freeze the coefficients of C^Φ at x and consider the pure second-order operator with constant coefficients $C^\Phi(x)$. Choose $\lambda \in \mathbb{C}$ such that

$$(C^\Phi)_{dd}(x)\, \lambda = \sum_{k=1}^{d} \langle \frac{f_k\, (D\Phi)_{dk}}{|\det(D\Phi)|} \circ \Phi^{-1} \rangle_{E^-(x,R)} + \langle \Theta\, (h \circ \Phi^{-1}) \rangle_{P(x,R)}.$$

Define $T \colon E^- \to \mathbb{C}$ by $T(y) = \lambda\, y_d$. By the Lax-Milgram theorem and Lemma 3.19(b) there exists a unique $\hat{\chi} \in W^{1,2}_{P(x,R)}(E^-(x,R))$ such that

$$\sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x)\, (\partial_k \hat{\chi})\, \overline{\partial_l w} = \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x)\, (\partial_k (\tilde{u} - T))\, \overline{\partial_l w} \tag{3.21}$$

for all $w \in W^{1,2}_{P(x,R)}(E^-(x,R))$. Define $\chi \colon \Omega \to \mathbb{C}$ by

$$\chi(y) = \begin{cases} \hat{\chi}(\Phi(y)) & \text{if } y \in \Phi^{-1}(E^-(x,R)), \\ 0 & \text{if } y \in \Omega \setminus \Phi^{-1}(E^-(x,R)). \end{cases}$$

Then $\chi \in W^{1,2}(\Omega)$ by [ER15] Lemma 6.4. Set $\tilde{\chi} = \chi \circ \Phi^{-1}|_{E^-}$ and $\tilde{\eta} = \tilde{u} - T - \tilde{\chi}$. Clearly we have $\tilde{\eta} \in W^{1,2}(E^-(x,R))$. Moreover, $A_p^{C^\Phi(x)}(\tilde{\eta}) = 0$ weakly on $B(x,R)$ by (3.21) and

Lemma 2.8. Let $r \in (0, R]$. Using the De Giorgi estimates (3.9) of Corollary 3.16, one deduces that

$$\begin{aligned}
&\sum_{k=1}^{d} \int_{E^-(x,r)} |\partial_k \tilde{u} - \langle \partial_k \tilde{u} \rangle_{E^-(x,r)}|^2 \\
&\le \sum_{k=1}^{d} \int_{E^-(x,r)} |\partial_k \tilde{u} - \langle \partial_k \tilde{\eta} \rangle_{E^-(x,r)} - \langle \partial_k T \rangle_{E^-(x,r)}|^2 \\
&\le \sum_{k=1}^{d} 3 \int_{E^-(x,r)} |\partial_k \tilde{\eta} - \langle \partial_k \tilde{\eta} \rangle_{E^-(x,r)}|^2 + \sum_{k=1}^{d} 3 \int_{E^-(x,r)} |\partial_k T - \langle \partial_k T \rangle_{E^-(x,r)}|^2 + 3 \int_{E^-(x,r)} |\nabla \tilde{\chi}|^2 \\
&\le 3 c_{DG} \left(\frac{r}{R}\right)^{d+2} \sum_{k=1}^{d} \int_{E^-(x,R)} |\partial_k \tilde{\eta} - \langle \partial_k \tilde{\eta} \rangle_{E^-(x,R)}|^2 + 3 \int_{E^-(x,R)} |\nabla \tilde{\chi}|^2 \\
&\le 9 c_{DG} \left(\frac{r}{R}\right)^{d+2} \sum_{k=1}^{d} \int_{E^-(x,R)} |\partial_k \tilde{u} - \langle \partial_k \tilde{u} \rangle_{E^-(x,R)}|^2 + (9 c_{DG} + 3) \int_{E^-(x,R)} |\nabla \tilde{\chi}|^2.
\end{aligned}$$

So

$$\Psi(r) \le 9 c_{DG} \left(\frac{r}{R}\right)^{d+2} \Psi(R) + (9 c_{DG} + 3) \int_{E^-(x,R)} |\nabla \tilde{\chi}|^2. \tag{3.22}$$

We next estimate $\int_{E^-(x,R)} |\nabla \tilde{\chi}|^2$. Ellipticity, the equality $\tilde{\chi}|_{E^-(x,R)} = \hat{\chi}|_{E^-(x,R)}$ and (3.21) give

$$\begin{aligned}
&(d! K^{d+2})^{-1} \mu \int_{E^-(x,R)} |\nabla \tilde{\chi}|^2 \\
&\quad \le \operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) \, (\partial_k \hat{\chi}) \, \overline{\partial_l \hat{\chi}} \\
&\quad = \operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) \, (\partial_k (\tilde{u} - T)) \, \overline{\partial_l \hat{\chi}} \\
&\quad = \operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) \, (\partial_k \tilde{u}) \, \overline{\partial_l \hat{\chi}} - \operatorname{Re} \sum_{l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{dl}(x) \, \lambda \, \overline{\partial_l \hat{\chi}}
\end{aligned} \tag{3.23}$$

The first term of (3.23) can be further expanded by (3.20) and (3.19) to obtain

$$\begin{aligned}
&\operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl}(x) \, (\partial_k \tilde{u}) \, \overline{\partial_l \hat{\chi}} \\
&\quad = \operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} (C^\Phi)_{kl} \, (\partial_k \tilde{u}) \, \overline{\partial_l \tilde{\chi}} + \operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big((C^\Phi)_{kl}(x) - (C^\Phi)_{kl} \Big) \, (\partial_k \tilde{u}) \, \overline{\partial_l \tilde{\chi}}
\end{aligned}$$

$$= \operatorname{Re} \sum_{k,l=1}^{d} \int_\Omega c_{kl}\,(\partial_k u)\,\overline{\partial_l \chi} + \operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big((C^\Phi)_{kl}(x) - (C^\Phi)_{kl} \Big)\,(\partial_k \tilde u)\,\overline{\partial_l \tilde\chi}$$

$$= \operatorname{Re}(f,\chi)_{L_2(\Omega)} + \operatorname{Re} \sum_{k=1}^{d} (f_k, \partial_k \chi)_{L_2(\Omega)} + \operatorname{Re}(h, \operatorname{Tr}\chi)_{L_2(\Gamma)}$$

$$+ \operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big((C^\Phi)_{kl}(x) - (C^\Phi)_{kl} \Big)\,(\partial_k \tilde u)\,\overline{\partial_l \tilde\chi}.$$

Moreover,

$$\sum_{k=1}^{d} (f_k, \partial_k \chi)_{L_2(\Omega)} = \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big(\frac{f_k\,(D\Phi)_{lk}}{|\det(D\Phi)|} \circ \Phi^{-1} \Big)\,\overline{\partial_l \tilde\chi}$$

and using (3.10) one deduces that

$$(h, \operatorname{Tr}\chi)_{L_2(\Gamma)} = \int_{P(x,R)} \Theta\,(h \circ \Phi)\,\overline{\operatorname{Tr}\tilde\chi}.$$

If $l \in \{1, \ldots, d-1\}$, then $\int_{E^-(x,R)} \partial_l \tilde\chi = 0$. Alternatively, $\int_{E^-(x,R)} \partial_d \tilde\chi = \int_{P(x,R)} \operatorname{Tr}\tilde\chi$. Hence by the choice of λ one deduces that

$$(d! K^{d+2})^{-1} \mu \int_{E^-(x,R)} |\nabla \tilde\chi|^2 \tag{3.24}$$

$$\le \operatorname{Re}(f,\chi)_{L_2(\Omega)}$$

$$+ \operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big(\frac{f_k\,(D\Phi)_{lk}}{|\det(D\Phi)|} \circ \Phi^{-1} - \langle \frac{f_k\,(D\Phi)_{lk}}{|\det(D\Phi)|} \circ \Phi^{-1} \rangle_{E^-(x,R)} \Big)\,\overline{\partial_l \tilde\chi}$$

$$+ \operatorname{Re} \int_{P(x,R)} \Big(\Theta\,(h \circ \Phi^{-1}) - \langle \Theta\,(h \circ \Phi^{-1}) \rangle_{P(x,R)} \Big)\,\overline{\operatorname{Tr}\tilde\chi}$$

$$+ \operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big((C^\Phi)_{kl}(x) - (C^\Phi)_{kl} \Big)\,(\partial_k \tilde u)\,\overline{\partial_l \tilde\chi}.$$

We estimate the terms separately.

First

$$\operatorname{Re}(f,\chi)_{L_2(\Omega)} \le d!\,K^d \Big(\int_{E^-(x,R)} |f \circ \Phi^{-1}|^2 \Big)^{1/2} \Big(\int_{E^-(x,R)} |\hat\chi|^2 \Big)^{1/2}$$

$$\le d!\,K^d\,\|f \circ \Phi^{-1}\|_{M,\gamma-2,x,E^-,1}\,R^{\frac{\gamma-2}{2}}\;(2R) \Big(\int_{E^-(x,R)} |\nabla \hat\chi|^2 \Big)^{1/2}$$

$$= 2\,d!\,K^d\,\|f \circ \Phi^{-1}\|_{M,\gamma-2,x,E^-,1}\,R^{\frac{\gamma}{2}} \Big(\int_{E^-(x,R)} |\nabla \tilde\chi|^2 \Big)^{1/2},$$

where we used the Poincaré inequality of Lemma 3.19(b).

Secondly, recall that $\det(D\Phi)$ is constant. If $k, l \in \{1, \ldots, d\}$, then it follows from Lemma 3.23 that

$$\begin{aligned}
&\Big(\int_{E^-(x,R)} \Big| \frac{f_k\,(D\Phi)_{lk}}{|\det(D\Phi)|} \circ \Phi^{-1} - \langle \frac{f_k\,(D\Phi)_{lk}}{|\det(D\Phi)|} \circ \Phi^{-1} \rangle_{E^-(x,R)} \Big|^2 \Big)^{1/2} \\
&\quad \le R^{\gamma/2}\, |||\frac{f_k\,(D\Phi)_{lk}}{|\det(D\Phi)|} \circ \Phi^{-1}|||_{\mathcal{M},\gamma,x,E^-,1} \\
&\quad \le 2\, d!\, K^d\, R^{\gamma/2}\, \|(D\Phi)_{lk} \circ \Phi^{-1}\|_{L_\infty(E^-(x,1))}\, |||f_k \circ \Phi^{-1}|||_{\mathcal{M},\gamma,x,E^-,1} \\
&\qquad + 2^{d+\gamma}\, d!\, K^d\, R^{\gamma/2}\, |||(D\Phi)_{lk} \circ \Phi^{-1}|||_{C^{(\gamma-d)/2}(E^-)}\, \|f_k \circ \Phi^{-1}\|_{L_\infty(E^-(x,1))} \\
&\qquad + 2^{\gamma}\, d!\, K^d\, R^{\gamma/2}\, \|(D\Phi)_{lk} \circ \Phi^{-1}\|_{L_2(E^-(x,1))}\, \|f_k \circ \Phi^{-1}\|_{L_\infty(E^-)} \\
&\quad \le 2\, d!\, K^{d+1}\, R^{\gamma/2}\, |||f_k \circ \Phi^{-1}|||_{\mathcal{M},\gamma,x,E^-,1} \\
&\qquad + 2^{d+\gamma}\, d!\, K^{d+2}\, R^{\gamma/2}\, \|f_k \circ \Phi^{-1}\|_{L_\infty(E^-)} \\
&\qquad + 2^{2d+\gamma}\, d!\, K^{d+1}\, R^{\gamma/2}\, \|f_k \circ \Phi^{-1}\|_{L_\infty(E^-)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} &\Big(\frac{f_k\,(D\Phi)_{lk}}{|\det(D\Phi)|} \circ \Phi^{-1} - \langle \frac{f_k\,(D\Phi)_{lk}}{|\det(D\Phi)|} \circ \Phi^{-1} \rangle_{E^-(x,R)} \Big) \overline{\partial_l \tilde\chi} \\
&\le 2^{2d+\gamma}\, d!\, K^{d+2}\, R^{\gamma/2} \Big(\int_{E^-(x,R)} |\nabla \tilde\chi|^2 \Big)^{1/2} \cdot \\
&\qquad \cdot \Big(|||f_k \circ \Phi^{-1}|||_{\mathcal{M},\gamma,x,E^-,1} + \|f_k \circ \Phi^{-1}\|_{L_\infty(E^-)} \Big)
\end{aligned}$$

by the Cauchy–Schwarz inequality.

Thirdly, using Lemmas 3.24 and 3.17 one estimates

$$\begin{aligned}
&\operatorname{Re} \int_{P(x,R)} \Big(\Theta\,(h \circ \Phi^{-1}) - \langle \Theta\,(h \circ \Phi^{-1}) \rangle_{P(x,R)} \Big) \overline{\operatorname{Tr} \tilde\chi} \\
&\quad \le \Big(\int_{P(x,R)} \Big| \Theta\,(h \circ \Phi^{-1}) - \langle \Theta\,(h \circ \Phi^{-1}) \rangle_{P(x,R)} \Big|^2 \Big)^{1/2} \Big(\int_{P(x,R)} |\operatorname{Tr} \tilde\chi|^2 \Big)^{1/2} \\
&\quad \le 2^{\gamma-1}\, R^{(\gamma-1)/2} \Big(|||\Theta\,(h \circ \Phi^{-1})|||_{C^{(\gamma-d)/2}(P(x,1))} + \|\Theta\,(h \circ \Phi^{-1})\|_{L_\infty(P(x,1))} \Big) \cdot \\
&\qquad \cdot (2R)^{1/2} \Big(\int_{E^-(x,R)} |\nabla \tilde\chi|^2 \Big)^{1/2}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|||\Theta\,(h \circ \Phi^{-1})|||_{C^{(\gamma-d)/2}(P(x,1))} &\le \|\Theta\|_{L_\infty(P(x,1))}\, |||h \circ \Phi^{-1}|||_{C^{(\gamma-d)/2}(P(x,1))} \\
&\qquad + |||\Theta|||_{C^{(\gamma-d)/2}(P(x,1))}\, \|h \circ \Phi^{-1}\|_{L_\infty(P(x,1))} \\
&\le 2K^d\, |||h \circ \Phi^{-1}|||_{C^{(\gamma-d)/2}(P)} + K^{d+2}\, \|h \circ \Phi^{-1}\|_{L_\infty(P)}.
\end{aligned}$$

Therefore

$$\operatorname{Re} \int_{P(x,R)} \Big(\Theta\,(h \circ \Phi^{-1}) - \langle \Theta\,(h \circ \Phi^{-1}) \rangle_{P(x,R)} \Big)\, \overline{\operatorname{Tr} \tilde{\chi}}$$
$$\leq 2^{\gamma+2}\, K^{d+2}\, R^{\gamma/2}\, \big(|||h \circ \Phi^{-1}|||_{C^{(\gamma-d)/2}(P)} + \|h \circ \Phi^{-1}\|_{L_\infty(P)} \big) \Big(\int_{E^-(x,R)} |\nabla \tilde{\chi}|^2 \Big)^{1/2}.$$

Fourthly, again we have $|(C^\Phi)_{kl}(x) - (C^\Phi)_{kl}(y)| \leq |||(C^\Phi)_{kl}|||_{C^\kappa}\, |x-y|^\kappa \leq d!\, d^2\, K^{d+2}\, M\, R^\kappa$ for all $k,l \in \{1,\ldots,d\}$ and $y \in E^-(x,R)$, and therefore

$$\operatorname{Re} \sum_{k,l=1}^{d} \int_{E^-(x,R)} \Big((C^\Phi)_{kl}(x) - (C^\Phi)_{kl} \Big)\, (\partial_k \tilde{u})\, \overline{\partial_l \tilde{\chi}}$$
$$\leq d!\, d^2\, K^{d+2}\, M\, R^\kappa \Big(\int_{E^-(x,R)} |\nabla \tilde{u}|^2 \Big)^{1/2} \Big(\int_{E^-(x,R)} |\nabla \tilde{\chi}|^2 \Big)^{1/2}$$
$$\leq d!\, d^2\, K^{d+2}\, M\, R^{\frac{\gamma}{2}}\, \|\nabla \tilde{u}\|_{M,d-\delta,x,E^-,1} \Big(\int_{E^-(x,R)} |\nabla \tilde{\chi}|^2 \Big)^{1/2}.$$

Together (3.24) gives

$$\Big(\int_{E^-(x,R)} |\nabla \tilde{\chi}|^2 \Big)^{1/2} \leq c_0\, c_1\, R^{\frac{\gamma}{2}}, \tag{3.25}$$

where $c_1 = 2^{2d+\gamma}\, \mu^{-1}\, d!^2\, d^2\, K^{2d+4}(1+M)$ and

$$c_0 = \|f \circ \Phi^{-1}\|_{M,\gamma-2,x,E^-,1} + \sum_{k=1}^{d} \big(|||f_k \circ \Phi^{-1}|||_{\mathcal{M},\gamma,x,E^-,1} + \|f_k \circ \Phi^{-1}\|_{L_\infty(E^-)} \big)$$
$$+ |||h \circ \Phi^{-1}|||_{C^{(\gamma-d)/2}(P)} + \|h \circ \Phi^{-1}\|_{L_\infty(P)} + \|\nabla \tilde{u}\|_{M,d-\delta,x,E^-,1}.$$

It follows from (3.22) and (3.25) that

$$\Psi(r) \leq 9 c_{DG} \Big(\frac{r}{R} \Big)^{d+2} \Psi(R) + (9 c_{DG} + 3)\, c_0^2\, c_1^2\, R^\gamma$$

for all $0 < r \leq R \leq 1$. Again these bounds can be improved by Lemma 3.18. It follows that there exists an $a > 0$, depending only of c_{DG}, γ and d, such that

$$\Psi(r) \leq a \Big(\frac{r}{R} \Big)^{\gamma} \Psi(R) + a\,(9 c_{DG} + 3)\, c_0^2\, c_1^2\, r^\gamma$$

for all $0 < r \leq R \leq 1$. By taking $R = 1$ one obtains

$$\begin{aligned} r^{-\gamma}\, \Psi(r) &\leq a\, \Psi(1) + a\,(9 c_{DG} + 3)\, c_0^2\, c_1^2 \\ &\leq a\, \|\nabla \tilde{u}\|_{M,d-\delta,x,E^-,1}^2 + a\,(9 c_{DG} + 3)\, c_0^2\, c_1^2 \\ &\leq a\,(9 c_{DG} + 4)\, c_0^2\, c_1^2 \end{aligned}$$

for all $r \in (0,1]$, as desired. □

We also state the interior version here.

Proposition 3.26. *Let $\kappa \in (0,1)$, $\delta \in [0,\kappa]$, $\mu, M > 0$ and $R_e \in (0,1]$. Then there exists a $c \geq 1$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $C \in \mathcal{E}_p^\kappa(\Omega,\mu,M)$, $u \in W^{1,2}(\Omega)$, $f, f_1, \ldots, f_d \in L_2(\Omega)$ and $h \in L_2(\Gamma)$. Suppose that

$$\int_\Omega \sum_{k,l=1}^d c_{kl}\,(\partial_k u)\,\overline{\partial_l v} = (f,v)_{L_2(\Omega)} + \sum_{k=1}^d (f_k, \partial_k v)_{L_2(\Omega)} + (h, \operatorname{Tr} v)_{L_\infty(\Gamma)}$$

for all $v \in W^{1,2}(\Omega)$. Then

$$|||\nabla u|||_{\mathcal{M},\gamma,x,\Omega,R_e} \leq c\left(\|f\|_{M,\gamma-2,x,\Omega,R_e} + \sum_{k=1}^d |||f_k|||_{\mathcal{M},\gamma,x,\Omega,R_e} + \|\nabla u\|_{M,d-\delta,x,\Omega,R_e}\right)$$

for all $x \in \Omega$ such that $d_\Gamma(x) > R_e$, where $\gamma = d + 2\kappa - \delta$.

Proof. The proof is similar to the proof of Proposition 3.25, using the De Giorgi estimate (3.15) of Proposition 3.7 instead of (3.9) in Corollary 3.16, where no transformation is needed. Since $\int_{E^-(x,R)} \partial_l \tilde{\chi} = 0$ for all $l \in \{1, \ldots, d\}$ the proof is (much) easier. □

3.4 Regularity improvement: the Lipschitz case

We can obtain similar estimates as long as we have a corresponding De Giorgi estimate like Corollary 3.16. For the rest of this section, we give the regularity estimate for Lipschitz domains and operators with real measurable principal coefficients.

Proposition 3.27. *Let $K \geq 1$ and $\mu, M > 0$. There exists a $\kappa \in (0,1)$ such that for all $\gamma, \tilde{\gamma} \in [0,d)$ and $\delta \in (0,2]$ with $\gamma + \delta < d - 2 + 2\kappa$ and $\gamma + \delta \leq \tilde{\gamma}$, there is a $c > 0$ such that the following is valid.*

*Let $\Omega \subset \mathbb{R}^d$ be open. Let $U \subset \mathbb{R}^d$ be an open set and Φ a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\Phi(U) = E$ and $\Phi(\Omega \cap U) = E^-$. Suppose also that K is larger than the Lipschitz constant of $\Phi|_{\Omega\cap U}$ and $\Phi^{-1}|_{E^-}$. Let $C \in \mathcal{E}_p(\Omega,\mu,M)$ be **real** measurable, $u, g \in W^{1,2}(\Omega)$, $\beta \in L_\infty(\Gamma)$ with $\|\beta\|_{L_\infty(\Gamma)} \leq M$ and $f, f_1, \ldots, f_d \in L_2(\Omega)$. Suppose that*

$$\mathfrak{a}_p(u,v) = (f,v)_{L_2(\Omega)} + \sum_{i=1}^d (f_i, \partial_i v)_{L_2(\Omega)} + \int_\Gamma \beta \operatorname{Tr} g\, \overline{\operatorname{Tr} v} \tag{3.26}$$

for all $v \in W^{1,2}(\Omega)$. *Then*

$$\|\nabla \tilde{u}\|_{M,\gamma+\delta,x,E^-,1} \leq c\Big(\varepsilon^{2-\delta}\|f\circ\Phi^{-1}\|_{M,\gamma,x,E^-,1} + \sum_{k=1}^{d}\|f_k\circ\Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} + \varepsilon^{2-\delta}\,\|\nabla(g\circ\Phi^{-1})\|_{M,\gamma,x,E^-,1} + \varepsilon^{\tilde{\gamma}-\gamma-\delta}\|g\circ\Phi^{-1}\|_{M,\tilde{\gamma},x,E^-,1} + \varepsilon^{-(\gamma+\delta)}\|\nabla u\|_{L_2(\Omega)}\Big)$$

for all $x \in \frac{1}{2}E^-$ *and* $\varepsilon \in (0,1]$.

Proof. The transformed coefficients C^Φ is measurable from (3.18). Using [ER15] Lemma 5.1 there exist $\kappa \in (0,1)$ and $c_{DG} > 0$ such that

$$\int_{E^-(x,r)} |\nabla u|^2 \leq c_{DG}\Big(\frac{r}{R}\Big)^{d-2+2\kappa}\int_{E^-(x,R)}|\nabla u|^2$$

for all $x \in \frac{1}{2}E^-$, $0 < r \leq R \leq 1$ and $u \in W^{1,2}(E^-(x,R))$ such that

$$\sum_{k,l=1}^{d}\int_{E^-(x,R)}(C^\Phi)_{kl}\,\partial_k u\,\overline{\partial_l v} = 0$$

for all $v \in W^{1,2}_{P(x,R)}(E^-(x,R))$. The rest of the proof is similar to Proposition 3.22, by using the above De Giorgi estimate instead of Corollary 3.16. □

Again we also give the interior estimate as below.

