How Primary School Students' Arguments Develop: Taking Initial Steps in a Deductive Discourse

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Abstract

A wealth of studies has shown that students are inclined to use example-based reasoning—*empirical arguments*—which are insufficient for proving universal statements. Conversely, formal mathematical proofs typically use definitions and axioms and take logical, deductive steps to reach generalised conclusions. While expecting young students to provide formal proofs might be inappropriate, some research has shown that moving students beyond using empirical arguments to prove universal statements is achievable. Nevertheless, it remains unclear *how* students arrive at particular arguments, and, in the absence of an explicitly expressed deductive proof, what might constitute smaller steps of learning and *how* young students can be supported to take these steps.

A commognitive perspective is taken to examine these gaps in the research. The research focuses on Year 4 students' discourses about odd and even numbers and their substantiations for claims about the sums of odd and even numbers. Four studies are presented, each of which uses elements of commognitive theory to look closely at students' discourse. The first study identifies the ways in which students use examples. It finds that students' example-use can be more nuanced than an empiric-generic dichotomy, and that discursive markers of example-use can be detected in students' discourse that implicitly signal generality.

The three subsequent studies look closely at the discourse of one group of four students, using commognitive tools to reveal the character of the students' argumentation and examine their first steps in a deductive discourse. The commognitive stance had maintained that learners' initiation in a new discourse would be socially motivated and process-oriented (*ritualised*) rather than intrinsically motivated and outcome-oriented (*explorative*). However, the studies in this thesis show that students were able to take explorative first steps in a new deductive discourse. This has implications for curriculum and task design and pedagogy, as well as for the commognitive perspective on learning mathematics. The findings further suggest that the mechanism of conflict among discourses, commonly suggested as motivation to adopt a new discourse, might

be insufficient to prompt learning. Young children in this study required a mathematical reason – not just a social reason – to change their discursive ways.

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CHAPTER 1: Introduction

In this chapter, I describe my motivation for researching this particular area, including my personal background and experiences, and my initial struggles with (i) defining proof, (ii) evaluating the relevance and appropriateness of proving in the primary[1] classroom, (iii) figuring out what proof might look like in the primary classroom, and (iv) considering how young students might be supported to understand and use proof[2]. I conclude by summarising how these questions and my struggle to work through them led me to my current research focus. I then provide an overview of the structure of my thesis, briefly explaining how I arrived at my specific research questions and how the parts of my thesis fit according to the research questions.

1.1 Motivation

1.1.1 My Background and Experiences

My motivation for researching this particular area began as I considered mathematical proof for the very first time during a postgraduate course about mathematical processes. During my 20-year career in primary school education, I had been a classroom teacher at many year levels (for students aged 5–13) and a mathematics specialist teacher, providing specialist support to extend students who showed aptitude in the subject and accelerate learning for those who struggled in mathematics. I had also spent more than ten years leading professional development in schools, working with teachers to improve teaching and learning experiences. Nevertheless, despite this extensive experience, I found myself sitting in a postgraduate class learning about mathematical proof and the process of proving for the very first time.

I had many questions! What exactly counted as mathematical proof and what did not? If mathematical proof was so important, how could I have acquired so many years of experience in the field of mathematics education without ever encountering

[1] The term *primary* is used in New Zealand to refer to education in Years 1-6 for students aged from age 5-11 years of age.
[2] This part of the chapter is based on a paper submitted to NZCER Set journal in 2017. Permission to reuse the content of this paper is included in Appendix A.

mathematical proof? Was it because proving was inappropriate for the primary classroom? I acknowledged that proving was a *process*—it was not mathematical *content*—and I wondered if the optimal time to introduce processes versus content was different? In regard to mathematical content, it seemed evident that a certain readiness would be required to understand various branches of mathematics such as calculus, quadratic equations, or trigonometric relationships. It would be inappropriate, therefore, to teach such content in the early years of primary schooling. However, I wondered whether a similar readiness was required to introduce the process of proving and use of mathematical proof. To answer my questions, I realised I needed to evaluate the relevance and appropriateness of proving for students in primary schools. I needed to consider what proof might look like in the primary classroom and how young students might be supported to understand and use proof. Although I did not realise it at the time, these questions represented the beginnings of my doctoral research.

1.1.2 What is Mathematical Proof?

Schoenfeld (1994) stated proof "is an essential component of doing, communicating, and recording mathematics" (p. 76). But what is proof? Being personally unfamiliar with the field of mathematics beyond the school system, I believed that there was a particular set of universal proofs, similar to scientific laws that showed mathematical theorems to be true. However, as I began reading relevant literature, I discovered that what constitutes proof had been debated for decades, and continues to be debated, and that there was no universal consensus as to what constituted proof. I found that proof came in different forms (e.g., direct proof, proof by induction, proof by contradiction, proof by exhaustion, two-column, flow-chart). And I realised the vague idea of proof that I originally held mostly aligned with a *formal proof*, in the form of logical deductive steps that are expressed algebraically to prove traditional theories (Hales, 2008). I intuitively knew that formal, deductive proofs would be beyond typical primary school students' understanding and, since this resembled my original concept of proof, this was one reason I had failed to consider proof having a place in the primary school classroom.

Nevertheless, as I continued with my inquiry, I learned that proving need not necessarily demand the rigor or fussiness associated with formal deductive proofs.

Some mathematicians and scholars have claimed that a proof could simply be a physical demonstration or an explanation to remove or create doubts about a claim, which is accepted by a community at a given time (e.g., Balacheff, 1987; Harel & Sowder, 2007). Through considering different versions of proof, I realised that, although the fundamental purpose of a mathematical proof is to know whether a mathematical assertion or idea is true or false (Cadwallader Olsker, 2011; Hanna & Jahnke, 1996), proving also has a more practical role in *convincing* others about our statements or theorems. Moreover, this practical role could promote *understanding.* Indeed, Hanna and Jahnke (1996) argued that using proof in teaching allows students to communicate their thinking and enhance their mathematical understanding. They added that "a proof actually becomes legitimate and convincing to a mathematician only when it leads to real mathematical understanding." (p. 878). As I experienced these personal revelations—associating proving with enhanced understanding—I also realised that proving and proof deserve a place in the primary classroom.

1.1.3 Where is Proof in the Curriculum?

When reflecting on why I had never encountered mathematical proof before, I realised that the blame lay partly in its invisibility within the New Zealand Curriculum document (Ministry of Education, 2007). The National Council of Teachers (NCTM) standards (NCTM, 2000) describe both content and process standards and explicitly include the words "reasoning" and "proof" in the Pre-K-12 mathematics standards. Here, proofs are considered to "offer powerful ways of developing and expressing insights" and "are formal ways of expressing particular kinds of reasoning and justification," through which "students should see and expect that mathematics makes sense." (p. 4). In contrast, the word "proof" does not feature at all in the New Zealand Curriculum document: the achievement objectives describe only mathematical content, not processes. *What* should be learned is clear, but *how* students should learn and be engaged in the learning is less obvious.

The New Zealand curriculum does not seem to be alone in its omission of proving, though. Hanna and Jahnke (1996) noted a general absence of proof in many countries' school curricula. This provides insight as to why proving may be missing from school classrooms. As to the impact of the omission of proof in curricula at the primary school level, there is concern about a lack of transitional experiences of

proof and proving within the school system: students experience a sudden and unfamiliar introduction to proving practices in secondary school (A. J. Stylianides, 2005). In addition, Mamona-Downs and Downs (2015) expressed concern over the struggle tertiary level students encounter, given the abrupt change in the nature of mathematics through the formalisation of proof, noting that the "standards set for argumentation … are largely foreign to the school experience" (p. 239). They, too, suggested that the transition of experiences with proving and proof should occur in a more continuous manner.

It is clear from the NCTM standards (NCTM, 2000) that proof is considered to have a place at all levels with "different expectations of sophistication" (p. 4). Even so, deciding how to translate these different levels of sophistication into the classroom was not necessarily straightforward for me. Hanna and Jahnke (1996) acknowledged how proof might feature implicitly in curricula. They noted terms like "make and test conjectures", "follow… [and] construct… arguments", and "judge the validity of arguments", were used as references to proof and proving in younger grades (p. 900). On closer inspection of the New Zealand Curriculum document, I noticed similar implicit references; for example, the document describes how students "learn to… conjecture, to justify and verify, and to seek patterns and generalisations" (Ministry of Education, 2007, p. 26). Admittedly, these processes might be viewed as the foundations for proof, and I considered that the inclusion of these "softer" terms in the curriculum, in favour of the words "proof" and "proving", might have the advantage of being more accessible, achievable, and desirable at *all* levels of the curriculum. Certainly, providing students with opportunities to *conjecture*, *justify* and *verify* and to *seek patterns and generalisations* in primary school may address the concerns of A. J. Stylianides (2005) and Mamona-Downs and Downs (2015) by better preparing students for more formal experiences of proof in the secondary and tertiary years. However, one drawback of avoiding using the words "proving" and "proof" in curricula is that the alternatives (conjecture, justify, reason, verify, generalise) might be seen as substantially different. The alternative words might reinforce the tendency to view proof as irrelevant for the primary classroom (indeed, this had been the case for me). Hence, I came full circle and concluded that including the words "proof" and "proving" in curricula is important in signalling that proof *is* relevant for even our youngest students and that primary teachers should include

this in their mathematics lessons. Nevertheless, I acknowledged that this move alone would not alleviate ambiguity about what it entailed and constituted across different education levels.

1.1.4 What Might Constitute Proof in the Primary Classroom?

Semadeni (1984) argued that "the concrete-operational child is not capable of hypothetical reasoning, of deduction expressed in words and symbols" (p. 32). I agreed that introducing primary school-aged students to *formal* proofs and demanding *algebraically* expressed reasoning would be inappropriate. From my observations, primary students tended to rely on confirming examples to accept a statement—something that is termed (I came to learn) an *empirical argument* (Chazan, 1993). An example of empirical proof would be where students become convinced that the angles in any triangle add to 180° by measuring the angles in a few different triangles or tearing their vertices off and rearranging them to establish that they make a straight (and therefore 180°) line. The problem with this form of reasoning is that one cannot be sure that the examples that have been measured do not just happen to be special in some way that satisfies the statement and that other examples may prove it to be false. However, although it is insufficient to simply use examples to prove a conjecture, a single counterexample *is* sufficient to demonstrate that it is *not true* for all cases. Since this type of proof is still proof by example, I considered that primary students might be reasonably able to use counterexamples to show that a conjecture is not true. Even here, though, I realised care needs to be taken, as one inaccurate counterexample may lead students to arrive at false conclusions. For example, if students inaccurately measure triangles, they may deduce that triangles' interior angles do not always add to 180°. Indeed, Chazan (1993) cited studies, which showed example-based reasoning is also favoured by secondary students and even tertiary students and teachers. This left me wondering: if secondary and tertiary students struggle with understanding deductive proof, might it seem unrealistic to expect primary students to move towards more generalised reasoning?

However, with regards to Semadeni's perspective that young students are incapable of, "hypothetical reasoning, of deduction expressed in words" (not symbols), A. J. Stylianides (2005) included excerpts from a classroom discussion illustrating how a

primary school-aged student provided convincing proofs for "odd + odd = even" and "even + odd = odd". Even though the student did not use algebraic symbols, her arguments were not based on specific examples (i.e., they were not empirical) and could be mapped onto a typical algebraically represented deductive proof. The arguments described by A. J. Stylianides (2005) resemble what Semadeni (1984) termed *action proofs*. Semadeni described action proofs as falling somewhere, "between an intuitive explanation and a deductive argument" and involving a physical action or the manipulation of objects so that, "children can convince themselves of the validity of a statement." (p. 32). Action proofs do not demand formal proof expressed in symbols but can help students to understand *why* a statement is true or false. These perspectives of proof enabled me to see that it might be feasible for students to reason deductively, but their reasoning might be expressed in everyday language (rather than algebraically) and with the support of manipulatives or diagrams.

Interestingly though, some students in the primary school class described by A. J. Stylianides (2005) seemed convinced of the truth in the argument presented to them but still believed that every single case would still need to be checked to make sure it was true! This finding concurs with Chazan's (1993) earlier observation that even when students are aware of deductive proofs, they do not appreciate that the proofs could be generalised to all examples. I pondered whether teachers should, therefore, not be concerned that young students might struggle to accept universal truths—rather they should anticipate these struggles and utilise them as learning opportunities. Instead of avoiding proving activity in the primary classroom for fear of these struggles and issues, I began to believe that opportunities to debate empirical, example-based reasoning versus deductive, generalised proofs might provide the foundations for mathematically worthwhile arguments. Moreover, I felt if students were exposed to, and began to consider the merits of, deductive proofs over empirical proofs at an early age, then they may find it easier at secondary and tertiary level to formalise proof and to accept the generalisability of deductive proof.

1.1.5 What is Required in Order for Proof to Be Incorporated Into the Primary Classroom Successfully?

Despite being convinced of the place of proof in the primary classroom, I realised that there would be certain requirements in order for the process to engender greater mathematical understanding. Mathematical experiences requiring proof would need to be appropriately pitched and designed. Students would need to be able to build new understandings from existing knowledge and, according to Kitcher (1984), one of the primary functions of proof would also be to generate new knowledge or to have an element of mathematical discovery. Hence, learning associated with proving tasks would need to be within students' *zone of proximal development* (ZPD) (Vygotsky, 1978), and students would need to experience a degree of uncertainty about the truth of a conjecture.

Drawing from personal experience, I also appreciated that the way we teach can influence what is learned. Hanna and Jahnke (1996) warned that, even if proof features in the curriculum, teaching proof procedurally as a set of steps to be memorised or having students simply memorise proofs would not achieve the goal of conceptual understanding. For example, students who experience an "odd + odd = even" lesson by just being given the rule to memorise, or by being *taught* an action proof procedurally by the teacher, would have impoverished experiences of proving and proof compared with those who are given opportunities to explore the conjecture and discover truths for themselves. Even at the tertiary level, Mamona-Downs and Downs (2015) claimed that there was too much emphasis on procedural work and memorising proofs rather than on reinventing proof. I considered procedural learning of proofs to be even less desirable in the primary classroom. I reflected that in order for proving to promote understanding, a pedagogical shift would be needed: from focusing on *finding* answers and procedurally *applying* rules or solution paths, to provoking students to *discover patterns* and *make conjectures* for themselves and requiring students to *justify* their *reasoning*. Correspondingly, there would be a need to consider mathematical tasks and how they might be designed to prompt and provoke such proving behaviours.

I realised that what was being signalled here was the need to redefine and renegotiate what was mathematically acceptable in the primary mathematics

classroom, in order for proof to be both relevant and achievable. This realisation aligned with recent calls from within the New Zealand educational context to transform the socio-mathematical norms within classrooms (J. Hunter & Anthony, 2008; R. Hunter & Anthony, 2011) and foster inquiry-based classrooms. Indeed A. J. Stylianides (2005) also stressed that the evidence he used to advocate for proof in the primary classroom is largely reliant on the fact that the teacher in his study had established her classroom as, "a community of mathematical discourse", where ideas were validated through mathematical argument rather than the authority of the teacher (p. 22). In efforts to be economical with time in a "crowded curriculum", I knew all too well that it was tempting and easy for a teacher to give the game away in a show and tell or to prematurely correct a student's thinking. And I knew mathematical understandings could not be negotiated, refuted, or clarified in this way. Hence, if proving were to be part of the primary classroom, there would be a requirement for teachers to both develop mathematical argumentation as a norm and to support and expect students to be capable of thinking and communicating mathematically.

Being mindful of the increasing demands that seem to be placed on teachers, I considered how feasible it would be to implement the educational goals of proof. Thus, I considered some achievable first steps that could be taken. To encourage justification and generalisation, teachers might simply ask: Is that always true? How do you know that it is true for all numbers/examples? How do you know that there is not some number/example that it will not work for? (Carpenter et al., 2003). Through pressing students to make conjectures, justify their thinking and to generalise, teachers could lay the foundations for proof so that, over time, students may begin to ask these questions independently. I recognised that, for teachers who had already incorporated mathematical inquiry and mathematical argumentation as a norm within their classroom, the foundations of proof would already be building. For teachers working towards this goal, it would be important for proof to be seen not as a separate and additional process but as a part of everyday mathematical argumentation. The requirement for proof to be embedded in mathematics is reflected in the NCTM (2000) statement, "Reasoning and proof cannot simply be taught in a simple unit on logic… Reasoning mathematically is a habit of mind…it must be developed through consistent use in many contexts" (p. 56).

1.1.6 Summary

The initial questions I had when first encountering mathematical proof led to a point where I firmly believed that *mathematical proof* was not *childproof* and should not be kept out of reach of young students—it was a mathematical process that belonged at *all* levels within the education system. In reaching this position, I had needed to: reassess what proof was; consider what the educational goals of proof were and how they might be translated into the primary classroom; and appreciate that certain pedagogical considerations would be necessary for these goals to be realised. Although using and understanding formalised deductive proof might not be a realistic goal (or even a desirable one) within the primary mathematics classroom, I knew the foundations of proof and the educational goals of mathematical proof and proving certainly were achievable. These goals included: using knowledge to discover and make sense of new mathematics; conjecturing, reasoning, and justifying; and engaging in mathematical argumentation in an effort to convince others of a mathematical assertion. I was convinced that learning mathematics in a way that promoted proving had the potential for a greater understanding of mathematics and that engaging students in learning this way, at an early age, could foster future mathematical success. Of course, arriving at this point was not the destination of my journey—it was just the beginning. I had so many more questions that needed answering. Consequently, the final part of this chapter provides an overview of where my inquiry headed next, by outlining the structure of this thesis and my research aims within each findings chapter (Chapters 4–7).

1.2 Structure of the Thesis

The thesis begins in Chapter 2 with a review of relevant literature in order to provide a clearer picture of what is understood and established about proof and proving at the primary school level, and what remains unclear. The literature review provides a background for the present research, by covering: the occurrence of proof and reasons behind its marginalisation; views about what proof is; the role of proof and what proving may involve; and what is required for proof to be successfully incorporated into the primary classroom. I provide an overview of studies that have shown students' overreliance on example-based reasoning and the difficulties that they appear to have with transitioning to what are conventionally considered to be

valid proofs. Here, I point to the absence of wide-scale research in the area of proof and proving at the primary school level and specific gaps in the research. The review concludes with a justification of why the research outlined in this thesis is needed and states general aims of the research in relation to gaps identified.

In the *Theoretical Framework and Methodology* chapter (Chapter 3), I provide a general overview of Sfard's (2008) commognitive framework, including its main tenets and how it provides an appropriate theoretical lens through which to analyse and describe phenomena in question. However, some aspects of commognitive theory, including recent theoretical developments, are reserved for later chapters where their inclusion is more relevant. In this chapter, I also discuss the appropriateness of a qualitative research design and I outline my research aims. I then describe the work that was conducted to gather and analyse appropriate and sufficient data to address my research aims. The particular research methods used in this study are reviewed, including my decision to analyse my wider set of data to address my first research aim and then to focus my observations and analysis on just one group of students to address my other two research aims. I describe the research procedure including the participants, the tasks designed, how the data were collected and how it was analysed. Finally, I address matters pertaining to the integrity of my research including ethical issues.

I adopt Sfard's (2018) view of research as "the activity of telling stories about some aspects of reality" (p. 220) and present my research findings in Chapters 4–7 as self-contained studies. Each of these chapters begins with an introduction that includes the research aim(s) addressed. Each chapter also states the specific research question(s) of the study, provides an overview of the theoretical principles that are particularly relevant to the research questions in focus, and offers a brief account of how the study was conducted before the observations and analyses are presented and discussed. Given this structure, some repetitions are unavoidable especially in the background literature, theory, and methods. Yet, this format was deemed preferable in order to provide a cohesive and complete research story for the reader in each chapter.

In Chapter 4 I look across all my data (gathered from seven groups of students) to establish categories of example-use regarding odd and even numbers and examine

the space between empirical and generic use of examples. I look for discursive markers pointing towards whether a learner is referring to generality in their reasoning regarding their use of examples.

I then focus on one student group for the remaining three chapters. In Chapter 5, I introduce a new methodological tool—*The Grove of Realizations[3] tool*—as a means of assessing the effectiveness of interpersonal communication within a group of learners and visually exposing communicational breaches, as well as tracking the course of development for different learners. I use this tool in this chapter to provide an account of the students' pre-existing discourses on odd and even numbers. The grove of the students' pre-existing discourses in Chapter 5 then forms a baseline from which to compare learning (shown by growth within the grove) that occurs within the group session in Chapters 6 and 7.

In Chapter 6, I put Cooper and Lavie's (2020) theoretical principles for designing an *exploration-enabling task* into use and describe a task I designed to manipulate learners' interpretation of a task situation; the goal was to make it possible for their initial entry into a new deductive discourse on odd and even numbers to be intrinsically motivated and outcome-oriented (*explorative*) rather than socially motivated and process-oriented (*ritualistic*). Drawing on the notion of *interdiscursivity*—coalescing discursive elements from two discourses—I provide an empirical account of how two students from the focus group engaged with the task and examine characteristics of their performances to determine whether their performances might be considered ritualistic or explorative.

The focus shifts in Chapter 7 to the third student (Danny) in the group and builds from findings established in Chapters 5 where Danny's (sequence-based) talk was shown to be distinct from the others' (structure-based) and he appeared resistant to adopting new ways of talking. Using Sfard's (2019) *dialogic engagement* as a lens for explaining the relationship between learning and communication, I discuss Danny's reason for resistance to participating in structure-based talk and both the extent to which Danny participated in this new mode of talk and the reasons he did so.

[3] Where the thesis uses a British English language default, I adopt Sfard's spelling when I use the term realization (and its variations: realize; realized; realizing; realized) for the entire thesis. The term is defined in Section 4.2.

Finally, in Chapter 8, I conclude the thesis with a summary of the main findings of the studies, together with a critique of the limitations of the studies and recommendations for further research.

CHAPTER 2: Literature Review

This chapter reviews relevant literature to provide a clearer picture of what is understood and established about proof and proving at the primary school level, and what remains unclear. The literature review provides a background for the present research by covering: the occurrence of proof and reasons behind its marginalisation; views about what proof is; and the role of proof and what proving may involve. I then provide an overview of studies which have shown students' overreliance on invalid, example-based reasoning and the difficulties that they appear to have with transitioning to what are conventionally considered to be valid proofs. Here, I point to the absence of wide-scale research in the area of proof and proving at the primary school level and specific research gaps. The review concludes with a justification of why the research outlined in this thesis is needed and I articulate the general aims of my research.

2.1 Introduction

Bruner (2009) argued that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (p. 33). In other words, there should be some continuity between what a scholar does and what a child does in school. Accordingly, since mathematical proof is considered fundamental to the work of mathematicians, it follows that proof should also be a fundamental part of mathematics education (CadwalladerOlsker, 2011; Hanna, 1995; Schoenfeld, 1994; A. J. Stylianides, 2007). It seems appropriate, then, that the PISA 2021 (OECD, 2018) framework highlights the centrality of mathematical reasoning (both deductive and inductive). Reforms in some countries' curriculum documents also now require proof and proving to be taught at all levels (see NCTM, 2000; DfE, 2013). For instance, the NCTM (2000) standards refer to proofs as offering "powerful ways of developing and expressing insights" and recommend the process "in all content areas and — with different expectations of sophistication—at all grade levels" (p. 4).

However, mathematical proof and the process of proving have been marginalised in mathematics education (Ball et al., 2002; Bieda, 2010; Hiebert et al., 2003; Sears, 2012; A. J. Stylianides, 2016; G. J. Stylianides et al., 2017), especially in the primary

classroom. Furthermore, there is a lack of consensus about the meaning of the term "proof", what it entails, and how it differs from other terms such as *argumentation* (Staples et al., 2016; Steen, 1999)—what constitutes proof and proving at the primary level is especially unclear. There is also little research that explores *how* young students can be supported to develop valid arguments (Stylianides & Stylianides, 2017). Hence, there seems to be several factors hindering the occurrence of mathematical proof in the primary school sector.

2.2 Why is proof marginalised in the Primary Classroom?

Despite increasing calls for proving activity to be embedded in school mathematics (e.g., CadwalladerOlsker, 2011; DfE, 2013; 2011; Hanna, 1995; NCTM, 2000; OECD, 2018; Schoenfeld, 1994; A. J. Stylianides, 2007), studies have found the inclusion of proof to be severely limited in secondary schools around the world (Hiebert et al., 2003; Bieda, 2010; Sears, 2012), and there is scant documented evidence of its use in primary classrooms. Explanations for why proof may be uncommon in school mathematics include: *teacher beliefs* (Bieda, 2010; Ersen, 2016; Frasier, 2010; Iscimen, 2011; Knuth, 2002; A. J. Stylianides, 2016); *teacher knowledge* (Knuth, 2002; Martin & Harel, 1989; Stylianides & Ball, 2008; Stylianides & Stylianides, 2009); *high pedagogical demands* (A. J. Stylianides, 2016); and a *lack of curriuculm and teaching materials* (A. J. Stylianides, 2016; Stylianides & Stylianides, 2009). Each of these reasons is explored in the following sections.

2.2.1 Teacher Beliefs

Studies have shown that teachers' perceptions about the subject or topic they teach (e.g., Harlen & Holroyd, 1997), teachers' expectations of their students (e.g., Boaler, 2013; Dweck, 2007; Rosenthal & Jacobson, 1968), and their beliefs about who the subject is important and useful for (Bieda, 2010; Frasier, 2010; Iscimen, 2011; Knuth, 2002; Weber, 2012), translate into teaching in the classroom. Ersen (2016) found that 18% of the 192 pre-service mathematics teachers in her study had negative perceptions of mathematical proof, describing it as "hard to understand, unnecessary" and "the nightmare of students" (p. 94). For example, one teacher said, "Mathematical proof is like a foreign language. Just as we cannot understand a language that we don't know; the moment we see a proof, we … don't understand." (Ersen, 2016, p. 91). Another even more damning statement was, "Mathematical

proof is like murder. Because it slaughters the students." (Ersen, 2016, p. 91). If proof is perceived in such a negative way by teachers, it follows teachers may be less inclined to include it in their programmes, and, even if they do include it, students' learning experiences and their perceptions of proof are likely to be impoverished.

Several studies have revealed teachers' low expectations of students' capability in using and understanding proof. Studies in the U.S. (Bieda, 2010; Frasier, 2010; Knuth, 2002) and Turkey (Iscimen, 2011) found secondary teachers consider proof as being a process that only the most able students are capable of. For example, Frasier (2010) surveyed 374 high school maths teachers and found that a significant number viewed proof as being important for upper-level students but not important for mid- or lower-level students. Half of the teachers in the study believed that one out of every four students would be incapable of understanding proof in mathematics. Weber (2012) showed even some mathematics professors held similarly low expectations of tertiary level students' capacity to understand proof; they believed proving to be suitable for only some mathematics major students. There is an absence of studies on primary teachers' beliefs about students' capacities to understand and use proof. Nevertheless, it may seem logical to infer from studies of secondary and tertiary educators that primary teachers may also view their younger students as lacking the capacity to understand and use proof and would, therefore, consider it inappropriate to include proving in primary education.

2.2.2 Teacher Knowledge

Unless teachers understand a mathematical concept or process or have had opportunities to engage in proving activity themselves as mathematics learners, they cannot be expected to be effective in teaching it. Teacher knowledge about proof may include (but is not necessarily limited to): understanding the difference between valid and invalid modes of argumentation; ascertaining whether a conjecture can be verified or refuted (and, if so, how); and understanding the power of a single counterexample to refute a false statement. However, some studies have shown secondary teachers' (pre-service and in-service) knowledge about proof to be weak (Knuth, 2002; G. J. Stylianides et al., 2007). For example, Knuth (2002) found that all 16 in-service secondary teachers in his study incorrectly considered at least one non-proof to be proof (of the eight non-proof arguments presented).

Since primary teachers are not necessarily mathematics specialists and therefore might not have undertaken proof-related mathematics, they may be even less likely than secondary teachers to understand that valid proof necessarily relies on deductive reasoning. Indeed, research has shown that pre-service primary teachers themselves tend to use invalid arguments as proof (Goulding et al., 2002; Goulding & Suggate, 2001; Martin & Harel, 1989; Morris, 2002; Stylianides & Stylianides, 2009). Stylianides and Ball (2008) suggested that teachers' weak knowledge and misconceptions about proof renders them "inadequately prepared" to "cultivate proving in their classrooms" (p. 329) and is, to some extent, to blame for the difficulties with proof that students of all levels experience.

2.2.3 High Pedagogical Demands

Since cognitively demanding instruction requires students to think critically and deeply about the mathematics involved (Smith & Stein, 1998), proving might be considered to be inherently cognitively demanding. However, when using rich, cognitively demanding tasks, teachers tend to over-direct the learning, which results in students developing a passive approach to learning (Sullivan, 2011). Hence, despite the potential for rich learning experiences and deep understanding afforded through engaging in proving activity (Hanna & Jahnke, 1996), there is a danger that it may be reduced to procedural activity. Hanna and Jahnke (1996) warned that teaching proof procedurally as a set of steps to be memorised or having students memorise proofs is unlikely to enhance students' learning and understanding. Yet, even at tertiary level, Mamona-Downs and Downs (2015) claimed that there is too much emphasis on procedural work and memorising proofs rather than exploring and developing proof.

On the other hand, if students are completely unsupported when engaging in proving tasks and teachers do nothing with student ideas, they will likely *do nothing* to advance student mathematical thinking within the class. Simply having students take turns to share their thinking in a *show and tell* is also unlikely to advance the majority of student thinking, as the implicit message will be that all strategies or forms of argumentation are equally good (Smith & Stein, 2011). To support students to engage productively in proving activity and reason mathematically, Ball (1993) showed how establishing the classroom as "a community of mathematical discourse" (p. 388) is

important. Here, ideas are validated through strategic orchestration of mathematical argumentation rather than the authority of the teacher. Indeed, according to the NCTM (2000) standards (for Grades 3–5), it is a teacher's responsibility to establish the class as a "mathematical community", which is, "continually developing, testing and applying conjectures about mathematical relationships" (p. 191). The teacher's role is to build the capacity of students to "reason… and communicate mathematically" (NCTM, 2000, p. 197).

However, orchestrating student discussions so that they successfully build on student thinking is no easy task, and even experienced teachers may find such ambitious teaching very daunting (De Holton et al., 2001; Lev-Zamir & Leiken, 2011; Sherin & van Es, 2005; Swan & Burkhardt, 2014). Smith and Stein (2011) offered a framework to support such discussions through five practices: anticipating, monitoring, selecting, sequencing and connecting student thinking. Although this framework supports teachers to construct effective discussion, each practice is still demanding for teachers. The teacher is required to: consider where the majority of student thinking is at and make in-the-moment decisions about which thinking to share, in what order, and for what purpose; facilitate students' productive engagement with tasks; maintain the tasks' high cognitive demands; and orchestrate mathematical argumentation that capitalises on the diversity of emerging student thinking. These pedagogical demands offer another reason as to why proof may be uncommon in primary classrooms.

2.2.4 Curriculum and Teaching Materials

Instructional supports, such as curriculum documents and textbooks, can help teachers to engage their students in a range of mathematical content and processes. They can support teachers to identify content to be taught, select age- or level-appropriate strategies and can offer a selection of activities for teachers to select from to help reinforce learning. However, support for teaching proof is rarely provided in curriculum resources. Hanna and Jahnke (1996) noted a general absence of proof in many countries' school curricula. Muddying the waters somewhat is the fact that terms such as "argumentation" and "prove" (with the terms "argument" and "proof" as their counterpart products) are sometimes used interchangeably and inconsistently (Staples et al., 2016). Steen (1999) pointed to an avoidance of the word

"proof" in many curriculum documents in favour of *euphemisms* like "argument", "justify", "validate" (p. 274) as indicative of uncertainty about the role of proof in school mathematics.

In the case of the New Zealand Curriculum (Ministry of Education, 2007), which only alludes to proving when describing mathematics as a subject in which students learn "to conjecture, to justify and verify, and to seek patterns and generalisations" (p. 26); the word "proof" does not feature at all. Since policy frames practice, the invisibility of proof in the New Zealand Curriculum may well reduce the occurrence of proof in New Zealand classrooms. Indeed, several researchers have suggested that an absence of proof in the curriculum may contribute to a lack of proving activity within the school system and poor transitional experiences of proof (e.g., Mamona-Downs & Downs, 2015; Mariotti, 2000; Sowder & Harel, 1998; A. J. Stylianides, 2007).

However, mathematical proof has become an increasingly more common feature in some overseas curriculum documents, including those for primary education. For example, the National Council of Teachers (NCTM) revised their (1989) standards from recommending that proving be part of the mathematical experiences only for high-school students who intended to go to college, to recommending reasoning and proof be part of instructional programmes for students "from pre-kindergarten through grade 12" (NCTM, 2000, p. 56). The NCTM (2000) standards suggest that reasoning and proof should enable "each and every student" to "recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof" (p. 56). Admittedly, simply including the word "proof" in reformed curricula in favour of *softer* words (e.g., "argumentation", "justify") does not necessarily alleviate ambiguity about what it entails. Still, it does represent a huge step forward in signalling that proof *is* relevant for even our youngest students and primary teachers should include proof in their mathematics lessons.

Nevertheless, even when policy documents signal proof as something to be learned, little (if any) guidance is offered as to *how* students should learn and engage in the process. Textbooks, too, seem to offer inadequate support for teachers. For example, Otten et al. (2014) found few supports for students to construct a proof in their

analysis of six U.S. geometry textbooks. Similarly, Thompson et al. (2012) found that most of the 20 U.S. secondary textbooks they examined had very few proof-related reasoning exercises. Secondary teachers themselves confirmed this issue in an online survey (Frasier, 2010), reporting dissatisfaction with textbooks in helping students learn about mathematical proof and identifying textbooks as a deterrent from including proof in their programmes. Thompson et al. (2012) argued that textbooks are "a particularly critical link between the intended and attained curriculum in mathematics" (p. 254). In other words, while textbooks could serve to mitigate some of the difficulties that teachers have in productively engaging students in proving activity, if textbooks do not present such opportunities, it is less likely that these activities will appear in the classroom.