Proposition 3.28. *Let* $K \geq 1$ *and* $\mu, M > 0$. *There exists a* $\kappa \in (0,1)$ *such that for all* $\gamma \in [0,d)$, $\delta \in (0,2]$ *with* $\gamma+\delta < d-2+2\kappa$ *and* $R_e \in (0,1]$, *there exists a* $c > 0$ *such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ *be open. Let* $C \in \mathcal{E}_p(\Omega,\mu,M)$ *be* ***real*** *measurable,* $u, g \in W^{1,2}(\Omega)$, $\beta \in L_\infty(\Gamma)$ *with* $\|\beta\|_{L_\infty(\Gamma)} \leq M$ *and* $f, f_1, \ldots, f_d \in L_2(\Omega)$. *Suppose that*

$$\mathfrak{a}_p(u,v) = (f,v)_{L_2(\Omega)} + \sum_{i=1}^{d}(f_i,\partial_i v)_{L_2(\Omega)} + \int_\Gamma \beta\,\mathrm{Tr}\,g\,\overline{\mathrm{Tr}\,v} \tag{3.27}$$

for all $v \in W^{1,2}(\Omega)$. *Then*

$$\|\nabla u\|_{M,\gamma+\delta,x,\Omega,R_e} \leq c\Big(\varepsilon^{2-\delta}\|f\|_{M,\gamma,x,\Omega,R_e} + \sum_{k=1}^{d}\|f_k\|_{M,\gamma+\delta,x,\Omega,R_e} + \varepsilon^{-(\gamma+\delta)}\|\nabla u\|_{L_2(\Omega)}\Big).$$

for all $x \in \Omega$ *such that* $d_\Gamma(x) > R_e$ *and* $\varepsilon \in (0,1]$.

3.5 Regularity improvement: the C^1 case

We finish this chapter with another similar estimate, when the domain is C^1 with uniformly continuous principal coefficients. The technique is similar to Proposition 3.22 and Proposition 3.25, but using the full extent of Lemma 3.18.

In this section, assume that $C \in \mathcal{E}_p(\Omega, \mu, M)$ is uniformly continuous. Define the modulus of continuity

$$\omega_C(r) = \sup \{|c_{kl}(x) - c_{kl}(y)| \mid x, y \in \Omega , |x-y| \in (0,r] , k, l \in \{1, \dots d\}\}$$

for all $r \in (0,1)$. Then $\lim_{r\to 0} \omega_C(r) = 0$. The modulus of continuity for the transformed coefficients ω_{C^Φ} is defined similarly.

Again we obtain a regularity estimate near the boundary.

Proposition 3.29. *Let $K \geq 1$, $\mu, M > 0$, $\gamma \in [0,d)$ and $\delta \in (0,2]$ with $\gamma + \delta < d$. There exists a $c > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$. Let $U \subset \mathbb{R}^d$ be open and Φ be a C^1-diffeomorphism such that $\Phi(U) = E$ and $\Phi(\Omega \cap U) = E^-$. Suppose that $\|\Phi\|_{C^1} \leq K$ and $\|\Phi^{-1}\|_{C^1} \leq K$. Let $C \in \mathcal{E}_p(\Omega, \mu, M)$ be uniformly continuous, $u, g \in W^{1,2}(\Omega)$, $\beta \in L_\infty(\Gamma)$ with $\|\beta\|_{L_\infty(\Gamma)} \leq M$ and $f, f_1, \dots, f_d \in L_2(\Omega)$. Furthermore suppose that

$$\mathfrak{a}_p(u,v) = (f,v)_{L_2(\Omega)} + \sum_{i=1}^d (f_i, \partial_i v)_{L_2(\Omega)} + \int_\Gamma \beta \operatorname{Tr} g \, \overline{\operatorname{Tr} v} \tag{3.28}$$

for all $v \in W^{1,2}(\Omega)$. Then

$$\begin{aligned}\|\nabla \tilde{u}\|_{M,\gamma+\delta,x,E^-,1} \leq c \Big(& \varepsilon^{2-\delta} \|f \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} + \sum_{k=1}^d \|f_k \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} \\ & + \varepsilon^{2-\delta} \|\nabla (g \circ \Phi^{-1})\|_{M,\gamma,x,E^-,1} + \|g \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} \\ & + \varepsilon^{-(\gamma+\delta)} \|\nabla u\|_{L_2(\Omega)} \Big)\end{aligned}$$

for all $x \in \frac{1}{2}E^-$ and $\varepsilon \in (0,1]$.

Proof. The proof is the same as in Proposition 3.22.

Define C^Φ as in (3.18). Since Φ is C^1 we know that the transformed coefficients are also uniformly continuous with modulus of continuity $\omega_{C^\Phi}(r) \leq d! \, d\, K^{d+2} \omega_C(Kr)$ for all $r \in (0, \frac{1}{K})$. Let $c_{DG} > 0$ be as in the De Giorgi estimates (3.8) of Corollary 3.16 respective to C^Φ.

Let ε_0 be as in Lemma 3.18 with respective to the parameters $(a, \alpha, \beta) = (4c_{DG}, d, \gamma+\delta)$. We can choose $R_0 \in (0,1)$ such that $w_C(R_0)$ is small enough with respective to ε_0 such that Lemma 3.18 is applicable. The rest is identical to Proposition 3.22. □

Finally, we give the interior estimate as before.

Proposition 3.30. *Let $\mu, M > 0$, $\gamma \in [0, d)$, $\delta \in (0, 2]$ with $\gamma + \delta < d$ and $R_e \in (0, 1]$. Then there exists a $c > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be open. Let $C \in \mathcal{E}_p(\Omega, \mu, M)$ be uniformly continuous, $u, g \in W^{1,2}(\Omega)$, $\beta \in L_\infty(\Gamma)$ with $\|\beta\|_{L_\infty(\Gamma)} \le M$ and $f, f_1, \ldots, f_d \in L_2(\Omega)$. Furthermore suppose that

$$\mathfrak{a}_p(u, v) = (f, v)_{L_2(\Omega)} + \sum_{i=1}^{d} (f_i, \partial_i v)_{L_2(\Omega)} + \int_\Gamma \beta \operatorname{Tr} g \, \overline{\operatorname{Tr} v} \tag{3.29}$$

for all $v \in W^{1,2}(\Omega)$. Then

$$\|\nabla u\|_{M,\gamma+\delta,x,\Omega,R_0} \le c \left(\varepsilon^{2-\delta} \|f\|_{M,\gamma,x,\Omega,R_0} + \sum_{k=1}^{d} \|f_k\|_{M,\gamma+\delta,x,\Omega,R_0} + \varepsilon^{-(\gamma+\delta)} \|\nabla u\|_{L_2(\Omega)} \right)$$

for all $x \in \Omega$ such that $d_\Gamma(x) > R_e$ and $\varepsilon \in (0, 1]$.

Proof. We use the De Giorgi estimate (3.6) of Proposition 3.15 instead of (3.8) of Corollary 3.16. The rest is the same as in Proposition 3.29. Without the transformation the choice of $R_0 \in (0, 1)$ such that $\omega_C(R_0) \le \sqrt{2\, c_{DG}\, \varepsilon_0}\, \mu$ would suffice. □

Chapter 4

Kernel estimates

We can now give the kernel bounds for the elliptic operators using the regularity estimates in Chapter 3. This is done by estimating the semigroups with a Davies perturbation. The aim of this chapter is to obtain **Gaussian kernel bounds** of the form

$$|K_t(x,y)| \leq c\, t^{-\frac{d}{2}}\, e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t}$$

for some $b, c, \omega > 0$, and **Gaussian Hölder kernel bounds** of the form

$$|K_t(x,y) - K_t(x',y')| \leq c\, t^{-\frac{d}{2}} \Big(\frac{|x-x'| + |y-y'|}{t^{1/2} + |x-y|}\Big)^{\kappa} e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t}$$

for some $b, c > 0$ and $\kappa \in (0,1)$. We emphasize that the factor $e^{\omega t}$ cannot be avoided due to the lower order terms as well as the boundary conditions.

Roughly speaking, we will use the Campanato estimate we obtained using tools in Chapter 3 to obtain the L_∞ and C^κ estimates of the semigroup, and that gives the kernel's existence with corresponding bounds.

In this chapter, we will first describe Davies' perturbation and estimate the perturbed semigroups, then we will give the kernel estimates under Lipschitz condition, followed by the estimates with $C^{1+\kappa}$ domain and C^1 domain respectively.

4.1 Davies perturbation

Define

$$\mathcal{D} = \big\{\psi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}) \mid \|\partial_k \psi\|_\infty \leq 1 \text{ and } \|\partial_k\, \partial_l \psi\|_\infty \leq 1 \text{ for all } k, l \in \{1, \dots, d\}\big\}.$$

Given $\psi \in \mathcal{D}$, define a multiplication operator $U_\rho = e^{-\rho\psi}$, then define the perturbed semigroup S^ρ by $S_t^\rho = U_\rho S_t U_{-\rho}$. In order to find its generator let $u \in W^{1,2}(\Omega)$. Then

$$\lim_{t\to 0^+} \frac{1}{t}(S_t^\rho u - u) = \lim_{t\to 0^+} \frac{1}{t}(e^{-\rho\psi} S_t e^{\rho\psi} u - u) = e^{-\rho\psi} \lim_{t\to 0^+} \frac{1}{t}(S_t e^{\rho\psi} u - e^{\rho\psi} u) = U_\rho A U_{-\rho} u.$$

Therefore S^ρ is generated by $-A^{(\rho)} = -U_\rho A U_{-\rho}$. From here we can find an explicit formula for the perturbed form associated with the perturbed operator.

Proposition 4.1. *The operator $A^{(\rho)}$ is the operator associated with the form $\mathfrak{a}_\beta^{(\rho)}$ with form domain $D(\mathfrak{a}_\beta^{(\rho)}) = W^{1,2}(\Omega)$ and*

$$\mathfrak{a}_\beta^{(\rho)}(u,v) = \mathfrak{a}_p(u,v) + \int_\Omega \sum_{k=1}^d \Big(a_k^{(\rho)}\, (\partial_k u)\, \overline{v} + b_k^{(\rho)}\, u\, \overline{(\partial_k v)} \Big) + \int_\Omega a_0^{(\rho)}\, u\, \overline{v} + \int_\Gamma \beta\, (\operatorname{Tr} u)\, \overline{\operatorname{Tr} v}$$

with

$$a_k^{(\rho)} = a_k - \rho \sum_{l=1}^d c_{kl}\, \partial_l \psi \quad , \quad b_k^{(\rho)} = b_k + \rho \sum_{l=1}^d c_{lk}\, \partial_l \psi$$

and

$$a_0^{(\rho)} = a_0 - \rho^2 \sum_{k,l=1}^d c_{kl}\, (\partial_k \psi)\, \partial_l \psi + \rho \sum_{k=1}^d a_k\, \partial_k \psi - \rho \sum_{k=1}^d b_k\, \partial_k \psi.$$

Proof. Note that U_ρ is just a multiplicative operator and is invertible, hence $U_\rho W^{1,2}(\Omega) = W^{1,2}(\Omega)$. Thus $\mathfrak{a}_\beta^{(\rho)}$ is well-defined in $W^{1,2}(\Omega)$.

Again since U_ρ is the multiplication operator with a real smooth function, we have $\mathfrak{a}_\beta^{(\rho)}(u,v) = \mathfrak{a}_\beta(U_{-\rho}u, U_\rho v)$ for all $u, v \in W^{1,2}(\Omega)$. The definition of the associated operator gives the desired formula. □

The point of using such perturbation is to obtain the exponential decay by optimizing ρ and ψ. It turns out that the perturbed semigroup has similar estimates. We start this section with L_2-estimates of the perturbed semigroups.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^d$ be open bounded with Lipschitz boundary Γ. Then for all $\mu, M > 0$, there exist $c_0, \omega_0 > 0$ such that*

$$\begin{aligned} \|S_t^\rho u\|_{L_2(\Omega)} &\le e^{\omega_0(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \\ \|\nabla S_t^\rho u\|_{L_2(\Omega)} &\le c_0\, t^{-1/2} e^{\omega_0(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \\ \|A^{(\rho)} S_t^\rho u\|_{L_2(\Omega)} &\le c_0\, t^{-1}\, e^{\omega_0(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \end{aligned} \tag{4.1}$$

for all $(C, a, b, a_0, \beta) \in \mathcal{E}(\Omega, \mu, M)$, $\beta \in L_\infty(\Gamma)$, $t > 0$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$ with $\|\beta\|_{L_\infty(\Gamma)} \le M$, where S^ρ is the semigroup generated by $-A^{(\rho)}$.

Proof. Without lost of generality we may assume that $\mu \le 1$. By [Neč12] Theorem 2.4.2 there exists a $c_1 > 0$ so that $\|\operatorname{Tr} v\|_{L_1(\Gamma)} \le c_1\, \|v\|_{W^{1,1}(\Omega)}$ for all $v \in W^{1,1}(\Omega)$. Let $u \in L_2(\Omega)$. Then the boundary term can be estimated by

$$\Big| \int_\Gamma \beta\, (\operatorname{Tr} S_t^\rho u)\, \overline{\operatorname{Tr} S_t^\rho u} \Big| \le c_1\, \|\beta\|_{L_\infty(\Gamma)}\, \|(S_t^\rho u)\, \overline{S_t^\rho u}\, \|_{W^{1,1}(\Omega)}$$

$$\leq c_2\,(\|S_t^\rho u\|_{L_2(\Omega)}^2 + 2\|\nabla S_t^\rho u\|_{L_2(\Omega)}\,\|S_t^\rho u\|_{L_2(\Omega)}),$$

where $c_2 = c_1\,M$. Now by ellipticity

$$\begin{aligned}
\mu\,\|\nabla S_t^\rho u\|_{L_2(\Omega)}^2 &\leq \operatorname{Re}\mathfrak{a}_p(S_t^\rho u)\\
&\leq \operatorname{Re}\mathfrak{a}_\beta^{(\rho)}(S_t^\rho u) + 2d\,M\,(1+|\rho|)\,\|\nabla S_t^\rho u\|_{L_2(\Omega)}\,\|S_t^\rho u\|_{L_2(\Omega)}\\
&\qquad + M\,(1+|\rho|)^2\,\|S_t^\rho u\|_{L_2(\Omega)}^2 + c_2\,(\|S_t^\rho u\|^2 + 2\|\nabla S_t^\rho u\|_{L_2(\Omega)}\,\|S_t^\rho u\|_{L_2(\Omega)})\\
&\leq \operatorname{Re}\mathfrak{a}_\beta^{(\rho)}(S_t^\rho u) + 2(1+|\rho|)(d\,M + c_2)\,\|\nabla S_t^\rho u\|_{L_2(\Omega)}\,\|S_t^\rho u\|_{L_2(\Omega)}\\
&\qquad + (1+|\rho|)^2\,(M + c_2)\,\|S_t^\rho u\|_{L_2(\Omega)}^2\\
&\leq \operatorname{Re}\mathfrak{a}_\beta^{(\rho)}(S_t^\rho u) + \frac{\mu}{2}\,\|\nabla S_t^\rho u\|_{L_2(\Omega)}^2 + \omega_1\,(1+\rho^2)\,\|S_t^\rho u\|_{L_2(\Omega)}^2
\end{aligned}$$

for all $t > 0$, where $\omega_1 = \frac{4}{\mu}\,(d\,M + c_2)^2 + 2(M + c_2)$. Therefore

$$\frac{1}{2}\,\mu\,\|\nabla S_t^\rho u\|_{L_2(\Omega)}^2 \leq \operatorname{Re}\mathfrak{a}_\beta^{(\rho)}(S_t^\rho u) + \omega_1\,(1+\rho^2)\,\|S_t^\rho u\|_{L_2(\Omega)}^2. \tag{4.2}$$

Differentiating $\|S_t^\rho u\|_{L_2(\Omega)}^2$ and using (4.2) gives

$$\begin{aligned}
\frac{d}{dt}\,\|S_t^\rho u\|_{L_2(\Omega)}^2 &= -2\operatorname{Re}(A^{(\rho)}\,S_t^\rho u, S_t^\rho u)_{L_2(\Omega)}\\
&= -2\operatorname{Re}\mathfrak{a}_\beta^{(\rho)}(S_t^\rho u) \leq 2\omega_1\,(1+\rho^2)\,\|S_t^\rho u\|_{L_2(\Omega)}^2.
\end{aligned}$$

Hence by Gronwall's lemma

$$\|S_t^\rho u\|_{L_2(\Omega)} \leq e^{\omega_1(1+\rho^2)t}\,\|u\|_{L_2(\Omega)}$$

for all $t > 0$.

For the estimates of $\|A^{(\rho)}\,S_t u\|_{L_2(\Omega)}$ and $\|\nabla S_t u\|_{L_2(\Omega)}$ we follow as in [ER15] Lemma 7.1. We firstly need to prove that S^ρ is holomorphic in a sector. Let $\varphi_0 \in (0, \frac{\pi}{2})$ be such that $\mu\cos\varphi_0 - M\sin\varphi_0 = \frac{1}{2}\mu$. Then for all $|\varphi| \leq \varphi_0$ we estimate $e^{i\varphi}A$, which associates with $e^{i\varphi}\mathfrak{a}_\beta$:

$$\begin{aligned}
\operatorname{Re}(e^{i\varphi}\mathfrak{a}_\beta(u)) &\geq \cos\varphi\operatorname{Re}(e^{i\varphi}\mathfrak{a}_\beta(u)) - \sin\varphi|e^{i\varphi}\mathfrak{a}_\beta(u)|\\
&\geq (\mu\cos\varphi - M\sin\varphi)\|\nabla u\|_{L_2(\Omega)}^2 \geq \frac{1}{2}\mu\|\nabla u\|_{L_2(\Omega)}^2.
\end{aligned}$$

Therefore $e^{i\varphi}A$ is elliptic with ellipticity constant $\frac{1}{2}\mu$ (and bound M). Noting that $e^{i\varphi}A$ perturbs to $e^{i\varphi}A^{(\rho)}$ and generates the semigroup $t \mapsto e^{-t\,e^{i\varphi}A^{(\rho)}} = S_{t\,e^{i\varphi}}^\rho$, repeating the above gives the estimate

$$\|S_{t\,e^{i\varphi}}^\rho u\|_{L_2(\Omega)} \leq e^{2\omega_1(1+\rho^2)t}\,\|u\|_{L_2(\Omega)} \tag{4.3}$$

for all $|\varphi| \le \varphi_0$, $t > 0$ and $u \in L_2(\Omega)$. It follows that the semigroups $S^\rho_{t\,e^{i\varphi}}$ are all strongly continuous, and so S^ρ is holomorphic with angle at least φ_0 by [Kat80] Theorem IX.1.23.

Let $C(t)$ be the circle with centre t and radius $t \sin\varphi_0$. Cauchy's formula gives

$$A^{(\rho)} S^\rho_t = -\frac{1}{2\pi i} \int_{C(t)} \frac{1}{(z-t)^2} S^\rho_z \, dz.$$

Using (4.3) we have

$$\|A^{(\rho)} S^\rho_t u\|_{L_2(\Omega)} \le \frac{1}{t \sin\varphi_0} e^{4\omega_1 (1+\rho^2) t} \, \|u\|_{L_2(\Omega)}.$$

for all $u \in L_2(\Omega)$ and $t > 0$. Finally using (4.2) we get

$$\begin{aligned}
\frac{1}{2}\mu \, \|\nabla S^\rho_t u\|^2_{L_2(\Omega)} &\le \operatorname{Re} \mathfrak{a}^{(\rho)}_\beta (S^\rho_t u) + \omega_1 \, (1+\rho^2) \, \|S^\rho_t u\|^2_{L_2(\Omega)} \\
&\le \frac{1}{t \sin\varphi_0} e^{5\omega_1(1+\rho^2)t} \, \|u\|^2_{L_2(\Omega)} + \omega_1 (1+\rho^2) e^{2\omega_1(1+\rho^2)t} \, \|u\|^2_{L_2(\Omega)} \\
&\le \frac{2}{t \sin\varphi_0} e^{5\omega_1(1+\rho^2)t} \, \|u\|^2_{L_2(\Omega)}
\end{aligned}$$

for all $u \in L_2(\Omega)$ and $t > 0$ as claimed. □

4.2 Semigroup estimates of order $< d$

The layout for the rest of this chapter is pretty similar to Chapter 3. We first give the semigroup estimates for order less than d, then for order above d, all under the $C^{1+\kappa}$ assumption. Then we can get the desired kernel estimates for $C^{1+\kappa}$ domains. The kernel estimates for Lipschitz domain and C^1 domain will follow similarly.

If we have Morrey estimates for $\nabla S^\rho_t u$ of order $\gamma \in (d-2, d)$, then a Poincaré inequality gives Campanato estimates for $S^\rho_t u$ of order $\gamma + 2 \in (d, d+2)$, which gives Hölder continuity. For $C^{1+\kappa}$ domains we have Morrey estimates for all $\gamma \in [0, d)$, so that S^ρ_t maps into the space of ν-Hölder continuous functions for all $\nu \in (0,1)$.

We first give $L_2 \to L_\infty$ and Hölder estimates for the semigroup near Γ. This is by Morrey estimates on $S^\rho_t u$ and $\nabla S^\rho_t u$ of order $\gamma \in (d-2, d)$ by performing an induction from $\gamma = 0$.

Proposition 4.3. *Let $\kappa \in (0,1)$, $K \ge 1$, $\mu, M > 0$ and $\nu \in (0,1)$. Then there exist $c, \omega > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of class $C^{1+\kappa}$ with boundary Γ. Let $U \subset \mathbb{R}^d$ be open and Φ be a standard $C^{1+\kappa}$-diffeomorphism from U onto E such that $\Phi(U \cap \Omega) = E^-$. Suppose that K is larger than the Lipschitz constant for Φ and Φ^{-1}. Moreover, suppose that $|||(D\Phi)_{ij}|||_{C^\kappa} \le K$ and $|||(D(\Phi^{-1}))_{ij}|||_{C^\kappa} \le K$ for all $i, j \in \{1, \ldots, d\}$. Let $(C, a, b, a_0, \beta) \in \mathcal{E}^\kappa(\Omega, \mu, M)$. Then $(S^\rho_t u) \circ \Phi^{-1}$ is continuous on $\frac{1}{2} E^-$. Moreover,

$$|S^\rho_t u(x)| \le c \, t^{-d/4} \, e^{\omega(1+\rho^2)t} \, \|u\|_{L_2(\Omega)}$$

and

$$|(S_t^\rho u)(x) - (S_t^\rho u)(y)| \leq c\, t^{-d/4}\, t^{-\nu/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}\, |x-y|^\nu$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ *and* $x, y \in \Phi^{-1}(\frac{1}{2}\, E^-)$ *with* $|x-y| \leq \frac{1}{2K}$.

Proof. For all $\gamma \in [0, d)$ let $P(\gamma)$ be the hypothesis

> There exist $c, \omega > 0$, depending only on K, μ, M and c_{DG}, such that
>
> $$\|(S_t^\rho u) \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} \leq c\, t^{-\gamma/4}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \tag{4.4}$$
>
> and
>
> $$\|\nabla((S_t^\rho u) \circ \Phi^{-1})\|_{M,\gamma,x,E^-,1} \leq c\, t^{-\gamma/4}\, t^{-1/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \tag{4.5}$$
>
> for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x \in \frac{1}{2}\, E^-$.

Clearly $P(0)$ is valid by Lemma 4.2.

Lemma 4.4. *Let* $\gamma \in [0, d)$ *and suppose that* $P(\gamma)$ *is valid. Let* $\delta \in (0, 2]$ *and suppose that* $\gamma + \delta < d$. *Then* $P(\gamma + \delta)$ *is valid.*

Proof. Let $c_0, \omega_0 > 0$ be as in Lemma 4.2. Let $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x \in \frac{1}{2}\, E^-$. Note that

$$\|(S_t^\rho u) \circ \Phi^{-1}\|_{L_2(E^-)} \leq d!\, K^d\, \|S_t^\rho u\|_{L_2(\Omega)} \leq d!\, K^d\, e^{\omega_0(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}, \tag{4.6}$$

by Lemma 4.2.