2.2.5 Summary

The NCTM (2000) standards recommend the inclusion of proof at all levels of education with "different expectations of sophistication" (p. 4). However, the literature review so far shows that several barriers contribute to the marginalisation of proof, especially in primary school mathematics. To successfully incorporate proof into the primary classroom, what is meant by proof and how these different expectations of sophistication should be translated into the classroom needs to be clarified. Accordingly, the review now shifts from proof in a school context to research on levels of proof.

2.3 What is Mathematical Proof, and What Might it Look Like in the Primary School?

CadwalladerOlsker (2011) argued that proof is a "fundamental" part of mathematics; "we do not know whether a mathematical proposition is true or false until we have proved (or disproved) it" (p. 34). What constitutes proof, however, has been debated for decades and there is still no universal consensus as to what constitutes proof (Balacheff, 2002). Hanna (2002) noted the word "proof" is often synonymous with formal proof, and this is what students at secondary school typically are introduced to. Formal proofs, such as two-column methods, take the form of logical deductive steps and are often expressed algebraically to prove traditional theories (Hales, 2008). For example, the Pythagorean theorem states that in a right-angled triangle the sum of the squares of the two shorter sides is equal to the square of the

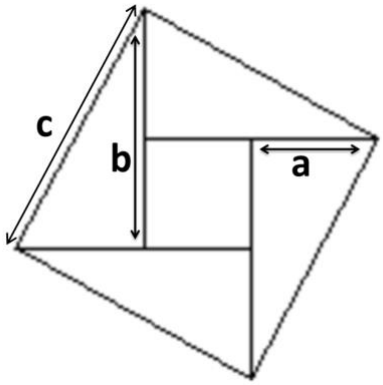
hypotenuse. The theorem can be proved by arranging four copies of a right-angle triangle with sides *a, b* and *c* arranged in a square with side *c* (Figure 1). The triangles are of similar area, while the smaller square has a side equal to *b-a* and therefore an area $(b-a)^2$. The proof uses deductive reasoning, expressed algebraically to show that the area of the large square, therefore, is

$$(b-a)^2 + 4\,ab/2$$
$$= (b-a)^2 + 2ab$$
$$= b^2 - 2ab + a^2 + 2ab$$
$$= a^2 + b^2$$

because this area occupies a square with side *c* and area c^2, $c^2=a^2+b^2$ (Bogomolny, 2017).

Figure 1

Proving Pythagorean Theorem (Bogomolny, 2017)



It is unlikely that a formal, deductive proof expressed algebraically such as this would be within typical primary school students' understanding. Indeed, referring to Piaget's (1936) theory of cognitive development, Semadeni (1984) claimed that proofs are not generally taught in primary level classes because young students lack developmental readiness. As noted in the previous section, however, there have been more recent calls recommending that mathematical proof and proving be included at *all* levels of education. Such calls suggest that engaging in proving activity and formulating a valid argument might not be completely out of the reach of young children. So, what might constitute proof in the primary classroom?

2.3.1 Empirical Arguments

While there is much debate as to what *does* constitute proof (Balacheff, 2002; CadwalladerOlsker, 2011), there seems to be a general consensus that empirical arguments *do not* (e.g., Balacheff, 1988; A. J. Stylianides, 2007; G. J. Stylianides 2008, 2009). Empirical arguments contrast with deductive proofs insofar as the latter use definitions and theorems to produce "logically rigorous deductions of conclusions from hypotheses" (NCTM 2000, p. 56) and are considered mathematically valid. Empirical arguments are example-based arguments and provide inconclusive evidence for mathematical generalisations: by verifying the truth of a universal statement on only a subset of all possible cases, they fail to eliminate the possibility of the existence of a counterexample. Therefore, mathematically speaking, these arguments are invalid in the sense that they show the truth of the statement for some but not necessarily all cases.

The selection of examples used and how they are used, however, can affect the status of an argument. Within the category of empiricism, Balacheff (1988) distinguished between *naïve empiricism*—validating an assertion on only a small number of specific cases—and *crucial experiment*—carefully selecting extreme cases to validate an assertion. For instance, in considering the conjecture *the sum of any two odd numbers is even*, a student might try a few pairs of odd numbers and would conclude this statement to be true based on their even result. This student would be described as using naïve empiricism. If, however, the student selected different kinds of pairs of odd numbers—some with small odd numbers, some with large odd numbers, some with the same odd numbers, some with different—their argument would be described as a crucial experiment. The intentional way in which a student selects empirical examples at the level of crucial experiment is considered to show a more sophisticated level of thought and skill with regard to proving than example selection and use at the level of naïve empiricism. However, both of these categories of example-based arguments provide inconclusive evidence for mathematical generalisations.

Even though empirical arguments are considered mathematically invalid, school students' (of all education levels) pervasive use of empiricism when proving has been well documented (e.g., Coe & Ruthven, 1994; Education Committee of the European

Mathematical Society, 2011; Healy & Hoyles, 2000; Knuth et al., 2002; Knuth et al., 2009; Martin & Harel, 1989; Porteous, 1990; G. J. Stylianides, 2009; Stylianides & Stylianides, 2017). Furthermore, this phenomenon has continued even after students have received explicit instruction about the nature of mathematical proof (Education Committee of the European Mathematical Society, 2011). Indeed, in a special issue of *Educational Studies in Mathematics* focusing on research in the area of proof, Stylianides and Stylianides (2017) summarised students' overreliance on empirical examples to prove the truth of a mathematical generalisation as a "key and persistent problem" (p. 124).

2.3.2 Action proofs

Back in 1984, Semadeni proposed an alternative form of proof, *action proof*, for use in the primary school setting to bridge the gap between empirical and algebraically expressed deductive reasoning. Semadeni described an action proof as falling somewhere "between an intuitive explanation and a deductive argument" and involving the performance of a physical action or manipulation of objects so that "children can convince themselves of the validity of a statement" (p. 32). A learning experience involving action proof for "odd + odd = even" might begin with the teacher using an arrangement of two rows of seven counters as concrete manipulatives. As Figure 2a shows, the teacher may then group the manipulatives to show that the two odd numbers each have one leftover when they are grouped in twos. Hence, with the two groups of seven, or two odd numbers, the teacher is able to demonstrate that the two remaining leftovers form an even number (see Figure 2b).

The learning experience would then move towards generalising this principle through using numbers that cannot be made with manipulatives, for example, 10,023+ 10,023. Semadeni (1984) noted that this action proof should not be confused with the procedure of calculating the answers for two odd numbers to prove the answers are even for a few examples and generalising it to all cases. Semadeni argued that while action proofs do not demand formal proof expressed in symbols, they can help students to understand *why* a statement is true. Nelson (1997) also concurred that pictures and diagrams (such as those in Figures 2a and 2b) could be useful tools for producing proofs with high explanatory power.

Figure 2

(a) Two Odd Sets of Counters Grouped to Show There is One Leftover in Each Set (Red) and (b) The Two Leftover Counters (Red) Form "Two", So the Two Odd Numbers Make an Even Number

Figure 2a

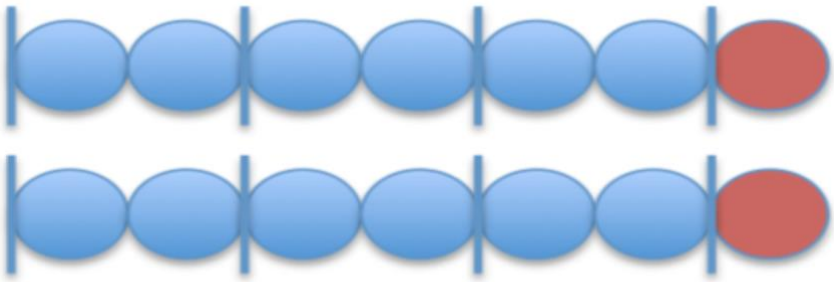
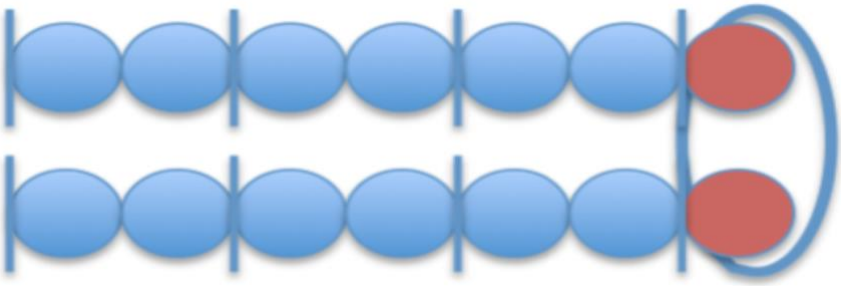


Figure 2b



The role proof plays in sense-making within Semadeni's (1984) action proof is echoed in the NCTM (2000) standards reference to proofs as "powerful ways" through which "students should see and expect that mathematics makes sense" (p. 4). However, while the explanatory power of such action proofs might be appealing, Semadeni's suggestion contradicts the NCTM (2000) recommendations that students should use forms of reasoning that do not rely on the teacher or other forms of authority figures. Rather, students should be involved in the process of proving and encouraged to create their own conjectures, author their own arguments and explore their validity. These recommendations point to a more active role in proving—*explaining* and *convincing* others about our statements or theorems—and it is this active role that may promote deep understanding (Hanna & Jahnke, 1996). Hanna and Jahnke (1996) argued that when incorporating proving in teaching and learning

experiences, students' mathematical understanding becomes enhanced through communicating their reasoning.

2.3.3. A. J. Stylianides' (2007) Definition of a Proof Deemed Appropriate for Primary School Students

A. J. Stylianides' (2007) research showed primary school students who were involved in the process of proving were able to author their own arguments. In fact, he showed that the nature of proofs derived in primary classrooms might be quite like what one might expect in more advanced educational settings. He provided excerpts of a classroom discussion illustrating how a student in Grade 3[4] demonstrated a convincing proof that "odd + odd = even" and "even + odd = odd". The student first drew two sets of seven lines and proceeded to circle them in twos and then articulated an argument, which is summarised in Table 1. Even though the student did not use algebraic symbols in her proof, the argument expressed in everyday language could easily be mapped onto a typical algebraically represented proof (as shown by the rows in Table 1). The student began with a numeric example (7 + 7), but her reasoning was not empirical—it was a generalised argument expressing deductive reasoning.

Table 1

Student Argument and Corresponding Algebraic Proof for Odd + Odd = Even, Presented by A. J. Stylianides (2007).

Student Argument	Standard Algebraic Proof
[All] odd numbers if you circle them by twos, there's one left over.	All odd numbers are numbers of the form $2n+1$, where n is a whole number.
So, if you… plus one, um, or if you plus another odd number, then the two ones left over will group together,	So, after adding two odd numbers we have: $(2k + 1) + (2m + 1)$ $= (2k + 2m) + (1 + 1)$ $= 2k + 2m + 2$ [or $2(k + m + 1)$],
and it will make an even number.	which is an even number.

[4] Grade 3 students are typically 8 and 9 years of age.

Based on these empirical observations, A. J. Stylianides (2007) aimed to achieve a definition of proof that would work at any level of the curriculum. He adopted an *intellectual-honesty principle* and a *continuum principle* to guide his conceptualisation of proof. The intellectual-honesty principle states that a proof should be honest to the discipline of mathematics while also considering what is known or accessible to learners. The continuum principle insists that there should be continuity throughout the educational levels regarding the conceptualisation of proof. With regard to the primary classroom, these principles ruled out empirical arguments as qualifying as proofs because they would violate both principles, while formal, algebraically expressed proofs would unlikely qualify as these would likely be inaccessible for most young learners.

A. J. Stylianides (2007) suggested that the proofs that young students are exposed to, or come to use, should mirror those endorsed by the mathematical community in terms of their *foundation*—they should be based on definitions or axioms—and *formulation*—they should involve a process of logical deduction. However, in terms of their *representation*, young students' proofs might be expressed in everyday language rather than algebraic symbols, and they might require the support of manipulatives or diagrams to provide a visual demonstration of a verbalised or written argument. These principles led A. J. Stylianides to suggest that a proof should entail a sequence of connected assertions that use statements, forms of reasoning and forms of expressions that are mathematically valid but are also within the conceptual reach of the classroom community. A. J. Stylianides' definition echoed the sentiments of Harel and Sowder (2007) that proof should be a powerful explanatory tool to help young students develop mathematical insights, but it should also account for the perspective of the mathematical community.

2.3.4 Generic Proofs

Whereas the use of examples in empirical arguments would not qualify as valid proofs, according to A. J. Stylianides' (2007) definition, arguments that make use of *generic* examples could. The idea of a generic example can be traced back to Mason and Pimm (1984), who defined it as "an actual example, but one expressed in a way as to bring out its intended role as the carrier of the general" (p. 287). Balacheff (1988) also distinguished generic examples from empirical examples, explaining how

the characteristic properties of a generic example could be used to describe the larger set of objects to prove general conjectures. Balacheff (1988) claimed that a generic example involved "making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of its class" (p. 219). In other words, unlike empirical example-use, a generic example removes the need to produce endless specific examples (Reid & Vallejo Vargas, 2018) by showing general (rather than particular) properties of the cases it exemplifies. Hence, the student illustration presented by A. J. Stylianides' (Table 1) might be considered a generic proof since the student made use of the *particular* example (two groups of seven) both to illustrate *general* properties of odd numbers (circling three groups of two sticks for each set to show one leftover in each) and *why* two odds make an even sum (circling the two leftover ones).

The usefulness of generic examples and their particular appropriateness for the primary school context have been widely acknowledged (e.g., Balacheff, 1988; Kempen & Biehler, 2019; Malek & Movshovitz-Hadar, 2011; Mason & Pimm, 1984; Reid & Vallejo Vargas, 2018; Rowland, 1988; A. J. Stylianides, 2007). Not only are the more accessible to young learners than formal proofs, but they also have explanatory potential. Hanna (2000) maintained that explanatory proofs (a category within which generic proofs lie) are of great importance in school mathematics to help support students "not only to see *that* it [a theorem] is true but also *why* it is true" (p. 8, emphasis added). Hanna's position is in keeping with Rowland's (1988) earlier claim that generic examples can provide mathematics learners of all ages with convincing insights as to why a proposition is true. Similarly, Russell et al. (2011) referred to a version of generic proof, which they called *representation-based proof,* as being instrumental for young learners to demonstrate and prove generality. Hence, at school levels where formal proofs (e.g., deductive proofs expressed algebraically; two-column proofs) are not appropriate, generic arguments have the capacity to bridge the gap from empirical toward more formal proofs (Karunakaran et al., 2014) and provide insights as to *why* a mathematical generalisation is true (Leron & Zaslavsky, 2013).

However, there are some inconsistencies regarding the terms "generic *example*" and "generic *proof*" (e.g., Balacheff, 1988; G. J. Stylianides, 2008, 2009; Zaslavsky, 2018) in terms of their status (e.g., Balacheff, 1988; Leron & Zaslavsky, 2013; Movshovitz-Hadar, 1988; Reid & Vallejo Vargas, 2018; G. J. Stylianides, 2008, 2009), and criteria for what counts as a generic proof are somewhat lacking (Reid & Vallejo Vargas, 2018; Yopp & Ely, 2015). The term *generic example* generally refers to the particular object or representation that is used in reasoning. Some have suggested that the example selected should be non-trivial and large enough to be viewed as general, but also small enough and not too complex to perform easily with (e.g., Leron & Zaslavsky, 2013; Rowland, 2002; Semadeni, 1984; G. J. Stylianides, 2008, 2009). Conversely, the term *generic proof* tends to be associated with the accompanying reasoning that makes use of a generic example. Zaslavsky (2018) distinguished between these two terms by pointing out that a generic proof always associates a generic example with a mathematical claim or conjecture to show that the claim is true for all cases (p. 290).

These terms, though, are not used consistently in the literature. In fact, Balacheff (1988) used the term *generic example* to describe what is generally considered a generic *argument* or generic *proof*; that is, he used it to describe the *reasoning* associated with generic example-use. However, Balacheff maintained that it is the "actions performed" (pp. 285) on the example that transform it from being empirical to a generic. Like Balacheff (1988), G. J. Stylianides (2008, 2009) used the term *generic example* to refer to reasoning involving a generic example rather than the example itself. Although Balacheff (1988), G. J. Stylianides (2008, 2009), and Zaslavsky (2018) used the terms inconsistently, they all make a case that using a generic example (i.e., the object) alone does not constitute its generality nor qualify as a proof. It is the reasoning accompanying the example-use that is important when classifying it as a generic or not. Yopp and Ely (2015) concurred that it is the reasoning performed on an example that changes it from a particular example to a generic example.

In terms of the reasoning accompanying the use of a generic example (i.e., object), G. J. Stylainides (2008) offered some criteria for generic example proofs and distinguished these from what he termed *rationales*. According to G. J. Stylianides

(2008, 2009), rationales include an argument that may make use of a generic example but fail to explicitly refer to some key truths (e.g., definitions) and, as such, represent *non-proofs*. For instance, the student argument in Table 1 might be considered a rationale rather than a generic argument if the student had omitted the first statement, "[All] odd numbers, if you circle them by twos, there's one left over". G. J. Stylianides (2008, 2009) considered both generic example proofs and rationales to be important ways for students to make sense of mathematical claims without needing to use formal methods or algebraic notation. However, in terms of a hierarchy of arguments, G. J. Stylianides (2008, 2009) placed empirical arguments (being invalid arguments) at the bottom, with rationales providing a transitionary step between these and generic proofs.

Not everyone is convinced, though, that a generic proof actually qualifies as a valid proof. For instance, whereas G. J. Stylianides (2008) classified a generic example as being valid proof, Balacheff (1988) classified the generic example in a similar way to empirical reasoning. Balacheff (1998) considered the first three levels in his hierarchy of proofs (naïve empiricism, crucial experiment and generic example) to all be pragmatic. Only at the level of a *thought experiment* (the highest level in his hierarchy) did he claim that learners shift from practical to conceptual understanding, where they show an awareness of characteristic properties and relationships within the given situation using formalised symbolic expressions. Similarly, for Movshovitz-Hadar (1988) there was "no replacement for the formal proof" (p. 18), and Leron and Zaslavsky (2013) contended that a generic proof lacks the rigour or "fussiness" of "full, formal, deductive proofs" (p. 27). These perspectives, which focus on rigour as a criterion of proofs, are referred to as "absolutist" (G. J. Stylianides et al., 2016, p. 20).

Pushing back on absolutist perspectives on proof, Rowland (1998) was critical of what he considered to be "pejorative" (p. 72) treatment of the status of generic proofs and he questioned Balacheff's (1998) classification of a generic example within the inferior pragmatic category. Rowland argued that generic proofs are of "a very different order of generality" (p. 70) than naïve empiricism and crucial experiments, and that the barrier between generic proofs and formal proofs (or Balacheff's thought experiment) is more about notation than the level of sophistication in thinking.

Similarly, G. J. Stylianides (2009) acknowledged that the difference between a student producing an argument that met the level of proof rather than rationale according to his criteria did not necessarily determine a higher level of understanding. That is to say, that requiring students to explicitly refer to "such intuitively obvious properties" would simply represent a change in emphasis when proving, "from sense-making to carrying out ritual procedures" (G. J. Stylianides, 2009, p. 267).

Reid and Vallejo Vargas (2018) also countered absolutist perspectives, but for a different reason to Rowland (1998); they argued that such a standpoint fails to consider the intentions of the author of the proof and different people's interpretations of it. Hence, although formal proofs are mathematically rigorous, the reasoning may not necessarily be convincing nor provide students with insights as to why a statement is true or false. Semadeni (1984), too, indicated that a student's production of a formal proof might simply be a result of replicating procedures prompted by the authority of the teacher rather than a reflection of genuine deductive understanding. For these reasons, Reid and Vallejo Vargas (2018) claimed that deciding whether an argument is proof or not, "cannot be independent of a mind" (the psychological processes occurring in the mind of the author or the reader) (p. 5). In other words, in addition to evidence of mathematical reasoning, Reid and Vallejo Vargas (2018) suggested that for an example-based argument to qualify as proof, there was a need for evidence that *the author* of the argument understands why it will hold for all cases.

The requirements of Reid and Vallejo Vargas (2018) to account for the author's intention and reader's interpretations raises the issue of how it can be possible to determine whether an example is being used (by the author of an argument) or viewed (by the recipient) generically (Yopp & Ely 2015; Reid & Vallejo Vargas, 2018). Indeed, it is quite possible that some may see the same two sets of seven sticks used as a generic example by the student in A. J. Stylianides' (2007) illustration (described earlier) as a specific example rather than a general one. Mason and Pimm (1984) themselves raised the question of how the genericity of an example could be exposed to someone who only sees its specificity, and they claimed that it is "impossible to tell" whether someone considers an example in the way it is intended (p. 287). In reply, Yopp and Ely (2015) offered criteria for judging the genericity of example-

based arguments, but only focused on the quality of the finished argument itself, rather than the perception of the arguer's intent. Reid and Vallejo Vargas (2018) also acknowledged that obtaining evidence that *the author* of the argument understands why it will hold for all cases was not a straightforward task. Nevertheless, Reid and Vallejo Vargas (2018) provided questions that teachers can ask students directly or ask them to consider in their written work to evidence if general deductive reasoning is occurring in the minds of their students. Yet even here, the exemplars provided by Reid and Vallejo Vargas (2018) are somewhat polished and comprehensively written arguments. In the absence of a fully formed and finished proof, and in situations where students are unaware of what constitutes a proof or the production of a written proof is inappropriate (e.g., with primary school students), it is still unclear how can we tell if students are seeing the particular or the general in the examples.

What is being indicated from all accounts is a dichotomy: example-based proofs are either generic (according to varying and inconsistent criteria) and are, therefore, considered to be valid; or they are non-proofs. However, could the space between an empirical argument and a generic proof be visualised as more of a continuum? Certainly, within the category of empiricism itself, Balacheff (1988) discriminated between a crucial experiment and naïve empiricism in terms of the former dealing more explicitly with the question of generality via the careful selection of non-particular cases. Furthermore, G. J. Stylianides (2008, 2009) considered a rationale to be a halfway step between an empirical argument and generic proof. Therefore, beyond empiricism, there might be value in considering the *extent* to which students' use of examples could be considered generic. Moreover, in the absence of an explicit formulation of a generic argument (i.e., one that would meet the requirements of G. J. Stylianides, 2008, 2009), might there be subtle signs that point to students recognising the genericity of an example, indicating that their particular example is intended to account for all cases, and, therefore, they are reasoning deductively?

Furthermore, there has been minimal research to provide educators with ways forward in terms of supporting students to transition from empirical reasoning to valid reasoning (Stylianides & Stylianides, 2017) (or indeed from a rationale to a generic proof). Offering some light in this area, Stylianides and Stylianides (2009) developed an instructional sequence of tasks to aid prospective elementary teachers

during an undergraduate mathematics course to realise both the limitations of empirical arguments and the "intellectual need" (Harel, 1998, p. 501) for valid proofs. Part of the success of the study was contingent on (i) the learners experiencing *cognitive conflict*—the existence of counterexamples challenged their own conceptions about proof—and (ii) the instructor explicitly explaining the conventional notion of proof (what does and does not count at proof). Stylianides and Stylianides (2009) described their successful intervention as "an optimistic message" that instruction can support learners to move from the "stubborn problem" of empiricism (p. 348). Nevertheless, once again, the implication here is that successful interventions mean transitioning from non-proof to conventional proof. With primary school students, a positive transition might simply be to recognise and explain *why* a pattern exists and how they know it will continue indefinitely, without necessarily and explicitly learning what constitutes a proof, and without the need to adhere to conventions of proof.

2.4 Conclusion and Significance of the Proposed Research

In terms of the present research, proving is considered to be a relevant process for all students, including our youngest school learners (Ball & Bass, 2000, 2003; CadwalladerOlsker, 2011; DfE, 2013; OECD, 2018; Hanna, 1995; NCTM, 2000; Schoenfeld, 1994; A. J. Stylianides, 2007), and engaging in proving activity is considered to enhance mathematical understanding (Hanna & Jahnke, 1996; Harel & Sowder, 2007; NCTM, 2000). To date, research has shown primary school students are capable of engaging in proving activity and providing valid arguments (A. J. Stylianides, 2007) and has offered definitions (and accompanying examples) of proof deemed appropriate for young learners (A. J. Stylianides, 2007). Yet, a larger body of research has pointed to the tendency of students to rely on a few confirming cases as proof of a mathematical generalisation (Education Committee of the European Mathematical Society, 2011), and there is little research providing insights into the processes of learning in this area nor instructional solutions to remedy the situation.

Indeed, in a summary of research-based interventions in the area of proof, Stylianides and Stylianides (2017) highlighted a particular need for further research into proof at the elementary school level. They called for such research to have a "narrow and well-defined scope" and an explanatory theoretical framework that explains how

interventions can work to improve student learning (Stylianides & Stylianides, 2017, p. 125). Stylianides and Stylianides (2017) argued that research of this kind is necessary in order to practically realise calls for an increased focus on proof in school education and to remedy the difficulties that post-primary students had with learning proof. If teachers were to perceive progress in terms of smaller, more implicit steps (rather than a one-step transition from non-proof to proof) and had access to more practical ways to support their students' learning, they may be inclined, to perceive the topic as something that is accessible and relevant for their classrooms and all their students. And, where students have previously experienced an abrupt change in the nature of proving experiences in the transition to secondary school or university, acknowledging smaller steps of progress might also mean that transitional learning experiences in proof occur in more continuous ways for students.

By shedding light on what smaller steps might be involved in learning within the area of proof and exemplifying mechanisms that might facilitate such learning, the present research has the potential to address some of the gaps in the research. Correspondingly, the present research aims to account for:

(i) How primary school students arrive at a particular proof.

(ii) How students' reasoning develops (within the space between an empirical argument and generic proof) and. therefore, what might count as learning.

(iii) How primary school students can be supported to develop their arguments.

(These research aims are further refined in Chapter 3).

CHAPTER 3: Theoretical Framework and Methodology

Chapters 4–7 are a series of research stories derived from four differently focused studies. This chapter provides an overview of the framing and the decision-making that guided all of the studies. It explains why I made the theoretical and practical choices that underpinned the studies and describes how I conducted them. In this chapter, I also outline how I designed a safe and appropriate research project.

First, in Section 3.1, I outline the main tenets of Sfard's commognitive framework and justify the appropriateness of it as both a theoretical approach for achieving my research aims and a tool for analysis. Note that appropriate parts of the commognitive framework are repeated and built upon in more detail in Chapters 4–7. In Section 3.2, I describe my methodology including my ontological and epistemological stance, and I discuss my decision to conduct qualitative research and use a case study method. In this section, I also state my research aims. In Section 3.3, I describe the research methods used in this research including the participants chosen, the setting, the design of the tasks used (including justifying the decision to use concept cartoons and a puppet) and how they were implemented, and the data collection process. In Section 3.4 I, briefly explain the process of analysing the data. More specific information regarding data analysis is given for each study in Chapters 4, 5, 6 and 7. Finally, in Section 3.5, I summarise matters pertaining to the integrity of the research and ethical issues.

3.1 A Commognitive Approach to the Problems Outlined in Chapter 2

The literature reviewed in Chapter 2 showed that transitioning from empirical to deductive reasoning is a persistent problem for students. Here, it was suggested that more research was needed (particularly in the primary school context) to understand *how* students arrive at a particular proof, how their reasoning develops, and how students can be supported to develop their arguments as they make the transition to deductive reasoning. The studies and accounts reviewed have tended to view proofs as being hierarchical (e.g., Balacheff, 1987; G. J. Stylianides, 2008; Stylianides & Stylianides, 2009), and the difficulties associated with proving have been characterised as students' inability to realise and internalise canonical ways of proving. It became apparent that there was a need to account for the reasons why

students consider the types of reasoning, which are deemed unacceptable by the mathematical community, as appropriate. For the present research, therefore, there was a need for a coherent theoretical lens through which to describe and explain how students arrive at a particular proof, if at all, and how and when young students' reasoning develops from empirical to deductive. Accordingly, I now summarise Sfard's (2008) commognitive framework and justify its appropriateness for my research.

3.1.1 Overview of the Commognitive Framework

Sfard's (2008) commognitive theory is located within a participationist research paradigm and has roots in Vygotskyan socio-cultural ideas which hold that "patterned, collective forms of distinctly human forms are developmentally prior to the activities of the individual" (Sfard, 2006, p. 157). It sits in contrast to traditional cognitivist, psychological or acquisitionist views of learning, where development is viewed as a series of inner changes in the learner. From the commognitive perspective, mathematics is a special type of discourse; learning mathematics involves participating in a historically established discourse of mathematics, and learning is a lasting change in one's discourse. The course of development depends on the learner's participation in discourse and the "exposure to the discourse he or she is supposed to individualize and the support he or she gets from other participants" (Lavie & Sfard, 2019, p. 458).

In addition to the socio-cultural roots of commognition, Wittgenstein's (1953; 1958) rejection of a thinking-speaking dichotomy towards communication is acknowledged in Sfard's unity of intrapersonal communication with oneself and interpersonal communication with others. Indeed, central to the word itself—*commognition*—is the non-dualist stance that *comm*unication and c*ognition* are "two processes that are different manifestations of the *same* phenomenon" (Sfard, 2008, p. 296). In other words, thinking is regarded as communicating with oneself in a way that is not very different from how one would talk with others. Thus, where the traditional cognitivist perspective tends to view communication and behaviour as a window to "mental schemes, conceptions, inner representations" (Sfard, 2016, p. 41), for the commognitivist, there is no internal mental scheme to look for: *discourse* is the object of investigation in its own right.

3.1.2 Mathematics as a Discourse

According to the commognitive framework, discourses are "different types of communication, set apart by their objects, the kinds of mediators used, and the rules followed by participants and thus defining different communities of communicating actors" (Sfard, 2008, p. 93). As such, mathematics is a special kind of discourse characterised by four interrelated features: *keywords (and their use)*; *visual mediators (and their use); narratives;* and *routines* (Sfard, 2008; 2012). Keywords are words such as "five", "square", and "sum", which may have everyday meanings but are used in uniquely mathematical ways in mathematical discourse. Visual mediators are "visible objects that are operated upon as a part of the process of communication" (Sfard, 2008, p. 133), such as numerals, symbols, diagrams, and graphs. Narratives are utterances that describe mathematical objects and their relationships, and processes such as theorems, definitions, proofs, and algorithms, which Sfard (2012) described as being "generally endorsed" by the mathematical community, or "competent participants of mathematical discourses" (p. 2). Routines refer to the patterned ways in which mathematical tasks are performed. For the mathematical community, routines include conjecturing, generalising, convincing and proving, which, in turn, can lead to the generation of an endorsed narrative or theory. Learners' personal discourses—the keywords and visual mediators (and their uses), routines and narratives—are likely to differ from the canonical mathematical discourse as they learn to participate in mathematics. The commognitive framework thus presents a means to discern learners' ways of proving (empirically) from proofs endorsed by the mathematical community (e.g., deductive) at a discursive level.

3.1.2.1 Routines. While discourse is the object of concern in commognitive research, the routine is considered to be the unit of analysis (Lavie et al., 2019). Lavie et al. (2019) considered *when* and *how* a routine is performed in their definition of a routine, stating a routine is a conceptual pair consisting of the performer's interpretation of what is called a *task situation* together with the procedure executed. A task situation is deemed to be "any setting in which a person considers herself bound to act—to do something" (Lavie et al., 2019, p. 159), whereas the term *task* denotes a person's interpretation of a task situation. The authors claim that when people encounter a task situation, they interpret it by drawing from their *precedent-search-space*—familiar actions that the learner considers relevant for the present

situation—to decide how to act. In order to decide which situations are relevant, the learner uses *precedent identifiers*—features from the task situation at hand that prompt the recollection of a past situation. The recollection of a sufficiently similar situation helps them to decide how to act. Accordingly, when presented with the same proving task situation, primary school students will undoubtedly be drawing from a different (less established) precedent-search-space than mathematicians. It is unsurprising, therefore, that the routines they perform are not aligned with those endorsed by the mathematical community. Hence, altering the precedent-search-space that students draw from might be seen as being instrumental in bringing about a change in students' proving routines.

3.1.3 Learning: How Discourses Develop

According to the commognitive framework, learning mathematics is "tantamount to becoming a member of a mathematical community" insofar as the learner's discourse becomes "closer to a historically established form of the discourse" (Sfard, 2012, p. 2). This process is observable via a lasting transformation in a learner's discourse; that is, it is identifiable by changes in one or more of the four characteristics of discourse (words, visual mediators, narratives, routines).

The commognitive framework posits that learning can occur either at the *object-level*, which results in a learner's "growth in the number and complexity of endorsed narratives" (Sfard, 2008, p. 300), or *meta-level*, when there is a change in the metarules of the discourse. Object-level learning is mainly accumulative and requires reflection on the objects within the discourse. For instance, a learner who endorses "one half" as $^1/_2$ and $^2/_4$, may continue to expand this discourse on halves as $^3/_6$, $^7/_{14}$…. However, meta-level learning requires "reflection about the discourse in its entirety" (Sfard, 2012, p. 3). For instance, when transitioning from integers to fractions or from unsigned to signed numbers, the discursive development entails a change in rules, not just an expansion of the discourse. The new discourse is incommensurable with the learner's existing one: its words, visual mediators or routines are used differently. For instance, when transitioning to a discourse that entails fractions, the word "number" no longer denotes whole amounts. Incommensurable discourses can also "allow for the endorsement of seemingly contradictory discourses" (Sfard, 2008, p. 156). For instance, changing from a

discourse that uses only unsigned to one that uses signed numbers will mean that a narrative such as "when multiplying a number, the product will be larger than the original number" no longer holds true.

3.1.3.1 Deductive Reasoning as a Meta-Discourse of an Empirical Discourse. The commognitive framework has previously shed light on the development of thinking and the process at other critical transitions in mathematical learning (Ben-Zvi & Sfard, 2007; Caspi & Sfard, 2011; Lavie & Sfard, 2009; Sfard 2007). Caspi and Sfard (2011) described algebra as a meta-discourse of arithmetic—a meta-arithmetic. This meta-arithmetic has two forms—informal and formal—the latter being more explicit and rigorous. Regardless of its form, though, Caspi and Sfard argued that algebraic thinking occurs whenever one reflects on "numerical relation and processes in search for generalization" (p. 471). The authors went on to explain that generalising requires a linguistic change—which they term *saming*—where specific mathematical objects, such as numbers (e.g., 3, 5, 7,…), are replaced with a single signifier ("odd number") so as "to turn infinitely many similarly structured arithmetic expressions (e.g., *the square of 3, the square of 5,* etc.) into a single meta-arithmetic expression (*the square of an odd number*)" (p. 476).

Similarly, deductive discourses on mathematical objects may be described as meta-discourses of empirical discourses. Commognitively speaking, the general aim of mathematics is "to produce narratives [about mathematical objects] that can be endorsed, labelled as true, and become known as "mathematical facts" (Sfard, 2008, p. 223). And a mathematist's *substantiation routine* is defined as the action that is taken in deciding whether or not to endorse a narrative.

In empirical discourses, learners converse about specific mathematical objects and use the words, "odd" and "even", as adjectives describing features of the objects (e.g., numbers or the parity of arrangements of dots). Moreover, to substantiate the endorsement of a universal narrative (e.g., "the square of an odd number is odd"), learners participating in an empirical discourse will use a direct identification routine that relies on perceptually accessible evidence: they will use numeric keywords (specific numbers) or visual mediators signifying specific numbers (in numeric form, e.g., 9, or other visual formats, e.g., three rows of three dots) to model numeric examples (such as for "the square of 3").

In contrast, a deductive proof is "a sequence of endorsed narratives each of which is deductively inferred from previous ones and the last of which is the narrative that is being endorsed (Sfard, 2008, p. 232). Accordingly, deductive discourses require generalisations to be made about infinitely many (similar) mathematical objects in order to endorse or reject universal narratives. The infinitely many odd and even objects are condensed into a single, abstract, generalised object, and the words "odd" and "even" operate as nouns that signify abstract objects in their own right (e.g., odd is "even + 1" or "2n + 1"). Moreover, where participants in empirical discourses use concrete representations of the focal signifiers (they seek evidence beyond the discourse), in deductive discourses evidence arises from the discourse itself. Here "substantiation is a purely discursive activity" (Sfard, 2008, p. 233); properties and definitions of mathematical objects are used to logically deduce propositions and reach a conclusion.

As such, deductive discourses on mathematical objects (e.g., "the square of an odd number") are incommensurable with empirical discourses: in deciding whether a universal statement (e.g., "the square of an odd number is odd") should be endorsed, the deductive substantiation routine uses different criteria to an empirical substantiation routine. The two discourses employ different meta-rules.

The process of learning mathematics is seen commognitively as *routinization*—individualising historically developed routines (Lavie et al., 2019). Hence, where previously the transition from empiricism to deduction has been described as a change in levels of proof (or schema) (e.g., Balacheff, 1987; G. J. Stylianides, 2008; Stylianides & Stylianides, 2009), the commognitive framework characterises the levels as different discourses—an empirical discourse and a deductive discourse, the latter using different meta-rules. The transition from one level of proof to the other is thus characterised as a change in discourse. Here, narratives determined by the world itself become superseded by a discourse about discourse and substantiation routines entailing direct identification come to be replaced with discursively mediated ones. What counts as convincing depends on the discourse in which one is participating. Hence, when viewed commognitively like this, students' over-reliance on empirical evidence as proof of universal statements and the difficulties they have with transitioning to deductive arguments are unsurprising.

3.1.3.2 Commognitive Conflicts as Mechanisms for Learning. Sfard (2008) posited that since object-level learning requires an expansion of the existing discourse and not a change in discourse, this simpler kind of learning is possible via self-talk (thinking). Meta-level learning, on the other hand, involves a change in both what is understood and the means by which that knowledge is established and held. In other words, a learner adopting a new incommensurable discourse needs to revise their previously endorsed narratives, and the *way* in which they would come to endorse narratives would also need to change. This level of learning means that a familiar task will be done in a new unfamiliar way, and Sfard (2008) maintained that it is "implausible" that learners would initiate this kind of change themselves (p. 256). Hence, while interpersonal talk (discussion with others) is considered optional for object-level learning, it is considered to be indispensable for meta-level learning.

Commognitive conflict is considered to be a key mechanism for triggering meta-level learning in a commognitive framework. Commognitive conflict is communicative interaction whereby participants use the same signifiers (words or symbols) in different ways or perform the same mathematical tasks according to different rules (Sfard, 2008, p. 162). Usually, commognitive conflicts are revealed via different participants endorsing contradictory narratives. When this happens, participants need to negotiate meaning through discourse. Commognitive conflicts thus open a space for participants to consider new ways of talking. Resolving a commognitive conflict involves one of the interlocutors making sense of the other person's thinking and gradually accepting and adopting the new, initially incommensurable, discourse and abandoning his or her own.

In terms of the present research, students participating in an empirical discourse may have no incentive to change their discursive ways in the absence of a commognitive conflict. From the students' perspective, their discourse is adequate and their direct identification substantiations are convincing enough. Facilitating commognitive conflict here would entail creating a situation whereby conflicting narratives are endorsed (e.g., *the square of an odd number is odd,* and *the square of an odd number is even*) and learners encounter participants performing the same task according to different rules (i.e.., participants who use discursively mediated substantiations to endorse narratives). The desirable outcome would entail the learner taking initial

steps into the new, unfamiliar deductive discourse and modifying their existing routine.

3.1.4 Rituals and Explorations

In situations like the one described above, the commognitive standpoint maintains that a learner would not be expected to appreciate the usefulness of the new meta-rules when encountering an unfamiliar discourse (e.g., Ben-Zvi & Sfard, 2007; Lavie, et al., 2019; Nachlieli & Tabach, 2019). Hence, a learner would only perform a ritual because they desired to both be like others and communicate with others, especially people of authority (e.g., the teacher). The nature of the performance of the routine here is described as *ritualistic*. Sfard (2017) defined ritualised participation in mathematical discourse as "rote implementation of memorized routines" (p. 44) whereby the participant is *socially motivated* to adhere to goals set by others (e.g., the teacher). The performance of a ritual is *process-oriented*: mathematical competence means remembering and rigidly applying a relevant procedure, and the learner will be focused on how to proceed rather than the end-result. However, the goal of teaching is for learners to be explorative. When performing a routine exploratively, participants are *intrinsically motivated* to participate and make sense about the mathematics involved. Rather than being concerned with how to proceed, performers of explorations seek to construct new narratives about mathematical objects, which makes their performance *outcome-oriented*.

Ritual performances of routines are expected to precede explorations (e.g., Ben-Zvi & Sfard, 2007; Lavie, et al., 2019; Nachlieli & Tabach, 2019). Rituals are considered to be unavoidable and a "necessary occurrence" in the process of individualising a routine (or learning) (Lavie et al., 2019, p. 167): they assist learners in taking their first steps when entering a new discourse. As attested by previous research, the process of de-ritualisation is often gradual and slow (Lavie & Sfard, 2016; Sfard & Lavie, 2005). Initially, the learner is expected to experience the new discourse as a "discourse for others": the learner will first observe, and then act as a "peripheral participant", with the amount of scaffolding required decreasing (Ben-Zvi & Sfard, 2007, p. 121). Thus, the role of the teacher is to establish new routines and model participation in the new discourse and gradually reduce the amount of scaffolding provided to support learners in transforming rituals into explorations. As learning

proceeds, learners are expected to gradually de-ritualise these routines until, realising the usefulness of the discourse, they become intrinsically motivated to perform the routines exploratively: the discourse becomes a "discourse-for oneself" (Ben-Zvi & Sfard, 2007, p. 121) which the learner can independently utilise.

However, learners' participation in a particular learning situation is unlikely to be purely ritual or purely exploratory. Rather, performances can be perceived as fitting somewhere along a ritual-to-exploration continuum. Observing that a learner's focus has shifted from the performance to its outcome denotes a step in de-ritualising the routine (Lavie et al., 2019, p. 166). Although Lavie et al. acknowledged the difficulties in delineating explorative performance from ritual performance in terms of observable features, they provided a number of characteristics of a routine, which, if identified in a participant's performance, would suggest that the routine was more explorative than ritual: *flexibility; bondedness; applicability; performer's agentivity; objectification of the discourse*; and, *substantiability* (these terms are used in Chapter 6, Section 6.2, where a more detailed explanation of each term is provided).

3.1.5 Implications of Using the Commognitve Framework in the Present Research

The commognitive framework requires a fundamental change in researchers' thinking about why, and on what grounds, participants act as they do; it offers a participatory and discursive lens to examine phenomena that have puzzled researchers regarding students' struggles in transitioning to deductive reasoning. Commognitively-speaking, learning is synonymous with discursive development, which implies that the focus should be shifted to *when* and the *how* learners' discourses develop. Thus, commognitive research specifically focuses on interpersonal communication to account for what shapes, stimulates, or impedes the development of participants' discourse (and thus learning).

Correspondingly, for the present research, the commognitive framework presents a means to discern learners' ways of proving (empirically) from proofs endorsed by the mathematical community (e.g., deductive) at a discursive level. The commognitive framework characterises different levels of proof as different discourses—an empirical discourse and a deductive discourse—and highlights a change in meta-rules between the two. The transition from empirical reasoning to deductive reasoning thereby entails a change in discourse. In particular, narratives determined

by the world itself are replaced with a discourse about discourse and direct identification substantiation routines are replaced with discursively mediated ones.

In terms of the present research, the framework enables examination of the empirical-deductive space and transitions made in this space. Such transitions are unlikely to be quick nor are they expected to occur in one leap. Indeed, when learning is perceived as individualising collective and historically established ways of talking and doing, during which learners negotiate new and complex rules of the game, it is unsurprising that transitioning from one discourse to another (e.g., from empirical discourses on mathematical objects to deductive discourses) is neither an easy nor quick matter. Within the commognitive framework, such struggles are "not only understandable, but are to be expected" (Felton & Nathan, 2009, p. 572). Accordingly, the commognitive framework provides tools to examine smaller steps in learning (keywords, visual mediators, narratives, and routines) and the nature of the learning (ritual-to-explorative), steps that might otherwise be too subtle to detect.

Adopting the commognitive framework has implications for research design. Looking at phenomena at a discursive level enables the researcher to zoom in on finer details regarding the origins of specific students' reasoning, and when and how their arguments unfold in the way they do. As such, commognitive research has the potential to yield a better understanding of what barriers may hinder students' development from an empirical discourse to a deductive one and what may help. To quote Sfard and Keiren (2001),

> "a close and detailed picture of one little sample may sometimes be more revealing than a lengthy study with hundreds of participants. When using a microscope, one may discover a whole new world of complex relationships and rich phenomena." (pp. 69–70).

The framework also has implications for designing tasks and environments for supporting students' discursive transitions. When armed with a commognitive notion of a routine, it seems obvious that different participants may use different precedent identifiers when faced with the same task situation. Here, the framework implies that altering the precedent-search-space that students draw from might be instrumental in bringing about a change in students' proving routines. Hence, it signals that

exploring how a change in students' precedent-search-space might be facilitated as a consideration for the present research.

Furthermore, in terms of the present research, the commognitive framework suggests students participating in an empirical discourse may have no intrinsic incentive to change their discourse; they are likely to view their discourse as being adequate and consider their direct identification substantiations as convincing enough. This indicates, then, that having students encounter a commognitive conflict might be instrumental as a design mechanism for the present research. Facilitating commognitive conflict here might entail creating a situation whereby conflicting narratives are endorsed or one where learners encounter different rules for performing the same task.

Adopting the framework has the potential to offer compelling insights into longstanding quandaries regarding learners' overreliance on empirical reasoning and their assumed "inability" to adopt and replicate deductive reasoning when faced with different tasks and in different contexts. It provides a theoretical basis for the present research to engage in discourse analysis of *how* individuals mathematise. As a theoretical approach, the commognitive has the potential to add clarity to the understanding of young learners' discourses as they engage in proving, which is the focal point of the four studies in this research. It could provide, "a connecting thread that ties together all of the parts of a research report into a coherent whole" (Cai et al., 2019, p. 218). For the present research, the commognitive framework provides a coherent theoretical lens through which descriptions and explanations about how learners arrive at a particular proof and how and when they take their first steps in a deductive discourse, are possible.

3.2 Methodology

3.2.1 Constructivist/Interpretivist Paradigm

A researcher's ontology and epistemology inform his or her methodology. A researcher's ontology, or view of the world, pertains to the assumptions they make about what exists and the nature of reality (Al-Saadi, 2014; Richards, 2003; Snape & Spencer, 2003). A researcher's epistemology, or view on knowledge, concerns how the researcher knows what they know or how it is possible to find out about the

world (Al-Saadi, 2014; Crotty, 1998; Snape & Spencer, 2003). Hence, it is important for me to articulate my ontological and epistemological position to situate my work and frame my subsequent methodological choices. My ontological position is humanist and I take a participationist epistemology perspective. These are explained in greater detail below.

In situating my research and myself ontologically, I align with a fallibilist epistemology, and I take a humanist stance (Hersh, 1997) whereby mathematical objects are considered to be human constructs and mathematics is considered to be value- and culture-laden. From this perspective, mathematical knowledge is seen as "subject to revolutionary change as much as other forms of knowledge" (Lerman, 1990, p. 55). I describe my research as being located in a participatory and socio-cultural paradigm (Vygotsky, 1978), whereby our thoughts are shaped by participating in the community we find ourselves in and the community's ways of seeing or doing. Rather than the "truth" being out there in the world waiting to be discovered, it is socially constructed (Vygotsky, 1978). This can be described as a participationist stance.

I view mathematics as a form of human activity; and learning mathematics means learning to participate in this activity. Teachers are therefore responsible for establishing the class as a community of learners, and their role is to facilitate changes in learners' participation. In short, learning is perceived as evolving participation in activities rather than as the acquisition of concepts. Research studies that adopt this worldview are characterised by investigations conducted in the classroom, where changes in learning-teaching practices are an integral part of learning, and learning is "changes in patterns of participation in discourse" (Sfard & Cobb, 2014, p. 14).

Adopting a participationist approach, which looks for changes in discourse as evidence of learning, means that research aims focus on what happens as people communicate about activities. In this paradigm, new insights are possible via thorough documentation of a learner's discursive participation and through detailed analyses. The participationist scrutinises discourse to delineate what discursive development entails and looks for implicit signs of learning which might otherwise have escaped our attention.

These ontological and epistemological commitments frame the theoretical framework and methodological choices for the four studies herein. Being located within a participationist research paradigm, the commognitive framework is thus an appropriate choice for the present research. The methodological choices are now described in Section 3.2.2.

3.2.2 A Qualitative Research Approach

The literature reviewed in Chapter 2 detailed students' overreliance on empirical arguments as proof and showed the difficulties with transitioning to proving deductively to be widespread among students in many educational contexts. Quantitative research provides "a general picture of trends, associations and relationships" (Creswell, 2007, p. 40). Hence, providing a further quantitative account of young students' difficulties in proving mathematically is not necessary; researchers already know *what* is happening. Qualitative research is more interested in understanding *how* and *why* phenomenon occur. In qualitative research, researchers study things in their natural settings and interpret or make sense of "phenomena in terms of the meanings people bring to them" (Denzin & Lincoln, 2005, p. 3). The gap in the literature on students' proving is knowing *how* students engage in proving activity. Understanding this requires "a complex, detailed understanding of the issue:" (Creswell, 2007, p. 40); therefore, a qualitative approach was considered appropriate for this research.

3.2.3 A Case Study Approach

Given the complexity of the phenomenon to be studied, I aimed to source findings from real-world settings where phenomena of interest unfold naturally (Patton, 2002). Within the commognitive framework, it is discourse that is of interest; thus, a research methodology that collects and privileges discourse is necessary to provide "detailed insights into the questions of how and why things happen (Sfard, 2018, p. 227). For my research, I needed to examine the discursive space between empirical and deductive discourses. I needed to capture the individual discourses of young students as they engage in proving activity to better understand how and why they participate as they do, and how and why and their discourses develop.

Hence, I opted for a case study, which involves "the study of an issue… explored through cases within a bounded system" (Creswell, 2007, p. 73). The particular issue

for this research was the development of primary school students' participation in a deductive discourse. By conducting an in-depth analysis of this case, within its context, the goal was to better understand the issue from the perspective of the students (Merriam, 2009; Stake, 2006; Yin, 2014). The case was bounded by being a discourse on odd and even numbers and was examined by gathering data from groups of primary school students working on a task together, which was designed to stimulate a need for, and students' entry into, a deductive discourse. The data from within this case are considered in different ways in each study to illuminate the particular phenomenon at the heart of the case.

3.2.4 Research Aims

In Chapter 2, I articulated three general aims that addressed gaps in the research literature:

(i) How primary school students arrive at a particular proof.

(ii) How students' reasoning develops (within the space between an empirical argument and generic proof) and, therefore, what might count as learning.

(iii) How primary school students can be supported to develop their arguments.

In light of the theoretical framework and methodology adopted in this research, these aims are now reframed within one overarching aim: *to explore primary school students' first steps in a deductive discourse*. This aim is broken down into several smaller and more specific aims:

(i) To characterise primary school students' use of examples in their substantiations of mathematical narratives within the empirical-generic space.

(ii) To examine possible sources of commognitive conflicts in students' transition from empirical to deductive discourses.

(iii) To explore whether students' first steps in a deductive discourse could be exploratory (rather than ritualistic).

(iv) To investigate what can help and hinder students' initiation into a new, deductive discourse.

3.3 Research Methods

In this section I provide details about the research methods undertaken. I describe the recruitment and selection of participants involved in the studies (Section 3.3.1),

the group-work setting for conducting the research (Section 3.3.2), and the tasks used in this research, including their design and implementation (Section 3.3.3). Finally, in Section 3.3.4, I account for how the data were collected.

3.3.1 The Participants

Data for the studies in this research were gathered from seven groups of four students in Year 4 (eight or nine years old), coming from two Auckland-based primary schools. The schools were selected for convenience as this study did not require sites with particular characteristics. Having students from two different schools not only provided more students who fit the criteria but also helped counter the effects of any peculiarities one school might have in terms of prior exposure to proving activity or understanding of odd and even numbers. To select students for the study, I used purposeful sampling (Patton, 2002). Participating schools were asked to provide a list of students who met the following criteria:

(i) They were working at or above the curriculum level expectations.

(ii) They had no specific language difficulties.

(iii) They had no specific behavioural challenges.

(iv) They could be reasonably expected to participate in discussions.

These criteria were important because: (i) the teaching sessions needed to be rich in discourse; and (ii) the Teacher-Researcher would have little time to establish rapport or participation norms.

The parents/guardians of all of these students then received participant information sheets (Appendix B), which included: information about anonymity; voluntary participation and withdrawal rights; data-use agreement; institutional policies/regulations on research; and consent forms seeking consent for their child to participate, be audio- and video-recorded and for the use of their child's work and image. Written consent for participation was given by the school and the parents (Appendix C). A list was created of students who had parental consent to participate. I visited these students to discuss the purpose of the study and to give an overview of the procedures, methods of recording, and ethical considerations (see Appendix B for a copy of the participant information sheet used in this discussion and Appendix D for the student assent form). The visit also provided an opportunity to have any questions answered. A final list was then created from students who assented to

participate in the study, with priority given to students who had given assent and whose parents had given consent for their images to be used in presentations. The class teachers were then asked to create groups of four students from this list based on whom they thought would work well together. This process was deemed an appropriate way to establish the best possible chance of students articulating their thinking and, therefore, an appropriate way of obtaining the discursive data necessary to address the research aims. The 28 students who participated in this study comprised three groups of four students from one school and four groups of four students from the second school.

3.3.2 The Setting

In considering the setting for group sessions, it was necessary for the students to feel comfortable enough to interact and articulate their ideas, and so working in the students' normal classroom environment would have been ideal. However, in order to capture the audio clearly and to avoid interruptions, it was necessary to conduct the sessions in a quiet separate workspace. With these considerations in mind, the focus sessions were conducted in a small withdrawal teaching space within each of the two schools, but outside of the students' normal classroom. The students sat on chairs around a desk or sat in a circle on the floor as they worked with me. Each group session was conducted during normal mathematics teaching time and lasted between 30 and 55 minutes.

3.3.3 The Tasks

In this section, I account for the tasks used in this research. I describe design considerations of the tasks (Section 3.3.3.1), including the characteristics of the task—the number of cases, the purpose, and the mathematical content (Section 3.3.3.2). Then I provide a brief overview of research literature on concept cartoons (Section 3.3.3.3) and the use of puppets (Section 3.3.3.4) before describing how I used these features in designing task situations for this research (Section 3.3.3.5) and describing their implementation (Section 3.3.3.6). However, more information about the theoretical considerations behind the task situations are provided in Chapter 6.

3.3.3.1 Task Design. Task design plays an important role in encouraging student thinking and facilitating students in articulating their reasoning (Ball & Bass, 2003; Smith & Stein, 2011). When students share their thinking, formative information as

well as opportunities to expose and address misconceptions and compare and contrast ideas and solutions are provided (Smith & Stein, 2011). To fulfil the aims of this research, it was important to develop a task that would enable young participants (eight- and nine-year-old students) to engage in interpersonal discourse. I needed to create opportunities for participants to demonstrate their use of words and narratives, their selection and use of visual mediators, and develop routine ways of proving (Kim, Choi, & Lim, 2017).

3.3.3.2 Characteristics of Tasks. The characteristics of proving task situations have the potential to affect the mathematical activity generated, and so there are important criteria to consider in their design (A. J. Stylianides, 2016). A. J. Stylianides (2016) provided two main mathematical ways of classifying proving tasks:

(i) The number of cases involved: a single case (e.g., "Prove that 23 is a prime number"); multiple but finitely many cases (e.g., "Prove that there are four prime numbers between 10 and 20"); or infinitely many cases (e.g., "Prove that the sum of any two multiples of 3 is a multiple of 3").

(ii) The purpose of the task: to justify (e.g., any of the above examples in (i)) or to refute a statement (e.g., "Disprove that 21 is a prime number"; "Disprove that there are as many prime numbers between 1 and 10 as there are between 21 and 30"; "Disprove that the sum of any two prime numbers is a prime number").

These characteristics were initial considerations when designing the task used in the present research. Since the difficulties cited in research most often seemed to involve the difficulty of generalising when bridging the empirical-deductive divide with regard to universal statements (e.g., Coe & Ruthven, 1994; Education Committee of the European Mathematical Society, 2011; Healy & Hoyles, 2000; Knuth et al., 2009; Stylianides & Stylianides, 2018), I chose to design tasks involving infinitely many cases. Additionally, refuting a statement involving infinitely many cases (e.g., "Disprove that the sum of any two prime numbers is a prime number") relies on the existence of a counterexample and therefore invites the use of empirical proof rather than a general argument. On this basis, I also decided to design a task that would require students to justify a universal statement, even if they also refuted a false claim.

In terms of the mathematical content involved, I decided to select tasks that involved selecting and justifying narratives about the sums of odd and even numbers. Parity (or oddness and evenness) is a topic that commonly features in elementary school curricula (Roberts & Stylianides, 2012); therefore, the mathematical content was deemed to be accessible for the age of the students involved in the study (Levenson et al., 2007; NCTM, 2000). Furthermore, prior research on the topic of odd and even numbers with young students (e.g., Ball, 1993; Ball & Bass, 2000; Ball et al., 2008; Frobisher & Nelson, 1992; Frobisher 1999; Roberts & G. J. Stylianides, 2013; A. J. Stylianides, 2007) and pre-service elementary school teachers (Zazkis, 1998) also provided an established basis to work from.

Frobisher (1999) concluded that the students in his study lacked a conceptual understanding of parity. In contrast, Ball (1993) showed students articulating their reasoning (including faulty reasoning) and debating their ideas about odd and even numbers, and A. J. Stylianides (2007) showed young students to be capable of providing valid arguments involving the sums of odds and evens. These studies suggested that the topic would provoke individual substantiations for rejecting or endorsing narratives for categorising odd and even numbers and for the sums of odd and even numbers from primary school aged children. However, where these studies showed young students to be capable of providing reasoning (both faulty and valid) to support particular narratives about the sums of odds and evens, it remained unclear *how* students arrived at a particular proof and *when and how* their reasoning developed. Hence, the present research had the potential to shed new light on established findings.

3.3.3.3 Concept Cartoons. Disagreements can be "an important catalyst" for learning (Chazan & Ball, 1999, p. 7), as the awareness of alternative ideas may prompt students to rethink and re-evaluate their ideas. Concept cartoons have been shown to be an effective way to present conflicting ideas and encourage students to communicate their ideas and reasoning (Keogh & Naylor, 1993; Sexton, et al., 2009). Essentially, concept cartoons are cartoon-style drawings depicting characters with differing viewpoints about a particular situation, designed to stimulate thinking and provoke discussion. Keogh and Naylor (1993) initially created concept cartoons as a tool to prompt and challenge learners' ideas, and support learners to develop an

understanding of scientific concepts. In the science classroom, they have been shown to be a useful way to challenge students' misconceptions and thus support students in resolving them (Naylor & Keogh, 2013). Hence, for the present research, concept cartoons were considered useful because they could present several points of view, including key pieces of faulty thinking, that may not emerge in a group. The cartoons could stimulate mathematical argumentation by inviting the students to "air, debate and test their ideas" (Kabapinar, 2005, p. 137) in relation to the prompts in the cartoon.

Over the years, the principles of concept cartoon design have been refined and developed. Naylor and Keogh (2013) suggested the following principles for such design:

- Ensuring the inclusion of multiple statements (rather than a single statement), which should include both the scientifically (or mathematically in this case) acceptable statement and common misconceptions or false (mathematical) statements.
- Ensuring alternative narratives are plausible (based on research evidence about student's ideas at particular ages) and have equal status.
- Using everyday situations to make the task less intimidating and more inviting for students to engage with the task.
- Using a minimal amount of text.

These principles are in keeping with recommendations made by Durand-Guerrier et al. (2012) regarding the use of tasks which produce didactic situations that stimulate proving activity and the development of mathematical arguments. These authors (2011) argued that students are rarely confronted with false statements when, in fact, processing false statements can prompt a desirable degree of scepticism leading to critical thinking and deeper thinking. When students are faced with statements, the truth or which is yet to be decided (such as the ones presented in concept cartoons), their need for understanding is engaged, and they are more likely to appreciate a need for proof and participate in argumentation (e.g., Legrand, 2001).

At a metacognitive level, when students consider others' ideas, it may help them with thinking about their own thinking (Evans & Dawson, 2017). Students encounter commognitive conflict (Sfard, 2008, p. 162) when they are required to consider

plausible alternatives, some of which they may not have thought of before. The concept cartoons prompt students to decide which arguments are sound and which are faulty and, by being pressed to justify their judgments, they are assisted in uncovering the underlying reasons for erroneous thinking. Thus, in this research, the use of concept cartoons was considered an effective way of establishing a learning environment conducive to bringing commognitive conflicts to the surface and provoking young students' interpersonal engagement in proving activity.

3.3.3.4 Use of Puppets. The use of puppets in play-based therapy is well established. Puppetry has been used as a play-based therapeutic tool in assessment, individual therapy, family work, paediatric consultation, and school-based counselling (Bromfield, 1994). In these contexts, puppets are valued for the role they play in facilitating talk through "their expansion of the communicative repertoire between child and therapist" (Bromfield, 1994, p. 444). It is the role that puppets have in facilitating communication that has been taken advantage of in a variety of educational settings too: for example, drama and social education, road safety, nutrition, dental hygiene, and language development (Epstein et al., 2007). Simon et al.'s (2008) study also suggested that the use of puppets could help reduce the adjustment period required for children to accept a teacher because they would perceive a puppet as having less status or authority than a teacher (even though they would realise that the teacher was operating the puppet).

The use of puppets was thus appealing for this research, which depended on interpersonal dialogue for data. Since my role in the group sessions would be as the students' teacher, I would be seen as an authoritative figure. This meant that students might be less confident than they would usually be in their group work and, therefore, less vocal. Furthermore, the fact that I would also be new and unfamiliar to the students would compound this issue. Hence, the use of a puppet was considered to be an innovative tool for reducing the formality of the setting and reducing student anxiety so that the students might interact and communicate more readily.

Additionally, and of particular interest to the present research, the use of puppets has also been shown to increase the kind of talk that helps reasoning. In the classroom, dialogue can be categorised as teacher-led or peer-group interaction (Mercer et al., 2004)—both types of interactions have features that are conducive to developing

reasoning, but both have disadvantages too. In teacher-led discussions, the teacher acts as the "more knowledgeable other" (Vygotsky, 1978) by providing guidance for students and advancing their thinking via questioning or demonstration, but students may be reluctant to expose their thinking and less inclined to engage in discussions in the presence of a teacher. On the other hand, students may be more inclined to participate in argumentation in the more relaxed social space that peer-group interaction offers. However, these situations may fail to advance the thinking of the participants if the right questions are not asked or if students' ideas are too similar to stimulate debate. Simon et al. (2008) proposed that the use of puppets creates a third type of interaction that optimises the advantages of the other two types because it "introduces a different style of interpersonal relationship" (p. 1245). These authors argued that the puppet is still the expert (since the teacher is effectively in control of the puppet), and so it can challenge students' thinking in their zone of proximal development (Vygotsky, 1978). Yet the puppet can simultaneously play the role of a novice or a sceptic in a way that a teacher would struggle to pull off. The puppet's need for help provides a space for the students to play the "experts" and a reason for them to work harder to explain their thinking.

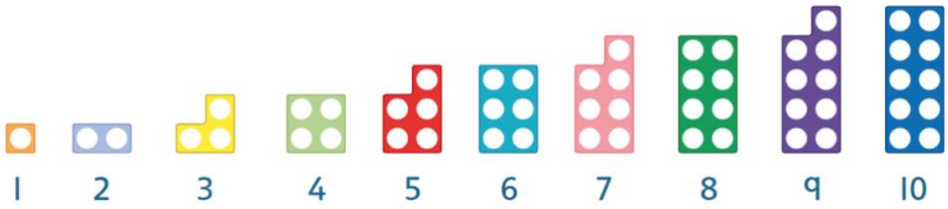
The usefulness of the puppet's ability to play the role of the sceptic falls in line with Mason et al.'s (2011) recommendations of a three-stage process for approaching mathematical proving: "convince yourself; convince a friend; convince a sceptic" (p. 87). Mason et al. (2011) argued that it is relatively easy for learners to convince themselves, but convincing a friend requires a level of metacognition through articulating and externalising what might seem obvious. However, it is the sceptic—who doubts and questions every statement—who may help the learner realise weaknesses in their argument or help them to see why their reasoning does, or does not, hold true by forcing them to double-question their reasoning for cracks or hidden assumptions. Hence, in this research, using a puppet to play the role of the sceptic maintained the teacher's role as questioner, rather than teacher. Working with the student groups in this way was considered to be useful for probing students' reasoning.

3.3.3.5 The Task Situations. For this research, the students were presented with three task situations: a classification task and two concept cartoons.

The Classification Task Situation. The students first classified numbers as odd and even. They took turns to turn over cards revealing numbers and sorted them into boxes labelled "Odd" and "Even." Each time they turned over a card, I asked the students to substantiate their classifications and to consider one another's questions and thinking. The first three cards (representing numbers six, nine and four) had Numicon tiles (Figure 3) printed on the reverse and were cut in the shape of the corresponding Numicon tile.[5] Cards featuring a numeral within the 1–10,000 range were then presented.

Figure 3

Numicon Tiles



The Concept Cartoon Task Situations. Following the classification task, the students were presented with two concept cartoons that I had designed: the "odd + odd" concept cartoon (Figure 4a); and the "even + odd" concept cartoon (Figure 4b). In keeping with Naylor and Keogh's (2013) four principles, the cartoons were presented simply, with minimal text, and the alternative viewpoints were plausible, had equal status and contained faulty narratives to create ambiguity and stir scepticism. Importantly, each cartoon featured three conflicting narratives: two are what I term *binary* narratives (e.g., "odd + odd = even" and "odd + odd = odd")—two mutually exclusive, dichotomous narratives; and the other narrative I term *bilateral* (e.g., "odd + odd = sometimes even and sometimes odd"). Both outcomes were considered possible. Having only binary options of "odd + odd = even" and "odd + odd = odd" would reinforce students' tendency to use empirical reasoning: if there were only the two binary options, students would likely assume that empirical evidence that disproved one conjecture would, by default, prove the other to be true. Therefore, the

[5] Numicon tiles are visual representations of numbers 1-10 presented as dots within a frameless 2 x 5 rectangle.

role of the bilateral narrative was to require the students to work harder to show why, for example, “odd + odd” is always even, and not just sometimes.

The students were encouraged to share which narratives they endorsed and to share their substantiations for endorsing these narratives in order to convince one another. Hence, the purpose of including the bilateral narrative was to raise the possibility of students experiencing uncertainty and require the students to work harder in considering whether observable patterns of odd or even sums are universal, how they could know this for sure, and how they could prove this to others.