We first prove the bounds (4.4) for $P(\gamma + \delta)$. Choose $\varepsilon = t^{1/4}\, e^{-t} \in (0, 1]$. Let c_1 be as in Lemma 3.20. Then it follows from Lemma 3.20, (4.5) and (4.6) that

$$\begin{aligned}
&\|(S_t^\rho u) \circ \Phi^{-1}\|_{\mathcal{M},\gamma+\delta,x,E^-,1} \\
&\quad \leq c_1 (\varepsilon^{2-\delta}\, \|\nabla((S_t^\rho u) \circ \Phi^{-1})\|_{M,\gamma,x,E^-,1} + \varepsilon^{-(\gamma+\delta)}\, \|(S_t^\rho u) \circ \Phi^{-1}\|_{L_2(E^-)}) \\
&\quad \leq c_1\, (\varepsilon^{2-\delta}\, c\, t^{-\gamma/4}\, t^{-1/2}\, e^{\omega(1+\rho^2)t} + \varepsilon^{-(\gamma+\delta)}\, d!\, K^d\, e^{\omega_0(1+\rho^2)t})\, \|u\|_{L_2(\Omega)} \\
&\quad \leq c_2\, t^{-(\gamma+\delta)/4}\, e^{\omega_1(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}
\end{aligned}$$

where $c_2 = c_1\, (c + d!\, K^d)$ and $\omega_1 = \omega_0 + \omega + \gamma + \delta$. By Proposition 2.18 there exists a $c_3 > 0$ such that

$$\|v\|_{M,\gamma+\delta,x,E^-,1} \leq c_3(\|v\|_{\mathcal{M},\gamma+\delta,x,E^-,1} + \|v\|_{L_2(E^-)})$$

for all $x \in \frac{1}{2}\, E^-$ and $v \in L_2(E^-)$. Hence

$$\begin{aligned}
\|(S_t^\rho u) \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} &\leq c_2\, c_3\, t^{-(\gamma+\delta)/4}\, e^{\omega_1(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} + c_3\, d!\, K^d\, e^{\omega_0(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \\
&\leq c_4\, t^{-(\gamma+\delta)/4}\, e^{\omega_2(1+\rho^2)t}\, \|u\|_{L_2(\Omega)},
\end{aligned} \tag{4.7}$$

where $c_4 = c_3(c_2 + d!\,K^d)$ and $\omega_2 = \omega_0 + \omega_1 + d$. This gives the bound (4.4) for $P(\gamma + \delta)$.

In order to obtain (4.5), we use Proposition 3.22. Note that

$$\mathfrak{a}_\beta^{(\rho)}(S_{2t}^\rho u, v) = (S_t^\rho\, A^{(\rho)}\, S_t^\rho u, v)_{L_2(\Omega)}$$

for all $t > 0$ and $v \in W^{1,2}(\Omega)$. It follows from Proposition 4.1 that

$$\mathfrak{a}_p(S_{2t}^\rho u, v) = (f, v)_{L_2(\Omega)} - \sum_{i=1}^d (f_i, \partial_i v)_{L_2(\Omega)} - \int_\Gamma \beta\, (\mathrm{Tr}\, S_{2t}^\rho u)\, \overline{\mathrm{Tr}\, v}$$

for all $v \in W^{1,2}(\Omega)$, where $f_i = b_i^{(\rho)}\, S_{2t} u$ and

$$f = S_t^\rho\, A^{(\rho)}\, S_t^\rho u - a_0^{(\rho)}\, S_{2t}^\rho u - \sum_{i=1}^d a_i^{(\rho)}\, \partial_i\, S_{2t}^\rho u.$$

Apply Proposition 3.22 with $\varepsilon = t^{1/4}\, e^{-t} \in (0, 1]$. The three terms in f are approximated separately using Lemma 4.2 with $\tilde{\gamma} = \gamma + \delta$. First,

$$\begin{aligned}
\varepsilon^{2-\delta}\, \|(S_t^\rho A^{(\rho)} S_t^\rho u) \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} &\le t^{(2-\delta)/4}\, c\, t^{-\gamma/4}\, e^{\omega(1+\rho^2)t}\, \|A^{(\rho)}\, S_t^\rho u\|_{L_2(\Omega)} \\
&\le c_0\, t^{-1}\, e^{\omega_0(1+\rho^2)t}\, t^{(2-\delta)/4}\, c\, t^{-\gamma/4}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \\
&\le c_0\, c\, t^{-(\gamma+\delta)/4}\, t^{-1/2}\, e^{(\omega_0+\omega)(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}.
\end{aligned}$$

Secondly,

$$\begin{aligned}
\varepsilon^{2-\delta}\, \|(a_0^{(\rho)}\, S_{2t}^\rho u) \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} &\le t^{(2-\delta)/4}\, 4M\, (1 + \rho^2)\, c\, (2t)^{-\gamma/4}\, e^{\omega(1+\rho^2)(2t)}\, \|u\|_{L_2(\Omega)} \\
&\le 4c\, M\, t^{-(\gamma+\delta)/4}\, t^{-1/2}\, e^{(2\omega+1)(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}.
\end{aligned}$$

Thirdly,

$$\begin{aligned}
&\varepsilon^{2-\delta}\, \|(\sum_{i=1}^d a_i^{(\rho)}\, \partial_i S_{2t} u) \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} \\
&\qquad \le t^{(2-\delta)/4}\, M(1 + |\rho|)\, \|(\nabla S_{2t}^\rho u) \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} \\
&\qquad \le 2t^{(2-\delta)/4}\, M\, t^{-1/2}\, e^{(1+\rho^2)t}\, K\, \|\nabla((S_{2t}^\rho u) \circ \Phi^{-1})\|_{M,\gamma,x,E^-,1} \\
&\qquad \le 2c\, K\, M\, t^{-(\gamma+\delta)/4}\, t^{-1/2}\, e^{(2\omega+1)(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}.
\end{aligned}$$

The terms with f_i in Proposition 3.22 can be estimated by

$$\begin{aligned}
\sum_{i=1}^d \|(b_i^{(\rho)}\, S_{2t} u) \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} &\le d\, M\, (1 + |\rho|)\, \|(S_{2t}^\rho u) \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} \\
&\le 2d\, M\, c_4\, t^{-(\gamma+\delta)/4}\, t^{-1/2}\, e^{(1+\rho^2)t}\, e^{\omega_2(1+\rho^2)(2t)}\, \|u\|_{L_2(\Omega)},
\end{aligned}$$

where we used (4.7) in the last step. Next,

$$\begin{aligned}\varepsilon^{-(\gamma+\delta)}\,\|\nabla S^\rho_{2t}u\|_{L_2(\Omega)} &\le t^{-(\gamma+\delta)/4}\,e^{(\gamma+\delta)t}\,c_0\,(2t)^{-1/2}\,e^{\omega_0(1+\rho^2)(2t)}\,\|u\|_{L_2(\Omega)}\\ &\le c_0\,t^{-(\gamma+\delta)/4}\,t^{-1/2}\,e^{(2\omega_0+d)(1+\rho^2)t}\,\|u\|_{L_2(\Omega)}.\end{aligned}$$

Finally for the new terms,

$$\begin{aligned}\varepsilon^{2-\delta}\,\|\nabla((S^\rho_{2t}u)\circ\Phi^{-1})\|_{M,\gamma,x,E^-,1} &\le t^{(2-\delta)/4}\,c\,(2t)^{-\gamma/4}\,(2t)^{-1/2}\,e^{\omega(1+\rho^2)(2t)}\,\|u\|_{L_2(\Omega)}\\ &\le c\,t^{(\gamma+\delta)/4}\,t^{-1/2}\,e^{(2\omega+1)(1+\rho^2)t}\,\|u\|_{L_2(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\|(S^\rho_{2t}u)\circ\Phi^{-1}\|_{M,\tilde\gamma,x,E^-,1} &\le c_4\,(2t)^{-(\gamma+\delta)/4}\,e^{\omega_2(1+\rho^2)(2t)}\,\|u\|_{L_2(\Omega)}\\ &\le c_4\,t^{-(\gamma+\delta)/4}\,t^{-1/2}\,e^{(2\omega_2+1)(1+\rho^2)t}\,\|u\|_{L_2(\Omega)},\end{aligned}$$

where (4.5) and (4.7) are used. Now (4.5) for $P(\gamma+\delta)$ follows from Proposition 3.22. □

End of proof of Proposition 4.3. This the same as the proof of Proposition 7.2 in [ER15].

Let $\kappa \in (0,1)$. From $P(d-2+2\nu)$ there exists $c,\omega > 0$ such that

$$\|\nabla((S^\rho_t u)\circ\Phi^{-1})\|_{M,d-2+2\nu,x,E^-,1} \le c\,t^{-(d-2+2\nu)/4}\,t^{-1/2}\,e^{\omega(1+\rho^2)t}\,\|u\|_{L_2(\Omega)}$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x \in \frac{1}{2}E^-$. By the Neumann-type Poincaré inequality in Lemma 3.20 one has

$$\|(S^\rho_t u)\circ\Phi^{-1}\|_{\mathcal{M},d+2\nu,x,E^-,1} \le c_1\,t^{-(d+2\nu)/4}\,e^{\omega(1+\rho^2)t}\,\|u\|_{L_2(\Omega)}$$

where $c_1 = c\sqrt{2c_N}$ and c_N is the constant for Neumann-type Poincaré inequality. Since this is uniform for all $x \in \frac{1}{2}E^-$, we have $(S^\rho_t u)\circ\Phi^{-1}|_{\frac{1}{2}E^-} \in \mathcal{M}_{d+2\nu}(\frac{1}{2}E^-)$ and hence it is continuous. Now we can use Proposition 2.19 to obtain the L^∞-estimate. by choosing $R = t^{1/2}e^{-t} \le 1$, there exists a $c_2 > 0$ only depending on κ and d such that

$$\begin{aligned}|((S^\rho_t u)\circ\Phi)(x)| &\le c_2\,R^\kappa\|(S^\rho_t u)\circ\Phi^{-1}\|_{\mathcal{M},d+2\nu,x,E^-,1} + |\langle (S^\rho_t u)\circ\Phi^{-1}\rangle_{E^-(x,R)}|\\ &\le c_2\,t^{\nu/2}e^{-\nu t}\,c'\,e^{-(d+2\nu)/4}e^{\omega(1+\rho^2)t}\|u\|_{L_2(\Omega)} + \omega_d^{-1/2}R^{-d/2}\|(S^\rho_t u)\circ\Phi\|_{L_2(E^-)}\end{aligned}$$

for all $t > 0$ and $x \in \frac{1}{2}E^-$. Therefore the L^∞-estimate follows from (4.6).

Similarly the Hölder estimate follows from Proposition 2.20: there exists a $c_3 > 0$ such that

$$|((S^\rho_t u)\circ\Phi^{-1})(x) - ((S^\rho_t u)\circ\Phi^{-1})(y)| \le c_3\,t^{-d/4}\,t^{-\nu/2}\,e^{\omega(1+\rho^2)t}\,\|u\|_{L_2(\Omega)}|x-y|^\nu$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x,y \in \frac{1}{2}E^-$ with $|x-y| < \frac{1}{2}$. The claim follows since Φ has Lipschitz constant at most K. □

We now combine with the interior bounds to obtain global bounds for $S^\rho_t u$.

Proposition 4.5. *Adopt the notation and assumptions of Proposition 4.3. Then* $S_t L_2(\Omega) \subset C(\overline{\Omega})$ *for all* $t > 0$*. Moreover, for all* $\nu \in (0,1)$ *there exist* $c, \omega > 0$*, depending only on* ν*,* κ*,* K*,* μ *and* M *such that*

$$|(S^\rho_t u)(x)| \le c\, t^{-d/4}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \tag{4.8}$$

and

$$|(S^\rho_t u)(x) - (S^\rho_t u)(y)| \le c\, t^{-d/4}\, t^{-\nu/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}\, |x-y|^\nu \tag{4.9}$$

for all $t > 0$*,* $u \in L_2(\Omega)$*,* $\rho \in \mathbb{R}$*,* $\psi \in \mathcal{D}$ *and* $x, y \in \Omega$ *with* $|x-y| \le 1$*.*

Proof. The estimate close to the boundary is as in Proposition 4.3 for all $x, y \in \Omega(x_0, \frac{1}{4K})$ for some $x_0 \in \Gamma$. For the interior we set $R_e = \frac{1}{8K}$ so that (4.8) and (4.9) holds for all $x, y \in \Omega$ such that $d_\Gamma(x), d_\Gamma(y) > \frac{1}{8K}$ and $|x-y| \le \frac{1}{16K}$. Combining the boundary and interior estimates we conclude that (4.8) and (4.9) holds for all $|x-y| \le \frac{1}{16K}$, so the claim follows. □

We finish this section by proving existence of the kernel as well as its bounds. First we need a lemma from [ER99b].

Lemma 4.6. *Let* $\Omega \subset \mathbb{R}^d$ *be open. Suppose* $\kappa, \tau' \in (0,1)$*,* $\gamma \in (0,\kappa)$ *and* $b, \omega, \tau > 0$ *. Then there exist* $b', M > 0$ *such that the following is valid.*

Let $c, t > 0$ *and* $f\colon \Omega \times \Omega \to \mathbb{C}$ *a function satisfying*

$$|f(x,y)| \le c\, e^{-b\frac{|x-y|^2}{t}}$$

and

$$|f(x',y') - f(x,y)| \le c\left(\left(\frac{|x-x'|}{\sqrt{t}}\right)^\kappa + \left(\frac{|y-y'|}{\sqrt{t}}\right)^\kappa \right) \tag{4.10}$$

uniformly for all $x, y, x', y' \in \Omega$*. Then one has the intermediate bounds*

$$|f(x',y') - f(x,y)| \le cM \left(\frac{|x-x'| + |y-y'|}{\tau\sqrt{t} + \tau'|x-y|} \right)^\gamma e^{-b'\frac{|x-y|^2}{t}} \tag{4.11}$$

uniformly for all $x, y, x', y' \in \Omega$ *such that* $|x-x'| + |y-y'| \le \tau\sqrt{t} + \tau'|x-y|$*.*

Proof. Let $x, y \in \Omega$, $h, k \in \mathbb{R}^d$ such that $x', y' \in \Omega$ and $|x-x'| + |y-y'| \le \tau\sqrt{t} + \tau'|x-y|$. Then

$$\begin{aligned}
|x-y|^2 &\le (|x'-y'| + |x-x'| + |y-y'|)^2 \\
&\le (|x'-y'| + \tau\sqrt{t} + \tau'|x-y|)^2 \\
&\le \tau(1+\varepsilon)|x-y|^2 + 2(1+\varepsilon^{-1})(|x'-y'| + \tau^2 t)
\end{aligned}$$

for all $\varepsilon > 0$. Set $\varepsilon = (1-\tau^2)(1+\tau^2)^{-1}$ then

$$-|(x') - (y')|^2 \le -\varepsilon(2(1+\varepsilon^{-1}))^{-1}|x-y|^2 + \tau^2 t.$$

Hence we have

$$|f(x',y') - f(x,y)| \le c\, e^{-b\frac{|(x')-(y')|^2}{t}} \le c\, M_1\, e^{-b'\frac{|x-y|^2}{t}}, \tag{4.12}$$

where $M_1 = 1 + e^{b\tau^2}$ ane $b' = \varepsilon\, b(2(1+\varepsilon^{-1}))^{-1}$.

Next, for all $\delta > 0$ we have $|x-y|t^{-1/2} \le a_\delta\, e^{\delta|x-y|^2/t}$, where $a_\delta = 2^{-1}\delta^{-1/2}$, then

$$t^{-1/2}(\sqrt{t} + |x-y|) = 1 + t^{-1/2}|x-y| \le (1+a_\delta)e^{\delta\frac{|x-y|^2}{t}}.$$

Therefore we have

$$\left(\frac{|x-x'|}{\sqrt{t}}\right)^\kappa + \left(\frac{|y-y'|}{\sqrt{t}}\right)^\kappa \le 2(1+a_\delta)\left(\frac{|x-x'|+|y-y'|}{\sqrt{t}+|x-y|}\right)^\kappa e^{\delta\frac{|x-y|^2}{t}}.$$

Combining the above with (4.10) one establishes

$$|f(x',y') - f(x,y)| \le cM_\delta\left(\frac{|x-x'|+|y-y'|}{\sqrt{t}+|x-y|}\right)^\kappa e^{2\delta\frac{|x-y|^2}{t}}, \tag{4.13}$$

where $M_\delta = 2(1+a_\delta) + (1+a_\delta)^2(\tau+\tau')^\kappa$.

Now the lemma follows by an interpolation between (4.12) and (4.13). Let $\theta = \gamma\kappa^{-1}$ and set $\delta = (1-\theta)b'/(4\theta)$. Then

$$\begin{aligned}|f(x',y') - f(x,y)| &= |f(x',y') - f(x,y)|^{1-\theta}|f(x',y') - f(x,y)|^\theta \\ &\le cM_1^{1-\theta}M_\delta^\theta\left(\frac{|x-x'|+|y-y'|}{\sqrt{t}+|x-y|}\right)^{\theta\kappa} e^{-b''\frac{|x-y|^2}{t}}\end{aligned}$$

where $b'' = b'(1-\theta) - 2\delta\theta > 0$. Note that the bound is not uniform with respect to γ because a_δ diverges as $\gamma \to \kappa$. $\square$

Since we have Hölder estimates for all $\nu \in (0,1)$ we do not lose any Hölder index after all. The appendix of [EO15] provided an alternative which no Hölder index is lost, and we will use that in the next section.

Proposition 4.7. *Adopt the notation and assumptions of Proposition 4.3. Then the semigroup S has a kernel K mapping from $(0,\infty)\times\Omega\times\Omega$ into $\mathbb{C}$ that is continuous. Moreover there exist $b, c, \omega > 0$ such that*

$$|K_t(x,y)| \le c\, t^{-d/2}\, e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t} \tag{4.14}$$

and

$$|K_t(x,y) - K_t(x',y')| \le c\, t^{-d/2}\left(\frac{|x-x'|+|y-y'|}{t^{1/2}+|x-y|}\right)^\nu e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t} \tag{4.15}$$

for all $x, x', y, y' \in \Omega$ and $t > 0$ with $|x-x'| + |y-y'| \le \tau\, t^{1/2} + \tau'\,|x-y|$.

Proof. We first prove the existence of the kernel and prove the L_∞-bounds. Let $c, \omega > 0$ be as in Proposition 4.5.

From Lemma 2.7 we extrapolate using the dual semigroup: $\|S_t^\rho\|_{1\to 2} = \|(S_t^\rho)^*\|_{2\to\infty} \le c\, t^{-d/4}\, e^{\omega(1+\rho^2)t}\, e^{\omega t}$. Combining the two we get the $L_1 \to L_\infty$ estimate:

$$\|S_{2t}^\rho\|_{1\to\infty} \le \|S_t^\rho\|_{2\to\infty}\|S_t^\rho\|_{1\to 2} \le c^2\, t^{-d/2}\, e^{2\omega(1+\rho^2)t} \tag{4.16}$$

for all $t > 0$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$. Dunford-Pettis gives the existence of a kernel $K_t \in L_\infty(\Omega \times \Omega)$ such that

$$(S_t u)(x) = \int_\Omega K_t(x, y) u(y)\, dy \tag{4.17}$$

for all $t > 0$ and $u \in L_1(\Omega) \cap L_2(\Omega)$. Meanwhile, S_{2t}^ρ also has kernel $(x, y) \mapsto K_{2t}(x, y) e^{-\rho(\psi(x)-\psi(y))}$, so we estimate

$$|K_{2t}(x, y)| \le \|S_{2t}^\rho\|_{1\to\infty} e^{\rho(\psi(x)-\psi(y))} \le c^2\, t^{-d/2}\, e^{2\omega(1+\rho^2)t}\, e^{-\rho(\psi(x)-\psi(y))}$$

for all $\rho \in [0, \infty)$. There exists a $c_0 > 0$ such that

$$\sup\{\psi(x) - \psi(y) \colon \psi \in \mathcal{D}\} \ge c_0\, |x - y|$$

for all $x, y \in \mathbb{R}^d$, so

$$|K_{2t}(x, y)| \le c^2\, t^{-d/2}\, e^{2\omega(1+\rho^2)t}\, e^{-\rho\, c_0 |x-y|}$$

for all $x, y \in \Omega$. Optimizing over $\rho \in [0, \infty)$ gives

$$|K_{2t}(x, y)| \le c^2\, t^{-d/2}\, e^{-\frac{c_0^2 |x-y|^2}{8\omega t}}\, e^{2\omega t},$$

which gives the Gaussian bound of the kernel.

To obtain the Hölder bound we first perturb the first coordinate. Let $\nu' = \frac{1+\nu}{2}$. For $x, x' \in \Omega$ we have

$$\begin{aligned} |(S_{2t})u(x) - (S_{2t})u(x')| &\le c\, t^{-d/4}\, t^{-\nu'/2}\, e^{\omega t}\, |x - x'|^{\nu'}\, \|S_t u\|_{L_2(\Omega)} \\ &\le c^2\, t^{-d/2}\, t^{-\nu'/2}\, e^{2\omega t}\, |x - x'|^{\nu'}\, \|u\|_{L_1(\Omega)} \end{aligned}$$

for all $u \in L_1(\Omega) \cap L_2(\Omega)$. Dunford-Pettis then gives

$$|K_{2t}(x, y) - K_{2t}(x', y)| \le c^2\, t^{-d/2}\, t^{-\nu'/2}\, e^{2\omega t}\, |x - x'|^{\nu'}$$

In order to perturb the second variable note that the dual semigroup S_t^* admits the kernel given by $(t, x, y) \mapsto \overline{K_t(y, x)}$. To see this let $u, v \in L_2(\Omega)$. By Fubini's theorem we have

$$(S_t^* u, v)_{L_2(\Omega)} = (u, S_t v)_{L_2(\Omega)} = \int_\Omega u(x) \int_\Omega K_t(x, y)\, \overline{g(y)}\, dy\, dx = \int_\Omega \int_\Omega \overline{K_t(x, y)} f(x)\, dx\, g(y)\, dy$$

for all $u, v \in L_2(\Omega)$. By estimating the kernel for S_t^* we have

$$|K_{2t}(x,y') - K_{2t}(x,y')| \le c^2\, t^{-d/2}\, t^{-\nu'/2}\, e^{2\omega t}\, |y-y'|^{\nu'}$$

Now apply Lemma 4.6 on the function $t^{d/2}\, e^{-2\omega t} K_{2t}$. There exist $b', M > 0$ such that

$$|K_{2t}(x',y') - K_{2t}(x,y)| \le c^2\, M\, t^{-d/2} \left(\frac{|x-x'| + |y-y'|}{\tau\sqrt{t} + \tau'|x-y|} \right)^{\nu} e^{-b' \frac{|x-y|^2}{t}}\, e^{2\omega t}$$

for all $x, y, x', y' \in \Omega$ such that $|x-x'| + |y-y'| \le \tau\sqrt{t} + \tau'|x-y|$. □

4.3 Semigroup estimates: the $C^{1+\kappa}$ case

In order to prove that K_t is differentiable with Hölder continuous derivatives, we need the Campanato estimates on ∇S_t^ρ of order $\gamma \in (d, d+2\kappa]$. This is derived from Proposition 3.25 and 3.26.

Proposition 4.8. *Let $\kappa \in (0,1)$, $K \ge 1$ and $\mu, M > 0$. There exist $c, \omega > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of class $C^{1+\kappa}$ with boundary Γ. Let $U \subset \mathbb{R}^d$ be open and Φ be a standard $C^{1+\kappa}$-diffeomorphism from U onto E such that $\Phi(U \cap \Omega) = E^-$. Suppose that K is larger than the Lipschitz constant for Φ and Φ^{-1}. Moreover, suppose that $|||(D\Phi)_{ij}|||_{C^\kappa} \le K$ and $|||(D(\Phi^{-1}))_{ij}|||_{C^\kappa} \le K$ for all $i, j \in \{1, \ldots, d\}$. Let $(C, a, b, a_0, \beta) \in \mathcal{E}^\kappa(\Omega, \mu, M)$. Then $\nabla((S_t^\rho u) \circ \Phi^{-1})$ is continuous on $\frac{1}{2}\, E^-$. Moreover we have

$$|\nabla S_t^\rho u(x)|_{L_\infty(\Phi^{-1}(\frac{1}{2}\, E^-))} \le c\, t^{-d/4}\, t^{-1/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}$$

and

$$|(\nabla S_t^\rho u)(x) - (\nabla S_t^\rho u)(y)| \le c\, t^{-d/4}\, t^{-1/2}\, t^{-\kappa/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}\, |x-y|^\kappa$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \Phi^{-1}(\frac{1}{2}\, E^-)$ with $|x-y| \le \frac{1}{2K}$.

Proof. We use Proposition 3.25. Note that

$$\mathfrak{a}_\beta^{(\rho)}(S_{2t}^\rho u, v) = (S_t^\rho\, A^{(\rho)}\, S_t^\rho u, v)_{L_2(\Omega)}$$

for all $v \in W^{1,2}(\Omega)$ and

$$\mathfrak{a}_p(S_{2t}^\rho u, v) = (f, v)_{L_2(\Omega)} - \sum_{k=1}^{d} (f_k, \partial_k v)_{L_2(\Omega)} - \int_\Gamma \beta\, (\operatorname{Tr} S_{2t}^\rho u)\, \overline{\operatorname{Tr} v}$$

for all $v \in W^{1,2}(\Omega)$, where $f_k = b_k^{(\rho)}\, S_{2t} u$ and

$$f = S_t^\rho\, A^{(\rho)}\, S_t^\rho u - a_0^{(\rho)}\, S_{2t}^\rho u - \sum_{k=1}^{d} a_k^{(\rho)}\, \partial_k\, S_{2t}^\rho u.$$

Moreover, $S^\rho_{2t}u$ extends to a Hölder continuous function on $\overline{\Omega}$ for any exponent up to 1 by (4.15). Let $\delta \in [0,\kappa]$. By Proposition 3.25 there is a $c_1 > 0$, depending only on κ, K, μ, M and δ, such that

$$
\begin{aligned}
&|||\nabla((S^\rho_{2t}u) \circ \Phi^{-1})|||_{\mathcal{M},\gamma,x,E^-,1} \qquad (4.18)\\
&\leq c_1 \Big(\|f \circ \Phi^{-1}\|_{M,\gamma-2,x,E^-,1} + \sum_{k=1}^d (|||f_k \circ \Phi^{-1}|||_{\mathcal{M},\gamma,x,E^-,1} + \|f_k \circ \Phi^{-1}\|_{L_\infty(E^-)})\\
&\qquad + |||h|||_{C^{(\gamma-d)/2}(\Gamma)} + \|h\|_{L_\infty(\Gamma)} + \|\nabla((S^\rho_{2t}u) \circ \Phi^{-1})\|_{M,d-\delta,x,E^-,1}\Big)
\end{aligned}
$$

for all $x \in \frac{1}{2}E^-$, where $\gamma = d + 2\kappa - \delta$ and $h = \beta \operatorname{Tr} S^\rho_{2t}u$.

It follows from Proposition 4.5 that

$$
\begin{aligned}
|||h|||_{C^{(\gamma-d)/2}(\Gamma)} &\leq \|\beta\|_{L_\infty(\Gamma)} |||\operatorname{Tr} S^\rho_{2t}u|||_{C^{(\gamma-d)/2}(\Gamma)} + |||\beta|||_{C^{(\gamma-d)/2}(\Gamma)} \|\operatorname{Tr} S^\rho_{2t}u\|_{L_\infty(\Gamma)}\\
&\leq M\,c\,(2t)^{-d/4}\,(2t)^{-(\gamma-d)/4}\,e^{\omega(1+\rho^2)(2t)}\,\|u\|_{L_2(\Omega)} + M\,c\,(2t)^{-d/4}\,e^{\omega(1+\rho^2)(2t)}\,\|u\|_{L_2(\Omega)}\\
&\leq 2M\,t^{-d/4}\,t^{-(\gamma-d)/4}\,e^{(2\omega+1)(1+\rho^2)t}
\end{aligned}
$$

since β is κ-Hölder continuous.

Similarly we have

$$\|h\|_{L_\infty(\Gamma)} \leq M\,t^{-d/4}\,t^{-(\gamma-d)/4}\,e^{(2\omega+1)(1+\rho^2)t}.$$

The other terms in (4.18) can be estimated similarly as in the proof of Proposition 4.3. Note that the term with $|||f_k \circ \Phi^{-1}|||_{\mathcal{M},\gamma,x,E^-,1}$ requires that the b_k are Hölder continuous and the ψ are twice differentiable. Hence there are suitable $c_2, \omega_2 > 0$ such that

$$|||\nabla((S^\rho_{2t}u) \circ \Phi^{-1})|||_{\mathcal{M},\gamma,x,E^-,1} \leq c_2\,t^{-d/4}\,t^{-(\gamma-d)/4}\,e^{\omega_2(1+\rho^2)t}$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x \in \frac{1}{2}E^-$. The claim follows in similar argument at the end of Proposition 4.3. □

Similar to Section 4.2, we can now obtain the L_∞ and Hölder estimates of ∇S_t.

Proposition 4.9. *Adopt the notation and assumptions of Proposition 4.8. Then $S_tL_2(\Omega) \subset C^1(\overline{\Omega})$ for all $t > 0$. Moreover, there exist $c, \omega > 0$, depending only on γ, κ, K, μ and M, such that*

$$|(\partial_k S^\rho_t u)(x)| \leq c\,t^{-d/4}\,t^{-1/2}\,e^{\omega(1+\rho^2)t}\,\|u\|_{L_2(\Omega)} \qquad (4.19)$$

and

$$|(\partial_k S^\rho_t u)(x) - (\partial_k S^\rho_t u)(y)| \leq c\,t^{-d/4}\,t^{-1/2}\,t^{-\gamma/2}\,e^{\omega(1+\rho^2)t}\,\|u\|_{L_2(\Omega)}\,|x-y|^\gamma \qquad (4.20)$$

for all $k \in \{1,\ldots,d\}$, $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \Omega$ with $|x-y| \leq 1$.

Proof. Proposition 3.26 can be used to obtain the interior estimates for ∇S^ρ_t like Proposition 4.8. The bounds can be combined just like Proposition 4.5. □

Corollary 4.10. *Adopt the notation and assumptions of Theorem 4.8. Then there are $c, \omega > 0$, depending only on κ, K, μ and M, such that*

$$|(U_\rho\, \partial_k S_t\, U_{-\rho} u)(x)| \leq c\, t^{-d/4}\, t^{-1/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \tag{4.21}$$

and

$$|(U_\rho\, \partial_k S_t\, U_{-\rho} u)(x) - (U_\rho\, \partial_k S_t\, U_{-\rho} u)(y)| \leq c\, t^{-d/4}\, t^{-1/2}\, e^{\omega(1+\rho^2)t} \Big(\Big(\frac{|x-y|}{\sqrt{t}}\Big)^\kappa + |x-y| \Big)\, \|u\|_{L_2(\Omega)} \tag{4.22}$$

for all $k \in \{1, \ldots, d\}$, $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \Omega$.

Proof. First $U_\rho\, \partial_k S_t\, U_{-\rho} = \partial_k S^\rho_t + \rho\, (\partial_k \psi)\, S^\rho_t$. Moreover,

$$\begin{aligned} &|((\partial_k\psi)\, S^\rho_t u)(x) - ((\partial_k\psi)\, S^\rho_t u)(y)| \\ &\quad \leq |(\partial_k\psi)(x) - (\partial_k\psi)(y)|\, |(S^\rho_t u)(x)| + |(\partial_k\psi)(y)|\, |(S^\rho_t u)(x) - (S^\rho_t u)(y)|. \end{aligned}$$

Using the elementary estimates $|\rho| \leq t^{-1/2}\, e^{\rho^2 t}$, $\|\partial_k\psi\|_\infty \leq 1$ and $|(\partial_k\psi)(x) - (\partial_k\psi)(y)| \leq d\,|x-y|$ for all $t > 0$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \Omega$ with $|x-y| \leq 1$, the corollary follows from Propositions 4.5 and 4.9. □

Finally we are able to prove the main kernel estimate. In order to recover the full Hölder exponent, we cannot repeat the proof for Proposition 4.7 any more. Instead we express the kernel explicitly using [AE19]. After that the kernel estimate follows from the proof for the appendix of [EO15].

Theorem 4.11. *Let $\kappa \in (0,1)$, $K \geq 1$, $\mu, M, \tau > 0$, $\tau', \nu, \nu^* \in (0,1)$ and α, β be multi-indices. Suppose that $|\alpha| + \nu \leq 1 + \kappa$ and $|\beta| + \nu^* \leq 1 + \kappa$. Then here exist $b, c, \omega > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of class $C^{1+\kappa}$ with boundary Γ. Suppose for all $x \in \Gamma$ there exist open $U \subset \mathbb{R}^d$ and a standard $C^{1+\kappa}$-diffeomorphism Φ from U onto E such that $\Phi(\Omega \cap U) = E^-$ and $\Phi(x) = 0$. Suppose that K is larger than the Lipschitz constant for $\Phi|_{\Omega\cap U}$ and $\Phi^{-1}|_{E^-}$. Moreover, suppose that $|||(D\Phi)_{ij}|||_{C^\kappa} \leq K$ and $|||(D(\Phi^{-1}))_{ij}|||_{C^\kappa} \leq K$ for all $i, j \in \{1, \ldots, d\}$. Let $(C, a, b, a_0, \beta) \in \mathcal{E}^\kappa(\Omega, \mu, M)$. Let A be the operator associated with the form $\mathfrak{a}_\beta$ given in (2.2). Then there exists a function $(t, x, y) \mapsto K_t(x, y)$ from $(0, \infty) \times \Omega \times \Omega$ into $\mathbb{C}$ such that the following is valid.

I. *The function $(t, x, y) \mapsto K_t(x, y)$ is continuous from $(0, \infty) \times \Omega \times \Omega$ into $\mathbb{C}$.*

II. *For all $t \in (0, \infty)$ the function K_t is the kernel of the operator e^{-tA}.*

III. *For all $t \in (0,\infty)$ the function K_t is once differentiable in each variable and the derivative with respect to one variable is differentiable in the other variable. Moreover,*

$$|(\partial_x^\alpha \, \partial_y^\beta \, K_t)(x,y)| \leq c \, t^{-d/2} \, t^{-(|\alpha|+|\beta|)/2} \, e^{-b \, \frac{|x-y|^2}{t}} \, e^{\omega t}$$

and

$$\begin{aligned} &|(\partial_x^\alpha \, \partial_y^\beta \, K_t)(x,y) - (\partial_x^\alpha \, \partial_y^\beta \, K_t)(x',y')| \\ &\qquad \leq c \, t^{-d/2} \, t^{-(|\alpha|+|\beta|)/2} \left(\left(\frac{|x-x'|}{t^{1/2} + |x-y|} \right)^{\nu} + \left(\frac{|y-y'|}{t^{1/2} + |x-y|} \right)^{\nu^*} \right) e^{-b \frac{|x-y|^2}{t}} \, e^{\omega t} \end{aligned}$$

for all $x, y, x', y' \in \Omega$ and $t > 0$ with $|x-x'| + |y-y'| \leq \tau \sqrt{t} + \tau' \, |x-y|$.

We emphasize that one may choose $|\alpha| = |\beta| = 1$ with $\nu = \kappa$. This is the best possible for $C^{1+\kappa}$-domains.

Proof. It follows from Proposition 4.3 that $S_t L_2(\Omega) \subset C(\overline{\Omega})$ for all $t > 0$ and similarly for the dual semigroup. Hence for all $x \in \Omega$ and $t > 0$ there exist $k_{x,t}, \tilde{k}_{x,t} \in L_2(\Omega)$ such that

$$(S_t u)(x) = (u, k_{x,t})_{L_2(\Omega)} \quad \text{and} \quad (S_t^* u)(x) = (u, \tilde{k}_{x,t})_{L_2(\Omega)}$$

for all $u \in L_2(\Omega)$. For all $t > 0$ define $K_{2t} \colon \Omega \times \Omega \to \mathbb{C}$ by

$$K_{2t}(x,y) = (\tilde{k}_{y,t}, k_{x,t})_{L_2(\Omega)}.$$

Then $(S_{2t} u)(x) = (S_t \, (S_t^*)^* u)(x) = \int_\Omega K_{2t}(x,y) \, u(y) \, dy$ for all $x \in \Omega$ and $u \in L_2(\Omega)$ by Proposition 2.3 and (1) in [AE19]. It follows again from Proposition 4.3 that there are $c, \omega > 0$ such that

$$\|k_{x,t}\|_{L_2(\Omega)} \leq c \, t^{-d/4} \, e^{\omega t} \quad \text{and} \quad \|k_{x,t} - k_{y,t}\|_{L_2(\Omega)} \leq c \, t^{-(d+1)/4} \, e^{\omega t} \, |x-y|^{1/2} \tag{4.23}$$

for all $x, y \in \Omega$ and $t > 0$ with $|x-y| \leq 1$. By Lemma 4.2 there exist $c_1, \omega_1 > 0$ such that $\|A \, S_t\|_{L_2(\Omega) \to L_2(\Omega)} \leq c_1 \, t^{-1} \, e^{\omega_1 t}$ for all $t > 0$. Let J be a compact interval with $J \subset (0,\infty)$. Because $\frac{d}{dt}(S_{2t} u)(x) = -(S_t \, A \, S_t u)(x)$ one deduces from the mean value theorem that there exists a $c_2 > 0$ such that

$$\|k_{x,t} - k_{x,s}\|_{L_2(\Omega)} \leq c_2 \, |s-t| \tag{4.24}$$

for all $s, t \in J$ and $x \in \Omega$. Now let $s, t \in J$ and $x, x', y, y' \in \Omega$ with $|x - x'| \leq 1$ and $|y - y'| \leq 1$. Then obviously

$$\begin{aligned} K_{2t}(x,y) - K_{2s}(x',y') = {} & (\tilde{k}_{y,t} - \tilde{k}_{y,s}, k_{x,t})_{L_2(\Omega)} + (\tilde{k}_{y,s} - \tilde{k}_{y',s}, k_{x,t})_{L_2(\Omega)} \\ & + (\tilde{k}_{y',s}, k_{x,t} - k_{x',t})_{L_2(\Omega)} + (\tilde{k}_{y',s}, k_{x',t} - k_{x',s})_{L_2(\Omega)}. \end{aligned}$$

Using (4.23) and (4.24) and similar estimates for $\tilde{k}$, it follows that the map $(t,x,y) \mapsto K_{2t}(x,y)$ from $J \times \Omega \times \Omega$ into $\mathbb{C}$ is continuous. This proves Statements I and II.

The last statement requires much more work. Let $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$, $t > 0$ and $x \in \Omega$ Then it follows from Propositions 4.5, Corollary 4.10 and the Riesz representation theorem that there exist $k^{\alpha,\rho}_{x,t}, \tilde{k}^{\alpha,\rho}_{x,t} \in L_2(\Omega)$ such that

$$(U_\rho \, \partial^\alpha S_t \, U_{-\rho} u)(x) = (u, k^{\alpha,\rho}_{x,t})_{L_2(\Omega)} \quad \text{and} \quad (U_\rho \, \partial^\beta S^*_t \, U_{-\rho} u)(x) = (u, \tilde{k}^{\beta,\rho}_{x,t})_{L_2(\Omega)}$$

for all $u \in L_2(\Omega)$. Moreover, it follows from Proposition 4.5 and Corollary 4.10 that there exist $c, \omega > 0$, depending only on κ, K, μ, M, ν and ν^*, such that

$$\|k^{\alpha,\rho}_{x,t}\|_{L_2(\Omega)} \leq c \, t^{-d/4} \, t^{-|\alpha|/2} \, e^{\omega(1+\rho^2)t} \tag{4.25}$$

and

$$\|k^{\alpha,\rho}_{x,t} - k^{\alpha,\rho}_{y,t}\|_{L_2(\Omega)} \leq c \, t^{-d/4} \, t^{-|\alpha|/2} \, e^{\omega(1+\rho^2)t} \Big(\Big(\frac{|x-y|}{\sqrt{t}} \Big)^\nu + |x-y| \Big) \tag{4.26}$$

for all $t > 0$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \Omega$, with similar estimates for the dual semigroup.

Let $t > 0$. By [AE19] Proposition 2.3 there exists a separately continuous function $K^{(\alpha,\beta)}_{2t} \colon \overline{\Omega} \times \overline{\Omega} \to \mathbb{C}$ such that

$$\Big((\partial^\alpha S_t)(\partial^\beta S^*_t)^* u \Big)(x) = \int_\Omega K^{(\alpha,\beta)}_{2t}(x,y) \, u(y) \, dy$$

for all $u \in L_2(\Omega)$ and $x \in \overline{\Omega}$. Next we use the Davies perturbation. Let $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$. Then the kernel of $U_\rho \, (\partial^\alpha S_t)(\partial^\beta S^*_t)^* \, U_{-\rho}$ is the function $(x,y) \mapsto e^{-\rho\psi(x)} \, K^{(\alpha,\beta)}_{2t}(x,y) \, e^{\rho\psi(y)}$. It follows from (1) in [AE19] that $(x,y) \mapsto (\tilde{k}^{\beta,-\rho}_{y,t}, \tilde{k}^{\alpha,\rho}_{x,t})_{L_2(\Omega)}$ is the kernel of the operator $(U_\rho \, \partial^\alpha S_t \, U_{-\rho}) \, (U_{-\rho} \, \partial^\beta S^*_t \, U_\rho)^*$. Since

$$U_\rho \, (\partial^\alpha S_t)(\partial^\beta S^*_t)^* \, U_{-\rho} = (U_\rho \, \partial^\alpha S_t \, U_{-\rho}) \, (U_{-\rho} \, \partial^\beta S^*_t \, U_\rho)^*$$

one deduces that

$$e^{-\rho\psi(x)} \, K^{(\alpha,\beta)}_{2t}(x,y) \, e^{\rho\psi(y)} = (\tilde{k}^{\beta,-\rho}_{y,t}, \tilde{k}^{\alpha,\rho}_{x,t})_{L_2(\Omega)}$$

and consequently

$$K^{(\alpha,\beta)}_{2t}(x,y) = e^{\rho(\psi(x)-\psi(y))} \, (\tilde{k}^{\beta,-\rho}_{y,t}, \tilde{k}^{\alpha,\rho}_{x,t})_{L_2(\Omega)} \tag{4.27}$$

for all $x, y \in \Omega$ and $t > 0$.

Using the bounds (4.25) and similar ones for the dual semigroup, one deduces that there are $c_1, \omega_1 > 0$, depending only on κ, K, μ and M, such that

$$|K^{(\alpha,\beta)}_{2t}(x,y)| \leq e^{\rho(\psi(x)-\psi(y))} \, c_1 \, t^{-d/2} \, t^{-(|\alpha|+\beta|)/2} \, e^{\omega_1(1+\rho^2)t}$$

for all $t > 0$, $x, y \in \Omega$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$. There exists a $c_0 > 0$ such that

$$\sup\{\psi(x) - \psi(y) \colon \psi \in \mathcal{D}\} \geq c_0 \, |x-y| \tag{4.28}$$

for all $x, y \in \mathbb{R}^d$. Then

$$|K^{(\alpha,\beta)}_{2t}(x,y)| \leq c_1 \, t^{-d/2} \, t^{-(|\alpha|+\beta|)/2} \, e^{-\rho c_0 |x-y|} e^{\omega_1(1+\rho^2)t}$$

for all $t > 0$, $x, y \in \Omega$ and $\rho \in [0, \infty)$. Choosing $\rho = \frac{c_0 |x-y|}{2\omega_1 t}$ gives

$$|K_{2t}^{(\alpha,\beta)}(x,y)| \le c_1 \, t^{-d/2} \, t^{-(|\alpha|+\beta|)/2} \, e^{-\frac{c_0^2 |x-y|^2}{4\omega_1 t}} \, e^{\omega_1 t} \tag{4.29}$$

for all $t > 0$ and $x, y \in \Omega$.

Next we turn to Hölder estimates. Using once more (4.27) one deduces that

$$\begin{aligned}
&|K_{2t}^{(\alpha,\beta)}(x,y) - K_{2t}^{(\alpha,\beta)}(x',y')| \\
&\quad = e^{\rho(\psi(x)-\psi(y))} \Big((\tilde{k}_{y,t}^{\beta,-\rho} - \tilde{k}_{y',t}^{\beta,-\rho}, \tilde{k}_{x,t}^{\alpha,\rho})_{L_2(\Omega)} + (\tilde{k}_{y',t}^{\beta,-\rho}, \tilde{k}_{x,t}^{\alpha,\rho} - \tilde{k}_{x',t}^{\alpha,\rho})_{L_2(\Omega)} \Big) \\
&\qquad + (\tilde{k}_{y',t}^{\beta,-\rho}, \tilde{k}_{x',t}^{\alpha,\rho})_{L_2(\Omega)} \, e^{\rho(\psi(x')-\psi(y'))} \Big(e^{\rho((\psi(x)-\psi(x'))-(\psi(y)-\psi(y')))} - 1 \Big)
\end{aligned}$$

for all $t > 0$, $x, y, x', y' \in \Omega$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$. Clearly

$$\Big| e^{\rho((\psi(x')-\psi(x))-(\psi(y')-\psi(x')))} - 1 \Big| \le |\rho| \, (|x-x'| + |y-y'|) \, e^{|\rho| \, (|x-x'|+|y-y'|)}$$

and $(\tilde{k}_{y',t}^{\beta,-\rho}, \tilde{k}_{x',t}^{\alpha,\rho})_{L_2(\Omega)} \, e^{\rho(\psi(x')-\psi(y'))} = K_{2t}^{(\alpha,\beta)}(x',y')$ Using the bounds (4.25) and (4.26), and similar ones for the dual semigroup, one deduces that there are $c_2, \omega_2 > 0$, depending only on κ, K, μ, M, ν and ν^*, such that

$$\begin{aligned}
&|K_{2t}^{(\alpha,\beta)}(x,y) - K_{2t}^{(\alpha,\beta)}(x',y')| \\
&\le e^{\rho(\psi(x)-\psi(y))} \, c_2 \, t^{-d/2} \, t^{-(|\alpha|+\beta|)/2} \, e^{\omega_2(1+\rho^2)t} \Big(\Big(\frac{|x-x'|}{\sqrt{t}} \Big)^{\nu} + |x-x'| + \Big(\frac{|y-y'|}{\sqrt{t}} \Big)^{\nu^*} + |y-y'| \Big) \\
&\qquad + |\rho| \, (|x-x'| + |y-y'|) \, e^{|\rho| \, (|x-x'|+|y-y'|)} \, |K_{2t}^{(\alpha,\beta)}(x',y')|
\end{aligned}$$

for all $t > 0$, $x, y, x', y' \in \Omega$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}$. If $\rho \ge 0$, then minimising over ψ gives

$$\begin{aligned}
&|K_{2t}^{(\alpha,\beta)}(x,y) - K_{2t}^{(\alpha,\beta)}(x',y')| \\
&\le e^{-\rho c_0 |x-y|} \, c_2 \, t^{-d/2} \, t^{-(|\alpha|+\beta|)/2} \, e^{\omega_2(1+\rho^2)t} \Big(\Big(\frac{|x-x'|}{\sqrt{t}} \Big)^{\nu} + |x-x'| + \Big(\frac{|y-y'|}{\sqrt{t}} \Big)^{\nu^*} + |y-y'| \Big) \\
&\qquad + |\rho| \, (|x-x'| + |y-y'|) \, e^{|\rho| \, (|x-x'|+|y-y'|)} \, |K_{2t}^{(\alpha,\beta)}(x',y')|
\end{aligned} \tag{4.30}$$

for all $t > 0$, $x, y, x', y' \in \Omega$ and $\rho \in [0, \infty)$, where we used (4.28).