Figure 4

(a) *The “Odd + Odd” Concept Cartoon and (b) The “Even + Odd” Concept Cartoon*

Figure 4a

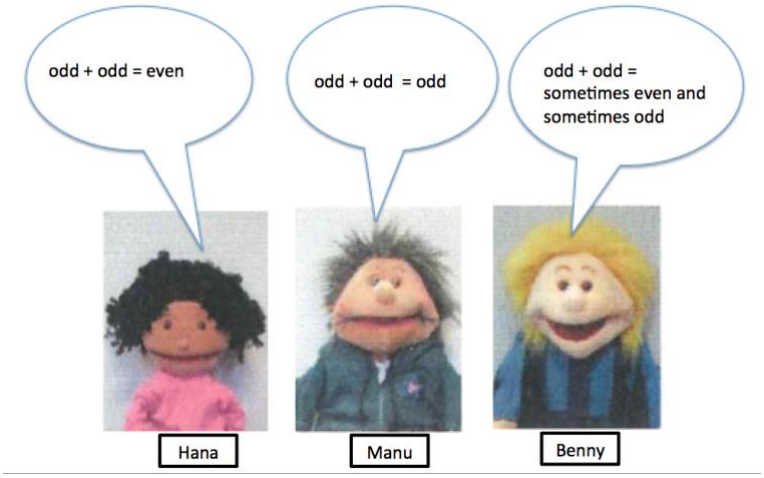
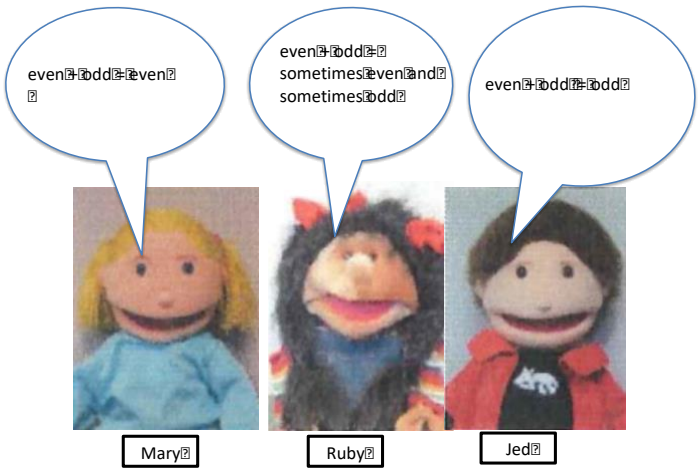


Figure 4b



3.3.3.6 The Implementation of the Tasks. I began the session by introducing a puppet called Jerry (Figure 5) and a fictitious scenario that he was present because he needed help to settle a mathematical disagreement that some of his friends were having. The intent here was to position the students as "the more knowledgeable others" rather than as novices being assessed by an outsider. Jerry (under the guise of the Teacher-Researcher—me) had a legitimate reason to press the students for more reasoning by playing the role of an unconvinced sceptic. He could be engineered to ask probing questions, such as, "How do you know there isn't a pair of odds that make an odd number?" or "I don't really understand what you are saying. Could you show me what you mean in a drawing?"

Each group of students participated in a session which was structured using the three task situations described above: (i) classification of numbers as even or odd; (ii) consideration of either the "odd + odd" concept cartoon or the "even + odd" concept cartoon (with Jerry the puppet); and (iii) consideration of whichever concept cartoon had not been considered in part (ii) (without Jerry the puppet).

Figure 5

Jerry the Puppet

To support their communication, students were able to use pens and paper, counters, Numicon tiles (Figure 3) and "odd" and "even" Numicon strips[6] (Figures 6a and 6b show the strips are similar in design to Numicon tiles in terms of the paired configuration of dots, but they are printed on long, folded, strips of paper with both ends exposed).

Figure 6

(a) The "Odd" Numicon Strip and (b) The "Even" Numicon Strip

Figure 6a

Figure 6b

3.3.4 Data Collection

Data collected for this research comprised video and audio recordings, transcripts of the group sessions, and students' written work. Two video cameras were used to clearly capture the group interactions: one recorded the teacher face-on to capture her actions, Jerry's involvement, and the student sitting alongside her; and another recorded from the opposite angle to capture the other students' interactions. A voice recorder was also placed on the group table to capture clear audio and in case of camera malfunction.

[6] More details about the design and use of Numicon strips are given in Chapter 6, Section 6.3.2.

Each group session was transcribed in its entirety. Once the transcription for each session was completed, the video-recorded sessions were replayed several times to check for accuracy. The transcribed data indicated the sequence of turns in the session, speaker, and non-verbal language to ensure it was clear what was said, to whom it was directed, and how it was said (e.g., silence, gestures, facial expressions, body language). Student names were replaced with pseudonyms rather than codes (e.g., letters) to preserve a sense of the individual participants involved. The role of the commognitive analyst involves "examining the use of language and non-language elements, and how these two interact to become a discourse" (Kim et al., 2017, p. 485). Hence, any visual mediators that students used or produced themselves in the form of written work played an important role as tools for communication. Correspondingly, student work was collected at the end of the session and photos of their work were added to transcripts where appropriate.

3.4 Data Analysis

While more detail regarding data analysis for each study is provided in Chapter 4–7, this section gives a general overview of the conventions I used when presenting episodes of dialogue in each of the studies and which data analysis tools were used to address the first three aims of the thesis.

I used the following conventions in presenting excerpts of the transcript in the findings sections:

- The turn number is included to provide the reader with an idea of turns elapsing between each episode and to enables me to refer to key turns within my analysis.
- The column to the right of the dialogue includes participants' actions as they spoke (including photos of their use of visual mediators or written work), and information about how the utterances were spoken, facial expressions, and body language.
- Words that were spoken with emphasis are in bold and italics.
- Three dots represent hesitation by an interlocutor.
- Where three dots appear in all four columns of the episodes of the transcript simultaneously, they indicate the omission of one or more turns (the number of which is indicated by the gap in the turn numbers).

- Where necessary, square brackets indicate who participants were addressing in their utterances, or what they were referring to.
- Round brackets were used to indicate where dialogue was inaudible or to add additional relevant information.
- Ungrammatical or colloquial language was not edited.

Since I had the dual role of Teacher-Researcher in the group sessions, I reference myself as the teacher in the third person for clarity. This enables me to distinguish my actions and intentions during the lesson as the teacher from my intentions as the researcher and observations as the analyst (when I use first-person).

The analytical framework must be aligned with research aims and questions (Kim et al., 2017) and its theoretical perspective. The commognitive framework comes with ready-made tools with which researchers can analyse their data: the four features of discourse (words and word-use, visual mediators, narratives, and routines); and Lavie et al.'s characteristics of explorative performance of routines (flexibility, bondedness, applicability, performer's agentivity, objectification of the discourse, and substantiability). In addressing my first research aim (to characterise primary school students' use of examples in their substantiations of mathematical narratives within the empirical-generic space), discriminating the four discursive features of the commognitive framework (keywords, visual mediators, narratives and routines) in learners' discourses was paramount. To address the second aim (to examine possible sources of commognitive conflicts in students' transition from empirical to deductive discourses) these four discursive features were also important in characterising differences between the group members at the object-level and the meta-level. However, I also introduce a new methodological tool—the Grove of Realizations tool—in Chapter 5 to help visually expose sources of commognitive conflict. Lavie et al.'s (2019) characteristics of routine performance were used with regard to research aim (to explore the extent to which students' first steps in a deductive discourse could be exploratory (rather than ritualistic)).

3.5 Integrity of the Research

This interpretivist and qualitative research, which uses a commognitive theoretical and analytical frame, has affordances for the research aims. However, it also presents

some threats to the credibility, transferability, dependability, and confirmability of the research. Each of these is discussed in more depth in the following sections.

3.5.1 Credibility

The criterion of credibility involves demonstrating truth in the findings and establishing that they are credible or believable from the view of the participants (Tobin & Begley, 2004). The commognitive researcher's intention is to "talk about participants' talk on understanding" (Sfard, 2008, p. 272) rather than directly commenting about participant's understanding; hence the work of the commognitive researcher is more transparent than would be the case in selective and interpretive presentations of qualitative research in other theoretical frames. Sfard (2012) insisted that all kinds of interaction are "events of learning" (p. 8) and so the principle of *contextuality* should be followed. This means interactions should be documented as fully as possible, and the researcher should maintain an awareness of the conversational context of participants' utterances.

Hence, in the present research I maximised the credibility of the findings by adhering to this principle by:

- Collecting data from a range of sources—student work, cameras showing two viewpoints, and voice recorder—in order to build an accurate picture of what occurred.
- Transcribing each group session in its entirety.
- Creating transcripts by repeated watching, cross checking against other sources, and checking parts I was uncertain about with a supervisor.
- Using a format for recording transcripts that simultaneously showed the number of the turn within each session, who was speaking and who they were addressing, a verbatim account of what was said, and what actions occurred.
- Keeping the context of the conversation and environment in mind when considering participants' utterances.

Credibility can also be established by adhering to Sfard's principle of *directness* (Sfard, 2012, p. 8), which stipulates that what was said and done by the participants should be presented in reports before the researcher's account of the data. Hence, when describing my findings (Chapters 4, 5, 6 & 7), I provide episodes of dialogue taken from the overall lesson transcript before the analysis of same; and all analysis

was linked to instances within the data. Correspondingly, the data was presented in high resolution. My regard to detail aimed to get the reader as close as possible to being in the room with the students as the Teacher-Researcher to make my findings and conclusions more credible.

Furthermore, when analysing data, Sfard (2012) maintained that the principle of *alternating perspectives* should guide the researcher. I adhered to this principle by remaining aware of the possibility of "incommensurability between [my] mathematical discourse and the discourse of the participants of [my] study" (Sfard, 2012, p. 8) and considering how participants' use of and meanings for words may be different from my own. I also involved a supervisor experienced in commognition as a critical friend to resolve interpretation dilemmas and to interrogate my assumptions. Thus, the stories I present "are not directly about the world, but about the participants' narratives about the world" (Sfard, 2012, p. 8).

3.5.2 Transferability

Transferability addresses how possible it might be to apply the results to other contexts (Tobin & Begley, 2004). With regard to this consideration:

- The particular students and mathematics used in this study are context-bound, but the commognitive framework used for analysis is transferable to other students, other mathematics, and other contexts.
- The study serves as an example of how commognitive analyses can provide insights when working with young children in a cultural context in a way that has not been done before.
- While the findings are particular, they can also be applied to other situations in which primary students are learning through argumentation. This is possible because the research used Sfard's (2008) features of discourse (words and word-use; visual mediators, narratives, and routines) and Lavie et al.'s (2019) characteristics of explorative performance of routines (see Section 3.4) for analysing discourse and shows these examples to have the qualities of mathematics discourse more broadly.

Given the rich description of the context provided in each of the four studies (including details regarding the participants, the setting, the tasks used, the role I took as Teacher-Researcher, the analysis of the data, and the theoretical lens adopted

through which the findings were interpreted), a reader should be able to determine the applicability of the findings to a different situation of their research interest (Lincoln & Guba, 1985).

3.5.3 Dependability

Dependability concerns how auditable the research is—to what degree there is a complete account of the various stages of research (Koch, 1994). Researchers are required to ensure their work is logical (e.g., there is alignment between the research aims, design, and process of analysis), verifiable, and well documented in order to achieve dependability (Tobin & Begley, 2004).

I had three supervisors oversee my thesis, which helped with establishing a clear rationale regarding theoretical and methodological decisions. In ensuring my research was verifiable and well-documented, the research process, including the rationale for research decisions made, were outlined in broad terms in this chapter (Sections 3.2, 3.3 and 3.4) for the reader. Furthermore, the inclusion of data extracts before the analyses, including the use of photos and student work, increases the authenticity and thereby increases the dependability of this research.

3.5.4 Confirmability

Confirmability of research deals with the issue of objectivity and with establishing that interpretations are derived from the data, rather than the researcher's subjective views (Tobin & Begley, 2004). The fact that only one person (myself) conducted the data collection, transcriptions, and discursive analysis for the research described in this thesis puts the findings and conclusions drawn at risk of being subjective. However, I worked closely with one of my supervisors when analysing the data. We established a process whereby I would select and analyse an episode of data which my supervisor would probe for further evidence or highlight instances where my assumptions had clouded my analysis. Furthermore, my three supervisors oversaw the reviewing of my findings, which helped to identify confusions and illuminate instances of subjectivity, distortion, and blind spots in the analysis process. This oversight helped to ensure that my interpretations were grounded in the data. Furthermore, the level of detail required in presenting any commognitive research (i.e., showing what was done and said by the students before analysing their talk)

meant that my interpretations were accessible to others, which means that instances of subjectivity will also be observable by the reader.

3.5.6 Ethical Considerations

The University of Auckland Human Participants Ethics Committee approved an application to conduct the study on 1 April 2019 (Ref: 019961). Potential ethical issues were identified and considered with reference to The University of Auckland Human Participants Ethics Guidelines. Participation in the study was voluntary, and the participants were informed that they would be able to withdraw from the study at any time. Assent from student participants was obtained, and their parents/legal guardians gave consent for their child to be video- and audio-recorded, have copies of their work used, and have records of the data kept securely for six years before being destroyed. Pseudonyms have been used for the school and participants to reduce the risk of compromising anonymity; however, participants understood that anonymity could not be guaranteed.

CHAPTER 4: Research Study 1:
How Can We Tell if Learners Are Referring to the General in the Particular? Discursive Markers of Generic Example-Use

4.1 Introduction

Arguments that make use of generic examples are more accessible for young learners than formal proofs, and they have high explanatory potential. However, determining whether learners are using examples generically and are therefore seeing the general rather than the particular in examples, is neither obvious nor straightforward (Mason & Pimm, 1984; Reid & Vallejo Vargas, 2018; Yopp & Ely, 2015). This is particularly difficult in the absence of a fully formed and finished proof and in situations where students are not aware of what constitutes a proof or the production of a written proof is not appropriate (e.g., with primary school students).

In situations where an explicit formulation of a generic argument is inappropriate or missing, can more subtle signs point to students implicitly recognising the genericity of an example and implicitly reasoning deductively? If so, what might be some signs that students are indicating their particular example is intended to account for all cases? Answers to these questions will not necessarily be found by examining students' finished products—the proof they developed as an endpoint. The answers to such questions require fine-grained analysis of students' working "in-the-moment" and, correspondingly, the use of an appropriate theoretical framework. Accordingly, this chapter addresses the first research aim of this thesis by *characterising primary school students' use of examples in their substantiation of mathematical narratives within the empirical-deductive space*.

4.2. The Commognitive Framework

Sfard (2008) defined mathematical discourse as a special form of communication, including self-communication (thinking), made distinguishable via four interrelated characteristics: keywords (e.g., numeric words like "three", "fourteen"); visual mediators (e.g., numerals, symbols, diagrams, pictures); narratives (e.g., definitions, proofs); and routines (repetitive patterns of mathematical actions according to which mathematical tasks are being performed). Learning is seen as a lasting

transformation in a learner's discourse, which is identifiable by changes in one or more of these four characteristics.

According to Sfard (2008), learning occurs both at the object-level and the meta-level. Object-level learning occurs through a growth in *realizations* [7] about mathematical objects. Where *signifiers* are the words or symbols functioning as nouns in participants' utterances, it is the "perceptually accessible objects" (Sfard, 2008, p.154) that serve as their realizations. Object-level learning is signalled by an expansion in the routines and endorsed narratives within a discourse. For example, when an individual's realizations of even expand from "numbers in the sequence 0,2,4,6,8…" to also realizing any "number ending in 0,2,4,6,8" as even or realizing "all evens are divisible by two". However, meta-level learning occurs when learners transition into a discourse that is incommensurable with their existing one, requiring a change in endorsing "propositions about the discourse rather than about its objects" (Sfard, 2007, p. 573).

Transitioning from endorsing empirical substantiations for universal statements to endorsing deductive substantiations represents a meta-level shift in learning. Whereas in empirical discourses learners converse about specific objects, in deductive discourses learners are required to converse about abstract objects. In commognitive terms, learners performing an empirical substantiation routine will use numeric keywords (specific numbers) or visual mediators signifying specific numbers (in numeric form, e.g., "6", or other visual formats, e.g., three pairs of dots for "six") to model the resulting sums they make, and they rely on the sums of such numeric examples (such as "3 + 5 = 8") to substantiate a universal narrative (e.g., "odd + odd = even"). In contrast, a deductive routine for substantiating a universal narrative (e.g., "odd + odd = even") relies on a series of propositions supported by definitions (e.g., odd is "even + 1" or "2n + 1"), theorems or axioms, whereby each proposition is logically deduced from the previous one in an organised way (e.g. [2n+1] + [2m+1] = 2n+2m+2...) (Cooper & Lavie, 2020). These two substantiation

[7] Sfard (2008) uses the term "realization" instead of the more common term "representation", to convey the fact that there is actually 'nothing' there to represent—the mathematical objects are discursively created.

routines are incommensurable: an empirical routine substantiating a universal statement cannot be endorsed in a deductive discourse and vice versa.

Although examples are not necessarily part of a deductive routine, there is space for them to be so if they are used generically: as "an actual example, but one expressed in a way as to bring out its intended role as the carrier of the general" (Mason & Pimm, 1984, p. 287). The use of generic examples, therefore, is considered as a way for learners to bridge the gap between empirical arguments and formal, deductive proofs. However, as indicated in the literature review (Chapter 2), deciding whether an example is being used empirically or generically has been considered difficult (Reid & Vallejo Vargas, 2018; Yopp & Ely, 2015) or even impossible (Mason & Pimm, 1984).

Commognitively speaking, determining whether an example is being used empirically or generically should be visible via some change in a substantiation routine (i.e., changes in keywords, visual mediators or narratives). Hence, in this chapter, I examine students' verbal responses and their accompanying actions using the commognitive framework to characterise primary school students' use of examples. I look for discursive indicators as subtle signs that examples are being used more empirically or generically.

Accordingly, this study aims to answer the following two questions:

(i) How can learners' use of examples in their reasoning be categorised?

(ii) What commognitive indicators are present in learners' reasoning that suggests that they are using examples of odd and even generically rather than empirically?

4.3. Conduct of the Study

In this section, I describe how the study was conducted in order to answer the research questions. I provide a brief overview of the research methods—participants, the task situations and their implementation (for more detail refer to Chapter 3, Section 3.3), and then I provide a detailed account of the data analysis. Here, I describe the discursive features that were used to analyse the data and how they were established. I conclude with an overview of how the discursive features were

used to form categories of example-use and how the data are presented in the findings section to exemplify these categories.

4.3.1 Data Collection

This study used data collected from twenty-eight Year-4 students from two New Zealand (NZ) schools. As the unit of investigation in this study was discourse, teachers selected students to work in groups of four according to whom they considered would be willing and able to engage in dialogue about mathematics. The students were first shown the "odd plus odd" concept cartoon dilemma (Chapter 3, Figure 4a) and later the "even plus odd" concept cartoon (Chapter 3, Figure 4b) (or vice versa). The cartoons featured three characters, each with a speech bubble containing differing narratives on the sums of odds and evens for students to reject or endorse and then substantiate their choices. In order to convince one another, the students were encouraged to share which narratives they endorsed and share their substantiations for endorsing these narratives.

4.3.2 Data Analysis

To establish a baseline of discursive features that point to an awareness of generality, I used an argument endorsed by A. J. Stylianides (2007) as being expressed in purely general terms and which he classified as being equivalent to Balacheff's (1988) highest level of proof—a thought experiment. The argument comes from a third-grade student, Betsy, to substantiate "odd + odd = even."

> Betsy: *[provides a visual demonstration of how two groups of seven sticks can be grouped in twos] … if you added another even one to an odd number, or another one to an odd number, then it would equal an even number, 'cause all odd numbers if you circle them … by twos, there's one left over, so if you ... plus one, um, or if you plus another odd number, then the two ones left over will group together, and it will make an even number.*

A. J. Stylianides (2007) compared Betsy's argument with a standard algebraic proof (as shown in the first two columns of Table 2) and pointed to features within her argument that, he claimed, justified it as being classified as a general argument. To also contrast Betsy's argument with an empirical argument, a third column is added in Table 2 showing a "hypothetical empirical argument". First, like the standard algebraic proof, Betsy used definitions (previously endorsed by her class) of even and

odd, which A. J. Stylianides referred to as the *foundation* of her argument (p 9). Note that no definitions of odd or even are provided in the hypothetical argument. Second, from these definitions, she formulated a logically connected sequence of assertions that concluded with the derivation of the statement "odd + odd = even". Thus, A. J. Stylianides claimed that the "formulation" of Betsy's argument was logical and deductive (p. 9). Importantly, this argument is expressed in everyday language—the language that the 8- and 9-year-old students in my study would be using.

Table 2

Correspondences Between a Standard Algebraic Proof, Betsy's General Argument, (as Presented in A. J. Stylianides, 2007), and (Hypothetical) Empirical Reasoning for "Odd + Odd = Even".

	Standard algebraic proof	**Betsy's argument**	**(Hypothetical) empirical argument**
Object	All odd numbers are numbers of the form $2n + 1$, where n is a whole number.	…all odd numbers if you circle them… by twos, there's one left over.	Five is odd and three is odd.
Substantiating narratives	So, the sum of two odds will be: $(2k+1) +(2m+1) = (2k+2m) +(1+1) = 2k+2m+2$ [or $2(k+m+1)$].	…if you plus another odd number, then the two ones left over will group together.	When I add them, I get eight,
	which is an even number.	…and it will make an even number.	…which is even.
	Deductive Reasoning		Inductive Reasoning

Using fine-grained discourse analysis that utilises Sfard's (2008) commognitive framework, I examined Betsy's reasoning (and example-use) for further distinguishing features that marked Betsy's reasoning as general—similar to the

"standard algebraic proof" and different to the "hypothetical empirical argument" (Table 2). I now describe several discursive features of Betsy's argument, which provided a basis of comparison for my student data.

4.3.2.1 The Mathematical Objects in Focus. The mathematical objects in focus within empirical arguments are numeric (and thus specific) examples of "odd + odd = even" (e.g., 5 + 3 = 8). In contrast, the object in focus in Betsy's argument is oddness and evenness (abstract objects)—the mathematical properties of odds and evens. Betsy uses the structure of two odd numbers within the sum as the object in her narratives: "…all odd numbers if you circle them… by twos, there's one left over." Accordingly, to assess where students in my study were seeing the general (rather than the particular) in the examples they use, I endeavoured to identify where learners were signifying mathematical properties or the general structure of even or odd rather than numeric examples as the objects in their narratives.

4.3.2.2 The Nature of the Substantiating Narratives Used. Where empirical arguments use the numbers generated in the sums in their examples to substantiate endorsing a universal narrative (e.g., "I get eight, which is even"), Betsy presents a sequence of logically connected narratives to deductively substantiate her endorsement of "odd + odd = even". She uses conditional "if…then" narratives to signify factual implications in hypothetical situations and their consequences; for example, "<u>if</u> you… added another one to an odd number, <u>then</u> it would equal an even number." Hence, Betsy's substantiating narratives correspond with a standard algebraic proof (as presented in A. J. Stylianides, 2007) and contrast with those in hypothetical substantiating narratives associated with empirical reasoning, as shown in Table 2. Accordingly, I examined whether the students in my study were using logically connected narratives (rather than simply using the numbers generated in the sums in their examples to substantiate endorsing a universal narrative) as further signs that they were using examples generically rather than empirically.

4.3.2.3 Use of Keywords and Visual Mediators. Betsy visually mediates a specific example of "odd + odd" with her drawing of two sets of seven sticks, but she does not use specific number words; she uses the words "by twos" for "even," and "one(s) left over" for "odd." Where empirical arguments use numeric keywords ("five," "three,"

"eight"), Betsy's more generic keywords signify realizations of general properties of the generic mathematical structure of even and odd numbers.

Correspondingly, in terms of my data analysis, I looked for evidence of learners using generic keywords (or generic visual mediators) in favour of numeric ones to provide further indications that they may be seeing the general in the examples they used.

4.3.2.4 Tense. Tense is the way we speak about time, and different tenses change the meaning of a sentence. The hypothetical empirical argument in Table 2 uses the present tense: "Five is odd", "three is odd", "I add", "I get", "which is even". However, having provided her visual demonstration of an example for "odd + odd," Betsy uses the future tense to describe a hypothetical event: "it would equal;" "if you plus another number;" the two ones will group together;" "it will make an even number." The auxiliary verb "will" is used in the English language to make predictions or state facts about the future; for example, "The sun will rise tomorrow." Hence, Betsy's use of this auxiliary verb suggests that she is indicating that the even outcome "will *always*" occur with any combination of two odds. She also uses the word "would" in her precursory sentence implying her use of this example is intended to potentially signify imaginary or hypothetical situations occurring when particular conditions are satisfied (i.e., situations involving "even plus odd" addends): "if you added another even one to an odd number, or another one to an odd number, then it would equal an even number." Had Betsy spoken the same narratives in the present tense without the auxiliary verbs "will" or "would" (e.g., "I am adding another even one to …and odd one and it equals an even number"), it would signify the situation as relating to the particular example in focus. Instead, her use of future tense changes the interpretation of the mathematical situation she is describing as being something more general and more permanent. For the reasons described here, I examined the tense of the students' narratives. I looked for evidence of narratives using the future-tense (including the use of auxiliary verbs such as "will" or "would") as providing further indications of the students in my study using examples to signify *general* situations occurring under particular conditions.

4.3.2.5 Use of Determiners. In English, determiners (including articles, demonstratives, and quantifiers) are words that introduce nouns. The use of articles and demonstratives changes whether the speaker is signifying specific nouns or

general nouns. Where indefinite articles (i.e., “a” and “an”) are used in talk about a general noun, the definite article “the” shows the speaker is referring to a specific noun. The use of demonstrative pronouns (e.g., “this,” “that,” “these,” “those”) makes nouns even more specific as a speaker can point to the noun. And quantifiers (e.g., “some,” “any,” “all”) signify how much or little of the noun is being discussed.

In contrast to the empirical argument that only uses specific nouns for numbers (“five,” “three,” “eight”) and the determiner “which”—signalling the specific selection of this example—Betsy frequently uses indefinite articles in her argument: “another even,” “an odd number,” “another one,” “another odd number,” “an even number.” Her use of these indefinite articles signals she is talking *generally* about a nonspecific odd and even rather than a specific case. Betsy also uses the quantifying determiner “all”— “all odd numbers”—which signals her realization is raised to the general level, accounting for the infinite set of odd numbers. She only uses definite articles to describe the generality of having two ones when odds are grouped in twos: “*the* two ones left over.”

Accordingly, when analysing my data, I examined students’ use of determiners in their substantiating narratives to provide further discursive markers of realizations of generality versus particularity.

4.3.2.6 Actors. Note that in the standard algebraic proof (Table 2) there are no actors (e.g., “the sum of two odds will be…”). Sfard (2008) defined this as “alienation”—the act of presenting phenomena as occurring independently, that is, without humans (p. 44). Alienation is achieved via the use of the passive voice and by using the given noun (e.g., “the sum”) as the grammatical subject in narratives and eliminating personal pronouns (“I”, “you”, “we”). Alienation has the effect of giving nouns (e.g., “the sum,” “odd,” “even”) “a life of their own” (Sfard, 2008, p. 50) and conveys an assumption of permanence in the patterns observed.

In comparison, Betsy uses the active voice and the second-person personal pronoun “you”, which takes the role of the grammatical subject in her narratives: “if <u>you</u> added,” “if <u>you</u> circle them,” “if <u>you</u> plus another.” The inclusion of an actor in her narratives reduces objectivity. However, in comparison to the empirical argument (Table 2), which uses the first-person pronoun “I”, Betsy’s use of the second-person

suggests that she has already made this realization personally and her substantiating narratives are intended to persuade others. Furthermore, some of Betsy's narratives are in the passive voice and are agent-free. For example, instead of saying, "you can group the two left over ones together," she uses the passive voice and avoids personal pronouns: "the two ones left over will group together." In other words, there is evidence of some degree of alienation in Betsy's narratives. Accordingly, when analysing learners' use of examples in my data, attention to actors in narratives (indicated by use or absence of personal pronouns and active or passive voice) were considered as discursive markers of alienation (which is part of the process of objectification) and, therefore, generality.

4.3.3 Summary

When analysing my data for the above discursive features, I looked for routine ways in which learners reasoned with their example-use. According to Lavie and Sfard (2019), routines are "repetitive forms of actions one undertakes to produce narratives about numbers and to make sure they would be endorsed by the community of the discourse" (p. 424). Accordingly, I considered observing a learner using the same or similar way of using examples (when substantiating the endorsement of a universal narrative) more than once as constituting a routine. I looked for a diversity of routines within the data gathered from the 28 learners that fell within the space between empiric example-use and Betsy's general argument. That is, I aimed to detect and describe categories of routines that were empiric or were general (like Betsy's), but also ones that were neither truly empiric nor truly general.

In Section 4.4, to exemplify each category of example-use that emerged from the data analysis, I present episodes of dialogue alongside the speaker's name. The column to the right of the dialogue includes participants' actions as they speak and information about how the utterances were spoken and to whom they were directed (wherever relevant). I include the turn number to provide the reader with an idea of turns elapsing and to enable me to refer to key turns within my analysis. Following each episode, I examine and describe students' use of examples by utilising Sfard's (2008) commognitive framework to provide discursive markers signalling leaners' realizations of particularity or generality in the examples they use. Based on the

analysis of Betsy's general argument, the six discursive features were used to examine the learners' use of examples and their narratives in the data I gathered: (i) the mathematical object in focus; (ii) the nature of the substantiation narratives used; (iii) use of keywords and visual mediators; (iv) tense; (v) use of nouns or determiners; and (vi) use or absence of actors (active or passive voice).

4.4 Findings

From the analysis of my data, I was able to classify all comprehensible instances of students' example-use into four categories according to whether the example-use involved numeric or generic talk regarding odd and even numbers and whether the substantiations the students offered were inductive or deductive. The categories are exemplified in the following four subsections.

4.4.1 Numeric Talk With Examples That Are Used Inductively

The first category that emerged was students using numeric examples inductively, more commonly referred to as empirical example-use. In this category, there were no discursive markers to suggest a learner was seeing the general in the particular. Only numeric keywords (specific numbers) and visual mediators signifying specific numbers were used. Learners used numeric results from specific sums as the object within their substantiating narratives that they used to endorse a universal narrative.

This routine was observed in all seven student groups. The following episode pertains to the statement of "odd + odd = even" coming from four learners in one of the student groups.

Episode 1

Numeric Talk With Examples That Are Used Inductively to Endorse "Odd + Odd =Even."

Turn	Speaker	What was said	What was done
264.	ADAM:	Ah, this one's right. Hanna's right. Hanna's right.	Points to Hanna ["odd + odd = even"] in the concept cartoon.
265.	TOBY:	Yeah, because five plus five equals ten. Seven plus seven equals fourteen. Six plus six equals twelve.	
266.	ADAM:	And seventeen plus seventeen equals an even number anyway.	
267.	HEMI:	Yeah, yeah, yeah. Hanna's right. Hannah's right. [Teacher-Researcher (T.R.) invites the students to use Numicon shapes if they want to]	
275.	HEMI:	It equals six. This equals six.	Takes two Numicon 3 tiles and places them together to make a 2 x 3 array.
276.	ADAM:	Yeah, this is easy. Ooo yeah! [Speaking to the group]. Odd and odd and add them together. Easy!	Takes Numicon 5 and Numicon 7 tiles. Places them together to make a 6 x 2 array.
278.	TOBY:	And these ones. I'm gonna use these ones.	Silently counts the array in twos. Takes Numicon 9 and 7 tiles.
279.	ERIN:	Voila! Madam!	Takes Numicon 5 and 9 and tiles and forms a 7 x 2 array.

280.	T.R.:	…Has anyone found an odd plus odd equals odd?	
281.	STUDENTS:	No.	
282.	ERIN:	Yes! Two four… Oh yes, yes.	Starts counting her 7 x 2 array in twos.
283.	T.R.:	You have? Where?	
284.	ERIN:	Oh, I think.	
285.	ADAM:	No, no.	Counts Erin's arrangement in ones to himself.
286	ERIN:		Starts to count her array in ones.
287.	TOBY:	No. Because two, four, six, eight, ten, twelve. Fourteen.	Reaches over to count Erin's 9 + 5 strip; counting in twos.
288.	ERIN:	So, it's even.	
289.	TOBY:	Yeah	

This episode shows all four learners using numeric examples inductively. Toby and Adam initially recall or calculate the sums of pairs of specific odd numbers to substantiate endorsing "odd + odd = even." Then, when invited to use the Numicon tiles, all the students visually mediate specific instances of "odd + odd = even." Their keywords are entirely numeric — "five;" "ten;" "six;" "twelve;" "seventeen;" "two;" "four;" "eight;" "fourteen;" "nine;" "eighteen"—and at no point do any of the students use words signifying realizations of mathematical structure. Correspondingly, they use the resulting number of the specific sums they have selected (and not the mathematical structure of the odd addends and even sum) as the object in their substantiations for "odd + odd = even." And when the students physically form symmetrical structures (a paired array) with their two odd (asymmetrical) tiles, it is the number rather than the shape that signifies evenness [275, 276, 286–288]. Even when Adam refers to his selected Numicon tiles using fewer specific nouns "odd and odd," [276] he silently counts the resulting sum (twelve) to substantiate the example's evenness rather than using the symmetrical structure (i.e., a *2 x 6* array). The students' disregard of the even structure of the pairs of odd Numicon tiles they use is most evident when Erin endorses her selection of Numicon 9 and 5 tiles arranged in a *2 x 7* array as an example of "odd + odd = odd" [282]. Erin, Adam and Toby then substantiate rejecting it as an instance of "odd + odd = odd" on the basis of confirming the count is "fourteen," and therefore even [285–289]. Again, it is the

number "fourteen," rather than the even (symmetrical) *shape* that is the object in these substantiations.

Note also that the students do not use words such as "like" (e.g., "[*like*] seventeen plus seventeen equals an even number" [266]) or "for example" (e.g., "because, [*for example*], five plus five equals ten" [265]) to indicate they are using their examples for illustrative purposes: their examples alone *are* their substantiations used to endorse "odd + odd = even". The students' use of numeric keywords and inductive substantiations, along with an absence of discursive markers to indicate the students may be illustrating a hypothetical (and thus potentially general) occurrence, suggest that these students are apprehending the particular, not the general, in their example-use.

4.4.2 Hybrid Numeric-Generic Talk With Examples That Are Used Inductively

The second category was for example-use that contained some discursive markers of generic realizations of odd and even numbers but still used inductive substantiations. In addition to (or instead of) numeric keywords, learners used words signifying more generic features, such as an "extra one" or, "one missing" (for odd). And while they used visual mediators of specific numbers, the way in which they used these visual mediators reflected their realizations of the generic symmetrical structure of even and asymmetrical structure of odd. The generic shape of the resulting specific sum (rather than the numeric result) was the object used to substantiate its oddness or evenness. However, in endorsing a universal narrative, substantiations were inductively generated from confirming results rather than deductively using the generic structure of the addends.

Episode 2 illustrates this category: Sadie uses generic talk in regard to her numeric examples, which she uses inductively to endorse the universal narrative "even + odd = odd".