Now let $t > 0$, $x, y, x', y' \in \Omega$ and suppose that $|x-x'| + |y-y'| \le \tau \sqrt{t} + \tau' \, |x-y|$. Let $\varepsilon \in (0,1]$ and $\lambda \in [1, \infty)$, to be chosen later. Choose $\rho = \frac{c_0 |x-y|}{2\lambda\omega_2 t}$. Then

$$-\rho \, c_0 \, |x-y| + \omega_2 \, \rho^2 \, t = -\frac{c_0^2 \, |x-y|^2}{\omega_2 \, t \, \lambda} \Big(\frac{1}{2} - \frac{1}{4\lambda} \Big) \le -\frac{c_0^2 \, |x-y|^2}{4\omega_2 \, t \, \lambda}.$$

Note that $\frac{|x-y|}{\sqrt{t}} \le \varepsilon^{-1/2} \, e^{\varepsilon \frac{|x-y|^2}{t}}$, so

$$t^{1/2} + |x-y| = t^{1/2}(1 + \frac{|x-y|}{\sqrt{t}}) \le 2t^{1/2} \, \varepsilon^{-1/2} \, e^{\varepsilon \frac{|x-y|^2}{t}}$$

and

$$\frac{1}{t^{1/2}} \le 2\frac{1}{t^{1/2}+|x-y|}\,\varepsilon^{-1/2}\,e^{\varepsilon\frac{|x-y|^2}{t}}. \tag{4.31}$$

Hence

$$|x-x'|+|y-y'| \le \tau\sqrt{t}+\tau'\,|x-y| \le 2e^t\,(\tau+\tau')\,\varepsilon^{-1/2}\,e^{\varepsilon\frac{|x-y|^2}{t}}$$

for all $\varepsilon\in(0,1]$. Then

$$|x-x'| = \Big(\frac{|x-x'|}{\sqrt{t}}\Big)^{\nu} t^{\nu/2}\,|x-x'|^{1-\nu} \le 2\Big(\frac{|x-x'|}{\sqrt{t}}\Big)^{\nu} e^{2t}(\tau+\tau')^{1-\nu}\,\varepsilon^{-1/2}\,e^{\varepsilon\frac{|x-y|^2}{t}} \tag{4.32}$$

Therefore the first term in (4.30) can be bounded by

$$\begin{aligned}
&2c_2\,t^{-d/2}\,t^{-(|\alpha|+\beta|)/2}\,e^{-\frac{c_0^2|x-y|^2}{4\omega_2 t\lambda}}\,e^{(\omega_2+2)t}\Big(\Big(\frac{|x-x'|}{\sqrt{t}}\Big)^{\nu}+\Big(\frac{|y-y'|}{\sqrt{t}}\Big)^{\nu^*}\Big)(1+\tau+\tau')\,\varepsilon^{-1/2}\,e^{\varepsilon\frac{|x-y|^2}{t}}\\
&\quad\le 4c_2\,(1+\tau+\tau')\,\varepsilon^{-1}\,t^{-d/2}\,t^{-(|\alpha|+\beta|)/2}\,\cdot\\
&\qquad\cdot\Bigg(\Big(\frac{|x-x'|}{t^{1/2}+|x-y|}\Big)^{\nu}+\Big(\frac{|y-y'|}{t^{1/2}+|x-y|}\Big)^{\nu^*}\Bigg)e^{-\frac{c_0^2}{4\omega_2\lambda}\frac{|x-y|^2}{t}}\,e^{2\varepsilon\frac{|x-y|^2}{t}}\,e^{(\omega_2+2)t},
\end{aligned}$$

where we used (4.31) in the last step.

The second term in (4.30) can be estimated as in the appendix of [EO15]. For the convenience of the reader we include the proof. By (4.29) there are $b_3, c_3, \omega_3 > 0$, depending only of K, μ and M, such that

$$|K_{2t}^{(\alpha,\beta)}(x',y')| \le c_3\,t^{-d/2}\,t^{-(|\alpha|+\beta|)/2}\,e^{-b_3\frac{|x'-y'|^2}{t}}\,e^{\omega_3 t}.$$

Since $\tau'<1$, there are $\delta,\eta>0$ such that $(1+\eta)(1+\delta)(\tau')^2<1$. Then $|x-x'-(y-y')| \le \tau\sqrt{t}+\tau'\,|x-y|$ and

$$\begin{aligned}
|x-y|^2 &\le (1+\eta^{-1})\,|x'-y'|^2+(1+\eta)\,|x-x'-(y-y')|^2\\
&\le (1+\eta^{-1})\,|x'-y'|^2+(1+\eta)\,\Big((1+\delta)\,(\tau')^2\,|x-y|^2+(1+\delta^{-1})\,\tau^2 t\Big).
\end{aligned}$$

So

$$\Big(1-(1+\eta)(1+\delta)(\tau')^2\Big)|x-y|^2 \le (1+\eta^{-1})\,|x'-y'|^2+(1+\eta)\,(1+\delta^{-1})\,\tau^2 t$$

and

$$-b_3\,\frac{|x'-y'|^2}{t} \le -\tilde b\,\frac{|x-y|^2}{t}+\tilde\omega,$$

where $\tilde b = b_3\,(1-(1+\eta)(1+\delta)(\tau')^2)(1+\eta^{-1})^{-1}$ and $\tilde\omega = (1+\eta)\,(1+\delta^{-1})\,\tau^2\,(1+\eta^{-1})^{-1}$.

Next (4.32) and (4.31) give

$$\begin{aligned}
&|\rho|\,(|x-x'|+|y-y'|)\\
&\quad= \frac{c_0\,|x-y|}{2\lambda\,\omega_2\,t}(|x-x'|+|y-y'|)
\end{aligned}$$

$$\leq \frac{c_0}{2\lambda\,\omega_2}\,\varepsilon^{-1/2}\,e^{\varepsilon\frac{|x-y|^2}{t}}\;2e^{2t}\,(1+\tau+\tau')\,\varepsilon^{-1/2}e^{\varepsilon\frac{|x-y|^2}{t}}\,\Big(\Big(\frac{|x-x'|}{\sqrt{t}}\Big)^{\nu}+\Big(\frac{|y-y'|}{\sqrt{t}}\Big)^{\nu^*}\Big)$$

$$\leq \frac{2c_0}{\lambda\,\omega_2}\,\varepsilon^{-3/2}\,(1+\tau+\tau')\,e^{2t}\,e^{3\varepsilon\frac{|x-y|^2}{t}}\,\left(\Big(\frac{|x-x'|}{t^{1/2}+|x-y|}\Big)^{\nu}+\Big(\frac{|y-y'|}{t^{1/2}+|x-y|}\Big)^{\nu^*}\right).$$

Alternatively,

$$|\rho|\,(|x-x'|+|y-y'|)\leq \frac{c_0\,|x-y|}{2\lambda\,\omega_2\,t}\,(\tau\sqrt{t}+\tau'\,|x-y|)$$

$$\leq \frac{c_0}{2\lambda\,\omega_2}\,(\tau'+\tau)\,\frac{|x-y|^2}{t}+\frac{c_0\,\tau}{8\lambda\,\omega_2}.$$

Hence the second term can be bounded by

$$c_4\,\varepsilon^{-3/2}\,e^{\frac{c_0\tau}{8\lambda\,\omega_2}}\,t^{-d/2}\,t^{-(|\alpha|+|\beta|)/2}\cdot$$

$$\cdot\left(\Big(\frac{|x-x'|}{t^{1/2}+|x-y|}\Big)^{\nu}+\Big(\frac{|y-y'|}{t^{1/2}+|x-y|}\Big)^{\nu^*}\right)e^{-\tilde{b}\,\frac{|x-y|^2}{t}}\,e^{3\varepsilon\,\frac{|x-y|^2}{t}}\,e^{\frac{c_0}{2\lambda\,\omega_2}(\tau'+\tau)\,\frac{|x-y|^2}{t}}\,e^{\omega_4 t},$$

where $c_4=\frac{2c_0\,c_3}{\omega_2}\,e^{\tilde{\omega}}\,(1+\tau+\tau')$ and $\omega_4=\omega_3+2$. Choose $\lambda\geq 1$ and $\varepsilon\in(0,1]$ suitably, it follows that there are $b,c,\omega>0$, depending only on κ, K, μ, M, ν and ν^*, such that

$$|(K_{2t}^{(\alpha,\beta)})(x,y)-(K_{2t}^{(\alpha,\beta)})(x',y')|$$

$$\leq c\,t^{-d/2}\,t^{-(|\alpha|+|\beta|)/2}\left(\Big(\frac{|x-x'|}{t^{1/2}+|x-y|}\Big)^{\nu}+\Big(\frac{|y-y'|}{t^{1/2}+|x-y|}\Big)^{\nu^*}\right)e^{-b\frac{|x-y|^2}{t}}\,e^{\omega t}.$$

In particular, the kernel $K_{2t}^{(\alpha,\beta)}$ is (Hölder) continuous on $\Omega\times\Omega$.

Let $t>0$, $v\in C_c^\infty(\Omega)$ and $u\in L_2(\Omega)$. Then $(S_t\partial^\beta v,u)_{L_2(\Omega)}=(\partial^\beta v,S_t^*u)_{L_2(\Omega)}=(-1)^{|\beta|}(v,\partial^\beta S_t^*u)_{L_2(\Omega)}=(-1)^{|\beta|}((\partial^\beta S_t^*)^*v,u)_{L_2(\Omega)}$. So $S_t\partial^\beta v=(-1)^{|\beta|}(\partial^\beta S_t^*)^*v$. Now suppose that $u\in C_c^\infty(\Omega)$. Then

$$(-1)^{|\alpha|+|\beta|}\int_\Omega\int_\Omega K_{2t}(x,y)\,(\partial^\alpha u)(x)\,(\partial^\beta v)(y)\,dx\,dy=(-1)^{|\alpha|+|\beta|}\,(S_{2t}\,\partial^\beta v,\overline{\partial^\alpha u})_{L_2(\Omega)}$$

$$=((\partial^\alpha S_t)(\partial^\beta S_t^*)^*v,\overline{u})_{L_2(\Omega)}$$

$$=\int_\Omega\int_\Omega K_{2t}^{(\alpha,\beta)}(x,y)\,u(x)\,v(y)\,dx\,dy.$$

Hence by density

$$(-1)^{|\alpha|+|\beta|}\int_{\Omega\times\Omega}K_{2t}(x,y)\,(\partial_x^\alpha\partial_y^\beta w)(x,y)\,d(x,y)=\int_{\Omega\times\Omega}K_{2t}^{(\alpha,\beta)}(x,y)\,w(x,y)\,d(x,y)$$

for all $w \in C_c^\infty(\Omega \times \Omega)$. Therefore $K_{2t}^{(\alpha,\beta)}$ is the appropriate distributional derivative of K_{2t}. Since all the $K_{2t}^{(\alpha,\beta)}$ are continuous, it follows from the lemma of Du Bois–Reymond that K_{2t} is differentiable in each variable and that the derivative is differentiable in the other variable. Moreover, $\partial_x^\alpha \partial_y^\beta K_{2t} = K_{2t}^{(\alpha,\beta)}$. This completes the proof of Statement III. □

Remark 4.12. We give a remark about boundedness of the domain here. The main reason that we need bounded domains is the need to estimate the trace integral. The parameter K in Theorem 4.11 is uniformly bounded for all $x \in \Gamma$ by compactness. However the theorem still holds for unbounded Lipschitz domain by an exhaustive argument, if the parameter K can be uniformly bounded on Γ.

4.4 Semigroup estimates: the Lipschitz case

For Lipschitz domain the estimates are not much different. We proved in Section 3.4 that there exists a $\kappa \in (0,1)$ such that $S_t u$ is κ-Hölder continuous. We proceed here that the kernel is also Hölder continuous.

Proposition 4.13. *Let $K \geq 1$ and $\mu, M > 0$. There exist $\kappa \in (0,1)$ and $c, \omega > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be a bounded, open and Lipschitz with boundary Γ. Let $U \subset \mathbb{R}^d$ be an open set and Φ a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\Phi(U) = E$ and $\Phi(\Omega \cap U) = E^-$. Suppose also that K is larger than the Lipschitz constant of $\Phi|_{\Omega \cap U}$ and $\Phi^{-1}|_{E^-}$. Suppose $(C, a, b, a_0, \beta) \in \mathcal{E}(\Omega, \mu, M)$ where C is real measurable. Then $S_t L_2(\Omega) \subset C(\overline{\Omega})$ for all $t > 0$. Moreover,

$$|(S_t^\rho u)(x)| \leq c\, t^{-d/4}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \tag{4.33}$$

and

$$|(S_t^\rho u)(x) - (S_t^\rho u)(y)| \leq c\, t^{-d/4}\, t^{-\kappa/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}\, |x-y|^\kappa \tag{4.34}$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \Omega$ with $|x - y| \leq 1$.

Proof. The boundary estimate follows from the proof of Proposition 4.3, using Proposition 3.27 instead of Proposition 3.22. The L_∞-estimate is the same. Let $\kappa \in (0,1)$ be as in Proposition 3.27. The Hölder estimate follows from the Morrey estimate of order $d - 2 + 2(\kappa - \varepsilon)$ for any small $\varepsilon > 0$.

The interior estimate is obtained similarly using Proposition 3.28, then the two estimates combines like Proposition 4.5. □

We now state the kernel estimates for elliptic operators with real principal coefficients in Lipschitz domains.

Theorem 4.14. *For all $K \geq 1$, $\mu, M, \tau > 0$ and $\tau' \in (0,1)$ there exist $\kappa \in (0,1)$ and $b, c, \omega > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be a bounded, open and Lipschitz with boundary Γ. Suppose for all $x \in \Gamma$ there exist open $U \subset \mathbb{R}^d$ and a bi-Lipschitz Φ from U onto E such that $\Phi(U \cap \Omega) = E^-$ and $\Phi(x) = 0$. Suppose that K is larger than the Lipschitz constant for $\Phi|_{\Omega \cap U}$ and $\Phi^{-1}|_{E^-}$. Let $(C, a, b, a_0, \beta) \in \mathcal{E}(\Omega, \mu, M)$ where C is real measurable. Let A be the operator associated with the form $\mathfrak{a}_\beta$ given in (2.2). Then there exists a function $(t, x, y) \mapsto K_t(x, y)$ from $(0, \infty) \times \Omega \times \Omega$ into $\mathbb{C}$ such that the following is valid.

I. *The function $(t, x, y) \mapsto K_t(x, y)$ is continuous from $(0, \infty) \times \Omega \times \Omega$ into $\mathbb{C}$.*

II. *For all $t \in (0, \infty)$ the function K_t is the kernel of the operator e^{-tA}.*

III. *For all $t \in (0, \infty)$ the function K_t is κ-Hölder continuous. Moreover,*

$$|K_t(x, y)| \le c\, t^{-d/2}\, e^{-b \frac{|x-y|^2}{t}}\, e^{\omega t}$$

and

$$|K_t(x, y) - K_t(x', y')| \le c\, t^{-d/2} \left(\frac{|x - x'| + |y - y'|}{t^{1/2} + |x - y|} \right)^\kappa e^{-b \frac{|x-y|^2}{t}}\, e^{\omega t}$$

for all $x, y, x', y' \in \Omega$ and $t > 0$ with $|x - x'| + |y - y'| \le \tau \sqrt{t} + \tau' |x - y|$.

Proof. It suffices to prove the Hölder estimate and the rest is Proposition 4.7. Let κ_0 be as in Proposition 4.13 and choose $\kappa \in (0, \kappa_0)$. It follows that K_t is κ'-Hölder continuous, where $\kappa' = \frac{\kappa + \kappa_0}{2}$. The desired estimate follows from interpolation of Lemma 4.6. □

Again, the boundedness of Ω can be relaxed to Lipschitz domain Ω such that for each $x \in \partial\Omega$ there exists a bi-Lipschitz map Φ that maps an open set U, containing x, onto E and $\Phi(x) = 0$, with uniform bound K.

4.5 Semigroup estimates: the C^1 case

Finally we state estimates when the domain is C^1 and the principal coefficients are uniformly continuous. They can be proved using similar techniques by using estimates from Section 3.5 instead.

Proposition 4.15. *Let $K \ge 1$, $\mu, M > 0$ and $\nu \in (0, 1)$. Then there exist $c, \omega > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of class C^1 with boundary Γ. Let $U \subset \mathbb{R}^d$ be open and Φ be a C^1-diffeomorphism from U onto E such that $\Phi(U \cap \Omega) = E^-$. Suppose that $\|\Phi\|_{C^1} \le K$ and $\|\Phi^{-1}\|_{C^1} \le K$. Let $(C, a, b, a_0, \beta) \in \mathcal{E}(\Omega, \mu, M)$ where C is uniformly continuous. Then $S_t L_2(\Omega) \subset C(\overline{\Omega})$ for all $t > 0$. Moreover,

$$|(S_t^\rho u)(x)| \le c\, t^{-d/4}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)} \tag{4.35}$$

and

$$|(S_t^\rho u)(x) - (S_t^\rho u)(y)| \le c\, t^{-d/4}\, t^{-\nu/2}\, e^{\omega(1+\rho^2)t}\, \|u\|_{L_2(\Omega)}\, |x - y|^\nu \tag{4.36}$$

for all $t > 0$, $u \in L_2(\Omega)$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}$ and $x, y \in \Omega$ with $|x - y| \le \frac{1}{2K}$.

Proof. This is the identical to Proposition 4.5 using Proposition 3.29 instead. □

We again state the kernel estimates for elliptic operator in C^1 domain with uniformly continuous coefficients.

Theorem 4.16. *For all $K \geq 1$, $\mu, M, \tau > 0$, $\tau' \in (0,1)$ and $\kappa \in (0,1)$ there exist $b, c, \omega > 0$ such that the following is valid.*

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of class C^1 with boundary Γ. Suppose for all $x \in \Gamma$ there exist open $U \subset \mathbb{R}^d$ and a C^1-diffeomorphism Φ from U onto E such that $\Phi(U \cap \Omega) = E^-$ and $\Phi(x) = 0$. Suppose that $\|\Phi\|_{C^1} \leq K$ and $\|\Phi^{-1}\|_{C^1} \leq K$. Let $(C, a, b, a_0, \beta) \in \mathcal{E}(\Omega, \mu, M)$ where C is uniformly continuous. Let A be the operator associated with the form $\mathfrak{a}_\beta$ given in (2.2). Then there exists a function $(t, x, y) \mapsto K_t(x, y)$ from $(0, \infty) \times \Omega \times \Omega$ into $\mathbb{C}$ such that the following is valid.

I. *The function $(t, x, y) \mapsto K_t(x, y)$ is continuous from $(0, \infty) \times \Omega \times \Omega$ into $\mathbb{C}$.*

II. *For all $t \in (0, \infty)$ the function K_t is the kernel of the operator e^{-tA}.*

III. *For all $t \in (0, \infty)$ the function K_t is κ-Hölder continuous. Moreover,*

$$|K_t(x, y)| \leq c\, t^{-d/2}\, e^{-b\, \frac{|x-y|^2}{t}}\, e^{\omega t}$$

and

$$|K_t(x, y) - K_t(x', y')| \leq c\, t^{-d/2} \left(\frac{|x - x'| + |y - y'|}{t^{1/2} + |x - y|} \right)^\kappa e^{-b \frac{|x-y|^2}{t}}\, e^{\omega t}$$

for all $x, y, x', y' \in \Omega$ and $t > 0$ with $|x - x'| + |y - y'| \leq \tau \sqrt{t} + \tau' |x - y|$.

Proof. This is identical to Proposition 4.7, using Proposition 4.15 instead. □

Chapter 5

Consequences of the kernel estimates

5.1 Related estimates and irreducibility

Using various estimates for the kernel we obtained in Chapter 4, there are some immediate consequences we can make. In this chapter we show that the elliptic operator generates a holomorphic C_0-semigroup not just in $L_2(\Omega)$ but also a holomorphic C_0-semigroup in $L_p(\Omega)$, where $p \in [1, \infty)$. If the operator is symmetric and all the coefficients are real, then one also obtains a Gaussian lower bound. Moreover we give irreducibility results.

Most results are a direct consequence of the Gaussian kernel estimates. We consider the three cases of $C^{1+\kappa}$, C^1 and Lipschitz domains. We state in each result the regularity required, but we will not repeat the proof for different boundary regularity if it is similar. We first present the semigroup estimates between $L_p(\Omega)$ and Hölder spaces.

Proposition 5.1. *Let $\kappa \in (0,1)$ and $\Omega \subset \mathbb{R}^d$ be a bounded open set of class $C^{1+\kappa}$. Let $\mu, M > 0$ and $(C, a, b, a_0, \beta) \in \mathcal{E}^\kappa(\Omega, \mu, M)$. Let A be the operator associated with the form $\mathfrak{a}_\beta$ given in (2.2). Let S be the semigroup generated by $-A$. Let α be a multi-index and $\nu \in (0,1)$ with $|\alpha| + \nu \leq 1 + \kappa$. Then there exist $c, \omega > 0$ such that*

$$\|\partial^\alpha S_t\|_{p \to q} \leq c\, t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}\, t^{-|\alpha|/2}\, e^{\omega t} \tag{5.1}$$

and

$$\|\partial^\alpha S_t\|_{L_p(\Omega) \to C^\nu(\Omega)} \leq c\, t^{-\frac{d}{2p}}\, t^{-|\alpha|/2}\, t^{-\nu/2}\, e^{\omega t} \tag{5.2}$$

for all $t > 0$ and $1 \leq p \leq q \leq \infty$.

Proof. Let K be the kernel of S. By Theorem 4.11 there are $b, c, \omega > 0$ such that

$$|(\partial_x^\alpha K_t)(x,y)| \leq c\, t^{-d/2}\, t^{-|\alpha|/2}\, e^{-b\, \frac{|x-y|^2}{t}}\, e^{\omega t}$$

and

$$|(\partial_x^\alpha K_t)(x,y) - (\partial_x^\alpha K_t)(x',y)| \leq c\, t^{-d/2}\, t^{-|\alpha|/2} \Big(\frac{|x - x'|}{\sqrt{t}}\Big)^\nu e^{-b\frac{|x-y|^2}{t}}\, e^{\omega t}$$

for all $x, y, x' \in \Omega$ and $t > 0$ with $|x - x'| \le \sqrt{t}$. Let $u \in L_p(\Omega)$. Suppose $p \ne 1$. Let q be the dual exponent of p. Then Hölder's inequality yields

$$\begin{aligned}
|(\partial^\alpha S_t u)(x)| &\le c\, t^{-d/2}\, t^{-|\alpha|/2}\, e^{\omega t} \int_\Omega e^{-b\,\frac{|x-y|^2}{t}}\, |u(y)|\, dy \\
&\le c\, t^{-d/2}\, t^{-|\alpha|/2}\, e^{\omega t} \Big(\int_{\mathbb{R}^d} e^{-b\,q\,\frac{|x-y|^2}{t}}\, dy \Big)^{1/q} \|u\|_{L_p(\Omega)} \\
&\le c_1\, t^{-d/(2p)}\, t^{-|\alpha|/2}\, e^{\omega t}\, \|u\|_{L_p(\Omega)} \qquad (5.3)
\end{aligned}$$

for all $x \in \Omega$ and $t > 0$, where $c_1 = c\,(\pi\, b^{-1})^{d/2}$. In particular, $\|\partial^\alpha S_t\|_{\infty\to\infty} \le c_1\, t^{-|\alpha|/2}\, e^{\omega t}$ and $\|\partial^\alpha S_t\|_{1\to\infty} \le c_1\, t^{-d/2}\, t^{-|\alpha|/2}\, e^{\omega t}$. Now for all $u \in L_1(\Omega)$,

$$\begin{aligned}
\int_\Omega |(\partial^\alpha S_t u)(x)|\, dx &\le c\, t^{-d/2}\, t^{-|\alpha|/2}\, e^{\omega t} \int_\Omega \int_\Omega e^{-b\,\frac{|x-y|^2}{t}}\, |u(y)|\, dy\, dx \\
&\le c\, t^{-d/2}\, t^{-|\alpha|/2}\, e^{\omega t} \int_\Omega |u(y)| \int_\Omega e^{-b\,\frac{|x-y|^2}{t}}\, dx\, dy \\
&\le c_1\, t^{-|\alpha|/2}\, e^{\omega t}\, \|u\|_{L_1(\Omega)},
\end{aligned}$$

so $\|\partial^\alpha S_t\|_{1\to 1} \le c_1\, t^{-|\alpha|/2}\, e^{\omega t}$ as well. Thus (5.1) follows from Riesz–Thorin.

If $t > 0$ and $x, x' \in \Omega$ with $|x - x'| \le \sqrt{t}$, then

$$|(\partial^\alpha S_t u)(x) - (\partial^\alpha S_t u)(x')| \le c_1\, t^{-d/(2p)}\, t^{-|\alpha|/2} \Big(\frac{|x - x'|}{\sqrt{t}} \Big)^\nu e^{\omega t}\, \|u\|_{L_p(\Omega)}.$$

Then (5.2) follows by combining the above with (5.3). □

One also retrieves the above bounds for the Lipschitz or C^1 cases, which we state as below.

Proposition 5.2. *Let $\Omega \subset \mathbb{R}^d$ be bounded, open and Lipschitz. Let $\mu, M > 0$ and $(C, a, b, a_0, \beta) \in \mathcal{E}(\Omega, \mu, M)$ where C is real measurable. Let A be the operator associated with the form $\mathfrak{a}_\beta$ given in (2.2). Let S be the semigroup generated by $-A$. Then there exist $\kappa \in (0, 1)$ and $c, \omega > 0$ such that*

$$\|S_t\|_{p\to q} \le c\, t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}\, e^{\omega t}$$
$$\|S_t\|_{L_p(\Omega)\to C^\nu(\Omega)} \le c\, t^{-\frac{d}{2p}}\, t^{-\nu/2}\, e^{\omega t}$$

for all $t > 0$, $1 \le p \le q \le \infty$ and $\nu \in (0, \kappa)$.

Proposition 5.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of class C^1. Let $\mu, M > 0$ and $(C, a, b, a_0, \beta) \in \mathcal{E}(\Omega, \mu, M)$ where C is uniformly continuous. Let A be the operator associated with the form $\mathfrak{a}_\beta$ given in (2.2). Let S be the semigroup generated by $-A$. Then for each $\nu \in (0, 1)$ there exist $c, \omega > 0$ such that*

$$\|S_t\|_{p\to q} \le c\, t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})}\, e^{\omega t}$$

$$\|S_t\|_{L_p(\Omega)\to C^\nu(\Omega)} \le c\, t^{-\frac{d}{2p}}\, t^{-\nu/2}\, e^{\omega t}$$

for all $t > 0$ *and* $1 \le p \le q \le \infty$.

Using these bounds we can show that the semigroup is holomorphic and strongly continuous on $L_p(\Omega)$ for all $p \in [1, \infty)$. We also show that there is a kernel for the semigroup on a sector with Gaussian bounds.

Proposition 5.4. *Adopt assumptions and notation from either Proposition 5.1, 5.2 or 5.3. Let* $\theta \in (0, \pi/2]$ *be such that* S *is holomorphic in* $L_2(\Omega)$ *with angle* θ. *Then* S *is also a holomorphic* C_0*-semigroup on* $L_p(\Omega)$ *of angle* θ *for all* $p \in [1, \infty)$. *Furthermore,* S_z *has a kernel* K_z *on* Σ°_θ. *For every* $\theta' \in (0, \theta)$ *there exist* $b, c, \omega > 0$ *such that*

$$|K_z(x, y)| \le c\, (\operatorname{Re} z)^{-d/2}\, e^{-b\, \frac{|x-y|^2}{|z|}}\, e^{\omega |z|}$$

for all $z \in \Sigma^\circ_{\theta'}$.

Proof. Take assumptions from Proposition 5.1. Then $S_t(L_1(\Omega) \cap L_2(\Omega)) \subset L_1(\Omega)$. It also implies that there exists a $c > 0$ such that $\|S_t u\|_{L_1(\Omega)} \le c\|u\|_{L_1(\Omega)}$ uniformly for all $u \in L_1(\Omega) \cap L_2(\Omega)$ and $t \in (0, 1]$. Let K^Δ_t be the heat kernel of the semigroup generated by Δ. Then the kernel K can be bounded by K^Δ as follows:

$$|K_t(x, y)| \le c\, (\pi b^{-1})^{d/2}\, e^{\omega t} K^\Delta_{(4b)^{-1} t}(x, y)$$

for all $x, y \in \Omega$ and $t > 0$. Using Lemma 2.1(v) of [AE97] it follows that S is a C_0-semigroup on $L_p(\Omega)$ for all $p \in [1, 2]$. By duality the result also holds for all $p \in [2, \infty)$.

Finally, the holomorphy and complex kernel bounds follows from the argument in Theorem 5.3 of [AE97] using the estimate (4.3) on the sector. □

With the kernel estimate we can also give bounds to the operator.

Proposition 5.5. *Let* $\kappa \in (0, 1)$ *and* $\Omega \subset \mathbb{R}^d$ *be a bounded open set of class* $C^{1+\kappa}$. *Let* $\mu, M > 0$ *and* $(C, a, b, a_0, \beta) \in \mathcal{E}^\kappa(\Omega, \mu, M)$. *Let* A *be the operator associated with the form* $\mathfrak{a}_\beta$ *given in (2.2). Suppose that* $0 \notin \sigma(A)$. *Let* $p \in (d, \infty)$ *and* $\nu \in (0, \kappa]$. *Suppose that* $\frac{d}{p} + \nu < 1$. *Then the operator* $\partial_k A^{-1}$ *is bounded from* $L_p(\Omega)$ *into* $C^\nu(\Omega)$ *for all* $k \in \{1, \ldots, d\}$.

Proof. It follows from Proposition 5.1 that there are $c, \omega > 0$ such that $\|\partial_k S_t\|_{L_p(\Omega)\to C^\nu(\Omega)} \le c\, t^{-d/(2p)}\, t^{-1/2}\, t^{-\nu/2}\, e^{(\omega-1)t}$ for all $t > 0$. Note that $\frac{d}{2p} + \frac{1}{2} + \frac{\nu}{2} < 1$. Hence a Laplace transform gives that the operator $\partial_k\, (\omega I + A)^{-1}$ is bounded from $L_p(\Omega)$ into $C^\nu(\Omega)$. If $-A_p$ denotes the generator of the semigroup consistent with S on $L_p(\Omega)$, then $0 \notin \sigma(A_p)$ by [KV07] Proposition 4. Then also $\partial_k A^{-1} = \partial_k\, (\omega I + A)^{-1}\, (\omega A_p^{-1} + I)$ is bounded from $L_p(\Omega)$ into $C^\nu(\Omega)$. □

Next, we give irreducibility results. It turns out that S_t does not only map into $C(\overline{\Omega})$, but the restriction of it to $C(\overline{\Omega})$ also forms a C_0-semigroup.

Corollary 5.6. *Let $\Omega \subset \mathbb{R}^d$ be an open set. Let $\mu, M > 0$ and $(C, a, b, a_0, \beta) \in \mathcal{E}(\Omega, \mu, M)$. Suppose one of the following is true:*

I. *c_{kl} is real measurable for all $k, l \in \{1, \ldots, d\}$ and Ω is Lipschitz, or*

II. *$c_{kl} \in C(\overline{\Omega})$ for all $k, l \in \{1, \ldots, d\}$ and Ω is C^1.*

For all $t > 0$ let $T_t \colon C(\overline{\Omega}) \to C(\overline{\Omega})$ be the restriction of S_t to $C(\overline{\Omega})$. Then T is a holomorphic C_0-semigroup.

Proof. This follows as in the proof of Theorem 4.3 in [Nit11], since Theorem 4.14 or 4.16 shows that S_t maps into the space of Hölder continuous functions. (Note that there is a gap in the proof of Theorem 4.3 in [Nit11] in case the condition $\beta \geq 0$ is not valid.) □

For the proof of the irreducibility of T we use arguments from [Are06] and [Ouh05]. We first give a technical lemma.

Lemma 5.7. *Let $\Omega \subset \mathbb{R}^d$ be open and connected. Let $S \subset \Omega$ be Borel measurable. Suppose that $|S| > 0$ and $|\Omega \setminus S| > 0$. Then there exist an $x_0 \in \Omega$ such that $|S \cap B(x_0, r)| > 0$ and $|B(x_0, r) \setminus S| > 0$ for all $r > 0$.*

Proof. The proof is as in [Are06] Lemma 11.1.1. Suppose not and define

$$\Omega_1 = \{x \in \Omega \mid \text{there exists an } r > 0 \text{ such that } B(x, r) \subset \Omega \text{ and } |S \cap B(x, r)| = 0\}$$

and

$$\Omega_2 = \{x \in \Omega \mid \text{there exists an } r > 0 \text{ such that } B(x, r) \subset \Omega \text{ and } |B(x, r) \setminus S| = 0\}.$$

Then Ω_1 and Ω_2 forms a partition of Ω. Since Ω is connected one deduces that $\Omega_1 = \Omega$ or $\Omega_2 = \Omega$.

Suppose that $\Omega_1 = \Omega$. (The case $\Omega_2 = \Omega$ is similar.) Then for all $x \in \Omega$ there exists an $r > 0$ such that $|S \cap B(x, r)| = 0$. Let $K \subset S$ be compact. Then in particular for all $x \in K$ there exists an $r > 0$ such that $|S \cap B(x, r)| = 0$. By compactness, one can cover K with a finite number of those balls, so that $|K| = 0$. Then the regularity of the Lebesgue measure implies that $|S| = \sup\{|K| \colon K \subset S \text{ compact}\} = 0$. This is a contradiction. □

Proposition 5.8. *Let $\Omega \subset \mathbb{R}^d$ be open and connected. Let $S \subset \Omega$ be Borel measurable. Suppose that $|S| > 0$ and $|\Omega \setminus S| > 0$. Then there exists a $u \in C_c^\infty(\Omega)$ such that $\mathbb{1}_S \, u \notin W^{1,2}(\Omega)$.*

Proof. By Lemma 5.7 there exists an $x_0 \in \Omega$ such that $|S \cap B(x_0, r)| > 0$ and $|B(x_0, r) \setminus S| > 0$ for all $r > 0$. Since Ω is open there exists an $r_0 > 0$ such that $B(x_0, 2r_0) \subset \Omega$. There exists a $u \in C_c^\infty(B(x_0, 2r_0))$ such that $u|_{B(x_0, r_0)} = \mathbb{1}$. Now suppose that $\mathbb{1}_S \, u \in W^{1,2}(\Omega)$.

Let $k \in \{1, \ldots, d\}$. There exists a $w_k \in \mathcal{L}_2(\Omega)$ such that $-(\mathbb{1}_S \, u, D_k v)_{L_2(\Omega)} = (w_k, v)_{L_2(\Omega)}$ for all $v \in C_c^\infty(\Omega)$. Then [GT83] Lemma 7.7 implies that $w_k(x) = 0$ for almost every

$x \in [\mathbb{1}_S\, u = 0]$ and also that $w_k(x) = 0$ for almost every $x \in [\mathbb{1}_S\, u = 1]$. So $w_k(x) = 0$ for almost every $x \in B(x_0, r_0)$. Therefore $-(\mathbb{1}_S\, u, D_k v)_{L_2(\Omega)} = 0$ for all $v \in C_c^\infty(B(x_0, r_0))$. Consequently

$$\mathbb{1}_{S\cap B(x_0,r_0)} = (\mathbb{1}_S\, u)|_{B(x_0,r_0)} \in W^{1,\infty}(B(x_0, r_0)).$$

Hence there exists a (Lipschitz) continuous function $v\colon B(x_0, r_0) \to \mathbb{C}$ such that $\mathbb{1}_{S\cap B(x_0,r_0)} = v$ almost everywhere on $B(x_0, r_0)$. So there exists a Borel measurable set $N \subset \mathbb{R}^d$ such that $\mathbb{1}_{S\cap B(x_0,r_0)}(x) = v(x)$ for all $x \in B(x_0, r_0) \setminus N$. Let $n \in \mathbb{N}$ with $n > r_0^{-1}$. Since $|S \cap B(x_0, \frac{1}{n}) \cap N^{\rm c}| > 0$ there exists an $x_n \in S \cap B(x_0, \frac{1}{n}) \cap N^{\rm c}$. Similarly there exists a $y_n \in B(x_0, \frac{1}{n}) \cap N^{\rm c} \setminus S$. Then $v(x_n) = 1$ and $v(y_n) = 0$ for all large $n \in \mathbb{N}$. Consequently $1 = \lim v(x_n) = v(x_0) = \lim v(y_n) = 0$, which is a contradiction. □

Proposition 5.9. *Suppose that Ω is connected. Let $\Omega_1 \subset \Omega$ be measurable. Suppose that $S_t L_2(\Omega_1) \subset L_2(\Omega_1)$. Then $|\Omega_1| = 0$ or $|\Omega \setminus \Omega_1| = 0$.*

Proof. [Ouh05] Theorem 2.2 says that $S_t L_2(\Omega_1) \subset L_2(\Omega_1)$ implies $\mathbb{1}_{\Omega_1} u \in W^{1,2}(\Omega)$. Then Proposition 5.8 then implies that $|\Omega_1|$ or $|\Omega \setminus \Omega_1| = 0$. □

Now we are able to show that the semigroup on $C(\overline{\Omega})$ is irreducible .

Proposition 5.10. *Suppose that Ω is connected. Let T be the C_0-semigroup on $C(\overline{\Omega})$ as in Corollary 5.6. Let $F \subset \overline{\Omega}$ be closed and suppose that $T_t I \subset I$ for all $t > 0$, where $I = \{u \in C(\overline{\Omega})\colon u|_F = 0\}$. Then $F = \varnothing$ or $F = \overline{\Omega}$.*

Proof. Suppose that $F \neq \varnothing$ and $F \neq \overline{\Omega}$. Define $f \in C(\overline{\Omega})$ by $f(x) = d(x, F)$. Let $t > 0$ and $x \in F$. If $\tau \in C(\overline{\Omega})$, then $f\,\tau \in I$, so $0 = (T_t(f\,\tau))(x) = \int_\Omega K_t(x, y)\, f(y)\, \tau(y)\, dy$. Hence $K_t(x, y)\, f(y) = 0$ for almost every $y \in \Omega$ and by continuity for all $y \in \Omega$. Therefore $K_t(x, y) = 0$ for all $y \in \overline{\Omega} \setminus F$ and by continuity for all $y \in \overline{\overline{\Omega} \setminus F}$, where the closure is in $\mathbb{R}^d$. Let F° denote the interior of F in $\mathbb{R}^d$. It is elementary to show that $\Omega \setminus (F^\circ) \subset \overline{\overline{\Omega} \setminus F}$. Hence we proved that $K_t(x, y) = 0$ for all $x \in F$, $y \in \Omega \setminus (F^\circ)$ and $t > 0$.

Let $J = \{u \in L_2(\Omega) : u|_F = 0 \text{ a.e.}\}$. If $u \in J$, $t > 0$ and $x \in F$, then

$$(S_t u)(x) = \int_\Omega K_t(x, y)\, u(y)\, dy = \int_{\Omega\setminus F} K_t(x, y)\, u(y)\, dy = 0.$$

So $S_t J \subset J$ for all $t > 0$. Since S is irreducible by Proposition 5.9, it follows that $|F| = 0$ or $|\overline{\Omega} \setminus F| = 0$. Since $F \neq \overline{\Omega}$ there exists an $x \in \overline{\Omega}$ and $r > 0$ such that $B(x, r) \subset \mathbb{R}^d \setminus F$. Then $0 < |\Omega(x, r)| \le |\overline{\Omega} \setminus F|$. Hence $|F| = 0$. Then also $|F^\circ| = 0$ and consequently $F^\circ = \varnothing$. Therefore $\Omega \setminus (F^\circ) = \Omega$. It follows that $K_t(x, y) = 0$ for all $t > 0$, $x \in F$ and $y \in \Omega$, and then by continuity for all $y \in \overline{\Omega}$. Then $1 = \lim_{t\downarrow 0}(T_t \mathbb{1}_{\overline{\Omega}})(x) = \lim_{t\downarrow 0} \int_\Omega K_t(x, y)\, dy = 0$ for all $x \in F$. This is a contradiction since $F \neq \varnothing$. □

5.2 Gaussian lower bounds

In light of [ER98], we are also able to give Gaussian lower bounds for the kernel. This is, however, limited to self-adjoint elliptic operators with real coefficients.

In this section, we first define the chain condition and prove that it is valid for Lipschitz boundary. We then use the chain condition to extend local lower bounds for small time t to global lower bounds.

The main aim of this section is to prove the result below.

Theorem 5.11. *Adopt assumptions and notations from Theorem 4.14. Assume in extra that a_k, b_k, a_0 and β are real-valued and that A is self-adjoint. Then there are $b, c, \omega > 0$ such that*

$$K_t(x,y) \geq c\, t^{-d/2}\, e^{-b\,\frac{|x-y|^2}{t}}\, e^{-\omega t}$$

for all $x, y \in \Omega$ and $t > 0$.

We first define the chain condition.

Definition 5.12. Let $\Omega \subset \mathbb{R}^d$ be open and connected. We say that Ω satisfies the **chain condition** if there exists a $c > 0$ such that for all $x, y \in \Omega$ and $n \in \mathbb{N}$ there are $x_0, \ldots, x_n \in \Omega$ such that $x_0 = x$, $x_n = y$ and $|x_{k+1} - x_k| \leq \frac{c}{n}\,|x - y|$ for all $k \in \{0, \ldots, n-1\}$.

In general open connected sets do not necessarily satisfy the chain condition. Here we would like to show that connected Lipschitz domains indeed satisfy the chain condition. To prove this we first define the geometric distance on a connected set.

Definition 5.13. Let $\Omega \subset \mathbb{R}^d$ be open bounded connected with Lipschitz boundary. If $T > 0$ and $\gamma\colon [0,T] \to \Omega$ is a Lipschitz curve, then γ is differentiable almost everywhere. We define the **length** of γ by $\ell(\gamma) = \int_0^T |\gamma'(t)|\, dt$. Define the **geometric distance** $d\colon \Omega \times \Omega \to [0,\infty)$ by $d(x,y)$ is the infimum of $\ell(\gamma)$, where $T > 0$ and $\gamma\colon [0,T] \to \Omega$ is a Lipschitz curve with $\gamma(0) = x$ and $\gamma(T) = y$.

The chain condition is clearly true with respect to the geometric distance for open connected sets, so it suffices to show that the two metrics are equivalent.

We first consider a special Lipschitz chart.

Lemma 5.14. *Let $U \subset \mathbb{R}^d$ be an open set and Φ be a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\Phi(U) = E$ and $\Phi(\Omega \cap U) = E^-$. Then there are $c_1, c_2 > 0$ such that $d(x,y) \leq c_1\,|x-y|$ and $|x-y| \leq c_2$ for all $x, y \in \Omega \cap U$.*

Proof. Let $L \in \mathbb{R}$ be larger than both the Lipschitz constant for Φ and Φ^{-1}. Further, let $x, y \in \Omega \cap U$. Define $\gamma\colon [0,1] \to \Omega$ by $\gamma(t) = \Phi^{-1}((1-t)\,\Phi(x) + t\,\Phi(y))$. Then $\gamma(0) = x$ and $\gamma(1) = y$. Moreover, γ is Lipschitz continuous and $|\gamma'(t)| \leq L\,|\Phi(y) - \Phi(x)| \leq L^2\,|y - x|$ for almost every $t \in [0,1]$. So $d(x,y) \leq \ell(\gamma) \leq L^2\,|x-y|$. Also $|x-y| \leq L\,|\Phi(x) - \Phi(y)| \leq 2L$. □

Lemma 5.15. *Let $\Omega \subset \mathbb{R}^d$ be bounded, open and Lipschitz with boundary Γ. Then there exists a $c > 0$ such that $|x - y| \leq d(x, y) \leq c\,|x - y|$ for all $x, y \in \Omega$.*

Proof. By a compactness argument there are $N \in \mathbb{N}$ and for all $k \in \{1, \ldots, N\}$ there are open $U_k \subset \mathbb{R}^d$ and a bi-Lipschitz map Φ_k from an open neighbourhood of $\overline{U_k}$ onto an open subset of $\mathbb{R}^d$ such that $\Phi_k(U_k) = E$ and $\Phi_k(\Omega \cap U_k) = E^-$; and moreover, $\Gamma \subset \bigcup_{k=1}^N U_k$. For all $k \in \{1, \ldots, N\}$ fix $w_k \in \Omega \cap U_k$. Again by compactness there are $N' \in \{N+1, N+2, \ldots\}$ and for all $k \in \{N+1, \ldots, N'\}$ there are $w_k \in \Omega$ and $r_k > 0$ such that $B(w_k, r_k) \subset \Omega$ and

$$\overline{\Omega} \subset \bigcup_{k=1}^{N} U_k \cup \bigcup_{k=N+1}^{N'} B(w_k, r_k).$$

By Lemma 5.14 there are $c_1, c_2 \geq 1$ such that $d(x, y) \leq c_1\,|x - y|$ and $|x - y| \leq c_2$ for all $k \in \{1, \ldots, N\}$ and $x, y \in \Omega \cap U_k$. Without loss of generality we may assume that $2r_k \leq c_2$ for all $k \in \{N+1, \ldots, N'\}$. For simplicity write $U_k = B(w_k, r_k)$ for all $k \in \{N+1, \ldots, N'\}$. Then $d(x, y) \leq c_1\,|x - y|$ and $|x - y| \leq c_2$ for all $k \in \{N+1, \ldots, N'\}$ and $x, y \in U_k$.

We next prove that the geometric distance d is bounded on Ω. Define $M = 2c_2 + \max\{d(w_k, w_l) : k, l \in \{1, \ldots, N'\}\}$. Let $x, y \in \Omega$. Then there are $k, l \in \{1, \ldots, N'\}$ such that $x \in U_k$ and $y \in U_l$. Hence $d(x, y) \leq d(x, w_k) + d(w_k, w_l) + d(w_l, y) \leq M$. Therefore d is bounded by M.

Finally suppose that there is no $c > 0$ such that $d(x, y) \leq c\,|x - y|$ for all $x, y \in \Omega$. Then for all $n \in \mathbb{N}$ there are $x_n, y_n \in \Omega$ such that $d(x_n, y_n) > n\,|x_n - y_n|$. It follows that $|x_n - y_n| \leq \frac{M}{n}$ for all $n \in \mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ is bounded since Ω is bounded. Passing to a subsequence if necessary, we may assume that the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent. Let $x = \lim_{n\to\infty} x_n$. Then $\lim_{n\to\infty} y_n = x$ and $x \in \overline{\Omega}$. Since $\overline{\Omega} \subset \bigcup_{k=1}^N U_k \cup \bigcup_{k=N+1}^{N'} B(w_k, r_k)$, there exists a $k \in \{1, \ldots, N'\}$ such that $x \in U_k$. Because U_k is open there exists an $N_0 \in \mathbb{N}$ such that $x_n \in U_k$ and $y_n \in U_k$ for all $n \in \mathbb{N}$ with $n \geq N_0$. Finally choose $n \in \mathbb{N}$ such that $n \geq \max\{N_0, c_1\}$. Then

$$n\,|x_n - y_n| < d(x_n, y_n) \leq c_1\,|x_n - y_n| \leq n\,|x_n - y_n|.$$

This is a contradiction. □

We can now show that the geometric distance is equivalent to the Euclidean distance.

Proposition 5.16. *Let $\Omega \subset \mathbb{R}^d$ be bounded, open and Lipschitz. Then Ω satisfies the chain condition.*

Proof. Let $c > 0$ be as in Lemma 5.15. Let $x, y \in \Omega$ and $n \in \mathbb{N}$. Since the case $x = y$ is trivial, we may assume that $x \neq y$. There exist $T > 0$ and a Lipschitz curve $\gamma\colon [0, T] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $\ell(\gamma) \leq 2d(x, y)$. For all $k \in \{1, \ldots, n-1\}$ let

$$t_k = \min\{t \in [0, T] : \ell(\gamma|_{[0,t]}) = \frac{k\,\ell(\gamma)}{n}\},$$

which exists by continuity. Set $x_k = \gamma(t_k)$. Further define $x_0 = x$ and $x_n = y$. Then

$$|x_{k+1} - x_k| \le d(x_{k+1}, x_k) \le \frac{\ell(\gamma)}{n} \le \frac{2d(x,y)}{n} \le \frac{2c}{n}\,|x-y|$$

for all $k \in \{0, \dots, n-1\}$, as required. $\square$

Finally we prove our main theorem.