Episode 2

Generic Talk With Numeric Examples That Are Used Inductively to Endorse "Even + Odd = Odd"

Turn	What Sadie said	What Sadie did
361.	[Speaking to herself] Ok. It equals odd. Because this one here is hanging off.	Takes Numicon 9 and 2 and places them together. Points to the extra one from the two.
367.	[Speaking to the Teacher-Researcher] So that one's even. Ok. And then if I put my two right here it, it would equal odd because this bit here is hanging off.	Holding up Numicon 2. Places it with Numicon 9 (as before). Points to the extra one on top of the arrangement.
369.	But can I have a experiment and see if Ruby's ["even + odd = sometimes even and sometimes odd"] right?	Selects Numicon 10 and 3 tiles. Rotates Numicon 3 around to see if it will "fit" with Numicon 10.
382	Yeah, I think it's Jed ["even + odd = even"]. This one's not working. It's still odd. Ok so I think it might be Jed that's right. Yeah. Yeah.	

Even though Sadie selects pairs of Numicon tiles signifying specific numbers, she does not tend to refer to these addends with number-names (only once does she refer to the Numicon 2 tile as "my *two*" [367]), nor does she use number-names for the sums in her examples. Instead, when Sadie selects Numicon tiles 2 and 9, she uses the words, "one…hanging off" [361], to substantiate the sum's oddness, pointing to the "one" as she does so. Her use of these keywords and visual mediators reveals Sadie's realization of the generic mathematical (asymmetrical) structure of oddness. However, Sadie only uses talk of mathematical structure to substantiate the odd *outcome* ("one…hanging off") for examples of "even + odd"; there are no narratives

substantiating why the "even (with no "ones…hanging off") + odd (with "one…hanging off'")" addends *together* make a shape with "one…hanging off," let alone why this would always be the case.

Sadie exhibits an ad hoc way of connecting the two Numicon pieces in each of her examples. With her first example, rather than connecting Numicon 2 with Numicon 9 horizontally to extend the even part of nine (i.e., 2 + [2 x 4 +1]), Sadie connects it vertically [361]. And, in the second example, Sadie tries different rotations of Numicon 3 to see if there is a way to connect it with Numicon 10 to make a symmetrical (and therefore even) shape and comments, "it's not working" [382], when she cannot. Accordingly, Sadie does not appear to be deductively using the generic structure of the odd and even addends to account for the resulting asymmetrical shape—where deductive use might be demonstrated by connecting the Numicon tiles *even-end-to-even-end* and accompanied with narratives substantiating how and why the structure of "even + odd" addends form a shape with "one…hanging off". Instead, her substantiation routine to endorse "even + odd = even" is: (i) more trial and check—she needs to "experiment" [369] with more than one pair of "even + odd" to check for oddness; (ii) absent of deductive narratives and actions; and (iii) based on empirical observations of the two confirming asymmetrical (odd) outcomes.

Note also that Sadie generally speaks in the present tense (e.g., "It equal<u>s</u> odd. Because this one here <u>is</u> hanging off." [361]), and she uses the personal pronoun "I" in her narratives (e.g., "if <u>I</u> put my two right here" [367]). When taken with her desire to "experiment" with examples, these two discursive features suggest that she has not yet formed a general realization that she is ready to persuade others of. Rather, her substantiations are for the benefit of herself—she is experimenting to establish patterns of results and is in the process of convincing herself that "even + odd" is always odd.

In summary, Sadie's use of visual mediators and keywords provides discursive markers of generic realizations of oddness and evenness, which point towards generic example-use. But Sadie's use of the personal pronoun "I" and her inductive substantiation routine, which uses empirical observations of the resulting sums' generic structure to endorse a universal narrative, shows she does not yet realize the general way in which (symmetrical) "even" plus (asymmetrical) "odd" addends

together form an asymmetric (odd) structure. The combination of these discursive markers, therefore, indicates she realizes the general structure of odd numbers and even numbers but is not yet seeing the generality in her examples of "odd plus even" addends within "odd".

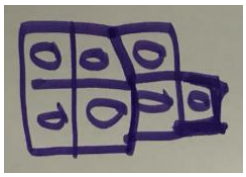
4.4.3 Hybrid Numeric-Generic Talk With Numeric Examples That Are Used Deductively

A further category that emerged was similar to the first category, in that it involved students' use of numeric examples, but the *way* in which students used these examples was quite different. In this category, numeric keywords (specific numbers) and visual mediators signifying specific numbers were used in conjunction with generic keywords, in ways that indicated generic realizations of the structure of odd and even. And, whereas in the first two categories, substantiations were inductive, in this category *deductive* substantiations were used to endorse universal narratives, which revealed learners' realizations of how the generic structures of the even or odd addends combine to make the particular example's odd or even outcome. Furthermore, (i) the use of indefinite determiners, (ii) the use of future tense, and (iii) a shift in personal pronoun use (from first-person to second-person or plural) all provided further discursive markers signalling generic use of a numeric example, while (iv) the use of expressions such as "like" or "for example" and "for instance" were used to indicate the introduction of a specific example for the purpose of explaining or confirming a generality.

Episode 3 illustrates two students (Jane and Zara) from the same student-group using numeric-generic talk with deductive substantiations to endorse "even + odd = odd".

Episode 3

Hybrid Numeric-Generic Talk With Examples That Are Used Deductively to Endorse "Even + Odd = Odd".

Turn	Speaker	What was said	What was done
528.	INTERVIEWER:	You think Jed ["even + odd = odd"] is right- why?	
529.	JANE:	Because there's four and there's gonna be three. It's gonna be odd because there's one more left.	Draws a Numicon-like 4 and 3 shapes, connected. 
		If that [extra one] wasn't there, it'd be even because it would be three and three and that's six. But if that's there it's seven so it's not even.	Pointing to the extra one on the bottom-right. Covers up the bottom-right square and then uncovers it again.
…	…	…	…
557.	ZARA:	Yeah, because if you have like two plus one, it will equal three. And there's an even and there's a one- and there's an odd and it would still equal an odd. Cos there'd be that one extra.	Picks up Numicon tiles 2 and 1.
…	…	…	…
571.	JANE:	For example, if there was six and one then we will have one still sticking out. Whatever odd number it is we'll still have one more sticking out.	Takes Numicon tiles 6 and 1 and connects them. Changes Numicon 1 for Numicon 3 and connects it to Numicon 6. Then picks up Numicon 10 and 7.

At first glance, these utterances suggest that Zara and Jane are substantiating their endorsement of "even + odd = odd" with numeric examples and, as such, their example-use could be confused with that described in the first category (numeric realizations with inductive substantiations). They use numeric keywords, visually mediated numeric examples (e.g., Jane's drawing of "four plus three" [529]; Jane and Zara's use of Numicon tiles [557; 571]), and narratives that make use of these

numeric examples—all of which are features consistent with the empirical example-use illustrated in the first category. However, there are a number of features that distinguish their example-use as being different to the first category and point towards generic example-use. First, in addition to using numeric keywords, both Jane and Zara also use generic keywords related to the mathematical structure of odd: "one more left" [529], "extra one" [557], and "one still/more sticking out" [571]. Second, Jane visually mediates the resulting shape from adding different pairs of even and odd Numicon tiles [529; 571] and specifically points to the "extra one" by covering it (to indicate evenness) and uncovering it (to indicate oddness) [529]. And third, their narratives and actions indicate that they are using these examples for illustrative purposes: their narratives include the words "like" (Zara: "if you have <u>like</u> two plus one" [557]), and "for example" (Jane: "<u>For example</u>, if there was six and one" [571]), and Jane proceeds to interchange Numicon tiles [571] to illustrate her substantiating narrative, "one sticking out", holds for other examples.

Further, and even subtler, discursive markers, which also suggest that Zara and Jane are using numeric examples generically (rather than empirically), come from: (i) their use of determiners; (ii) a shift in tense; and (iii) a shift from personal pronoun use. With regards to use of determiners, Zara switches from using numeric words ("two," "one," "three") to using indefinite articles ("<u>an</u> odd," <u>an</u> even") [557]. Had Zara used more demonstrative determiners such as "this," "that," or the specific pronoun "it"—for example, "<u>this</u> odd," or "<u>that</u> even," or "<u>it's</u> even" and "<u>it's</u> odd"—her example usage would suggest that the identity of the even and odd tiles she was referring to was known and were specifically the ones (i.e., "two" and "one") she had chosen. Instead, Zara's use of the indefinite article "an" implies that the identity of the even and odd addends is neither known nor obvious and serves to make these nouns more general. In a similar way, Jane initially uses numeric words ("four," "three," "six," "one") and specific pronouns ("it," "that") to refer to the addends and sums in her example [529], but she then switches to using the determiner "whatever" when describing an odd addend with its generic "one sticking out" [571]. This determiner is even more obvious in conveying a sense of generality than Zara's use of indefinite articles; it implies her example holds for *any* odd number.

Both girls' switch in tense—from present to future—strengthens the sense one gets of them moving from the specific to the general in their example-use. When Zara describes the combination of "two plus one" [557], she begins in the present tense to describe these two Numicon pieces: "there's an even and there's an odd". She then switches to future tense: "it would still equal an odd. Cos there'd be that one extra." This switch suggests that Zara may be using this particular numeric example to signify what *would* happen with *any* combination of "odd + even." Note also in Zara's narrative her use of the word "would" instead of "will"; where the word "will" is generally used to make predictions or state facts about the future, "would" implies an imaginary or hypothetical situation that is dependent on particular conditions. Similarly, Jane switches from using the present-tense, to describe her particular numeric example of "four plus three" [529], to the future tense, when she presents the example of "six and one" and says, "… we will have one sticking out" [571]. Her switch in tense for this numeric example mirrors the future tense she then uses in an even more generic narrative, "whatever odd number it is we'll still have one sticking out" [571], and therefore provides further support to claim she is using this example generically to show what happens with any such pair of "even + odd". Both the switch to future tense and the use of more general determiners distinguish Zara's and Jane's generic talk from that observed in the second category (generic talk with inductive substantiations).

In terms of the students' personal pronoun use, whereas in the previous category Sadie was the actor—she used the first-person pronoun "I" in her narratives (e.g., "if I put my two right here")— in Zara and Jane's narratives the actor is different: Zara uses the second-person pronoun "you" (e.g., "if you have like two plus one" [557]) and Jane uses the first plural pronoun "we" (e.g., "we will have one still sticking out" [571]). These discursive markers suggest that the two students' substantiations are not for them, that is, they are not in the process of ascertaining a realization. Rather, their substantiations are intended to communicate their realization to others—they are intended to persuade. Furthermore, there are a number of narratives which are devoid of personal pronouns. For example, Jane says, "It's gonna be odd because there's one more left" rather than "…we'll have one more left" [593]. And, similarly, instead of using "I [would have]," "you [would have]", or "we [would have]," Zara says:

> *"there's an even and there's a one- and there's an odd and it would still equal an odd. Cos there'd be that one extra." [557].*

An absence of personal pronouns erases the actor in these narratives. This serves to objectify "odd + odd = even" as a mathematical occurrence that is independent of a person's actions and conveys a sense of permanence.

Zara and Jane's generic talk is similar to the second category (generic talk with inductive substantiations) in that the object that both students use in their substantiations to endorse "even + odd = odd" is the mathematical (asymmetrical) structure of the sum—not the numeric result. However, where the substantiations in the second category were inductive (they use empirical observations of the resulting sums' generic structure to endorse a universal narrative), here they are deductive and reveal realizations of how the generic structures of the even or odd *addends* combine to make the particular example's odd or even outcome. Zara and Jane's use of "if…then" narratives provide evidence of logical and conditional substantiations in hypothetical situations. For example, regarding her use of Numicon tiles for "four" and "three," Jane says:

> *"If that [extra one] wasn't there [then] it'd be even because it would be 'three and three' and that's six. But if that's [the extra one] there [then] it's seven so it's [the sum] not even." [529].*

Jane's use of "if" denotes the hypothetical nature of her example and "would" forms a conditional sentence. Here, Jane is showing that she understands the difference between hypothetical situations like "four plus two" (even + even), which "[would] be even" because there are no "extra ones", and hypothetical situations similar to her example of "four plus three" (even + odd) where the "one more left" is the reason there is "one more left" in the sum. In short, Zara and Jane's substantiating narratives make use of generic structural features to account for any hypothetical or potential case of "even + odd."

The combination of all these discursive markers show Jane and Zara are using numeric examples to illuminate mathematically structural features (i.e., symmetrical or asymmetrical), representative of the classes of even and odd, to deductively substantiate endorsing the universal statement "even + odd = odd." These features

indicate that Jane and Zara are indeed seeing the general in the particular examples they use.

4.4.4 Nonspecific-Generic Talk With Examples That Are Used Deductively

In the final category, *nonspecific* examples rather than numeric examples were used. Only generic keywords were used and, in place of visual mediators of specific or easily identifiable numbers, nonspecific visual mediators of odd and even were used or created. As with the previous category, the endorsement of a universal narrative was deductively substantiated, and the substantiation revealed realizations of how the generic structures of the even or odd addends combine to make an odd or even outcome. The inclusion of conditional "if-then" statements suggested that the example was being used as a hypothetical situation with particular conditions and logical consequences. Further discursive markers of genericity in example usage (also in keeping with the previous category) included: (i) the use of words such as "like" or "for example"; (ii) the use of general rather than specific determiners; (iii) narratives in future tense rather than present tense; and (iv) the use of second-person pronouns ("you") or plural first-person pronouns ("we") rather than first-person pronouns ("I"). However, the absence of any numeric keywords or specific visual mediators in favour of generic keywords and nonspecific visual mediators in these substantiation routines signalled a more abstract realization of the mathematical situation than in the previous category.

Episode 4 features the same two students (Jane and Zara) who featured in the previous category, but this time they are using generic talk about nonspecific examples with deductive substantiations for endorsing the universal narrative "odd + odd = even".

Episode 4

Generic Talk With Nonspecific Examples That Are Used Deductively to Endorse "Odd + Odd = Even".

Turn	Speaker	What was said	What was done
341.	ZARA:	Yes, so if you have something like a square. If you have something like this.	Draws an oblong rectangle.
342.	JANE:	A rectangle.	
343.	ZARA:	Yes, it's an oblong. So, if you have like two circles on each it will be even.	Draws two circles in the rectangle.
		And just keep on going down.	Draws two lines going down from each of the circles.
		But if you added on an extra one here, then it wouldn't be even.	Draws the extra circle (bottom-right).
…	…	…	…
347.	ZARA:	So, if you put like another one ["one" is taken here to mean another "odd"] there [referring to her drawing] then it would be even.	
…	…	…	…
356.	T.R.:	Jane was going to add onto that to show how when we add on another odd number, it doesn't matter what it is…	Jane draws one extra purple circle in a square to complete the rectangle.
357.	ZARA:	It's even.	Jane continues to add pairs of purple circles in squares to extend the figure.

Unlike the previous category, in this episode the students use no numeric keywords, nor do they signify specific numbers with their use of visual mediators. Instead, Zara draws a rectangle signifying her realization of symmetry in a nonspecific even [343]. Zara's use of the word "square" and "oblong" [341; 343] and Jane's use of the

keyword "rectangle" [342] refer to the generic symmetrical structure of "even", evident in Zara's drawing. These keywords and visual mediators are consistent with generic example-use.

To emphasise evenness, Zara draws a pair of dots inside the rectangle and draws lines from each of the dots that "just keep on going down" [343]. While her drawing resembles the Numicon tiles (which signify specific numbers) that they had worked with previously, it also implies that the even number continues indefinitely, signifies *any* even, and highlights the "multiple of two" property in *any* even number. Zara then adds "an extra one" [343], which makes her drawing asymmetrical and therefore a realization of "*not* even." Thus, Zara's realization of a nonspecific odd embodies the same realizations of her former visually mediated nonspecific even: it implies that the odd number continues indefinitely, signifies *any* odd, and highlights the "multiple of two plus one" property in *any* odd number. With the Teacher-Researcher's prompt, Jane then adds to Zara's drawing to visually mediate Zara's substantiation. She initially draws one extra circle and then continues to add pairs of circles for the second generic odd. The visual mediation she provides is less abstract than Zara's: where Zara's two lines extending from just two circles signify a nonspecific even [343], Jane's repetition of "plus two circles" produces realizations of identifiable numeric odds (1,3,5,7,9…); although, it still gives a sense that the odds are extending indefinitely.

Zara's substantiation to endorse "odd + odd = even" deductively uses realizations of how the generic structures of the two odd addends combine to make an even outcome [347]. Note, as with the previous category, her use of "if…then…so" conditional statements and her use of "would" implies an imaginary or hypothetical situation that is dependent on these particular conditions. She starts by establishing the generic structure of even:

> *"<u>if</u> you have something like a square"* [341], and *"<u>if</u> you have like two circles on each <u>[then]</u> it <u>will be</u> even."* [343].

Then she establishes the generic structure of odd:

> *"<u>if</u> you added on an extra one here, <u>then</u> it <u>wouldn't be</u> even."* [343].

Finally, Zara uses these descriptions of evenness and oddness to show how the structures connect to form a generically even (symmetrical) structure:

"So, if you put like another one there then it would be even." [347].

Note, also, Zara's use of generic determiner "another one." Had Zara used numeric nouns (e.g., "if you put like five there"), demonstrative pronouns (e.g., "if you put this odd there"), or even if she had used a specific pronoun (e.g., "if you put it there"), the statement would have implied that the identity of the second odd she was referring to was known. Instead, her use of the indefinite article "another" implies the identity of the second odd addend (just like the first odd addend) is unknown—it is a generic odd. Furthermore, as with the previous category, Zara does not use the first-person personal pronoun "I" in her substantiating narratives, but consistently uses the second-person personal pronoun "you." These discursive markers signal that she is not the actor in her narratives—the people she is communicating with are—and suggests that she is not in the process of ascertaining a personal realization but intends to persuade others of the realization she has already ascertained.

The combination of all these discursive markers show Zara and, to a lesser degree, Jane are using nonspecific talk regarding the generic structure (i.e., symmetrical or asymmetrical) of even and odd to substantiate endorsing the universal statement "odd + odd = even." Their realizations are not dependent on numeric examples; they are abstract realizations. These discursive markers indicate that Jane and Zara are seeing the general in their nonspecific, and more abstract, examples.

4.4.5 Summary of Findings

In the above sections, I provided several subtle discursive markers that *implicitly* pointed towards generic, versus empiric, example-use. These features, which are also shown visually in Figure 7, are:

(i) Changes in keywords and visual mediators (from numeric to generic).

(ii) A switch in the mathematical object of focus in substantiating narratives—from the number generated in sums to the (symmetrical or asymmetrical) structure of the sum, to the structure of the addends within the sum.

(iii) A change from inductive substantiations to deductive substantiations, including the use of conditional statements ("if…then," "so").

(iv) Switching, first, from active voice using first-person personal pronoun ("I") to active voice using first-personal plural ("we") or second-person singular

("you") and, second, from active voice to passive and omitting personal pronouns altogether (alienation).

(v) A switch in tense (from present to future).

(vi) Changes in the use of determiners from the use of specific nouns or definite articles ("the") and demonstrative pronouns ("this," "that," "which") to the use of indefinite articles ("a," "an," "another") and quantifiers that refer to the whole extent of the particular group or situation in focus (e.g., "all," "any," "every").

(vii) The use of words such as "like" or "for example" to indicate the purpose of the example is to illustrate the general in the particular.

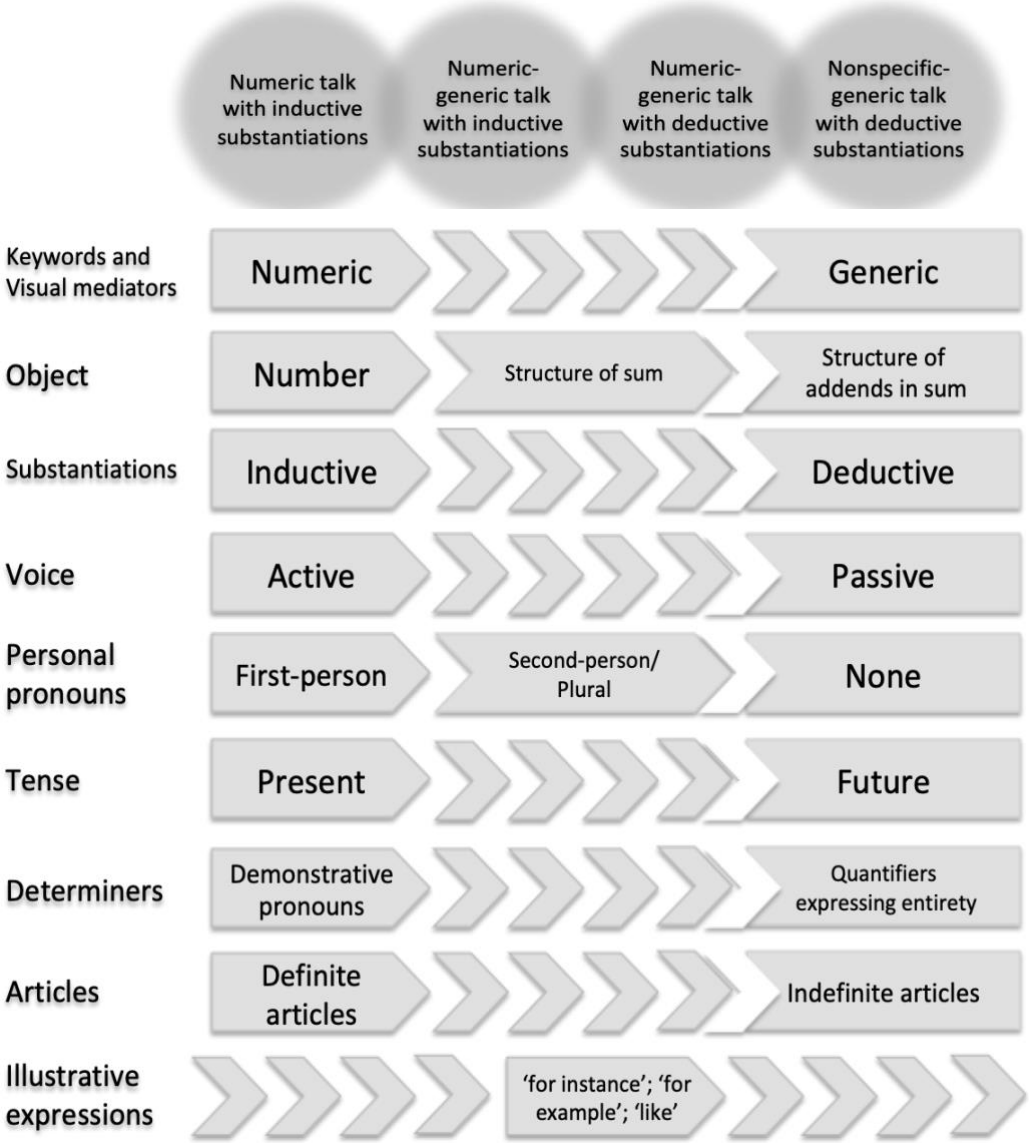
These discursive features helped in identifying and describing four different categories of example-use: (i) numeric talk with inductive substantiations; (ii) hybrid numeric-generic talk with inductive substantiations; (iii) hybrid numeric-generic talk with deductive substantiations; and (iv) nonspecific-generic talk with deductive substantiations. The first of my categories—numeric talk with inductive substantiations—aligns with traditional views of empirical example-use. In this category, learners used numeric examples, and the specific sums formed the object of the argument from which their endorsement of a universal narrative was inductively substantiated. Whereas there were no discursive markers to suggest that a learner was seeing the general in the particular in the first category of example-use, degrees of genericity were visible within all the other three categories. With the second category—hybrid numeric-generic talk with inductive substantiations—learners revealed realizations of evenness and oddness according to generic (symmetric or asymmetric) structure, but these realizations did not extend to how odd and even addends together form a sum's generic structure. Instead, learners inductively used repetitive observations of confirming results (e.g., repeated observations of an asymmetric structure occurring from "even + odd") to substantiate their endorsement of a universal narrative.

In keeping with the second category, learners in the third and fourth categories—hybrid numeric-generic talk with deductive substantiations, and nonspecific-generic talk with deductive substantiations—also revealed realizations of evenness and

oddness according to a generic (symmetric or asymmetric) structure. However, in endorsing universal narratives, they deductively substantiated how the generic structure of even or odd addends combine to make the generic structure of the sum, rather than inductively using repeated observations of confirming results. The third and fourth categories differ, though, in the degree of abstraction evident in example-use. While in the third "numeric-generic" category, numeric examples were used to illuminate mathematically structural features (i.e., symmetrical or asymmetrical), representative of the classes of even and odd, students in the fourth "nonspecific-generic" category were not dependent on numeric examples—they used nonspecific and therefore more abstract examples. The categories are intentionally presented as blurred and overlapping circles in Figure 7 to signal that they are not considered to be distinct categories with rigid boundaries. The categories represent a connected continuum from empirical example-use to generic example-use. Equally, the alignment of the discursive features with these categories is suggestive only to convey the idea that, as the categories move from empirical to more generic (from left to right), the features tend to change in the same way (from left to right).

Figure 7

Categories of Example-Use and Changes in Discursive Markers From Particularity to Genericity



4.5 Discussion

In this chapter, I used Sfard's (2008) commognitive framework to study students' use of examples when endorsing universal narratives about the sums of odds and evens. My goal was to identify discursive markers that might signal when examples were being used generically rather than empirically. My interest stemmed from previous studies pointing to the value of generic examples for learners (e.g., Balacheff, 1988; Kempen & Biehler, 2019; Malek & Movshovitz-Hadar, 2011; Mason & Pimm, 1984;

Reid & Vallejo Vargas, 2018; Rowland, 1988; Russell et al., 2011; Stylianides & Ball, 2008), but the difficulty in determining whether learners were seeing the general or the particular in examples used (Mason & Pimm, 1984; Reid & Vallejo Vargas, 2018; Yopp & Ely, 2016).

Previous research had pointed to the difficulty of knowing whether an author was aware of the general intent in an example (e.g., Reid & Vallejo Varga, 2018; Yopp & Ely, 2015), with Mason and Pimm (1984) claiming it was "impossible to tell" (p. 287). And for those who sought evidence of an awareness of generality in their example-use (e.g., Kempen & Biehler, 2019; Reid & Vallejo Vargas, 2018; Yopp & Ely, 2015), an *explicit* means of justifying why the exemplified situation would always hold for any case is required. Obtaining such explicit articulation, however, is not always possible nor appropriate (particularly with primary school students). However, utilising the commognitive framework in fine-grained discursive analysis of my student data provided a means of uncovering several subtle discursive markers that *implicitly* pointed towards generic, versus empiric, example-use (as shown in Figure 7).

Previously, example-use had been dichotomised as either empiric or generic (and non-proof or proof), but through identifying four categories of example-use, my findings have shown the occurrence of example-use to be more nuanced and multi-layered than previously thought. Although it is only with the third and fourth categories that I claimed learners were seeing the general in their particular example-use with regard endorsing the universal narratives in question, the second category fell somewhere in between what has previously been considered *either* empirical *or* generic example-use. In this category, learners were better placed than those in the first "numeric realization" category to potentially see the general in the particular because they had realized the generic structure (symmetric or asymmetric) of even and odd. Furthermore, where generic example-use had been viewed as a single category that served as a precursor to deductive proof expressed more formally, my findings showed two distinct categories of generic example-use distinguishable by the level of abstraction: "numeric-generic example-use" and "nonspecific-generic example-use."

As is suggested by the blurred and overlapping categories in Figure 7, it is quite possible that students' routine ways of using examples may fall somewhere between

two categories. For instance, a student may use a nonspecific visual mediator to signify an odd addend with a specific visual mediator to signify an even addend, thereby presenting hybrid "numeric-nonspecific-generic talk." It is also important to note that a learner's routine way of using examples may flux between categories. For the same reason, it is important to note that I am not claiming all the discursive features described within these categories will necessarily be, or need to be, present in learner's narratives to categorise learner's example-use as fitting within one of the categories. For instance, a student could use the active voice and first-person pronouns (rather than passive voice and no personal pronouns), and, if other discursive markers of genericity were evident, their use of examples might still fall within the "nonspecific-generic talk and deductive substantiations" category. Nor is it likely that one discursive feature will be sufficient to suggest that a learner is using examples generically. Hence, the intention behind observing discursive markers for distinguishing genericity from particularity in example-use is not so that they may be used as an exclusive "must-have" tick list, but for them to be used discerningly to strengthen one's conviction that a learner may be realizing generality in the example they use. Moreover, it is certainly not my intention to suggest that these discursive features should be somehow taught to students to convey they are seeing the general in the particular—they are features that researchers and educators may look for during lessons, or reflectively afterwards, as indicators of generic example-use.

Critics may argue that none of the students' reasoning illustrated within the categories of example-use in this study fully met the criteria of a generic example proof. Where I have claimed that students are seeing the general in their particular example-use (i.e., episodes presented for the third and fourth categories), G. J. Stylianides (2008), for example, would classify their reasoning as merely *rationales* (non-proof) on the grounds that these learners failed to explicitly refer to some key truths (e.g., definitions of odd and even). However, in this chapter, illustrating episodes of student reasoning that satisfied criteria of "proofiness" (i.e., demonstrating qualities of a proof such as rigour) was put aside. I aimed to uncover and illustrate ways in which educators and researchers may ascertain whether learners are referring to the general in the examples they use in their reasoning. Thus, where Reid and Vallejo Vargas (2018) and Yopp and Ely (2015) focused on students' finished products—their written proofs— for evidence that a student, as

the author of the argument, understood why the example would hold for all cases, I used students' inchoate reasoning captured in-the-moment. I looked at discourse among students, not at a particular individual student's products. Rather than looking for explicit signs of genericity in completed proofs, I was able to look for students' intentions when using examples by examining their routine ways of using examples in their substantiation narratives for more subtle signs of genericity. By doing so, I stepped away from the absolutist perspective shared by Movshovitz-Hadar (1988) and Leron and Zaslavsky (2013), where rigour is required as a criterion of proof, and adopted a subjectivist perspective.

Supporting learners at all levels of education to move beyond example-based arguments and to develop valid arguments has been recognised as an obstacle in moving from inductive to deductive reasoning (e.g., Harel, 2001; Fiallo & Gutiérrez, 2017; Hodds et al., 2014; Stylianides & Stylianides, 2009). And, supporting learners to endorse generality in a particular example has been seen as part of the issue. For instance, Mason and Pimm (1984) noted that although diagrams were a useful link, "transitions to a representation of every even number, in the form of 2n is not easy" (p. 283). However, by focusing on discursive markers of generic talk and deductive substantiations (rather than a finished written proof), the findings in this study suggest that generic reasoning per se may not be so challenging and rare even for primary school students. What may be the bigger hurdle is supporting students to see the value in, and adhere to, the fussiness and rigour demanded in formal proofs.

The present study is limited in its focus on universal narratives about the sums of odds and evens. The particular nature of odds and evens is accessible for primary school students, and these mathematical objects have useful visual accompaniments. Hence, the ability to capture students' use of generic examples and deductive reasoning in this context does not at all imply that the same students would do so in other contexts. Equally, the four categories that emerged in this study correlate with the specific task situation and are not necessarily transferrable to other task situations. However, by considering discourse as learning, and by capturing students' use of examples discursively, the study has brought otherwise unseen features to the fore. The discursive features I have provided are markers that could potentially be used to point to an awareness of generality in different task situations among

learners of different levels. Further work is required, and future studies would need to explore the way these discursive markers are present in learners' reasoning in different contexts and how they point to learners demonstrating and reasoning generality.

CHAPTER 5: Research Study 2:
Using a Grove of Realizations to Reveal Communicational Breaches

5.1 Introduction

What constitutes proof and proving at the primary level is not entirely clear. A. J. Stylianides (2007) defined a proof as an argument that uses and communicates reasoning in ways that are both endorsable by the wider mathematical community and within reach of (thus accepted by) the classroom community. As such, empirical arguments fail to qualify as proofs, but deductive arguments can qualify. Nevertheless, even with a working definition of primary-level proofs, there is little research that explores *where* primary school students' arguments stem from and *how* their arguments develop and become accepted within the classroom community.

Sfard (2008) described junctions such as the empirical-deductive one as incommensurable since they require a change in meta-level rules. Sfard suggests that the construct of a commognitive conflict may be useful for making sense of students' communication at these junctions. Accordingly, in this chapter I utilise Sfard's (2008) commognitive framework to provide insights into how young students' arguments unfold as the students substantiate (verify with evidence) their classifications of numbers as even or odd, and endorse or reject universal narratives about odd and even numbers. A new methodological tool called the *Grove of Realizations* is introduced for visually mapping the students' realizations of odd and even numbers and tracing disagreements within a group. In this way, the study in this chapter addresses the second research aim of this thesis and seeks *to examine possible sources of commognitive conflicts in students' transition from empirical-deductive discourses.*

5.2 The Commognitive Framework

As mathematical learning takes place, learners gradually begin to talk about mathematical objects as existing by themselves—a process called *objectification* (Sfard, 2008). The mathematical objects of interest in this study are odd and even numbers. From the commognitive standpoint, the words "odd" and "even" are

signifiers that operate as nouns, which are realized as "perceptually accessible objects" (Sfard, 2008, p. 154).

From the commognitive viewpoint, learners initially are expected to use the words "odd" and "even" as adjectives describing other perceptually accessible objects (e.g., numbers or arrangements of dots) rather than as nouns that signify abstract objects in their own right. These signifiers (odd and even) may be realized in a multitude of ways: infinitely many numbers (e.g., "odd" could be "one," "seventeen," "one billion and one" and "even" could be "two," "forty-six," "three million and eight") and each of these numbers could be realized as numerals (e.g., 1, 17; 2, 46) or icons (e.g., arrangements of dots). A person's realizations of odd or even may be observable in the ways in which they use these words, their visual mediators for odd and even, the narratives they endorse and generate regarding oddness and evenness, and their routines for substantiating universal narratives regarding odd and even numbers. For instance, learners who realize odd numbers as "numbers ending in 1, 3, 5, 7, 9" will endorse the narrative "27 is odd because it ends in seven" and will likely perform a "last digit check" procedure.