Proof of Theorem 5.11. Let T be the C_0-semigroup in $C(\overline{\Omega})$ as in Corollary 5.6. Then clearly $\lim_{t\downarrow 0} \|T_t \mathbb{1}_{\overline{\Omega}} - \mathbb{1}_{\overline{\Omega}}\|_{C(\overline{\Omega})} = 0$. Hence

$$\lim_{t\downarrow 0} \sup_{x\in\Omega} \Big|1 - \int_\Omega K_t(x,y)\,dy\Big| = 0.$$

It follows from [ER98] Theorem 2.1 that there are $c_1, c_2, t_0 > 0$ such that

$$K_t(x,y) \ge c_1\, t^{-d/2}$$

for all $x, y \in \Omega$ and $t \in (0, t_0]$ with $|x-y| \le c_2\, t^{1/2}$. Without loss of generality we may assume that $t_0 \le 1$. By Proposition 5.16 the set Ω satisfies the chain condition. That is, there exists a $c_3 > 0$ such that for all $x, y \in \Omega$ and $n \in \mathbb{N}$ there exist $x_0, \dots, x_n \in \Omega$ such that $x_0 = x$, $x_n = y$ and $|x_{k+1} - x_k| \le c_3 \frac{|x-y|}{n}$ for all $k \in \{0, \dots, n-1\}$. Since Ω is bounded and Lipschitz, there exists a $c_4 > 0$ such that $|\Omega(x,r)| \ge c_4\, r^d$ for all $x \in \Omega$ and $r \in (0,1]$.

Let $x, y \in \Omega$ and $t > 0$. Let $n \in \mathbb{N}$ be the smallest natural number such that

$$\frac{4c_3^2\,|x-y|^2}{c_2^2\, t} \le n \quad \text{and} \quad \frac{t}{t_0} \le n.$$

Then

$$n - 1 \le \frac{4c_3^2\,|x-y|^2}{c_2^2\, t} + \frac{t}{t_0}. \tag{5.4}$$

By the chain condition there exist $x_0, \dots, x_n \in \Omega$ such that $x_0 = x$, $x_n = y$ and $|x_{k+1} - x_k| \le \frac{c_3}{n}\,|x-y|$ for all $k \in \{0, \dots, n-1\}$. Then the semigroup property gives

$$\begin{aligned} K_t(x,y) &= \int_\Omega \dots \int_\Omega K_{\frac{t}{n}}(x,z_1)\, K_{\frac{t}{n}}(z_1,z_2) \dots K_{\frac{t}{n}}(z_{n-2},z_{n-1})\, K_{\frac{t}{n}}(z_{n-1},y)\, dz_1 \dots dz_{n-1} \\ &\ge \int_{B(x_1, \frac{c_2\sqrt{t}}{4\sqrt{n}})} \dots \int_{B(x_{n-1}, \frac{c_2\sqrt{t}}{4\sqrt{n}})} K_{\frac{t}{n}}(x,z_1)\, K_{\frac{t}{n}}(z_1,z_2) \dots K_{\frac{t}{n}}(z_{n-2},z_{n-1})\, K_{\frac{t}{n}}(z_{n-1},y) \\ &\qquad dz_1 \dots dz_{n-1}. \end{aligned}$$

If $z_k \in B(x_k, \frac{c_2\sqrt{t}}{4\sqrt{n}})$ for all $k \in \{1, \dots, n-1\}$ and we set $z_0 = x_0$ and $z_n = x_n$, then

$$|z_k - z_{k+1}| \le |x_k - x_{k+1}| + \frac{2c_2\sqrt{t}}{4\sqrt{n}} \le \frac{c_3}{n}\,|x-y| + \frac{c_2\sqrt{t}}{2\sqrt{n}} \le \frac{c_3}{n}\,\frac{c_2\sqrt{n}\sqrt{t}}{2c_3} + \frac{c_2\sqrt{t}}{2\sqrt{n}} = c_2\Big(\frac{t}{n}\Big)^{1/2}$$

for all $k \in \{0, \ldots, n-1\}$ and $\frac{t}{n} \le t_0$. Hence $K_{\frac{t}{n}}(z_k, z_{k+1}) \ge c_1\, n^{d/2}\, t^{-d/2}$ and

$$K_t(x,y) \ge \left(c_4 \left(\frac{c_2 \sqrt{t}}{4\sqrt{n}} \right)^d \right)^{n-1} \left(c_1\, n^{d/2}\, t^{-d/2} \right)^n$$
$$= c_1\, (c_1\, c_2^d\, c_4)^{n-1}\, n^{d/2}\, t^{-d/2} \ge c_1\, (c_1\, c_2^d\, c_4)^{n-1}\, t^{-d/2}.$$

Let $M \in [1, \infty)$ be such that $\frac{1}{M} \le c_1\, c_2^d\, c_4$. Then

$$(c_1\, c_2^d\, c_4)^{n-1} \ge \left(\frac{1}{M} \right)^{n-1} = e^{-(n-1)\log M} \ge e^{-(\log M) \left(\frac{4c_3^2\, |x-y|^2}{c_2^2\, t} + \frac{t}{t_0} \right)},$$

where we used (5.4). Now Theorem 5.11 follows. □

Chapter 6

Estimates for the Dirichlet-to-Neumann operator

In this chapter, we prove uniform estimates and Hölder continuity for the kernel of the semigroup generated by minus the Dirichlet-to-Neumann operator. The Dirichlet-to-Neumann operator can be associated to a second-order elliptic differential operator in divergence form. We obtain the Hölder continuity by an application of Proposition 3.27. If the domain has $C^{1+\kappa}$ boundary, the first-order coefficients are zero and the operator is self-adjoint, we combine with known Poisson bounds for the kernel to give Hölder Poisson kernel bounds as well.

First let us recall the definitions. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain with boundary Γ. Let $\mu, M > 0$. Let $C \colon \Omega \to \mathbb{R}^{d\times d}$ be measurable satisfying the ellipticity condition

$$\operatorname{Re} \sum_{k,l=1}^{d} c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geq \mu\, |\xi|^2$$

and

$$\|C(x)\| \leq M$$

for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. Define the lower terms $a, b \colon \Omega \to \mathbb{C}^d$ and $a_0 \colon \Omega \to \mathbb{C}$ be measurable such that $\|a(x)\|, \|b(x)\|, |a_0(x)| \leq M$ for all $x \in \Omega$. As in (2.1) the form $\mathfrak{a} \colon W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ is given by

$$\mathfrak{a}(u,v) = \int_\Omega \sum_{k,l=1}^{d} c_{kl}(\partial_k u)\, \overline{\partial_l v} + \int_\Omega \sum_{k=1}^{d} a_k\, (\partial_k u)\, \overline{v} + \int_\Omega \sum_{l=1}^{d} b_l\, u\, \overline{\partial_l\, v} + \int_\Omega a_0\, u\, \overline{v}.$$

Note that we have dropped the boundary integral.

Let A_D be the operator in $L_2(\Omega)$ associated with $\mathfrak{a}_D = \mathfrak{a}|_{W_0^{1,2}(\Omega)\times W_0^{1,2}(\Omega)}$. Then A_D is an elliptic operator with Dirichlet boundary conditions and

$$D(A_D) = \left\{ u \in W_0^{1,2}(\Omega) \mid \mathcal{A}u \in L_2(\Omega) \right\}$$

using similar arguments as in Lemma 2.2. It is also m-sectorial by Proposition 2.6. Throughout this section we assume that $0 \notin \sigma(A_D)$.

Under the above assumptions one can solve the Dirichlet problem.

Proposition 6.1. *Let* $\varphi \in \operatorname{Tr} W^{1,2}(\Omega)$*. Then there exists a unique* $u \in W^{1,2}(\Omega)$ *such that* $\mathcal{A}u = 0$ *and* $\operatorname{Tr} u = \varphi$*.*

Proof. We follow the argument as in [AE15] Lemma 2.1 and [BE21] Lemma 3.2(a).

Let $T \in B(W^{1,2}(\Omega))$ be the unique operator such that $(Tu, v)_{W_0^{1,2}(\Omega)} = \mathfrak{a}_D(u, v)$ for all $u, v \in W_0^{1,2}(\Omega)$. Note that $\ker A_D = \{0\}$ and $(Tu, v)_{W_0^{1,2}(\Omega)} = \mathfrak{a}_D(u, v) = (A_D u, v)_{L_2(\Omega)}$ for all $u, v \in W_0^{1,2}(\Omega)$, so T is injective. Note also that the inclusion $W_0^{1,2}(\Omega) \hookrightarrow L_2(\Omega)$ is compact. Therefore we can apply the Fredholm-Lax-Milgram lemma ([AEKS14] Lemma 4.1) to conclude that T is bijective.

Let $f_0 \in \operatorname{Tr} W^{1,2}(\Omega)$ be such that $\operatorname{Tr} f_0 = \varphi$. Since T is bijective we find $h \in W_0^{1,2}(\Omega)$ such that $(Th, g)_{W_0^{1,2}(\Omega)} = \mathfrak{a}(f_0, g)$ for all $g \in W_0^{1,2}(\Omega)$. Now $f = f_0 - h$ clearly satisfies $\operatorname{Tr} f = \varphi$ and

$$\begin{aligned}
\langle \mathcal{A}(f_0 - h), g\rangle_{(W_0^{1,2}(\Omega))^* \times W_0^{1,2}(\Omega)} &= \mathfrak{a}(f_0 - h, g) \\
&= \mathfrak{a}(f_0, g) - \mathfrak{a}_D(h, g) \\
&= \mathfrak{a}(f_0, g) - (Th, g)_{W_0^{1,2}(\Omega)} = 0
\end{aligned}$$

Hence $\mathcal{A}f = 0$ and the claim follows. □

We are now able to define the **Dirichlet-to-Neumann operator** $\mathcal{N}$. Let $\varphi, \psi \in L_2(\Gamma)$. Then we say that $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$ if there exists a $u \in W^{1,2}(\Omega)$ such that $\operatorname{Tr} u = \varphi$, $\mathcal{A}u = 0$ and $\partial_\nu u = \psi$. The operator $\mathcal{N}$ can be characterised by the form $\mathfrak{a}$.

Proposition 6.2. *Let* $\varphi, \psi \in L_2(\Gamma)$*. Then the following are equivalent.*

I. $\varphi \in D(\mathcal{N})$ *and* $\mathcal{N}\varphi = \psi$*.*

II. *There exists a* $u \in W^{1,2}(\Omega)$ *such that* $\operatorname{Tr} u = \varphi$ *and* $\mathfrak{a}(u, v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ *for all* $v \in W^{1,2}(\Omega)$*.*

Proof. Assume that $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$. Then there exists $u \in W^{1,2}(\Omega)$ such that $\operatorname{Tr} u = \varphi$, $\mathcal{A}u = 0$ and $\partial_\nu u = \psi$. Recall the definition of the weak co-normal derivative

$$\mathfrak{a}(u, v) - (\mathcal{A}u, v)_{L_2(\Omega)} = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)},$$

we conclude that $\mathfrak{a}(u, v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^{1,2}(\Omega)$. The converse is similar. □

If the form $\mathfrak{a}$ is symmetric, then the operator $\mathcal{N}$ is self-adjoint by [AEKS14] Theorem 4.5. The non-symmetric extension is as follows.

Proposition 6.3. *The operator* $\mathcal{N}$ *is m-sectorial.*

Proof. By ellipticity of $\mathfrak{a}$, let $\mu_1, \omega_1 > 0$ be such that

$$\operatorname{Re} \mathfrak{a}(u) \geq 2\mu_1 \|u\|^2_{W^{1,2}(\Omega)} - \omega_1 \|u\|^2_{L_2(\Omega)}$$

for all $u \in W^{1,2}(\Omega)$. By Proposition 6.1 we can define the map $\gamma_D \colon \operatorname{Tr} W^{1,2}(\Omega) \to W^{1,2}(\Omega)$ by $\gamma_D(\varphi) = u$, where $u \in W^{1,2}(\Omega)$ is such that $\mathcal{A}u = 0$ and $\operatorname{Tr} u = \varphi$. As in [AE12] Section 2 define $V(\mathfrak{a}) = \{u \in W^{1,2}(\Omega) \mid \mathfrak{a}(u, v) = 0 \text{ for all } v \in W^{1,2}_0(\Omega)\}$. Then $V(\mathfrak{a})$ is closed in $W^{1,2}(\Omega)$. If $u \in W^{1,2}(\Omega)$, then $u = \gamma_D(\operatorname{Tr} u) + (u - \gamma_D(\operatorname{Tr} u)) \in V(\mathfrak{a}) + W^{1,2}_0(\Omega)$. Therefore $W^{1,2}(\Omega) = V(\mathfrak{a}) + W^{1,2}_0(\Omega)$. Also $V(\mathfrak{a}) \cap W^{1,2}_0(\Omega) = \{0\}$ since $0 \in \rho(A_D)$. So $\operatorname{Tr}|_{V(\mathfrak{a})} \colon V(\mathfrak{a}) \to L_2(\Gamma)$ is injective. By Ehrling's lemma there exists a $c > 0$ such that

$$\|u\|^2_{L_2(\Omega)} \leq \frac{\mu_1}{\omega_1} \|u\|^2_{W^{1,2}(\Omega)} + c \, \|\operatorname{Tr} u\|^2_{L_2(\Gamma)}$$

for all $u \in V(\mathfrak{a})$. Then

$$\operatorname{Re} \mathfrak{a}(u) \geq \mu_1 \|u\|^2_{W^{1,2}(\Omega)} - c\, \omega_1 \|\operatorname{Tr} u\|^2_{L_2(\Gamma)} \tag{6.1}$$

for all $u \in V(\mathfrak{a})$. Now [AE12] Corollary 2.2 implies that the operator $\mathcal{N}$ as described in Proposition 6.2 is m-sectorial. □

Remark 6.4. Propositions 6.1, 6.2 and 6.3 remain valid if the principal coefficients C is complex valued. The proofs are word-by-word the same.

Let T be the semigroup generated by the operator $-\mathcal{N}$. Recall that we assume that $0 \notin \sigma(A_D)$. The main result of this section is the following theorem.

Theorem 6.5. *Suppose $d \geq 3$. Then there exist $\kappa \in (0, 1)$ and $c, \omega > 0$ such that $T_t \, L_2(\Gamma) \subset C^\kappa(\Gamma)$,*

$$\|T_t\|_{L_2(\Gamma) \to C^\kappa(\Gamma)} \leq c \, t^{-\frac{d-1}{2}} \, t^{-\kappa} \, e^{\omega t} \tag{6.2}$$

and

$$\|T_t\|_{2 \to \infty} \leq c \, t^{-\frac{d-1}{2}} \, e^{\omega t} \tag{6.3}$$

for all $t > 0$.

Suppose the bound (6.2) is valid. By the bounds $\|T_t\|_{2 \to 2} \leq c_1 \, e^{\omega_1 t}$ with the fact that $L_2(\Omega) \approx M_0(\Omega)$ and $C^\kappa(\Omega) \approx \mathcal{M}_{d+2\kappa}(\Omega)$, there exist $c_2, \omega_2 > 0$ such that

$$\begin{aligned} r^{-(d-1)} \int_{\Omega(x,r)} |T_t \varphi|^2 &\leq 2 r^{-(d-1)} \int_{\Omega(x,r)} (|\langle T_t \varphi \rangle_{\Omega(x,r)}|^2 + |u - \langle T_t \varphi \rangle_{\Omega(x,r)}|^2) \\ &\leq 2 r^{-d} \|T_t \varphi\|^2_{M_0(\Omega)} + 2 r^{2\kappa} \|T_t \varphi\|^2_{\mathcal{M}_{d+2\kappa}(\Omega)} \\ &\leq c_2 (c_1 + c) \, e^{\omega_2 t} (r^{-d} + r^{2\kappa} \, t^{-\frac{d-1}{2}} \, t^{-\kappa}) \end{aligned}$$

for all $t > 0$, $x \in \Gamma$, $r \in (0, 1]$ and $\varphi \in L_2(\Gamma)$. Choosing $r = t^{1/2} e^{-t} \in (0, 1]$ gives (6.3) as desired. So it remains to prove the Hölder bounds (6.2).

If $t \in (0,\infty)$ and $\varphi \in L_2(\Gamma)$, then $T_t\varphi \in D(\mathcal{N})$. Hence there exists a unique $u_{t,\varphi} \in W^{1,2}(\Omega)$ such that $\operatorname{Tr} u_{t,\varphi} = T_t\varphi$ and

$$\mathfrak{a}(u_{t,\varphi}, v) = (\mathcal{N}\, T_t\varphi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in W^{1,2}(\Omega)$. The key idea for the proof of (6.2) is to estimate $u_{t,\varphi}$.

Lemma 6.6. *There exist $\tilde{c}_0, \tilde{\omega}_0 > 0$ such that*

$$\|u_{t,\varphi}\|_{L_2(\Omega)} \le \tilde{c}_0\, t^{-1/2}\, e^{\tilde{\omega}_0 t}\, \|\varphi\|_{L_2(\Gamma)} \quad \text{and} \quad \|\nabla u_{t,\varphi}\|_{L_2(\Omega)} \le \tilde{c}_0\, t^{-1/2} e^{\tilde{\omega}_0 t}\, \|\varphi\|_{L_2(\Gamma)}$$

for all $t > 0$ and $\varphi \in L_2(\Gamma)$.

Proof. As in [AE12] Section 2 define $V(\mathfrak{a}) = \{u \in W^{1,2}(\Omega) \mid \mathfrak{a}(u,v) = 0 \text{ for all } v \in W_0^{1,2}(\Omega)\}$. Let $c, \mu_1, \omega_1 > 0$ be as in (6.1). Then

$$\mu_1\, \|u\|^2_{W^{1,2}(\Omega)} \le \operatorname{Re} \mathfrak{a}(u) + c\,\omega_1\, \|\operatorname{Tr} u\|^2_{L_2(\Gamma)}$$

for all $u \in V(\mathfrak{a})$. In particular

$$\mu_1\, \|u_{t,\varphi}\|^2_{W^{1,2}(\Omega)} \le \operatorname{Re}(\mathcal{N}\, T_t\varphi, T_t\varphi)_{L_2(\Gamma)} + c\,\omega_1\, \|T_t\varphi\|^2_{L_2(\Gamma)}. \tag{6.4}$$

Since $\mathcal{N}$ is m-sectorial, T_t is holomorphic in some sector. Repeating the proof of Lemma 4.2, there exist $c, \omega > 0$ such that $\|\mathcal{N}\, T_t\varphi\|_{L_2(\Gamma)} \le c\, t^{-1}\, e^{\omega t} \|\varphi\|_{L_2(\Gamma)}$ and $\|T_t\varphi\|_{L_2(\Gamma)} \le c\, e^{\omega t} \|\varphi\|_{L_2(\Gamma)}$ for all $t > 0$ and $\varphi \in L_2(\Gamma)$. Substituting into (6.4) gives the desired bounds. □

By a compactness argument Theorem 6.5 is a consequence of the next proposition.

Proposition 6.7. *Let $U \subset \mathbb{R}^d$ be an open set and Φ a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\Phi(U) = E$ and $\Phi(\Omega \cap U) = E^-$.*

I. *If $d \ge 3$, then there exist $c, \delta_0, \omega > 0$ and $\kappa \in (0,1)$ such that*

$$|(T_t\varphi)(x) - T_t\varphi)(y)| \le c\, t^{-\frac{d-1}{2}}\, t^{-\kappa}\, e^{\omega t}\, \|\varphi\|_{L_2(\Gamma)}\, |x-y|^\kappa$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x, y \in \Gamma \cap \Phi^{-1}(\frac{1}{2}E)$ with $|x-y| \le \delta_0$.

II. *If $d = 2$, then for all $\varepsilon > 0$ there exist $c, \delta_0, \omega > 0$ and $\kappa \in (0,1)$ such that*

$$|(T_t\varphi)(x) - T_t\varphi)(y)| \le c\, t^{-\frac{d-1}{2}}\, t^{-\kappa}\, t^{-\varepsilon}\, e^{\omega t}\, \|\varphi\|_{L_2(\Gamma)}\, |x-y|^\kappa$$

for all $t > 0$, $\phi \in L_2(\Gamma)$ and $x, y \in \Gamma \cap \Phi^{-1}(\frac{1}{2}E)$ with $|x-y| \le \delta_0$.

Proof. Let $\kappa \in (0,1)$ be as in Proposition 3.27. Let $K \in [1,\infty)$ be larger than the Lipschitz constant of $\Phi|_{\Omega\cap U}$ and $\Phi^{-1}|_{E^-}$. For all $\gamma \in [0, d-2+2\kappa)$ let $P(\gamma)$ be the hypothesis

There exist $c, \omega > 0$ such that

$$\|\nabla(u_{t,\varphi} \circ \Phi^{-1})\|_{M,\gamma,x,E^-,1} \leq c\, t^{-\frac{\gamma+1}{2}}\, e^{\omega t}\, \|\varphi\|_{L_2(\Gamma)}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2}\, E^-$.

Clearly $P(0)$ is valid by Lemma 6.6.

We need two lemmas.

Lemma 6.8. *There exists a $c_1 > 0$ such that*

$$\|u\|_{\mathcal{M},\gamma+\delta,x,E^-,1} \leq c_1 \left(\varepsilon^{2-\delta} \|\nabla u\|_{M,\gamma,x,E^-,1} + \varepsilon^{-(\gamma+\delta-2)} \|\nabla u\|_{L_2(E^-)} \right)$$

for all $\gamma \in [0, d)$ and $\delta \in [0, 2]$, $\varepsilon \in (0, 1]$, $u \in W^{1,2}(E^-)$ and $x \in \frac{1}{2}\, E^-$ with $\gamma + \delta \geq 2$.

Proof. By the Neumann type Poincaré inequality of Lemma 3.19(a) there exists a $c > 0$ such that

$$\int_{E^-(x_0,R)} |u - \langle u \rangle_{E^-(x_0,R)}|^2 \leq c\, R^2 \int_{E^-(x_0,R)} |\nabla u|^2 \tag{6.5}$$

for all $x_0 \in \frac{1}{2}\, E^-$, $R \in (0, \frac{1}{2}]$ and $u \in W^{1,2}(E^-)$.

Now we prove the lemma. If $r \in (0, \frac{1}{2}\, \varepsilon^2]$, then

$$r^{-(\gamma+\delta)} \int_{E^-(x,r)} |u - \langle u \rangle_{E^-(x,r)}|^2 \leq c\, r^{2-\delta} r^{-\gamma} \int_{E^-(x,r)} |\nabla u|^2 \leq c\, \varepsilon^{2(2-\delta)} \|\nabla u\|^2_{M,\gamma,x,E^-,1}$$

Alternatively,

$$\int_{E^-(x,r)} |u - \langle u \rangle_{E^-(x,r)}|^2 \leq c\, r^2 \int_{E^-(x,r)} |\nabla u|^2 \leq 2^{\gamma+\delta-2}\, \varepsilon^{-2(\gamma+\delta-2)}\, \|\nabla u\|^2_{L_2(E^-)}\, r^{\gamma+\delta}$$

if $r \in [\frac{1}{2}\, \varepsilon^2, \frac{1}{2}]$, from which the lemma follows. □

Lemma 6.9. *Adopt the assumptions and notation of Proposition 6.7.*

I. *If $\gamma \in [0, 2] \cap [0, d)$, then there exist $c, \omega > 0$ such that*

$$\|u_{t,\varphi} \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} \leq c\, t^{-1/2}\, e^{\omega t}\, \|\varphi\|_{L_2(\Gamma)}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2}\, E^-$.

II. *Let $\gamma \in [0, d)$ and $\delta \in [0, 2]$ with $\gamma + \delta < d$. Suppose that $P(\gamma)$ is valid. Then there exist $c, \omega > 0$ such that*

$$\|u_{t,\varphi} \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} \leq c\, t^{-\frac{1 \vee (\gamma+\delta-1)}{2}}\, e^{\omega t}\, \|\varphi\|_{L_2(\Gamma)}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2}\, E^-$.

Proof. 'I'. By Lemma 6.6 we may assume that $\gamma > 0$. By the second part of Lemma 3.20 there exists a $c' > 0$ such that

$$\begin{aligned}\|u_{t,\varphi} \circ \Phi^{-1}\|_{\mathcal{M},\gamma,x,E^-,1} &\leq c'\Big(\|\nabla(u_{t,\varphi} \circ \Phi^{-1})\|_{M,0,x,E^-,1} + \|u_{t,\varphi} \circ \Phi^{-1}\|_{L_2(E^-)}\Big) \\ &\leq 2c'\, d!\, K^{d+1}\, \tilde{c}_0\, t^{-1/2}\, e^{\tilde{\omega}_0 t}\, \|\varphi\|_{L_2(\Gamma)}\end{aligned}$$

for all $t > 0$ and $\varphi \in L_2(\Gamma)$, where $\tilde{c}_0, \tilde{\omega}_0 > 0$ are as in Lemma 6.6. Then by Proposition 2.18 there exist $c_2, c_3 > 0$ such that

$$\begin{aligned}\|u_{t,\varphi} \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} &\leq c_2\,(\|u_{t,\varphi} \circ \Phi^{-1}\|_{\mathcal{M},\gamma,x,E^-,1} + \|u_{t,\varphi} \circ \Phi^{-1}\|_{L_2(E^-)}) \\ &\leq c_3\, t^{-1/2}\, e^{\tilde{\omega}_0 t}\, \|\varphi\|_{L_2(\Gamma)}\end{aligned}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2}E^-$.