Initially, learners may have limited realizations for odd and even numbers, which will be observable via their narratives and routines. For example, a learner may realize even numbers as being "divisible by two" but may not yet realize that even numbers "end in the digit 0, 2, 4, 6, 8." This situation may be observed in a learner's struggle to endorse large numbers as even, if dividing larger numbers by two is beyond their capability. For more advanced learners, these two content-specific realizations (or definitions) of even (e.g., the divisibility of two and ending in the digits 0, 2, 4, 6, 8) would co-exist. Evidence of an expansion of realizations of odd and even numbers for a learner is shown via their endorsement of new narratives about these objects and modifications in their routine ways of substantiating narratives about them. Observing an expansion of realizations for an individual signals learning at the object-level. Since object-level learning is a matter of growing new realizations for mathematical objects, Sfard (2019) claimed that "in principle" this type of learning could be achieved via intrapersonal communication (thinking) (p. 92); that is to say, interpersonal conversation (verbal communication with a more knowledgeable) is optional rather than necessary.

Learning can also take place at the meta-level; learners may eventually use the words "odd" and "even" as signifying general terms or mathematical objects in their own right (i.e., as algebraic expressions $2n+1$ and $2n$). This process is called objectification, which involves both *reification*—replacing talk about processes with talk about objects—and *alienation*—presenting phenomena impersonally, as if they were occurring independently of human participation (Sfard, 2008, p. 44). For example, in the utterance, "twelve is even because you can share it equally between two people," an activity (sharing) is indicated, an actor ("you") performs the activity, and the word "even" acts as an adjective describing the number twelve. Whereas in the sentence "even is 2n", "even" has been objectified, there is an absence of actors, and the word "even" is used as a noun that encapsulates all even numbers and realizations of even into one set, giving it separation from any activity and more permanence. Objectification is necessary for a deductive discourse on odd and even numbers as it uses properties (or definitions) of odd and even as premises from which to draw general conclusions. In comparison, an empirical discourse on odd and even numbers inductively uses the sums of particular numeric examples (such as $3 + 5 = 8$). The deductive discourse is incommensurable with the empirical discourse because in deciding whether a universal statement (e.g., "odd + odd = even") should be endorsed, the deductive substantiation routine uses different criteria to an empirical substantiation routine. Shifting from empirical to deductive substantiations requires a change in meta-level rules, and discursive evidence of this thereby signals meta-level learning. Hence, where the expansion of realizations and endorsed narratives signals learning at the object-level, evidence of objectification denotes the development of discourse at the meta-level.

According to the commognitive framework, development occurs through the learner's exposure to, and participation in, the discourse he or she is supposed to individualise and the support he or she receives from other participants. Where interpersonal talk is considered optional for object-level learning, it is viewed as being "indispensable" for meta-level learning to occur (Sfard, 2019, p. 92). A learner adopting the new incommensurable discourse would need to revise narratives they have previously endorsed and the *way* in which they arrive at endorsement. Commognitive conflicts—communicative interaction whereby participants use the same signifiers (words or symbols) in different ways or perform the same

mathematical tasks according to different rules— (Sfard, 2008) are seen as playing an essential role in meta-level learning, as they provide space for participants to consider new ways of talking. Sfard (2008) suggests that resolving a commognitive conflict involves one of the interlocutors to gradually accept and adopt the incommensurable discourse of the other and abandon his or her own.

With these theoretical tenets in mind, I explore *communicational breaches* that occur at the junction between empirical and deductive discourses in the context of odd and even numbers. I consider communicational breaches to include object-level gaps (regarding odd and even numbers) and meta-level or commognitive conflicts (regarding empirical and deductive proofs). Accordingly, this study aims to answer the following research question:

> What are the sources of communicational conflict within the group with regards to their talk about odd and even numbers and their substantiating universal narratives involving the sums of odds and evens?

5.3 Conduct of the Study

In this section, I describe how the study was conducted in order to answer the research question. I provide a brief overview of the research methods—participants, the task situations, and task implementation (for more detail refer to Chapter 3, Section 3.3). I then provide a detailed account of how the data analysis was conducted. Here I introduce the new methodological tool—the Grove of Realizations.

5.3.1 Data Collection

For this study, the data comes from one of the seven student groups involved in my research. The students—Danny, Jane, Zara, and Robert—were all eight years old at the time and came from the same class. First, they took turns to turn over cards revealing numbers, which they then classified as odd and even by placing them into the respectively named boxes. Each time they turned over a card, I asked the students to substantiate their classifications and to consider one another's questions and thinking. The first three cards (representing numbers six, nine, and four) had Numicon tiles printed on the reverse and were cut in the shape of the corresponding Numicon tile. The next cards were rectangular and featured a numeral within the 1–10,000 range. Following the classification task, the students were presented with

"odd + odd" concept cartoon (Chapter 3, Figure 4a) and then the "even + odd" concept cartoon (Chapter 3, Figure 4b). They were encouraged to share which narratives they endorsed and to share their substantiations for endorsing these narratives in order to convince one another. To support their communication, students had access to pens and paper, counters, and Numicon tiles (see Chapter 3, Figure 3: *Numicon tiles*).

5.3.2 Data Analysis

The purpose of the data analysis presented in this chapter was to expose each student's pre-existing realizations (i.e., what they already knew about odd and even before any learning took place) regarding odd and even numbers to establish a baseline discourse for each student, and then examine the similarities and differences between their individual discourses as potential sources of commognitive conflict amongst the group. I utilised Sfard's (2008) commognitive framework to look for routines (well-defined, repetitive patterns) in the students' discourse regarding their uses of the words "odd" and "even", their substantiating narratives about oddness and evenness, and their substantiating procedures for endorsing universal narratives regarding the sums of odds and evens.

The dialogue is presented in its raw form alongside each speaker's name. The turn number is also included to provide the reader with an idea of turns elapsing within and between each episode and to enable me to refer to key turns within my analysis. Where relevant, the column to the right of the dialogue includes participants' actions as they speak and/or information about how the utterances were spoken and to whom they were directed. Following each episode of dialogue, I provide interpretative elaborations of what is occurring in the dialogue.

5.3.3 The Grove of Realizations Tool

The Grove of Realizations is the methodological tool I introduce in this chapter to examine the similarities and differences between participants' discourses and tracking their discursive developments. I took inspiration from Sfard's (2008) "realization trees" (p. 165), which she used to show the hierarchical nature of realizations. Sfard (2008) showed the signifier at the root of the tree branching into several realizations which, in turn, were signifiers of further realizations. Sfard contended that researchers could use realization trees to map an individual learner's

realizations to assess the quality of their discourse (e.g., by the richness and depth to which the branches extended) and to track the development of that individual's discourse over time.

Realization trees have since been used and adapted by researchers with various goals. Using realization trees as visual means to show the complexity of functions, Nachleili and Tabach (2012) showed how students initially used each of the object realizations separately, before saming them as signifying a single object. Weingarden et al. (2019) created a Realization Tree Assessment tool (RTA) to assess explorative participation in mathematics lessons on a large scale. Their RTA tool provides a visual means to tell a story about the mathematics in lessons (the objects explicitly discussed, and the links created between them) and the nature of participation (who authored the links—the teacher or students). Each version of the tool, however, has its own merits and limitations. For instance, Weingarden et al. (2019) acknowledged that the RTA tool did not capture how many students participated in a lesson, meaning that a lesson where many students had participated and made links between mathematical objects would look the same as one where one student had done all the work for the class. This was a concession deemed necessary by the authors to achieve their goal of creating an efficient assessment tool that produced a simple and clear image.

Since my research goal was to assess the effectiveness of interpersonal communication within a group of learners, I perceived the limitation of the RTA tool to be an area that I would need to address. I needed to adapt the use of realization trees to account for how the same signifier might be realized in different ways by different interlocutors. Sfard (2008) pointed to the possibility of being able to map students' realizations according to her own personal tree of realizations (p. 167) to address this same goal; that is, individual realization trees could be produced for each individual in comparison to the researcher's own realization tree. However, I wanted a means to show dominant realizations, consistencies and inconsistencies within a group, and the course of development for different group members within one collective visual.

With the above goals in mind, I developed the Grove of Realizations tool (Figure 8), which was attuned to the mathematical content of the group discussion so that each

participant's "tree" existed, and could be identified, within the group's grove of realizations. My goals in this section of the chapter are to describe how this tool is constructed and to highlight its affordances for: (i) providing a means of visually representing participants' realizations ;(ii) contrasting participants' realizations and tracing commognitive conflicts within a group; and (iii) for tracking the development of participants' discourses within group work.

5.3.3.1 An In-Depth Exemplification of the Grove of Realizations Coding. In what follows, I detail the coding of the Grove of Realizations tool for the focus group with regard to the students' responses when substantiating their classifications of the first nine cards (which were presented as Numicon tiles or numerals) as odd or even. I provide a summary of the students' substantiations for their classifications in Table 3 and show how these informed the initial stages of constructing the realization tree in Figure 8.

Table 3

Summary of Student Substantiations for Classifying Numbers as Odd or Even

Turn	*Speaker*	*What was said*
		Classifying Numicon 6
31.	JANE:	*Because three plus three equals six and that's even.*
		Classifying Numicon 9
43.	ZARA:	*It's got a one there. [Pointing to the single one at the top of the Numicon piece].*
45.	DANNY:	*So, even are always first, 'cos zero, two, four, six, eight, ten. And then the odd numbers are starting from one—one, three, five, seven, nine, eleven. It goes like that. So that nine is odd.*
		Classifying Numicon 4
55.	ROBERT:	*Because, two and two.*
69	DANNY:	*Yeah. It's just the sequence. Same as the Fibonacci sequence and the other sequences.*
73.	ZARA:	*If you had two people then you'd be able, they'd both get two each.*
		Classifying numeral 5
88.	JANE:	*Because a four and one.*
90.	JANE:	*Because the four is even, but five has like…*
91.	ZARA:	*Instead of adding two on, you add on one and then it wouldn't be even.*
94.	JANE:	*So, two, two and one.*
95.	ZARA:	*Yeah. So, so every time you have an even number it has to be in twos.*
100.	DANNY:	*So, everything is like even-odd, even-odd, then even, then odd, so that's odd. So, it's like five even or odds and it goes like in a pattern.*
		Classifying numeral 22
119.	JANE:	*Because eleven and eleven.*
120.	DANNY:	*All you need to know is like if it ends with a… if… This is like a simple way- if it ends with a zero, two, four, six or eight it is an even number…*
122.	DANNY:	*And if it's [ends in] one, three, five, seven, nine it's odd.*
		Classifying numeral 98
140.	DANNY:	*It ends with eight.*
148.	DANNY:	*The last digit matters.*
		Classifying numeral 53
160.	DANNY:	*It ends with a three.*
172.	DANNY:	*Cos the last digit is three.*
		Classifying numeral 173
181.	DANNY:	*The last digit matters.*
		Classifying numeral 724
208.	DANNY:	*The four is in the sequence of evens.*

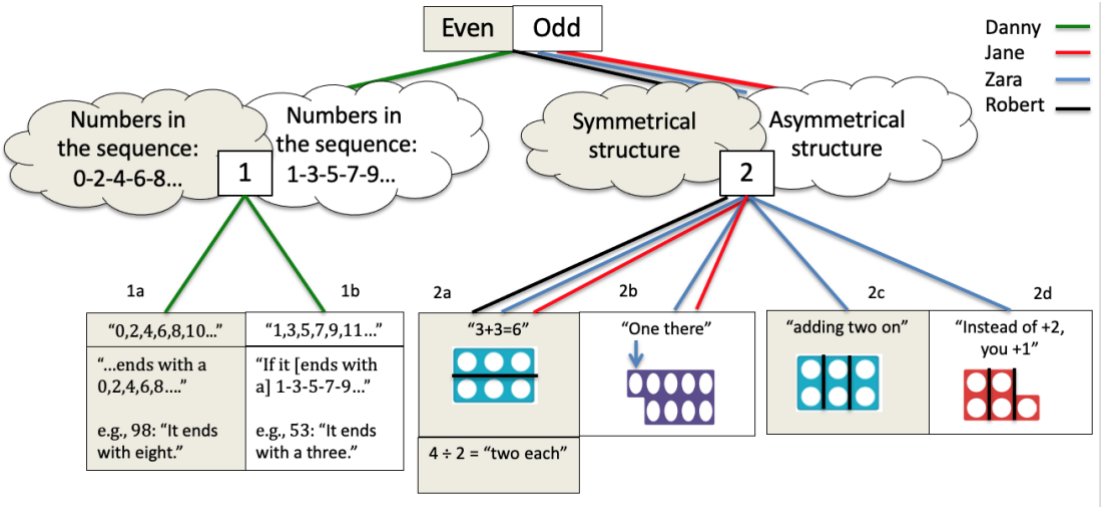
(i) Constructing the Root Node. Deciding what should be the root node was a primary decision in constructing the Grove of Realizations tool. As explained in the Section 5.2, an actual object does not exist because odd and even numbers are discursive objects, and so the decision should be made about what best and most conveniently signifies the object to be samed. Accordingly, I decided to simply use the words "odd" and "even" as the root node since the students would be responding to the words in the task situations.

(ii) Constructing the Realization Boxes. Although I could have constructed a personal tree using my personal realizations of odd and even as a base tree from which to record the students' realizations (Sfard's (2008) method), I looked for realizations made observable in students' discourses. For this, I focused on each student's participation, as they classified numbers as odd or even and substantiated their endorsement of narratives about the sums of odds and evens (when engaging in the concept cartoon tasks), to establish their pre-existing realizations for odd and even. This decision not only helped to reduce infinite realizations of odd and even to just the ones that were observed during the group session, but it also meant that the realizations within the grove grew organically from the data, rather than being pre-determined by an outsider (me) before the event. As and when a student articulated a new realization, I represented this in the grove via the addition of a realization box using, where appropriate, the student's narratives (presented in speech marks) and the visual tools they used.

(iii) Constructing the Branches. The ability to clearly represent which student articulated which realizations was a necessary feature within the tool and so, as Figure 8 shows, I used a different colour for each student within the group to form student-specific branches within the grove. By examining each student's use of the words "odd" and "even" according to the narratives they used and endorsed, the way they visually mediated odd and even, and their substantiating routines, I was able to establish each student's tree (the total number of observable realizations connected by student-specific branches) of realizations within the grove.

Figure 8

The Grove of Realizations Tool



(iv) Mapping the Discursive Terrain. I now discuss how I mapped the Grove of Realizations (Figure 8) for the focus student group according to the students' responses (Table 3) when classifying the cards.

The first realization box created in the grove came from Jane [31], who signalled symmetry in *six*— "three plus three"—to substantiate endorsing it as even. This realization is represented in box 2a on the tool with a red branch connecting to it to show it was Jane's realization. Similarly, Robert's substantiation for *four* [55] was of a similar nature to Jane's substantiation in that he also signalled the symmetry of "two plus two" as grounds to endorse four as even. However, to simplify the visual tool, I collapsed realizations of the same ilk into one *samed* realization box. Hence, although his utterance "two plus two" does not appear on the tool, it is implied that the realizations expressed in the boxes hold for other numbers, and so his branch (black) connects to box 2a because it is of the same type—they are both in the form of $n + n$ or $2n$ (although they were never expressed in this algebraic way).

Zara also shares this realization within the grove (her blue branch connects with box 2a) on account of her "fair shares" substantiation narrative [73]. Note though that Zara's realization was expressed differently—as a division (4 ÷ 2) rather than as a double (2 + 2)— but it was still considered to belong to the same category (i.e., in the form of $2n$) as Jane and Robert's. Hence, it is represented as an extension to this realization box on the tool. Other narratives endorsing, say, six as even, which would

have been considered as articulating the same realization (and thus would have been represented as further extensions to box 2a) include: *2 x 3*; double 3; two equal shares of three; having no remainder when shared between two; and a visual mediation of six objects split into two groups of three (which could have included other numbers too, not just six). Zara also articulated a different realization of even [91; 95], when she substantiated even numbers as being "in twos" [95] or "adding two on" [91]. This realization was considered to be different to those represented in box 2a because it was not in the form *2n* (e.g., where *3 + 3* or *2 x 3* substantiates six as even) but was more in the form *n*2* (e.g., where *2+2+2* or *3 x 2* substantiates six as even), and so it is represented in a new box (2a(i)), and Zara's branches connect with both boxes, 2a and 2a(i), for even.

With regard to odd numbers, for Zara these were not *fair shares* on account of (i) the extra "one there" [43], and (ii) they were grouped in twos but "you add on one" [91]; correspondingly, her branches connect with boxes 2b and 2b(i) for odd. It is unclear whether Jane's utterance "two, two and one" [94], when substantiating five as odd, refers to a realization of numbers that are formed from *two equivalent numbers plus one* (thus in the form *2n + 1*) or *repetitions of two plus one* (and in the form *n*2 + 1*). Had she been classifying a different odd number, for example *seven*, her realization would have been more apparent (e.g., Jane's would have either substantiated its oddness with *3+3+1* or *2+2+2+1*). Given Jane's tendency to use doubling narratives [31; 119] and her observation of five being a "four and one" [88], it seems more likely that her realization is of the form *2n +1,* so her branch connects with box 2b.

Realizations of even and odd represented in boxes 2a, 2b, 2a(i) and 2b(i) were all considered to be from the same *family realization* because they were all about the *structure* (the symmetry or asymmetry) of numbers. Therefore, these realizations all branch from the main *structure-based* family (box 2) of realizations for odd and even. Unlike the students' realizations, which are presented in boxes with speech marks because they are taken directly from the student dialogue, the main structure-based and sequence-based family realizations are my contributions to the grove. To distinguish my contributions from the students' articulated realizations, I present these in thought bubbles on the tool. In contrast, Danny's realizations for odd and even do not stem from this family of realizations. For Danny, odd and even was about

a number's place in a sequence of even-odd-even-odd consecutive numbers [45], so his branches (shown in green) stem from a *sequence-based* family of realizations (box 1) to realizations represented in boxes 1a and 1b. When classifying numbers with more than one digit, Danny elaborated his sequence-based substantiation. Since counting would be an inefficient way to check where larger numbers lie in an even-odd-even-odd sequence, Danny added a *check the last digit* procedure [120] to his original *check for membership in either sequence* procedure [45]. Note that I am not claiming here that the change in task situation resulted in new learning for Danny; it simply prompted Danny to elaborate his substantiation to account for how he was able to immediately classify larger numbers as odd or even. This is reflected in extensions to boxes 1a and 1b.

5.3.3.2 The Affordances of the Tool for This Chapter. For the study in this chapter, I continue the discursive analysis of the focus group, including furthering the creation of the grove of their pre-existing realizations to fulfil three goals. First, I aim to map the extent of an individual's pre-existing realizations for odd and even to produce their personal tree within the grove—the number of boxes with which branches of their given colour connected. For instance, in Figure 8, so far, Zara's tree of realizations extends to boxes 2a, 2b, 2a(i) and 2b(i). Second, I aim to visually represent realizations that are shared by more than one group members and realizations in which students' discourses differ from one another. Specifically, "thicker" branches, made from several colours, connecting with a box will indicate several members within the group share particular types of realizations. For instance, so far, three students (Jane, Zara, and Robert) share the realization represented in box 2a. Whereas a sole branch attached to a realization box indicates that an individual student held a realization that was different from the rest of the group. For instance, only Danny has realizations stemming from box 1, and, so far, only Zara has realizations represented in boxes 2b and 2a(i). Visually representing such discursive differences may help to explain breaches in interpersonal communication. Third, the analysis enables me to exemplify the affordances of the Grove of Realizations tool—regarding the scope of each student's realization tree and the consistencies and inconsistencies among the group's realizations—in real-time.

5.3.3.3 The Affordances of the Tool for Studies in Chapters 6 and 7. Since the group of students in this study continue to be students of focus in the studies in Chapters 6 and 7, their grove will provide a baseline in the form of a visual story regarding their realizations from which to compare their discursive developments (as will be shown in Chapters 6 and 7). Once the scope of the students' pre-existing discourses is established in this chapter, it is then possible to visually represent when learning takes place within a lesson by constructing an adapted grove showing growth in individual students' trees of realizations. The new realizations would become observable in the data by a participant endorsing a narrative that they had previously rejected or adopting a new narrative for the first time. The tool also has the means to show when divergent discourses begin to converge, and thus when groups are resolving communicational breaches. The resolution of a communicational breach would be shown on the tool as branches, which were previously more singular and spread out, converging and becoming thicker (multiple-coloured branches for multiple participants) for a realization. Finally, a comparison of *before* and *after* realization groves (i.e., comparing the realization grove established for students' pre-existing discourses with the realization grove at the end of the lesson) would provide a clear visual summary of the extent of learning within the group (via observing more realization boxes or more branches to boxes) and for whom this learning occurred (via observing the colours of branches of new growth).

5.4 Findings

The analysis of discursive data from the focus group revealed discursive gaps between Danny and the other students (Zara, Jane and Robert) both at the object-level and the meta-level. These two sources of interpersonal breaches in communication and the reasons behind their endurance are described separately.

5.4.1. Object-Level Differences in the Students' Realizations

5.4.1.1 Classifying Numbers as Odd or Even. Throughout the classification task, all four students correctly classified the numbers when it was their turn to place the cards on the boxes labelled "Even" and "Odd" and they agreed with the classification made by their group members. Indeed, as Episode 5 shows [27, 28, and 29 for Numicon 6 card; 53 and 55 for Numicon 4 card; and 86 for Numeral 5 card], if the

students had been asked to simply sort the cards into odd and even boxes with no justification, one might have prematurely assumed that they shared a mutual understanding about odd and even numbers. However, as has already been evidenced in Table 3 and visually presented in the students' grove of realizations (Figure 8), there were substantial differences in the students' articulated realizations of odd and even—Danny's were sequence-based, and those of the other students (Jane, Zara, and Robert) were structure-based. Of course, the absence of articulation of a realization does not necessarily denote the absence of such a realization for an individual. However, the following episode shows the first signs of Danny's rejection of the other students' structure-based substantiations of evenness and oddness. It thus indicates that his realizations of even and odd do not, in fact, include structure-based ones.

Episode 5

Danny's Rejection of Structure-Based Substantiations of Even and Odd

Turn	Speaker	What was said	What was done
		Classifying Numicon 6	
27.	JANE:	Six	
28.	ZARA:	Six. Even.	
29.	DANNY:	Six.	Reaches for it before the others have the chance to and puts it on the "Even" box.
30.	T.R.:	You think it's even. Why did that go in there?	
31.	JANE:	Because three plus three equals six and that's even.	
32.	DANNY:	What the heck! That's not right!	Gesturing towards Jane as a sign of disagreement.
…	…	…	…
34.	DANNY:	That's so not right… three plus three… how does the three come here?	
		Classifying Numicon 4	
52.	T.R.:	Do we all agree it's [Numicon 4] even?	
53.	ZARA, JANE, DANNY:	Yes, yeah.	
54.	T.R.:	Yes? Robert what do you think? Why is that one even?	
55.	ROBERT:	Because two and two.	
…	…	…	…
58.	DANNY:	But how did you get the two?	Screws his face up and shakes his head.
59.	ROBERT:	Mm… er, because two plus two.	
60.	DANNY:	How did you get the two?! Where's the two? How did you get the two?	Exasperation indicated in the tone of his voice.
		Classifying numeral 5	
86.	ZARA, JANE:	Odd.	Jane places the card on the odd box.
87.	T.R.:	How do you know it's odd?	
88.	JANE:	Because a four and one.	
89.	DANNY:	But how did you get the four and one?	Smiling.

Episode 5 shows the conflicting uses of the word "even" emerging within the group as soon as the very first card, showing Numicon 6, is turned over. Jane declares that Numicon 6 is even because, "three plus three, and that's even", showing that for her

the word even signifies a symmetrical amount or structure. The fact that the first number turned over is a Numicon tile in the form of a 2 x 3 array makes "double three" visibly salient in its structure, but Danny rejects (quite vehemently) this substantiation of evenness [32] and his comments [34; 58; 60; 89] suggest that he does not even see the "three plus three". Danny rejects the structure-based narratives, however, not because he does not "see" the symmetrical and asymmetrical configurations of the Numicon tiles (as will be evidenced in Episode 6), but because, for him, they are irrelevant with regard to his realization of evenness. Hence, his apparent bafflement regarding structure-based substantiation narratives can be attributed to his different realizations of even—they are sequence-based. For Danny, even and odd has nothing to do with the symmetry of a number's structure but is about a number's membership in an even-odd sequence.

What we see emerging between Danny and the other students (Jane, Zara, and Robert) is a breach in communication: the students are performing the same task according to different rules. However, in this particular situation, the conflict is not about which numbers are odd and even—this they agree on—it is about *how* the students substantiate a number's oddness and evenness. Furthermore, whereas a commognitive conflict is defined by Sfard (2008) as one that emerges at meta-level gaps in communication because the two discourses are incommensurable, this breach in communication is at the object-level. Here, the two specific ways of substantiating (structure-based and sequence-based) are not mathematically incommensurable since a consistent implementation of each of them yields the same conclusion. Hence, while each student's branches only extend to one of the families of realizations (Figure 8), for more advanced learners, these realizations would co-exist: the evenness of 18, for example, could be substantiated on account of it being divisible by 2 (structure-based realizations) *and* ending in 8 (sequence-based realization).

The origins of Danny's branches, stemming from the root nodes within the grove of combined realizations to the sequence-based family realization in Figure 8 [box 1], illustrate the object-level difference between Danny's realizations and those of the other group members. Jane, Zara, and Robert's realizations all stem from the structure-based family realization [box 2], and the dominance of structure-based

realizations of odd and even numbers within the group is visually illustrated by the thickness of this branch (Figure 8). Representing this discursive divide using the Grove of Realizations tool helps to visually expose the source of the ensuing communicational gap.

5.4.1.2 Danny Engages in Structure-Based Talk but Maintains Incommensurability Between the Modes of Talk. Despite Danny's rejection of structural-based substantiations of evenness and oddness, the following episode shows signs of Danny endorsing a structural difference between odd and even numbers (showing he can indeed see the difference). Yet, Danny maintains incommensurability between his sequence-based talk and the other students' structure-based talk.

Episode 6

Danny Maintains Incommensurability Between the Structure-Based Talk and His Sequence-Based Talk

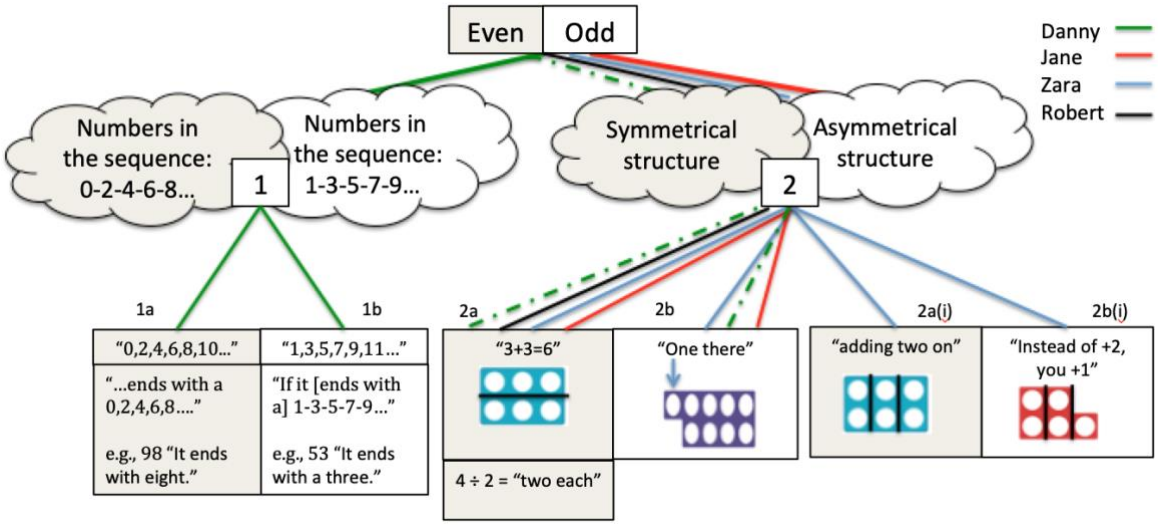
Turn	Speaker	What was said	What was done
		Classifying Numicon 4	
78.	T.R.:	Ok. Right so …, Zara said, "Odd is where it's unfair shares if we split it". And Danny, you're saying that it's like a sequence. We go "even, odd…"	
79.	DANNY:	No odd can, odd can still be a fair share 'cos you can split it up into decimals. Like with seven you can make three point five.	
		Classifying numeral 5	
101.	T.R.:	Right so you're saying the way to check if it's even or odd is to keep going up in the pattern…?	
102.	DANNY:	Yeah.	
103.	T.R.:	And you [directed to Danny] were a bit confused weren't you. When you said, "Where's the four and the one come from"? Is that right? When she said five is four and one.	
104.	DANNY:	I know I know. I know why. Cos five is like split and there's a four and there's a one but that doesn't make any sense in high-school maths, 'cos it goes like...	Points to the Numicon 5 piece.
105.	JANE:	We're not doing high school maths.	

106.	DANNY:	I know but, but it doesn't make any sense to like a really good, really good mathematicians 'cos it has no sense.	
		Classifying numeral 22	
119.	JANE:	Because eleven and eleven.	
120.	DANNY:	Cos always... I know, I know but that's a way you can check. But all you need to know is like if it ends with a… if this is like a simple way. If it ends with a zero, two, four, six or eight it is an even number or it ends…	Counts by tapping a finger on his other hand.

Danny's narratives within this episode are notable for two reasons. First, they demonstrate that Danny is quite capable of seeing a "four and one" structure in five [103] and the "eleven and eleven" in twenty-two [120] despite his utterances in Episode 5 suggesting otherwise. His words, "I know. I know. I know why," [104] indicate that he endorses the narrative "four and one" [103] as being a true statement about five; that is, he acknowledges that five can indeed be split into "a four and … a one". When Danny says, "odd can still be a fair share, 'cos you can split it up into decimals", [79] he endorses (by default) a structural difference between odd and even: odd fair shares involve decimals; even fair shares do not. In terms of representing his realizations via the Grove of Realizations tool, there is not yet a stable pattern of this mode of talk to indicate branches are growing from box 2 for Danny. In fact, despite making this structure-based distinction, Danny shows tenacious commitment to sequence-based substantiating narratives throughout the entire classification task. However, we might tentatively acknowledge this comment [79] as offering the potential for growth for him, and, accordingly, the dashed green lines towards boxes 2a and 2b in the Grove of Realizations (Figure 9) represent these potential realizations for Danny.

Figure 9

Tentative Growth of Branches to Structural Realizations for Danny



However, the second point of interest is that, while Danny endorses these narratives, he still manages to position them as being incommensurable with his own. His statement "but that's [just] a way you can check" [120] shows that he positions this structure-based substantiation procedure as substandard to his own superior and "simple" [120] (hence, more efficient) way. Furthermore, Danny claims that the other students' procedure "doesn't make sense" in "high school" (a more authoritative setting) or to "really good mathematicians" (people who have higher mathematical status) [104; 106]. In other words, he elevates his sequence-based substantiation routine as one that *does "*make sense," is simpler [120], and is endorsed by people and in places of mathematical authority. By doing so, Danny is able to both acknowledge the structure-based substantiation narratives as true facts about the numbers in question, but still maintain incommensurability between the two modes of talk because, according to his realizations for odd and even numbers, these structure-based substantiations are irrelevant.

5.4.2 Meta-Level Differences in the Students' Realizations

Having established object-level differences between Danny's realizations and talk and those of the other students, I now examine differences in realizations at the meta-level.

5.4.2.1 Danny's Objectified Talk. During the classification task, a sense of impatience develops in Danny's manner (see Episode 7). His impatience is evident as

early as five minutes into the task when the fourth card is turned over: he quietly comments “It’s easy. It’s just easy” [95]. As the task proceeds, his contributions to the discussion become much shorter, and there are many instances where Danny snatches cards as soon as they are turned over and places them immediately in the even or odd box without discussion.

Episode 7

Evidence of Danny's Frustration

Turn	Speaker	What was said	What was done
		Classifying numeral 5	
95.	ZARA:	Yeah. So, so every time you have an even number it has to be in twos.	[DANNY is listening but trying to interrupt. Quietly he says, "It's easy. It's just easy…"]
		After the first 6 cards	
151	T.R.:	Ok. Let's try another one. Danny do you want to turn the next one?	
152	DANNY:	No, I don't want to. Let's just get to the problem.	
		Classifying numeral 53	
157	ROBERT:	I think that's odd.	
158	T.R.:	You think that's odd. What makes you think it's odd?	
159	ZARA:	Because they…	
160	DANNY:	(Interrupts) It ends with a three!	
		Classifying numeral 428	
197	DANNY:	I can do all of these. I can do all of these in about fifteen seconds.	
		Classifying numeral 623	
210	DANNY:		Turns over 623 and puts it straight onto odd box without discussion.
211	T.R.:	Six hundred and twenty-three. Odd? Hold on Danny let's check everyone's happy.	Danny quickly turns over the next card [1,452] and attempts to place it.
		Classifying numeral 1,452	
215	T.R.:	All right, next one.	Danny snatches the card and puts it in even box, forcing Zara to move backwards out of the way.

Danny's frustrations demonstrated in this episode are interpreted as a further indication of his rejection of the other students' structure-based substantiation procedures. To explain this claim, I now underscore a further point of difference between their discourses in terms of the degree of objectification of odd and even. Jane, Robert, and Zara tend to use the words odd and even as labels assigned as

adjectives, describing the symmetrical (or asymmetrical) structures of properties of specific concrete objects (Numicon tiles) or numbers. For example, when Jane substantiates the evenness of six— "three plus three equals six and that's even" [31]—she describes the *even share* of three. Furthermore, their substantiating routines require an action or a process. According to their substantiation routines, to endorse a number as even, one is required to check if the specific numbers can: be *made* by a double [31; 55; 59; 119]; *make* two fair shares [73]; or *be grouped* in twos [95]. To endorse a number as odd, one is required to *show* an even result is not possible on account of a leftover one or unequal shares [43; 90; 91; 94].

In contrast, Danny's discourse replaces talk about processes involving concrete objects and numbers with talk about even and odd as mathematical objects existing in their own right, each as a condensed set of numbers reified from his known sequencing procedures. Take, for instance, Danny's narratives here, which illustrate how, when he uses the words "odd" and "even", there is an absence of action, and the words serve as nouns rather than adjectives:

> *"So, even are always first, 'cos zero, two, four, six, eight, ten." [45]*
>
> *"odd can still be a fair share" [79]*

In short, whereas for the other students these keywords appear as describing features derived from actions on specific numbers, for Danny infinite odd and even numbers have been encapsulated in two sequences, just like "the Fibonacci sequence" [Table 3: 69]. These discursive characteristics suggest that Danny's sequence-based discourse on odd and even is more objectified (and thus more developed and entrenched) than the structure-based discourse of the other students.

The level of objectivity sheds some light on Danny's frustration with his peers' structure-based discourse. From Danny's perspective, there is no need to waste time considering if numbers are symmetrical in structure as this requires a time-consuming check to see if they can (i) be formed from a double or pairs of numbers or (ii) make two whole number fair shares. For him, the last digit, simply and instantly, confirms a number's membership in the set of even or the set of odd numbers. Danny believes that his substantiation routine not only works and corroborates the other students' classifications but also is more developed and efficient. However, it is also the goal of the classification task—to classify numbers as

odd or even—that provides a further clue as to why all the students remained steadfastly committed to their own discourse. Had the students disagreed about *which* numbers were odd and even, the students would have had more incentive to consider one another's reasoning. In this case, though, the students agreed about their classifications, so there was no such incentive. Their own discourses sufficiently served them in achieving the goal of the task.

5.4.2.2 The Students' Engagement With the "Odd + Odd" Concept Cartoon. The focus now moves to the students' engagement with the "odd + odd" concept cartoon task regarding the narratives they endorse and their substantiating procedures (or proofs). The episodes and analysis provide more evidence of the objects odd and even being more objectified in Danny's talk, and thus shed further light on meta-level differences between his discourse and that of the other students in the group (note, Robert is largely silent for the remainder of the group session, so discursive comparisons are made between Danny's talk and the combined talk of Jane and Zara from this point forward).

(i) Jane and Zara's Empirical Routine. In Episode 8, Jane immediately both rejects the narrative "odd + odd = odd" [245] and endorses the narrative "odd + odd = even" [241], but then seems unsure after reading Benny's narrative "odd + odd = sometimes even and sometimes odd" [249] with regard to "really big" [253] numbers. Her quick endorsement of "odd + odd = even" and rejection of "odd + odd = odd" may have been dependent on her ability to visualise the structure of two specific, small, odd Numicon tiles or to quickly add two specific, small, odd numbers. If this is the case, her substantiation procedures are empirical in nature (number-specific) and would be limited to numbers she could easily visualise splitting into two equal, odd-number parts. This assumption makes sense when Jane's structure-based substantiations for classifying numbers as even (Episode 5) are considered. For example, Jane endorsed *six* as even and substantiated this endorsement with "three plus three…is even"—her substantiation was based on the ability to split a number into two equal (even) odd-number amounts. With regard to "big [even] numbers," Jane cannot quickly and easily perform this routine.

Episode 8

Jane's and Zara's Response to the "Odd + Odd" Cartoon

Turn	Speaker	What was said	What was done
240.	JANE:	Yeah, that's right. [Referring to Hana's narrative "odd + odd = even"]	
…	…	…	
242.	T.R.:	What about Manu? ["odd + odd = odd"]	
244.	JANE:	It's wrong.	
…	…	…	…
246.	T.R:	Whereas Benny says…	
247.	JANE:	Odd plus odd equals sometimes even and sometimes odd.	Reading Benny's statement in the cartoon.
249.	JANE:	I sometimes think that's right [referring to Danny's demonstration of "3 + 3 = 6"], because, um it might be… 'cos three and three… but what if there's another number and we add them together?	
…	…	…	…
253.	JANE:	So, because an odd and odd, like a three and three, equals odd [probably meant even] but if you put it together, they're a small number... But some odd numbers are like really big… we don't, we can't even count them… they might not, they might be odd but can't go even."	Danny shakes his head and quietly says "No".
…	…	…	
282.	ZARA:	Okay, so two odd numbers can sometimes make an even number.	
283.	T.R.:	Only sometimes?	
284.	ZARA:	We-elllll?	
285.	DANNY:	Always.	
….	…	…	….
288.	JANE:	Not **gotta** always	

Zara acknowledges instances of "odd + odd = even" [282], but the intonation in her response [284] suggests that she is also unsure as to whether "odd + odd" is always even or just sometimes. Zara does not give much away in terms of her substantiation

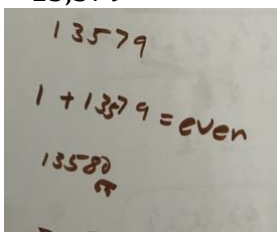
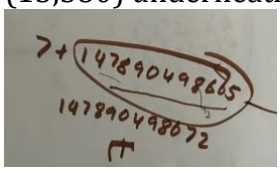
procedure, but, given her number-specific use of the words even and odd when classifying numbers was similar to Jane's, her uncertainty about which narrative to endorse may be explained in similar terms to Jane's—her substantiation procedure may also be empirical in nature (number-specific) and therefore limited to numbers she can easily visualise splitting into two equal odd-number parts.

(ii) Danny's Sequence-Based Proving Routine. In comparison to Jane and Zara's uncertainty, Danny's *to-the-point*, one-word endorsement of "odd + odd = even" [Episode 8: 285] signals more confidence. To substantiate this endorsement, in the next episode he performs a sequence-based proving routine, which draws from his sequence-based realizations of odd and even. First, he demonstrates that "1 + 13,579" is even [261]. He begins by writing the resulting sum, "13,580" and draws attention to the fact that it "end[s] in a zero", thereby drawing on his earlier substantiation that all even numbers end in zero, two, four, six, or eight. He then proceeds to demonstrate that this narrative holds for "even bigger" numbers, showing "7 + 147,890,498,665" produces a sum ending in two.

The routine of Danny's in Episode 9 is quite sophisticated in that, for one, it appears to address Jane's doubt regarding large numbers "we can't even count" [253], and for another, it also signals generality. If Jane wants proof for numbers she cannot even count to, Danny's sequence-based discourse avoids some of the difficulties that Jane would likely experience with just a structure-based discourse at her disposal. With the numbers that Danny chooses, Jane's substantiation of evenness or oddness would require a sophisticated calculation to be performed to figure out if a double exists to make 13,580 (his first example) or 147,890,498,672 (his second example) [261]. In comparison, his "check the last digit" proving routine is simple, quick, and efficient. Furthermore, Danny's selection of two seemingly random numbers in the thousands and then in the billions conveys an idea of generality; that is, Danny seems to be signifying that since this particular phenomenon ("odd + odd = even") happens with random and non-special numbers, then it is highly likely the same phenomenon occurs with all numbers like the ones picked (i.e., any odd number).

Episode 9

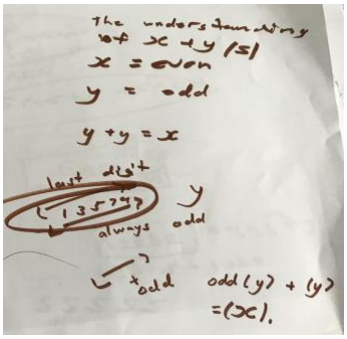
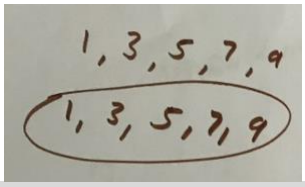
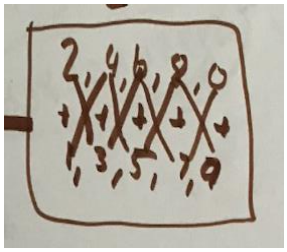
Danny's Sequence-Based Proving Routine

Turn	Speaker	What was said	What was done
259.	DANNY:	So, it's actually no matter how big the number is it's always the sequence. Cos it's like... if it's like odd, say a one… can I have a pen?	
…	…	…	…
261.	DANNY:	If it's bigger numbers, let's say like this is odd right? So, it's, one- three-five-seven-nine. So, the one is also odd right? So, one plus one-three-five-seven-nine is even. Because it's like, one, three, five, eight, zero, because that will end in a zero.	Writes 13,579 on a piece of paper. Underneath this, he writes 1 + 13,579  Writes the result of this (13,580) underneath.
		And if it's even bigger, you go um do seven plus, um 1-4-7-8-9-0-4-9-8-6-6 um 5 and if you add that together it equals 1-4-7-8-9-0-4-9-8-6… erm …7-2.	 Writes the result of 7 + 147,890,498,665 underneath (147,890,498,672).
263.	T.R.:	So that's a big number.	Jerry (the puppet) whispers.
264.	DANNY:	And it's **still** even.	
265.	T.R.:	He [Jerry] says, do we just need to keep trying lots of examples to prove it?	
266.	DANNY:	We don't. No, you don't, you don't because you just need the digits. The digits are like the most important.	Points to the two rows of 1-3-5-7-9.

Sometime later, Danny is invited by the Teacher-Researcher to share more of his work that, so far, he had only worked independently on. Here [Episode 10: 406], Danny's written work reveals an attempt at algebraic symbolism to substantiate endorsing "odd + odd = even": he substitutes the letter "*x*" for even and "*y*" for odd and replaces the narrative "odd + odd = even" with the sentence "$y + y = x$".

Episode 10

Danny's "Proofs-By Exhaustion"

Turn	Speaker	What was said	What was done
		Danny's "proof by exhaustion" for "odd + odd = even"	
406.	DANNY:	… this is ah like complicated, but so it's like, x is even, and y is odd. y + y is x.	Points to his written work. Points to "last digit 1, 3 5, 7, 9"
			
		Why? Cos last digit 1, 3, 5, 7, 9 is always odd: y. If this is plus odd, and then odd is y. And y is always odd, that equals x. So, if it's the last digit always that plus 9, plus 7, plus 5, plus 1, plus 3; then 9 + 3, 9 + 1, 9 + 7, 9 + 1… it always ends with like even digits.	Points to "y + y = x." Points to 9 on the top row, then alternates to the digits on the bottom row as he reads the sums.
			
		Danny's "proof by exhaustion" to endorse "odd + odd = even"	
553.	DANNY:	... There's like an easier way. Look at this thing. This is actually… when you learn this, this is really easy to do. Zero plus nine is odd, eight plus seven is odd, six plus five is odd, four plus three is odd, two plus one is odd, four plus one is odd, six plus three. Yeah, it's always odd. Look, just add these numbers. Just try to look at every pattern. It always will be odd.	Points to his written work.
			

Danny can be seen here as having performed a kind of proof by exhaustion to substantiate endorsing first "odd + odd = even" [406] and then "even + odd = odd" [553]. With regard to "odd + odd = even," he shows how adding *nine* (from the top row of his written work) with each and every other odd digit (in the bottom row of

his written work) produces a sum with an even last digit. Hence, Danny's use of the digit "9" suggests a generic realization of all naturals "10*k* + 9". Admittedly, Danny did not complete this proof by exhaustion (he only provided an exhaustion of summing "nine" with all other odd digits), but this is not to say that he could not have completed it had he been encouraged to do so. Then, with regard to "even + odd = odd," Danny performs a similar proving routine. Here, Danny writes 2,4,6,8, and 0 to represent all even numbers, and 1,3,5,7, and 9 to represent all odds.

What is notable about Danny's proving routines is that Danny manages to reduce odd and even (sets of infinite numbers) to two finite sets of numbers—numbers that end in 1,3,5,7,9 (for odd) and 0,2,4,6,8 (for even). Hence, his use of these digits suggests a generalization of all naturals being reduced to 10*k*, 10*k* +2, 10*k*+4, 10*k*+6, 10*k*+8 for even and 10*k*+1, 10*k*+3, 10*k*+5, 10*k*+7, 10*k*+9 for odd. Danny draws lines to connect the even digits with odd digits to show how the combinations of even and odd result in a sum that has an odd ones digit. His proving routine potentially reduces infinite cases of "odd + odd" to just fifteen possible cases ([1+1], [1+3], [1+5], [1+7], [1+9], [3+3], [3+5], [3+7], [3+9], [5+5], [5+7], [5+9], [7+7], [7+9], [9+9]), and 25 cases of "even + odd", making an exhaustion of cases achievable (although both proofs-by-exhaustion were incomplete).

5.4.2.3 Danny's Objectified Talk. Despite Danny's efforts to convince the other students in the group to endorse "odd + odd = even" and "even + odd = odd" on account of his sequence-based procedure, none of them seemed to take an interest in Danny's thinking. Following Danny's first attempt (Episode 10: 406), Jane queried what a sequence is but, seemingly dissatisfied with his response— "It's actually the most important thing" [273][8]—neither Jane nor Zara asked any further questions or adopted any part of his procedure. As with my analysis of the students' engagement with the classification task earlier, I now show how Danny's discourse on odd and even during the students' engagement with the concept cartoon tasks can be considered as more objectified than the other students.' This will help to shed light on the students' reluctance to endorse one another's discourses.

[8] This line is not shown in any Episode.

To highlight differences in terms of objectivity, I compare Danny's typical utterances regarding "odd + odd" with Jane and Zara's (Table 4). First, while Danny's algebraic symbolism did little more than restate the narrative "odd + odd = even" using letters, his reference to the infinite set of all odds and evens as one object provided a further marker that, for Danny, odd and even signified generalised mathematical objects. Zara and Jane, on the other hand, used the words as adjectives to describe specific concrete objects and numbers. Second, Danny's talk was mostly impersonal: he spoke in an authorless manner about odd and even as though these objects occurred independently of humans. In comparison, Jane and Zara's talk was mainly activity-based and personal: they focused on what actors—mainly second-person, "you" (underlined in Table 4)—could *do* with odd numbers and provided stories about the hypothetical actors' performances with numeric examples and the specific results that would emerge. In other words, through characteristics of reification and alienation, Danny's talk signified odd and even as having the permanence of "extradiscursive objects" (Sfard, 2008, p. 145); this was not evident in Jane and Zara's talk.

Table 4

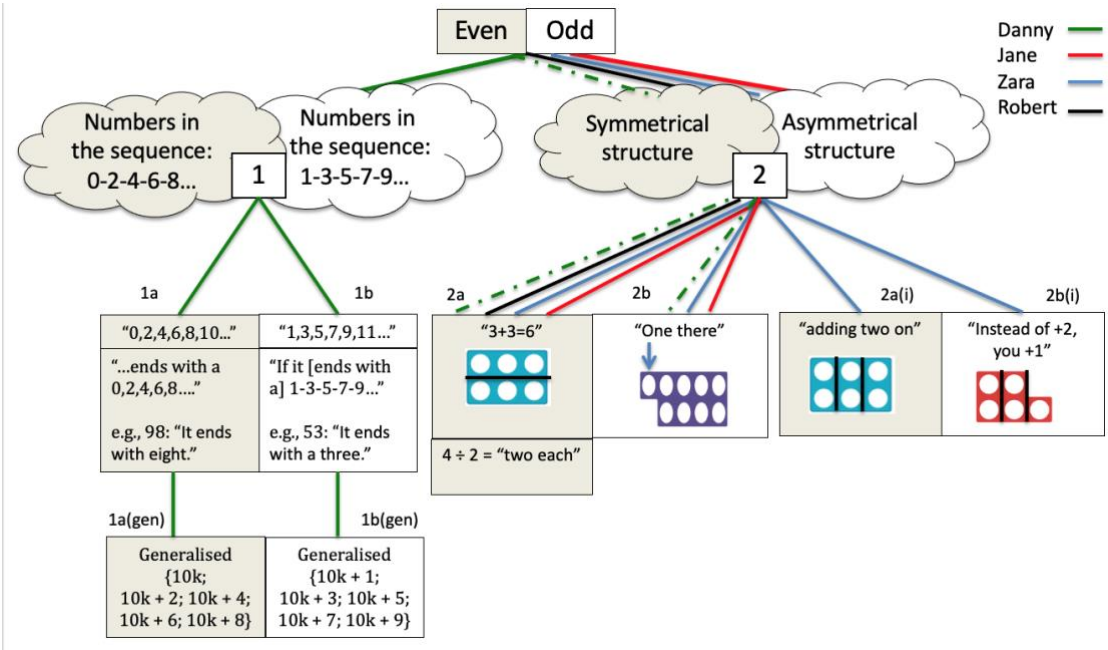
The Students' Talk on "Odd + Odd"

Danny	Jane	Zara
[406]. *x* is even and *y* is odd. *y* + *y* is x If this is plus odd, and then odd is *y,* and *y* is always odd, that equals *x.* …it's the last digit always that plus 9, plus 7, plus 5, plus 1, plus 3; then 9 + 3, 9 + 1, 9 + 7, 9 + 1… it always ends with like even digits	[253] if <u>you</u> put it together … they're going to be even. [467] if <u>you</u> have an odd number there's gonna be one, one extra but if <u>you</u> keep plussing a three or any other odd number it will go even. [481] if <u>you</u> actually put them together, they will actually be even because there's no, nothing pointing out and nothing uneven. [488] if you put another, the same [odd] piece together it'll be even, there's no bits there, <u>you</u> can share it, evenly.	[72] if <u>you</u> had two people then you'd be able, they'd both get two each. [397] if… <u>you</u> just put that one there and added this on the top, it wouldn't be even. [441] if <u>you</u> had two odd numbers like three and three and you can still make them into an even number. Six. [507] <u>you</u> could say three plus three equals six. Two odd numbers equal an even number…Three and three.

Hence, in terms of the students' pre-existing discourses, where Jane and Zara's empirical discourse rendered them short of discursive tools to decide which narrative to endorse, Danny already had appropriate discursive machinery to account for all cases of "odd + odd" in his substantiations of "odd + odd = even" (and all cases of "even + odd" in substantiations of "even + odd = odd"). In light of the aforementioned observations, I show that the branches of Danny's realization tree extend to generalised realizations of odd and even as (abstract) mathematical objects in their own right rather than as labels for many discrete concrete objects and numbers. These generalised realizations of even [1a(gen)] and odd [1b(gen)] are represented in Danny's tree within the Grove of Realizations in Figure 10. (Note that since Danny did not actually use the algebraic terms [10k; 10k+1 etc.] himself, these realizations on the tool are not shown with speech marks). Where additional object-level realizations that an individual has samed would be shown on the tool as *width* in their tree because they are additional object-level realizations, these realizations extend downwards from 1a and 1b on the tool and give his tree *depth*—they are meta-level realizations.

Figure 10

Danny's Generalized Realizations of Odd and Even



Danny continually rejected the other students' structure-based substantiation routines because, from his perspective, the structure-based substantiations required

a process (checking for symmetry in one way or another) and so were time-consuming and unnecessary when, with his substantiations, the last digit simply and instantly confirmed a number's membership in the set of even or the set of odd numbers. Danny's sequence-based routines not only worked and produced the same outcome; they also were more efficient than the alternatives. Danny's objectified, and thus more entrenched, talk helps to explain his dissatisfaction and frustrations with the other students' structure-based discourse and explain why, even with encouragement to makes sense of the structure-based discourse, he refused to adopt it. Equally, from Jane and Zara's perspective, their structure-based routines were supporting them to make sense of, and explain, generalisable properties of even and odd, and Danny's sequence-based substantiating procedures were too objectified for them for this purpose and were disconnected with their structure-based realizations.

5.5 Discussion

This chapter examined the sources of communicational breaches within the student group. The analysis of the students' discourse showed the ways in which four students substantiated their classifications of numbers as odd or even and then proceeded to prove universal narratives about the sums of odds and evens were both grounded in the students' personal realizations of odd and even. Since their personal realizations differed, the students exhibited divergent uses of the words "odd" and "even", divergent substantiations for classification of numbers, and divergent substantiation procedures. While the students attended to different features of oddness and evenness, they were able to agree on *which* numbers were odd and which were even though they disagreed about *why*. Furthermore, one student had discursive tools to account for all even and odd numbers with regard to universal narratives about the sums of odd and even numbers, while the others did not. Consequently, the students experienced breaches in communication at the object-level *and* the meta-level—they endorsed different narratives, acted according to different rules, and were unable to build on one another's ideas.

Examining the students' realizations according to their word use (or what others might call "word meaning") is important because "it is responsible for what the user is able to say about (and thus see) in the world" (Sfard, 2008, p. 133). The commognitive analysis in this chapter showed Danny's realizations diverged from the

other students' realizations: odd and even were sequence-based and generalised so they had the permanence of extra-discursive mathematical objects in their own right. Accordingly, his realizations determined the proving procedure that Danny performed, namely, a sequence-based (last digit) proof-by-exhaustion procedure to substantiate his endorsement of "odd + odd = even" and "even + odd = odd," which was deductive and accounted for all cases of odd and even. In contrast, Zara and Jane's realizations for odd and even were structure-based, and their uses of these words were not objectified like Danny's—they were used as adjectives to describe specific numbers. Zara and Jane's empirical, number-specific discourse meant that their proving procedures were also number-specific and, with regard to universal narratives about the sums of odds and evens, they could not account for all cases of odd and even numbers.

Utilising the commognitive framework to conduct discursive analysis was instrumental in unearthing each student's realizations and highlighted distinctions in the students' reasoning that may otherwise have been hard to see and explain. Furthermore, representing the students' discursive mathematical objects visually in the Grove of Realizations tool helped to visually clarify the effectiveness of the group's interpersonal communication and identify dual sources of communicational conflict—object-level and meta-level. At the object-level, Jane, Zara, and Robert's trees had branches connecting to shared structure-based realizations within the grove and were able to communicate their process of classifying odds and evens effectively with one another. Conversely, Danny's sequence-based realization tree was isolated in the grove, showing why he was unable to communicate effectively with the other three group members. The commognitive analysis also showed meta-level differences between Danny's and the other students: his realizations of even and odd were generalised and accounted for all even and odd numbers; the other students' realizations were number-specific. This was shown as depth on his realization tree, that which had not yet been established for the other students. This discursive analysis was visually represented in the Grove of Realizations tool, showing the branches of Danny's realization tree as disconnected from the other students, but more established.

For group members to resolve communicational conflicts, a "gradual mutual adjusting of their discursive ways" is required (Sfard, 2008, p. 145). However, during the entire classification task situation and most of the students' engagement with the concept cartoon task situations, there was little evidence to suggest that this was occurring. The stubbornness of the conflict within the group was attributed to two factors. First, during the classification task, the conflict was not about the *outcome* of classifying numbers as odd or even; it was about *how* the students substantiated oddness and evenness. Sfard (2007) noted that learners need a good reason to change their routines, and I suggest that the classification task did not present a reason for Danny, or the other students, to modify their substantiation routines. For the students, deciding *which* numbers are odd and even was the goal of the task, so they had no reason to resolve it because, on this, they agreed. Second, the alternative discourse did not serve either party well with regard to their goals. Danny's sequence-based substantiation procedure was too abstract and objectified for Jane and Zara, who were still making sense of generalisable properties of even and odd. And for Danny, his deductive discourse was entrenched and more efficient for him than Jane's and Zara's, and he could already account for all cases of odd and even numbers. Accordingly, even when Danny became attuned (albeit very slightly) to the structure-based discourse, he maintained incommensurability between this and his own by positioning his routines as superior ones that worked in more authoritative contexts and were favoured by people who had more authority.

In general, each student remained committed to his or her own discourse and showed little sign of adopting or even considering one another's reasoning; the commognitive breaches failed to give rise to a modification of discourse. In terms of commognitive theory, while meta-level transitions are expected to be slow and difficult, the stubbornness of the object-level divide—a content-specific division between their talks—was somewhat surprising. Sfard (2008) claimed that object-level learning did not necessarily require interpersonal talk. Yet, the findings here showed that even *with* interpersonal communication, the students were unwilling to adjust their discursive ways and resisted treating one another's realizations of even and odd as the same thing.

With regard to proof and proving at the primary school level, the findings suggest that there are advantages for pressing students for proof in terms of exposing sources of students' reasoning. In the absence of students' substantiations, the group consensus about an answer (in this case about whether numbers are even or odd) may have prematurely signalled shared reasoning. Hence, having students engage in proving activity and pressing students to publicly air their substantiations may assist in surfacing the differences in students' reasoning, those which may have otherwise remained hidden. However, the findings also serve as a warning that common pedagogical moves to take advantage of mathematical disagreements by encouraging students to make sense of one another's ideas may not necessarily result in students' adoption or even endorsement of them. Therefore, if a criterion for proof in the primary classroom is an argument accepted by the classroom community (Stylianides, 2007), the commognitive framework provides a useful lens to glean insights into barriers that a community may need to overcome to reach consensus.

These findings show how the content-specific nature of a learner's talk (whether their realizations are structure-based or sequence-based), combined with the sophistication of that discourse (whether the learner has objectified the focal entities of the discourse or not), affects their readiness to engage in deductive reasoning and determines the nature of their substantiations of universal statements. (see Table 5 for a visual summary (below). Hence, a learner who holds structure-based realizations of even and odd but has not yet objectified these mathematical objects (like Jane, Zara and Robert), may be expected to engage in empirical and structure-based substantiation routines. Whereas a learner who holds sequence-based realizations of these same objects and has objectified these as extra-discursive mathematical objects (like Danny) is supported to engage in deductive and sequence-based substantiating routines. Furthermore, a learner who is capable of realizing numbers in both ways, would be supported to engage in either (or both) deductive substantiating routines. These findings are significant for research and practice, as they point to how learners' progress in deductive reasoning can be stymied. Efforts to engage learners in deductive and structure-based substantiating routines may fail if the learners adhere only to sequence-based realizations of these mathematical objects (and vice versa). The content-specific nature of a learners' realizations of the

mathematical objects under observation and their degree of objectification are pivotal to learners' readiness to engage in deductive reasoning.

Table 5

How the Content-Specific Nature of Realizations and Objectification of Mathematical Objects Under Observation Affect a Learner's Substantiating Routines

Realizations	Objects	Substantiations for universal statements
Sequence-based	Not objectified	Empirical and sequence-based
	Objectified	Deductive and sequence-based
Structure-based	Not objectified	Empirical and structure-based
	Objectified	Deductive and structure-based

I conclude by summarising the affordances of the Grove of Realizations tool. For the study in this chapter, the tool was shown to be an effective means for the researcher to visually map the participants' realization trees regarding the mathematical objects in question—odd and even—and contrast their realizations within the grove to map and expose the sources of commognitive conflict within the group. This helped with communicating findings effectively and efficiently and served as a visual metaphor for the reader to makes sense of the analysis. From this point, it now provides a discursive baseline for tracking the discursive developments of the same students in Chapters 6 and 7. Of course, every methodological tool comes with both advantages and disadvantages, and I acknowledge both the affordances of the Grove of Realizations tool for further research and its limitations in Chapter 8.

CHAPTER 6: Research Study 3:
From Numeric Talk to Generic Talk: How an Interdiscursive Learning Environment Can Support Students' Explorative Entry into an Unfamiliar Discourse

6.1 Introduction

Transitioning from accepting and using empirical reasoning to using deductive proofs for universal narratives about the sums of odds and evens requires a change in meta-level rules: it calls for learners to view and use words like "odd" and "even" as general terms in their own right rather than as labels for particular numbers. Hence, meta-level learning is required to shift from a number-specific (empirical) discourse on odd and even numbers to a conflicting deductive one which requires abstract generalisation. At meta-level intersections such as this, learners require the guidance of an expert in taking their first steps in the emerging discourse. Previously, the commognitive stance has maintained that only ritualised (socially motivated and process-oriented) initiation into a new discourse is possible, as learners may have little intrinsic motivation to adopt its new rules. However, drawing on the notion of i*nterdiscursivity*—coalescing discursive elements from two discourses—Cooper and Lavie (2020) proposed *exploration-enabling* tasks as offering potential ways to support learners in taking exploratory (intrinsically motivated and outcome-oriented) first steps in a new discourse. Accordingly, the study in this chapter addresses two aims of this thesis: it investigates *what can help students' initiation into a deductive discourse on odd and even numbers* and seeks *to explore whether their first steps in a deductive discourse could be explorative.*

6.2 The Commognitive Framework

Sfard (2008) defined mathematical discourse as a special form of communication, including self-communication (thinking), made distinguishable via four interrelated characteristics: keywords and how they are used (e.g., how the words "even" and "odd" are used); visual mediators (e.g., numerals, symbols, diagrams, pictures); narratives (e.g., definitions, theorems); and routines (repetitive patterns of mathematical actions according to which mathematical tasks are performed). A

lasting change in at least one of these four features denotes that learning has occurred.

Learning can be at the object-level and the meta-level. Object-level learning occurs through a growth in realizations about mathematical objects and is signalled by an expansion in the routines and endorsed narratives within a discourse. An example of this is when an individual's understanding of "even" expands from being limited to last digit identification (i.e., numbers ending in 0,2,4,6,8) to include realizations of "divisibility of two". However, meta-level learning requires a change to endorsing "propositions about the discourse rather than about its objects" (Sfard, 2007, p. 573). Transitioning from endorsing and using empirical reasoning to using generic proofs represents such a meta-level shift. Whereas in empirical discourses learners converse about concrete objects, in deductive discourses learners are required to converse about abstract objects. Learners experience a commognitive conflict at such meta-level junctions (Sfard, 2008) since the new discourse with new rules does not fit (is incommensurable) with their existing discourse and rules; that is, it does not yet make sense to them. For learners, resolving the commognitive conflict and transitioning into the new discourse requires the support of a more knowledgeable other (e.g., the teacher). It is here that the teacher and learners need to negotiate a "leading discourse" (Ben-Zvi & Sfard, 2007) to be adopted.

In terms of the specific discourses examined in this study, a deductive discourse on odd and even numbers is seen commognitively as a meta-discourse which is built from an empirical discourse on odd and even numbers. An empirical discourse on odd and even numbers involves the use of numeric keywords (particular numbers) and/or visual mediators that signify particular numbers (e.g., in numeric form, "6", or other visual formats such as six objects or dots), and the sums of particular numeric examples (such as "3 + 5 = 8") are inductively used in substantiations for endorsing a universal narrative (e.g., "odd + odd = even"). In this discourse, the words "odd" and "even" are used as adjectives. They serve as labels for two mutually exclusive collections of perceptually accessible, mathematical objects (particular numbers) that are perceived to be similar or as belonging to the same family—a process that Sfard (2008) called "saming" (p. 112).

In contrast, a deductive discourse on odd and even numbers involves the use of generic keywords regarding the mathematical properties of even and odd numbers (e.g., "2n" or "in pairs" for evenness and "2n+1" or "extra one" for oddness) or visual mediators signifying these properties (e.g., diagrams of materials organised in "pairs" for evenness and in "pairs with an extra one" for oddness). Moreover, in this discourse, the properties of odd and even numbers—their definitions—are used in substantiations to logically deduce propositions and reach a conclusion in endorsing a universal narrative. Here, the words "even" and "odd" are no longer just labels for particular numbers but also discursive objects in their own right; they signify abstract mathematical constructs that encapsulate the properties of the classes of even and odd numbers. Sfard called this act of discursively turning processes into objects—turning the description of all odd and even numbers into nouns—"reification" (p. 43). Since the new deductive discourse uses properties (or definitions) of odd and even as premises from which to draw a general conclusion, it is incommensurable with the familiar discourse because, in deciding whether a universal statement (e.g., "odd + odd =even") should be endorsed, the deductive substantiation routine uses different criteria to an empirical substantiation routine. This shift in substantiation routine requires a change in meta-level rules, and discursive evidence of this thereby signals meta-level learning.

The process of learning mathematics is seen commognitively as "routinization"—individualising historically developed routines (Lavie et al., 2019, p. 154). The commognitive framework conceptualises routines as repetitive discursive patterns, according to which mathematical tasks are being performed (Sfard, 2008). A participant is likely to perform a routine in situations considered similar to ones where he or she previously performed the same routine. Routines can be performed as rituals or explorations. Sfard (2017) defined ritualized participation in mathematical discourse as "rote implementation of memorized routines" (p. 44). Here, the participant is socially motivated to adhere to goals set by others (e.g., the teacher) and the ritual performance of a routine is process-oriented—mathematical competence means remembering and rigidly applying a correct procedure, and the performer will be focused on *how* to proceed rather than the result. In contrast, participants who perform a routine exploratively are intrinsically motivated' to learn and make sense about the mathematics involved. Their performance is outcome-

oriented—they seek to perform the routine in order to construct new narratives about mathematical objects.

Pure rituals and explorations, however, are not expected; rather, it is more helpful to consider performances of a routine as fitting somewhere along a ritual-to-exploration continuum (Lavie et al., 2019). Observing a change in a learner's performance that shows their focus has shifted from the performance to its outcome denotes a step in de-ritualising the routine (Lavie et al., 2019). Lavie et al. admitted that delineating explorative performance from ritual performance in terms of observable features is not an easy and straightforward task. Nonetheless, Lavie et al. provided several characteristics of a routine, which, if identified in a participant's performance, would suggest that the routine is being de-ritualised (pp. 167–172):

- *Flexibility:* Initially, learners will only have one method for performing a task. When they realize there is now more than one way to perform the same task and produce the same outcome, the varied ways in which they complete similar tasks are indicative of explorative performance.
- *Bondedness:* At first, learners may not appreciate the relation between steps in a routine when they begin to imitate a performance. Even though they might try to perform all of the steps involved, they are unlikely to use the output of one step as the input of the next. Hence, when the performance becomes more explorative in nature, the steps are connected to one another and directed towards the outcome.
- *Applicability:* The precedents for routines are initially limited to the people and the environment in which they originated for the learner. Hence, learners may only perform a routine in identical contexts to begin with. A sign of explorative performance is when learners perform the routine in a range of task situations (e.g., on different occasions, in different environments, with different group members).
- *Performer's agentivity:* In ritual performances, when the performance is the task itself, learners are dependent on other people to decide how to respond to the assigned task situation, when to respond, and when the task is complete. Hence, periods when the participant independently performs the routine become longer and constitute evidence that a learner is de-ritualising a routine.
- *Objectification of the discourse:* Initially, concrete objects serve as realizations for particular words, and stories involving these words are remembered as being performed on concrete objects. In terms of the present study, Numicon tiles, counters, and specific numbers might serve as realizations for the words "odd" and

"even." Stories gradually become more abstract and learners no longer need to recall precedent task situations involving concrete objects that served as their realizations; they describe the routine as being performed on (abstract) mathematical objects rather than the concrete objects. In terms of the present study, the words "odd" and "even" will signify abstract mathematical constructs that encapsulate the properties of the class of even and odd numbers rather than specific numbers or concrete objects. The process of objectification involves reification (substituting talk about actions with the use of nouns) and alienation (addressing mathematical phenomena in an impersonal way). An increase in the use of nouns for (abstract) mathematical objects (rather than concrete objects) and the use of passive voice i indicates objectification.