'II'. By Statement I we may assume that $\gamma + \delta \geq 2$. Let $c_1 > 0$ be as in Lemma 6.8. Choose $\varepsilon = t^{1/2}\, e^{-t} \in (0, 1]$. Then

$$\begin{aligned}&\|u_{t,\varphi} \circ \Phi^{-1}\|_{\mathcal{M},\gamma+\delta,x,E^-,1} \\ &\leq c_1\Big(\varepsilon^{2-\delta}\, \|\nabla(u_{t,\varphi} \circ \Phi^{-1})\|_{M,\gamma,x,E^-,1} + \varepsilon^{-(\gamma+\delta-2)}\, \|\nabla(u_{t,\varphi} \circ \Phi^{-1})\|_{L_2(E^-)}\Big) \\ &\leq c_1\Big(t^{\frac{2-\delta}{2}}\, e^{-(2-\delta)t}\, c_\gamma\, t^{-\frac{\gamma+1}{2}}\, e^{\omega_\gamma t}\, \|\varphi\|_{L_2(\Gamma)} + t^{-\frac{\gamma+\delta-2}{2}}\, e^{(\gamma+\delta)t}\, d!\, K^{d+1}\, \tilde{c}_0\, t^{-\frac{1}{2}} e^{\tilde{\omega}_0 t}\, \|\varphi\|_{L_2(\Gamma)}\Big) \\ &= c_2\, t^{-\frac{\gamma+\delta-1}{2}}\, e^{\omega_1 t}\, \|\varphi\|_{L_2(\Gamma)}\end{aligned}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2}E^-$, with suitable $c_2, \omega_1 > 0$.

Finally, by Proposition 2.18 there exist $c_3, c_4, \omega_2 > 0$ such that

$$\begin{aligned}\|u_{t,\varphi} \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} &\leq c_3\,(\|u_{t,\varphi} \circ \Phi^{-1}\|_{\mathcal{M},\gamma+\delta,x,E^-,1} + \|u_{t,\varphi} \circ \Phi^{-1}\|_{L_2(E^-)}) \\ &\leq c_4\, t^{-\frac{\gamma+\delta-1}{2}}\, e^{\omega_2 t}\, \|\varphi\|_{L_2(\Gamma)}\end{aligned}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2}E^-$ and the lemma follows. □

Let $\kappa \in (0, 1)$ be as in Proposition 3.27.

Lemma 6.10. *Let $\gamma \in [0, d - 2 + 2\kappa)$ and suppose that $P(\gamma)$ is valid. Let $\delta \in (0, 2]$ and suppose that $\gamma + \delta < d - 2 + 2\kappa$. Then one has the following.*

I. *If $d \geq 3$, then $P(\gamma + \delta)$ is valid.*

II. *If $d = 2$, then for all $\eta > 0$ there exist $c, \omega > 0$ such that*

$$\|\nabla(u_{t,\varphi} \circ \Phi^{-1})\|_{M,\gamma+\delta,x,E^-,1} \leq c\, t^{-\frac{\gamma+\delta+1}{2}}\, t^{-\eta}\, e^{\omega t}\, \|\varphi\|_{L_2(\Gamma)}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2}\, E^-$.

Proof. Without loss of generality we may assume in case $d = 2$ that $\gamma + \delta \leq 2 - 2\eta$. Define $\tilde\gamma \in [\gamma + \delta, d)$ by

$$\tilde\gamma = \begin{cases} \gamma + \delta & \text{if } \gamma + \delta \geq 2, \\ 2 & \text{if } \gamma + \delta < 2 \text{ and } d \geq 3, \\ 2 - 2\eta & \text{if } d = 2. \end{cases}$$

Note that $\tilde\gamma \geq 2$ if $d \geq 3$. Let $c > 0$ be as in Proposition 3.27. By analyticity of T there exist $\tilde c, \tilde\omega > 0$ such that $\|\mathcal{N}\, T_t \varphi\|_{L_2(\Gamma)} \leq \tilde c\, t^{-1}\, e^{\tilde\omega t}\, \|\varphi\|_{L_2(\Gamma)}$ for all $t > 0$ and $\varphi \in L_2(\Gamma)$. By Lemma 6.9 there exist $\hat c, \hat\omega > 0$ such that

$$\|u_{t,\varphi} \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} \leq \hat c\, t^{-\frac{1 \vee (\gamma - 1)}{2}}\, e^{\hat\omega t}\, \|\varphi\|_{L_2(\Gamma)},$$

$$\|u_{t,\varphi} \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} \leq \hat c\, t^{-\frac{1 \vee (\gamma + \delta - 1)}{2}}\, e^{\hat\omega t}\, \|\varphi\|_{L_2(\Gamma)} \text{ and}$$

$$\|u_{t,\varphi} \circ \Phi^{-1}\|_{M,\tilde\gamma,x,E^-,1} \leq \hat c\, t^{-\frac{1 \vee (\tilde\gamma - 1)}{2}}\, e^{\hat\omega t}\, \|\varphi\|_{L_2(\Gamma)}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2}\, E^-$.

Let $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2} E^-$. Since $\mathcal{N}\, T_{2t} \varphi = T_t\, \mathcal{N}\, T_t \varphi$ it follows that

$$\begin{aligned}
\mathfrak{a}_p(u_{2t,\varphi}, v) &= \mathfrak{a}(u_{2t,\varphi}, v) - \int_\Omega \sum_{k=1}^d a_k\, (\partial_k u)\, \overline{v} - \int_\Omega \sum_{l=1}^d b_l\, u\, \overline{\partial_l\, v} - \int_\Omega a_0\, u\, \overline{v} \\
&= (\mathcal{N} T_{2t} \varphi, \operatorname{Tr} v)_{L_2(\Omega)} + (f, v)_{L_2(\Omega)} + \sum_{k=1}^d (f_k, \partial_k v)_{L_2(\Omega)} \\
&= (f, v)_{L_2(\Omega)} + \sum_{k=1}^d (f_k, \partial_k v)_{L_2(\Omega)} + \int_\Gamma (\operatorname{Tr} u_{t,\mathcal{N}\, T_t \varphi})\, \overline{\operatorname{Tr} v}
\end{aligned}$$

for all $v \in W^{1,2}(\Omega)$, where $f = -a_0\, u_{2t,\varphi} - \sum_{k=1}^d a_k\, \partial_k u_{2t,\varphi}$ and $f_k = -b_k\, u_{2t,\varphi}$. Hence Proposition 3.27 with the choice $\varepsilon = t^{1/2}\, e^{-t} \in (0, 1]$ gives

$$\begin{aligned}
&\|\nabla(u_{2t,\varphi} \circ \Phi^{-1})\|_{M,\gamma+\delta,x,E^-,1} \\
&\quad \leq c\Big(\varepsilon^{2-\delta}\, \|(a_0\, u_{2t,\varphi}) \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} + \varepsilon^{2-\delta} \sum_{k=1}^d \|(a_k\, \partial_k u_{2t,\varphi}) \circ \Phi^{-1}\|_{M,\gamma,x,E^-,1} \\
&\qquad + \sum_{k=1}^d \|(b_k\, u_{2t,\varphi}) \circ \Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} + \varepsilon^{-(\gamma+\delta)}\, \|\nabla u_{2t,\varphi}\|_{L_2(\Omega)} \\
&\qquad + \varepsilon^{2-\delta}\, \|\beta\|_{L_\infty(\Gamma)}\, \|\nabla(u_{t,\mathcal{N}\, T_t \varphi} \circ \Phi^{-1})\|_{M,\gamma,x,E^-,1} \\
&\qquad + \varepsilon^{\tilde\gamma - \gamma - \delta}\, \|\beta\|_{L_\infty(\Gamma)}\, \|u_{t,\mathcal{N}\, T_t \varphi} \circ \Phi^{-1}\|_{M,\tilde\gamma,x,E^-,1}\Big).
\end{aligned}$$

We estimate the terms.

First

$$\begin{aligned}\varepsilon^{2-\delta}\,\|(a_0\,u_{2t,\varphi})\circ\Phi^{-1}\|_{M,\gamma,x,E^-,1} &\le M\,t^{\frac{2-\delta}{2}}\,\hat{c}\,(2t)^{-\frac{1\vee(\gamma-1)}{2}}\,e^{2\hat{\omega}t}\,\|\varphi\|_{L_2(\Gamma)}\\ &\le \hat{c}\,M\,t^{\frac{2-\delta}{2}}\,t^{-\frac{1\vee(\gamma-1)}{2}}\,e^{2\hat{\omega}t}\,\|\varphi\|_{L_2(\Gamma)}\\ &= \hat{c}\,M\,t^{-\frac{\gamma+\delta+1}{2}}\,t^{\frac{2+(2\wedge\gamma)}{2}}\,e^{2\hat{\omega}t}\,\|\varphi\|_{L_2(\Gamma)}\\ &\le \hat{c}\,M\,t^{-\frac{\gamma+\delta+1}{2}}\,e^{(2\hat{\omega}+1+\gamma)t}\,\|\varphi\|_{L_2(\Gamma)}.\end{aligned}$$

Secondly,

$$\begin{aligned}\varepsilon^{2-\delta}\sum_{k=1}^{d}\|(a_k\,\partial_k u_{2t,\varphi})\circ\Phi^{-1}\|_{M,\gamma,x,E^-,1} &\le t^{\frac{2-\delta}{2}}\,d\,M\,\|\nabla(u_{2t,\varphi})\circ\Phi^{-1}\|_{M,\gamma,x,E^-,1}\\ &\le d\,M\,t^{\frac{2-\delta}{2}}\,K\,\|\nabla(u_{2t,\varphi}\circ\Phi^{-1})\|_{M,\gamma,x,E^-,1}\\ &\le d\,K\,M\,t^{\frac{2-\delta}{2}}\,c_\gamma\,(2t)^{-\frac{\gamma+1}{2}}\,e^{\omega_\gamma t}\,\|\varphi\|_{L_2(\Gamma)}\\ &= c_\gamma\,d\,K\,M\,t^{-\frac{\gamma+\delta+1}{2}}\,t\,e^{\omega_\gamma t}\,\|\varphi\|_{L_2(\Gamma)}\\ &\le c_\gamma\,d\,K\,M\,t^{-\frac{\gamma+\delta+1}{2}}\,e^{(\omega_\gamma+1)t}\,\|\varphi\|_{L_2(\Gamma)}.\end{aligned}$$

Thirdly,

$$\begin{aligned}\sum_{k=1}^{d}\|(b_k\,u_{2t,\varphi})\circ\Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1} &\le d\,M\,\|u_{2t,\varphi}\circ\Phi^{-1}\|_{M,\gamma+\delta,x,E^-,1}\\ &\le d\,M\,\hat{c}\,(2t)^{-\frac{1\vee(\gamma+\delta-1)}{2}}\,e^{\hat{\omega}t}\,\|\varphi\|_{L_2(\Gamma)}\\ &\le \hat{c}\,d\,M\,t^{-\frac{\gamma+\delta+1}{2}}\,t^{\frac{2\wedge(\gamma+\delta)}{2}}\,e^{\hat{\omega}t}\,\|\varphi\|_{L_2(\Gamma)}\\ &\le \hat{c}\,d\,M\,t^{-\frac{\gamma+\delta+1}{2}}\,e^{(\hat{\omega}+1+\gamma+\delta)t}\,\|\varphi\|_{L_2(\Gamma)}\end{aligned}$$

.

Fourthly,

$$\begin{aligned}\varepsilon^{-(\gamma+\delta)}\,\|\nabla u_{2t,\varphi}\|_{L_2(\Omega)} &\le t^{-\frac{\gamma+\delta}{2}}\,e^{(\gamma+\delta)t}\,\tilde{c}_0\,(2t)^{-1/2}\,e^{2\tilde{\omega}_0 t}\,\|\varphi\|_{L_2(\Gamma)}\\ &\le \tilde{c}_0\,t^{-\frac{\gamma+\delta+1}{2}}\,e^{(\gamma+\delta+2\tilde{\omega}_0)t}\,\|\varphi\|_{L_2(\Gamma)},\end{aligned}$$

where $\tilde{c}_0,\tilde{\omega}_0>0$ are as in Lemma 6.6. Fifthly,

$$\begin{aligned}\varepsilon^{2-\delta}\,\|\beta\|_{L_\infty(\Omega)}\,\|\nabla(u_{t,\mathcal{N}T_t\varphi}\circ\Phi^{-1})\|_{M,\gamma,x,E^-,1} &\le t^{\frac{2-\delta}{2}}\,M\,c_\gamma\,t^{-\frac{\gamma+1}{2}}\,e^{\omega_\gamma t}\,\|\mathcal{N}\,T_t\varphi\|_{L_2(\Gamma)}\\ &\le t^{\frac{2-\delta}{2}}\,M\,c_\gamma\,t^{-\frac{\gamma+1}{2}}\,e^{\omega_\gamma t}\,\tilde{c}\,t^{-1}\,e^{\tilde{\omega}t}\,\|\varphi\|_{L_2(\Gamma)}\end{aligned}$$

$$= \tilde{c}\, c_\gamma\, M\, t^{-\frac{\gamma+\delta+1}{2}}\, e^{(\omega_\gamma+\tilde{\omega})t}\, \|\varphi\|_{L_2(\Gamma)}.$$

Finally,

$$\begin{aligned}
\varepsilon^{\tilde{\gamma}-\gamma-\delta}\, \|\beta\|_{L_\infty(\Omega)}\, \|u_{t,\mathcal{N}\,T_t\varphi} \circ \Phi^{-1}\|_{M,\tilde{\gamma},x,E^-,1} &\leq t^{\frac{\tilde{\gamma}-\gamma-\delta}{2}}\, M\, \hat{c}\, t^{-\frac{1\vee(\tilde{\gamma}-1)}{2}}\, e^{\hat{\omega}t}\, \|\mathcal{N}\,T_t\varphi\|_{L_2(\Gamma)} \\
&\leq t^{\frac{\tilde{\gamma}-\gamma-\delta}{2}}\, M\, \hat{c}\, t^{-\frac{1\vee(\tilde{\gamma}-1)}{2}}\, e^{\hat{\omega}t}\, \tilde{c}\, t^{-1}\, e^{\tilde{\omega}t}\, \|\varphi\|_{L_2(\Gamma)} \\
&= \tilde{c}\, \hat{c}\, M\, t^{-\frac{\gamma+\delta+1}{2}}\, t^{-\frac{(2-\tilde{\gamma})\vee 0}{2}}\, e^{(\hat{\omega}+\tilde{\omega})t}\, \|\varphi\|_{L_2(\Gamma)}.
\end{aligned}$$

Note that $t^{-\frac{(2-\tilde{\gamma})\vee 0}{2}} = 1$ if $\tilde{\gamma} \geq 2$, so the lemma follows. □

Now we are able to complete the proof of Proposition 6.7.

End of proof of Proposition 6.7.

'I'. (Suppose that $d \geq 3$.) We know that $P(0)$ is valid. Then it follows by induction from Lemma 6.10.I that $P(\gamma)$ is valid for all $\gamma \in [0, d-2+2\kappa)$. In particular $P(d-2+\kappa)$ is valid. Hence by the Neumann type Poincaré inequality (6.5) one deduces that there are $c, \omega > 0$ such that

$$\|u_{t,\varphi} \circ \Phi^{-1}\|_{\mathcal{M},d+\kappa,x,E^-,1} \leq c\, t^{-\frac{d-1+\kappa}{2}}\, e^{\omega t}\, \|\varphi\|_{L_2(\Gamma)}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x \in \frac{1}{2}\, E^-$. Therefore the function $(u_{t,\varphi} \circ \Phi^{-1})|_{\frac{1}{2}\, E^-}$ has a continuous representative, which is Hölder continuous and it extends continuously to the closure of $\frac{1}{2}\, E^-$. By Proposition 2.20 there exists a $c' > 0$ such that

$$|(u_{t,\varphi} \circ \Phi^{-1})(x) - (u_{t,\varphi} \circ \Phi^{-1})(y)| \leq c'\, t^{-\frac{d-1+\kappa}{2}}\, e^{\omega t}\, \|\varphi\|_{L_2(\Gamma)}\, |x-y|^{\kappa/2}$$

for all $t > 0$, $\varphi \in L_2(\Gamma)$ and $x, y \in \frac{1}{2}\, E^-$ with $|x-y| < \frac{1}{2}$. The latter estimates extend to all $x, y \in \frac{1}{2}\, \overline{E^-}$ with $|x-y| \leq \frac{1}{4}$. Since Φ is bi-Lipschitz with Lipschitz constants K, it follows that

$$|u_{t,\varphi}(x) - u_{t,\varphi}(y)| \leq c'\, K^{\kappa/2}\, t^{-\frac{d-1+\kappa}{2}}\, e^{\omega t}\, \|\varphi\|_{L_2(\Gamma)}\, |x-y|^{\kappa/2}$$

for all $t > 0$, $u \in L_2(\Gamma)$ and $x, y \in \Gamma \cap \Phi^{-1}(\frac{1}{2}E)$ with $|x-y| \leq \frac{1}{2K}$ and Statement I follows.

The prove of Statement II is similar by using Lemma 6.10.II. □

The uniform Hölder bounds of Theorem 6.5 can be combined with the Poisson kernel bounds of [EO19] to obtain Hölder Poisson kernel bounds in case the domain Ω is of class $C^{1+\kappa}$ for some $\kappa > 0$.

Theorem 6.11. *Assume $d \geq 3$. Suppose there exists a $\kappa > 0$ such that Ω is of class $C^{1+\kappa}$. Suppose that $c_{kl} = c_{lk}$ is Hölder continuous for all $k, l \in \{1, \ldots, d\}$. Suppose that $a_k = b_k = 0$ for all $k \in \{1, \ldots, d\}$ and a_0 is real valued. Suppose that $0 \notin \sigma(A_D)$. Let*

K be the kernel of the semigroup on $L_2(\Gamma)$ generated by $-\mathcal{N}$, where $\mathcal{N}$ is the Dirichlet-to-Neumann operator. Then for all $\varepsilon, \tau' \in (0,1)$ and $\tau > 0$ there exist $c, \nu > 0$ such that

$$|K_t(x,y) - K_t(x',y')| \leq c\,(t \wedge 1)^{-(d-1)} \Big(\frac{|x-x'| + |y-y'|}{t + |x-y|}\Big)^{\nu} \frac{1}{\Big(1 + \dfrac{|x-y|}{t}\Big)^{d-\varepsilon}} (1+t)^{\nu}\, e^{-\lambda_1 t}$$

for all $x, y, x', y' \in \Gamma$ and $t > 0$ with $|x-x'|+|y-y'| \leq \tau\, t + \tau' \,|x-y|$, where $\lambda_1 = \min \sigma(\mathcal{N})$.

Proof. It follows from Theorem 6.5 and [EO19] Theorem 1.1 that there exist $c, \omega > 0$ and sufficiently small $\nu' \in (0,1)$ such that

$$|K_t(x,y)| \leq c\, t^{-(d-1)} \frac{1}{\Big(1 + \dfrac{|x-y|}{t}\Big)^{d}} e^{\omega t}$$

and

$$|K_t(x,y) - K_t(x',y)| \leq c\, t^{-(d-1)} \Big(\frac{|x-x'|}{t}\Big)^{\nu'} e^{\omega t} \tag{6.6}$$

for all $x, y, x' \in \Gamma$ and $t > 0$ with $|x - x'| \leq 1$. By duality, we obtain similarly, without loss of generality, that

$$|K_t(x,y) - K_t(x,y')| \leq c\, t^{-(d-1)} \Big(\frac{|y-y'|}{t}\Big)^{\nu'} e^{\omega t} \tag{6.7}$$

for all $x, y, y' \in \Gamma$ and $t > 0$ with $|y - y'| \leq 1$.

Now let $x, y, x', y' \in \Gamma$ and suppose that $|x-x'| + |y-y'| \leq \tau\, t + \tau' \,|x-y|$. Then $|x-y| \leq |x'-y'| + \tau\, t + \tau' \,|x-y|$, so $|x-y| \leq \frac{1}{1-\tau'} \,|x'-y'| + \frac{\tau}{1-\tau'}\, t$. Hence

$$1 + \frac{|x-y|}{t} \leq 1 + \frac{1}{1-\tau'} \,\frac{|x'-y'|}{t} + \frac{\tau}{1-\tau'} \leq \frac{1+\tau}{1-\tau'} \Big(1 + \frac{|x'-y'|}{t}\Big)$$

and

$$|K_t(x',y')| \leq c\, t^{-(d-1)} \frac{1}{\Big(1 + \dfrac{|x'-y'|}{t}\Big)^{d}} e^{\omega t} \leq c\, \frac{(1+\tau)^d}{(1-\tau')^d}\, t^{-(d-1)} \frac{1}{\Big(1 + \dfrac{|x-y|}{t}\Big)^{d}} e^{\omega t}.$$

Therefore

$$|K_t(x,y) - K_t(x',y')| \leq 2c\, \frac{(1+\tau)^d}{(1-\tau')^d}\, t^{-(d-1)} \frac{1}{\Big(1 + \dfrac{|x-y|}{t}\Big)^{d}} e^{\omega t}.$$

Next, it follows from (6.6) and (6.7) that

$$|K_t(x,y) - K_t(x',y')| \leq 2c\, t^{-(d-1)} \Big(\frac{|x-x'| + |y-y'|}{t}\Big)^{\nu'} e^{\omega t}.$$

Then

$$|K_t(x,y) - K_t(x',y')| \le c'\, t^{-(d-1)} \Big(\frac{|x-x'| + |y-y'|}{t} \Big)^{\nu' \varepsilon} \frac{1}{\Big(1 + \frac{|x-y|}{t} \Big)^{d(1-\varepsilon)}} \; e^{\omega t}$$

by interpolation, where $c' = 2c \, \frac{(1+\tau)^{d(1-\varepsilon)}}{(1-\tau')^{d(1-\varepsilon)}}$. Note that

$$\frac{1}{t} = \frac{1}{t + |x-y|} \Big(1 + \frac{|x-y|}{t} \Big).$$

Therefore

$$|K_t(x,y) - K_t(x',y')| \le c'\, t^{-(d-1)} \Big(\frac{|x-x'| + |y-y'|}{t + |x-y|} \Big)^{\nu' \varepsilon} \frac{1}{\Big(1 + \frac{|x-y|}{t} \Big)^{d(1-\varepsilon) - \nu'\varepsilon}} \; e^{\omega t}$$

and the required bounds follow if $t \in (0,3]$.

Finally, there exist $c > 0$ and $\nu \in (0,1)$ such that

$$\|T_t\|_{L_1(\Gamma) \to C^\nu(\Gamma)} \le \|T_1\|_{L_2(\Gamma) \to C^\nu(\Gamma)} \, \|T_{t-2}\|_{L_2(\Gamma) \to L_2(\Gamma)} \, \|T_1\|_{L_1(\Gamma) \to L_2(\Gamma)} \le c\, e^{-\lambda_1 t}$$

for all $t \ge 3$. Hence

$$|K_t(x,y) - K_t(x',y)| \le c\, e^{-\lambda_1 t} \, |x - x'|^\nu$$

for all $x, x', y \in \Gamma$ and $t \ge 3$ with $|x - x'| \le 1$. By duality there exists a $c' > 0$ such that

$$|K_t(x,y) - K_t(x',y')| \le c'\, e^{-\lambda_1 t} \, (|x-x'| + |y-y'|)^\nu$$

for all $x, x', y, y' \in \Gamma$ and $t \ge 3$ with $|x - x'| \le 1$ and $|y - y'| \le 1$. Since Γ is bounded, the required Hölder Poisson bounds follow for $t \ge 3$. □

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