- *Substantiability:* Learners performing a ritual will lack appropriate criteria to assess their performance. So, when pressed to substantiate their performance, they will often repeat their routine to show the procedure has been correctly carried out. Alternatively, they will resort to an authority figure (e.g., the teacher, a textbook) as a source of substantiation. But when performing an exploration, learners can substantiate their outcome by showing it holds when an alternate procedure produces the same outcome.

While the goal of teaching is for learners to be explorative, the traditional commognitive standpoint contends that routines must first be performed ritually before they can be performed exploratively (e.g., Lavie et al., 2019; Ben-Zvi & Sfard, 2007; Nachlieli & Tabach, 2019). The claim here is that, upon initially encountering a new discourse, a learner would not be expected to appreciate the usefulness of the new meta-rules so their motivation to perform them could only be socially motivated; that is, they desire to be like others and to communicate interpersonally with others, especially people of authority (e.g., the teacher). Rituals are considered to be an unavoidable and "necessary occurrence in the process of individualization" (or learning) (Lavie et al., 2019, p. 167); they support students in taking their first steps when entering a new discourse. The role of the teacher is to model the new discourse and help students in transforming rituals into explorations. At first, the student is expected to experience the new discourse as a "discourse for others": they will first observe, then act as a "peripheral participant" with the amount of scaffolding required decreasing (Ben-Zvi & Sfard, 2007, p. 121). As learning proceeds, learners are expected to gradually de-ritualise these routines until, appreciate the

usefulness of the discourse, they become intrinsically motivated to perform the routines exploratively; the discourse becomes a "discourse-for oneself" (Ben-Zvi & Sfard, 2007, p. 121) which the learner can independently utilise. As attested by earlier research, though, the process of de-ritualisation is often gradual and slow (Sfard & Lavie, 2005; Lavie & Sfard, 2019).

However, contrary to the traditional commognitivist perspective, Cooper and Lavie (2020) argued that explorative first steps in a new discourse may be conceivable via *exploration-enabling* opportunities to learn. To explain how this may be possible, Cooper and Lavie drew on Lavie et al.'s (2019) notion of a task situation—a situation where one feels compelled to act. Lavie et al. (2019) claimed that when learners encounter a task situation, they draw from their precedent-search-space—familiar actions that the learner considers relevant for the present situation—to decide how to act. In order to decide which situations are relevant, the learner uses precedent identifiers—features from the task at hand that prompt the recollection of a past situation deemed sufficiently similar as to warrant a similar action in the present one. Cooper and Lavie (2020) suggested that it might be possible to take advantage of key precedent identifiers to change a learner's interpretation of the task situation, which will influence his or her corresponding actions. They proposed interdiscursivity as a discursive mechanism whereby learners may be eased into a new and incommensurable discourse via "the blending of discursive elements from different discourses" (p. 1). The authors suggested that teachers can facilitate a change in learners' routines by using elements of the learner's existing discourse and their familiar routines in such a way that it carries meaning in the new discourse, but also the learner's action is deemed appropriate from the perspective of the new discourse. In this way, Cooper and Lavie (2020) claimed that interdiscursivity can support participation that is intrinsically motivated and outcome-oriented—thus explorative—rather than socially motivated and process-oriented—thus ritualistic.

What does this mean in terms of the meta-level learning required in moving from an empirical to a deductive discourse on odd and even numbers? Often when young children are required to prove a universal narrative, such as "odd + odd = even," they will verify it with several number-specific cases. When students first encounter such tasks, their precedent-search-space is likely to only include making decisions about

"Is it [the narrative] correct?" or (in the event of more than one narrative being presented to choose from) "Which is correct?" Furthermore, according to the rules of this discourse, the words (or precedent identifiers) "odd" and "even" are labels for particular numbers—concrete objects—so the students' precedent-search-space for these words will likely consist of specific numbers (i.e., 1,3,5,7,9… for odd, and 0,2,4,6,8… for even) and, therefore, their substantiation routine will entail trialling and testing appropriate specific cases. Correspondingly, a number of confirming cases (sometimes only one confirming case) are enough to complete the student's task (the task situation as interpreted by the student).

The new deductive discourse is incommensurable with the familiar discourse because the main figures are no longer concrete objects; they are abstract objects. The deductive discourse uses nonspecific properties (or definitions) of odd*ness* and even*ness* as premises from which to draw an abstract-generalised conclusion and explain why, for example, "odd + odd =even." In order to participate in a deductive discourse on odd and even numbers when completing the same task, students need to interpret the task differently and draw from a different precedent-search-space. Students need to interpret the task as, "How do I know for sure that "odd + odd = even" holds for *every* pair of odds and not just *some* (or many)?" Where the definitions or properties of odd and even are more salient than particular numbers within each class, the precedent-search-space that the students draw from will include the use of the words "odd" and "even" as signifiers of the class of all odds and all evens—oddness and evenness—and not just as labels for numbers.

In considering the extent to which a task can be considered as exploration-enabling, Cooper and Lavie (2020) considered three aspects of Sfard's (2008) notion of a routine: procedure, initiation, and closure. Initiation and closure are the conditions that prompt the performance of procedure and signal it is complete; they are associated with *when* the routine is performed. The procedure is *how* the routine is performed. According to Cooper and Lavie (2020), the design of exploration-requiring tasks should satisfy the following conditions:

1. Initiation: The student's goal should be to achieve something, not to complete a procedure.

2. Procedure: Students should be able to choose from alternatives making independent decisions, and not simply perform a rigid procedure.
3. Closure 1: The task has been completed when students provide a narrative [detailing the mathematical reasoning involved] not just a final answer.
4. Closure 2: The student [not the teacher] should be able to decide when the task has been successfully completed. (Cooper & Lavie, 2020, p. 4)

While Cooper and Lavie (2020) provided three hypothetical examples of exploration-enabling task situations, empirical studies are needed to show whether, and how, interdiscursivity can work in practice. Consequently, I used the principles of exploration-enabling task design to construct a task situation to support students' explorative entry into a new deductive discourse on odd and even numbers. The interdiscursive learning environment was designed to circumvent the commognitive conflict at the empirical-deductive divide. In particular, the task situation was designed to facilitate students in the discursive transition from endorsing substantiating narratives that use particular (numeric) examples to substantiate universal statements about the sums of odds and evens, to using a new rule that uses properties (or definitions) of odd and even in substantiating narratives to account for *all* cases—what others might consider as bridging the gap between using empirical proof and using generic proof (e.g., Fiallo & Gutiérrez, 2017; Stylianides & Stylianides, 2009).

6.3 Conduct of the Study

In this section, I describe the story so far in terms of the students' in focus and how the study described in this chapter was conducted. Since the design of the task situation is instrumental to this study, I provide a comprehensive overview of the task situations and their implementation (i.e., some details provided in the *Research Methods* in Chapter 3, Section 3.3 are repeated and elaborated on) and I provide a detailed account of the data analysis. I conclude the section by articulating the two research questions that guide this study.

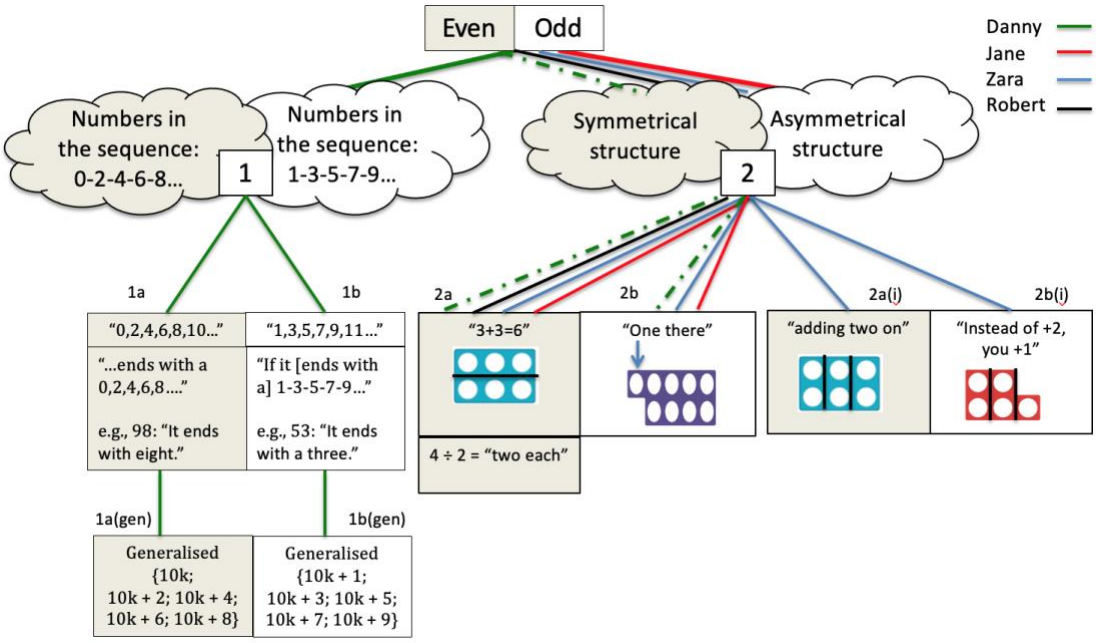
6.3.1 The Story So Far

For this study, the story returns to two of the students from Chapter 5—Jane and Zara. It picks up from where the students were shown the "odd + odd" concept cartoon dilemma and were encouraged to share which narratives they endorsed and

share their substantiations for endorsing these narratives in order to convince one another. In Chapter 5, I presented a new methodological tool, called the "Grove of Realizations", to visually map students' pre-existing realizations of odd and even numbers. The tool visually showed: (i) differences in realizations between participants (indicated by branches of different colours taking different courses within the grove); and (ii) more dominant realizations within the group (as shown by a realization having a branch formed with multiple colours in their grove). The grove of the students' pre-existing realizations (Figure 11) helped expose the source of the group's commognitive conflict. The grove showed that Jane and Zara held structure-based realizations of odd and even numbers based on numbers' asymmetry (odd) or symmetry (even) (their realizations all stemmed from branch 2 in the grove). In contrast, Danny held sequence-based realizations of odd and even (his realizations stemmed from branch 1), which were thus separated from Jane and Zara's realizations. Consequently, Danny's sequence-based substantiation routine for endorsing numbers' oddness and evenness conflicted with Jane and Zara's. The commognitive conflict was shown here to exist at the object-level.

Figure 11

The Student Group's Grove of Pre-Existing Realizations



The group's commognitive conflict proved to be resilient. Throughout the classification task, there was little evidence of the students adjusting their discursive ways to resolve their disagreement about *why* numbers were odd or even. This was

thought be on account of the students having no intrinsic need to build on one another's thinking. Their task was to classify numbers as odd or even—and on this they agreed—regardless of their different ways of substantiating their classifications. Once the task situation changed and the students were required to substantiate their endorsement of universal narratives about the sums of odd and even numbers, a meta-level division between Zara and Jane's talk and Danny's more objectified talk became more apparent. Danny's realizations of even and odd were generalised and accounted for all even and odd numbers. In contrast, Zara and Jane were shown to be lacking discursive tools to talk about all cases of even and all cases of odd.

Where Chapter 5 mapped the students pre-existing realizations (the realizations that each student brought to the lesson prior to any learning occurring within the lesson), this study examines if, when, and how their discourses *develop*.

6.3.2 Task Design

Although the task design was described in Chapter 3, it is elaborated on further in this section to explicate how it aligns with the aims of Cooper and Lavie's (2020) exploration-enabling task design principles to support students in taking their first steps in a deductive discourse on odd and even numbers.

6.3.2.1 The Concept Cartoons. Jane and Zara were presented with the "odd + odd" and "even + odd" concept cartoon task situations (Chapter 3, Figures 4a and 4b). The cartoons featured three characters, each with a speech bubble featuring differing narratives on the sums of odds and evens for students, working in small groups, to reject or endorse, and then substantiate their choices. Importantly, each cartoon featured three narratives: two are what I term "binary" narratives (e.g., "odd + odd = even" and "odd + odd = odd"), which have two mutually exclusive, dichotomous narratives; and the third I term *bilateral* (e.g., "odd + odd = sometimes even and sometimes odd"), where both outcomes are considered possible. Since the students' precedent-search-space (Lavie et al., 2019) regarding the task situation was likely to involve decisions about "Which one is correct?" having only binary options of "odd + odd = even" and "odd + odd = odd" would reinforce students' tendency to use empirical reasoning: if there were only the two binary options, evidence that disproved one conjecture would, by default, prove the other to be true. Hence, the

inclusion of the bilateral narrative requires the students to work harder to show why, for example, "odd + odd" is *always* even, and not just *sometimes*.

6.3.2.2 The Students' Prior Learning Experiences. Jane and Zara were already familiar with Numicon tiles (see Chapter 3, Figure 3: Visual representations of numbers 1–10 presented as dots within a frameless 2 x 5 rectangle), as they had previously used them when classifying numbers as odd and even. They had also provided substantiations that used the structure or properties of odd and even numbers to endorse classifying the tiles as odd or even. For example, in substantiating Numicon 5 tile as being odd, Zara had said:

> *"Instead of adding two on, you add on one and then it wouldn't be even… every time you have an even number it has to be in twos."*

And in substantiating Numicon 6 tile as being even, Jane had said:

> *"Because three plus three equals six and that's even."*

However, while the students had samed these objects (particular numbers) as belonging to the family of odds or evens according to similarities regarding the numbers' properties or definitions, they had not yet reified them as all-encompassing, abstract objects. Hence, when Jane and Zara initially substantiated their endorsement of the universal narratives presented in the concept cartoons (e.g., "odd + odd = even"), rather than drawing on properties (or definitions) of odd and even, I predicted that they would trial particular numeric examples (such as "3 + 5 = 8"). But, since they already endorsed structure-based substantiations of oddness and evenness, the students were in a favourable position to be supported to use definitions of odd and even with regard to universal narratives about the sums of odds and evens. In other words, Jane and Zara were in a favourable position to be supported to learn at the meta-level.

6.3.2.3 The Interdiscursive Potential of the New Visual Mediator: The "Numicon Strip." Drawing on Cooper and Lavie's (2020) notion of interdiscursivity as a mechanism to facilitate students' transition into a new incommensurable discourse, I aimed to use the students' familiar routines of trialling and testing appropriate specific cases from their existing number-specific discourses while introducing new visual mediators, which were grounded in this known discourse but could be used in ways that would "simultaneously [hold] meaning in an emerging [deductive]

discourse" (p. 8). Accordingly, I developed odd and even Numicon strips (Chapter 3, Figures 6a and 6b) as visual mediators. Numicon strips are similar to Numicon tiles in terms of the paired configuration of dots, but they are printed on long, folded, strips of paper with both ends exposed.

Paradoxically, yet importantly, the Numicon strips have the potential to simultaneously hold two different meanings from two distinct discourses: empirical and deductive. On the one hand, the strips can signify *particular* numbers—the strips are similar in design to (and can be treated in similar ways as) the Numicon tiles (representing the numbers 1–10) that the students had worked with previously, and it would be possible (although time-consuming) to determine the count of dots on each strip to confirm the precise number signified. But on the other hand, the strips can also signify *generic* oddness and evenness; they neither clearly nor instantly signify a particular number because the scale of the count required discourages a counting routine, and the folded presentation of the strips conceals the preciseness of the number each strip represents.

Interdiscursivity occurs when a learner reinterprets a task situation and uses a familiar routine from their current discourse in a way that is deemed appropriate in the eyes of the teacher ushering the learner into a target discourse that is new to them (Cooper & Lavie, 2020). In terms of substantiating universal narratives about the sums of odds and evens, students with only an empirical discourse at their disposal would, upon encountering the words "odd" and "even," draw from a precedent-search-space consisting only of numbers— they would have no means to account for all cases of odd and even numbers. However, I envisaged the introduction of the Numicon strips (offering a vague and less defined representation of a "number") might expand the learners' precedent-search-space to include the nonspecific strips as realizations of odd and even and would thereby suggest an alternate interpretation of the task situation for learners. That is to say, I hypothesised that the learners might consider these more abstract visual mediators of odd and even as being similar enough to their precedents of *particular* numeric odds and evens (when they had used odd and even Numicon tiles) but also, they would also see them as providing means to talk about odd numbers altogether and even numbers altogether. Hence, the Numicon strips would help soften the students'

transition towards a discourse that makes use of a generic odd and even. Expanding their discourse about odd and even in this way would lay the groundwork for students to use generic (and undefined) examples in explorative ways to generalise about the sums of odds and evens.

If this were to be the case, the learners would then be able to perform in similar ways with the Numicon strips as they had done with the Numicon tiles. By drawing on the precedent-search-space of selecting pairs of specific numbers (as they had done when working with Numicon tiles), such as "3 + 5 = 8" in substantiating "odd + odd = even", they would be able to use the nonspecific Numicon strips to show "odd *nonspecific number* + odd *nonspecific number* = even *nonspecific number*". By this simple visual mediation, I hoped the learners would be able to relate to the task situation with the familiar meaning of "odd (number) + odd (number) = even (number)." The intended mediation is interdiscursive in the sense that it holds two meanings of odd and even: for the learner it is still a call to perform a calculation for "odd + odd;" yet from the perspective of the emerging discourse of abstract generalisation it signifies "*any* odd + *any* odd = even." In Cooper and Lavie's (2020) words, "the familiar [number-specific] routine"—along with the words "odd" and "even", the visual mediators, and narratives that make up the task situation carries—"social meanings in the new [deductive] discourse [involving abstract generalisation]" (p. 3).

6.3.2.4 The Puppet. The task design also involved the Teacher-Researcher's use of Jerry, a puppet (Chapter 3, Figure 5), who remained a group member for the duration of the lesson. Jerry was introduced as a friend of the characters in the concept cartoon, and it was his job to report back to his friends and settle their disagreement following the help he received from the students. Jerry (under the guise of the Teacher-Researcher) had a legitimate reason to press the students for more reasoning by playing the role of an unconvinced sceptic. He could be engineered to ask probing questions, such as "How do you know there isn't a pair of odds that make an odd number?" or "I don't really understand what you are saying. Could you show me what you mean in a drawing?"

6.3.2.5 The "Exploration-Enabling" Nature of the Task Situation. I considered that, with appropriate mediation, the learners' explorative initiation into the

deductive discourse would be enabled according to Cooper and Lavie's (2020) criteria:

1. *Initiation:* In terms of the initiation, my aim was not to teach the students how to prove generally, and so the learners' goal would not be to complete a procedure according to some given proving criteria. Instead, they would be motivated to decide which narrative they should endorse, and which they should reject, in order to help Jerry in resolving his friends' disagreement (as shown in the concept cartoon). Had the cartoons featured just two binary narratives, learners would have been more susceptible to believing that their task could be completed with empirical evidence, where, by process of elimination, disproving one binary narrative would suggest the truth of the other. Hence, the inclusion of a bilateral narrative— "odd + odd = sometimes odd and sometimes even"—was to provide the students with a further motivation to help Jerry and friends understand why "odd + odd" is *always* even and not just *sometimes*.
2. *Procedure:* In terms of endorsing narratives about the sums of odds and evens, students would not be told which narrative was true or false. Instead, they would be charged with independently deciding which narrative to endorse and which to reject. Equally, in providing substantiation narratives for their decisions, students would not be given a procedure (substantiation routine) to apply ritualistically. While their existing numeric routine could still be used to reject invalid binary narratives, the students would potentially be able to draw on the precedent-search-space of selecting pairs of specific numbers (as they had done when working with Numicon tiles in substantiating "odd + odd = even" and, instead, perform this routine using the nonspecific Numicon strips). Hence, their story involving the Numicon strips would serve as a bridge between an empirical and a deductive discourse.
3. *Closure 1:* In terms of the closure, the completion of the task goes beyond simply selecting a character's narrative to endorse; the students would be required to provide reasoning to substantiate endorsing a narrative in order to convince themselves, one another, and Jerry the puppet (in the guise of the Teacher-Researcher).
4. *Closure 2:* The students (rather than the Teacher-Researcher) would determine whether and when the task had been completed successfully. This would be indicated by the group being in agreement and being able to substantiate why, for example, the narrative "odd + odd = even" should be endorsed, and not "sometimes even and sometimes odd" (i.e., they would be able to account for all

cases). The role of Jerry the puppet is especially useful here as the teacher has the capability of engineering Jerry's responses so that he can claim to be confused, ignorant, or sceptical. Since Jerry is positioned as a part of the student group (a peer), the students would need to wait for Jerry to be convinced by their reasoning in order to determine the task as being complete. In other words, while the students would determine when the task is complete, the situation would also be within the Teacher-Researcher's control.

Of course, the task only had the *potential* to be exploration-enabling. For instance, there was the possibility that students may struggle to endorse the Numicon strips as realizations of generic oddness and evenness and insist they are signifiers of particular odds and evens. And even if students endorsed the odd strips' features as being consistent with all odd numbers (and the even strips' features as being consistent with all even numbers), they might still insist on numeric substantiations for universal narratives. Or one student might observe that another student performed a routine with the Numicon strips that was sanctioned by the puppet (which they recognise as being under the guise of the Teacher-Researcher and therefore deem as having been sanctioned by an authoritative figure) and, upon invitation from the Teacher-Researcher to provide substantiation, simply replicate the procedure they had observed—perform a ritual. Hence, while I considered the design of the task to be exploration-*enabling*, I recognised that this did not make it exploration-*requiring*. The teacher would have a vital role in drawing out the interdiscursive features of the task to enable the students' discursive transformation via questioning, re-voicing, directing, and probing students' reasoning—rather than simply modelling a new discourse for the students to adopt for themselves. Notably, this view regarding the role of the teacher deviates from that of the traditional commognitivist viewpoint.

6.3.3 Data Analysis

As with the previous studies, I present three episodes of dialogue and utilise Sfard's (2008) commognitive framework to examine and describe the students' performances. In my analysis of the first episode (Episode 11), I also assess the degree to which the task is exploration-requiring according to Cooper and Lavie's (2020) criteria. And, in analysing the third episode (Episode 13), I look for indicators to suggest that the students were participating exploratively in the new deductive

discourse and were intrinsically motivated and outcome-oriented. Accordingly, I turn to Lavie et al.'s (2019) list of explorative characteristics in examining the extent to which students' routines can be considered exploratory rather than ritualistic: flexibility, bondedness, applicability, performer's agentivity, objectification of the discourse, and substantiability. However, given that the characteristic of "applicability" requires observing a routine being performed by a person in a range of task situations (on different occasions, in different environments, with different group members) and this chapter's study is concerned only with the students' initial performances in a new discourse, this characteristic was omitted.

Since pure rituals and explorations are not to be expected in mathematics classrooms (Lavie et al., 2019) I did not expect the students in my study to be purely explorative in the new deductive discourse, nor was it my goal to show such an occurrence within the student data I gathered. Rather, I expected my assessment of the students' performances to fall somewhere along a ritual-exploration continuum. Due to the interdiscursive nature of the learning environment in which the students were initiated into the new discourse, I hoped there would be more evidence to show that the students were intrinsically motivated to perform the routine in order to endorse a new narrative (outcome-oriented) than there would be evidence of the students' attention being on the performance itself (the procedure). That is to say, I hoped there would be explorative characteristics within the performances of the students' routines.

Observing the ways in which the students responded to the task situation would be crucial in determining whether and how the interdiscursive learning environment resulted in meta-level learning. Accordingly, the research questions asked in this study are:

(i) How do the students engage with exploration-enabling features of the task (e.g., the bilateral narrative and the Numicon strips) and how does this engagement result in a discursive change, and thus, meta-level learning?

(ii) What markers are there to suggest that the students' performance in the new deductive discourse is exploratory (intrinsically motivated and outcome-oriented) rather than ritualistic (socially motivated and process-oriented)?

6.4. Findings

The extracts and analyses I provide show three steps in an interdiscursive learning environment. The first episode (Episode 11) shows the emergence of a commognitive conflict within the group, where the students are in need of assistance in bridging the empirical-deductive gap in discourse and are motivated to consider a new discourse. I consider this to be a "sweet-spot" for interdiscursivity to occur. The analyses then focus on the participation of Jane and Zara, whose procedure can be seen to undergo a discursive transformation. The second episode (Episode 12) shows pivotal teaching moments where interdiscursivity—the intentional blending of discursive features (words and visual mediators) from two distinct discourses— is used by the teacher in assisting the students with communicating in more general terms about odd and even numbers. And the third episode (Episode 13) shows the students' uptake of these new discursive features in using a mid-way discourse with regards to substantiating "odd + odd = even."

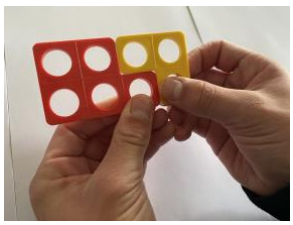
6.4.1 The Interdiscursive "Sweet-Spot"

The first section of the findings examines how a commognitive conflict arises as the students initially respond to the narratives presented in the first concept cartoon ("odd + odd"). In particular, the following exchanges provide evidence of the students initially engaging in empirical discourses and addresses the role that the bilateral narrative plays in the students' exchanges.

In Episode 11, the Teacher-Researcher has just introduced the "odd plus odd" concept cartoon to the students. As each character's narrative is read, both Danny and Jane endorse Hana's narrative "odd + odd = even", and Jane rejects Manu's "odd + odd = odd." However, when the students read Benny's bilateral narrative "odd plus odd = sometimes even and sometimes odd," Danny rejects it, and Jane becomes uncertain.

Episode 11

How a Bilateral Narrative Triggers Jane's Dissatisfaction With Empirical Substantiation

Turn	Speaker	What was said	What was done
240.	JANE:	Yeah, that's right. [Referring to Hana's narrative "odd + odd = even"]	
241.	DANNY:	I was right, it's right.	
242.	T.R.:	What about Manu? ["odd + odd = odd"]	
244.	JANE:	It's wrong.	
…	…	…	…
246.	T.R:	Whereas Benny says…	
247.	JANE:	Odd plus odd equals sometimes even and sometimes odd.	Reading Benny's statement in the cartoon.
248.	DANNY:	No wrong. See watch this.	Holds up two Numicon tiles of three and fits them together to make six.
249.	JANE:	I sometimes think that's right [Referring to Danny's demonstration of "odd + odd = even"], because, um it might be— 'cos three and three—but what if there's another number and we add them together?	
250.	DANNY:	Okay, watch this.	Takes one of the Numicon 3 tiles and rotates it to fit with Numicon 5. 
…	…	…	…

253.	JANE:	So, because an odd and odd, like a three and three, equals odd [probably meant even] but if you put it together, they're a small number... But some odd numbers are like really big—we don't, we can't even count them—they might not, they might be odd but can't go even."	
…	…	…	
282.	ZARA:	Okay, so two odd numbers can sometimes make an even number.	
283.	T.R.:	Only sometimes?	
284.	ZARA:	We-elllll?	
285.	DANNY:	Always.	
….	…	…	….
288.	JANE:	Not ***gotta*** always	
289.	DANNY:	Yes, it's gotta always …	
290.	JANE:	No, I mean it doesn't ***always*** have to.	
291.	DANNY:	Yes, it does.	
292.	JANE:	No, it doesn't.	
293.	DANNY:	Yes, it does.	
…	…	…	…
295.	DANNY:	It does. I'm smarter than you.	

This episode shows that in terms of the task's initiation and procedure, it appears that the students are invested in determining which narrative should be endorsed; they have not been told by the Teacher-Researcher which narrative is correct, nor have they been given a procedural way to follow in terms of substantiating their choices. The words "odd" and "even" trigger the students' precedent-search-space, which seemingly comprises these words as descriptors of particular numbers. Hence, in terms of Danny's task (the task situation as interpreted by him), and from his perspective, his substantiation, which entails presenting corresponding specific cases, is appropriate.

However, what we also see in this episode is the emergence of a commognitive conflict: Jane and Danny do not share criteria for endorsing universal narratives about the sums of two odds. To substantiate his endorsement of "odd + odd = even" [238] and rejection of "odd plus odd equals sometimes even and sometimes odd"

[248], Danny uses Numicon tiles to visually mediate the even sums of two pairs of specific odds [248; 250]. But with regard to the bilateral narrative, Danny's numeric examples only prove the "sometimeness" of "odd + odd = even" and are silent concerning possibilities where "odd + odd" turns out to be odd. For Jane, the bilateral narrative facilitates a need in her to question the binary narrative "odd + odd = even" and triggers her dissatisfaction with small-number, numeric examples. Her use of the word "sometimes" in her comment, "I *sometimes* think that's right" [249], mimics Benny's bilateral narrative in the concept cartoon and suggests that she is aware that Danny's demonstration does not account for *all* other cases and therefore fails to provide grounds for her to reject this narrative. Whereas Danny's demonstrations of two numeric cases of "odd + odd = even" are enough for him to endorse this universal narrative (and complete his task), for Jane, reliance on small number examples appears insufficient. The "rules of Danny's game" are not adequate for the game Jane wishes to play, which is informed by her concern around managing numbers that are "really big, we don't, can't even count them" [253]. It is this need to account for numbers beyond ones that are countable that opens the door for shifting away from an empirical discourse.

A debate between Jane and Danny over whether "odd + odd" is "always even" or only "sometimes even" continues [282–295]. In terms of closure, the students are evidently in control of deciding when the task is complete. The excerpt shows that it appears still incomplete for them because they do not agree on which narrative should be endorsed, nor do they agree on the endorsement criteria. With no discursive apparatus for them to talk about "really big numbers" [253], Danny directs the conversation towards who is smarter [295]. As noted by Sfard (2007), "entering into foreign forms of talk (and thought) requires a genuine interest" (p. 607), and, triggered by the bilateral narrative, the students now (at the end of Episode 11) seem to have a need to establish whether "odd + odd" is always even or just sometimes. Furthermore, they also have opened the space to consider another discourse with new rules, as their current discourse is limited in its productiveness to settle their disagreement. Jane, Zara, and Danny's lack of discursive apparatus to resolve the conflict creates the space for introducing, and a need for transitioning to, a new and foreign discourse—this is the sweet-spot for interdiscursive mediation to occur.

6.4.2 The Blending of Discursive Features

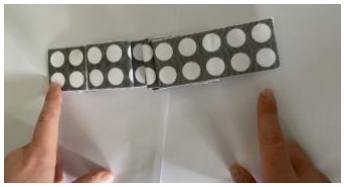
The previous section revealed the students' engagement in a number-specific (empirical) discourse on odd and even numbers. The students are at a point where meta-level learning is required because they need to enter a deductive discourse where general realizations of odd and even are required to make universally valid conclusions. However, "Left to their own inventiveness, the newcomers are likely to reach a dead end" (Ben-Zvi & Sfard, 2007, p. 131). The analysis presented in this section, therefore, illustrates a task situation that bridges the conflict as facilitated by the Teacher-Researcher. This section addresses the interdiscursive initiation into a deductive discourse for two students (Jane and Zara).

In Episode 12, the Teacher-Researcher takes advantage of Jane's uncertainty over whether the narrative "odd + odd = even" could be endorsed for numbers that are "really big, we don't, can't even count them" [253] as an opportunity to introduce an odd Numicon strip.

Episode 12

Introducing the Odd Numicon Strip

Turn	Speaker	What was said	What was done
298.	T.R.:	Is it an odd number or an even number?	Holds an odd Numicon strip.
299.	JANE:	Odd.	
…	…	…	
303.	T.R.:	So, Jane, you think it's an odd number. What about you Robert?	
304.	ROBERT:	Erm… yeah, yeah.	
305.	ZARA, JANE:	Yeah.	
306.	T.R.:	How do you [addressing the group] know it's an odd number?	
307.	ROBERT:	Because of that there.	Robert, Jane, and Zara point to the unpaired circle at the end.

308.	T.R.:	What are you pointing at Jane?	
309.	JANE:	'Cos that, it's like that. It's nine. And nine's an odd number.	Traces the extra one, then counts the circles visible on the top layer.
310.	T.R.:	It's nine. Did you need to count it to recognise it's odd?	
311.	JANE:	No.	
312.	T.R.:	What were you looking at, that showed you it's odd?	
313.	JANE:	Cos if it's like that, it would be even. But since there's one more.	Covers the end circle. Uncovers the end circle.
314.	T.R.:	Ah, so it's like the extra one that makes it odd, is that right?	
315.	JANE:	Yeah	
…	…	…	…
320.	T.R.:	What will another odd number look like?	
321.	JANE:	One.	
323.	DANNY:	It could be three. It could be five, it could be seven, it could be…	
…	…	…	…
328.	T.R.:	Would that be an odd number?	Points to the even Numicon strip.
329.	JANE, ZARA, ROBERT together:	No.	
330.	T.R.:	No. How did you know straight away?	
331.	JANE:	Because there're no extra bits.	Points to the ends of the even Numicon strip. 
332.	T.R.:	Ah. Okay so we'd need two that kinda look like this [i.e.,	Holds up the long odd Numicon (folded) strip.

		two odd Numicon strips to represent "odd + odd"]? Zara, are you happy with that?	
333.	ZARA:		Nods
334.	JANE:	Yes and (inaudible) there's the same amount of dots on each one [side],	Points to the pairs on the even Numicon strip.
		but that one there, there's one more on one side.	Points to the extra circle on the odd Numicon strip.

In this episode, instead of focusing on particular odd and even numbers (as the students had in Episode 11), there is evidence of the students (Jane, Zara, and Robert) attending to the general symmetrical and asymmetrical structure of even and odd numbers. Jane, Zara, and Robert all immediately attend to the "extra one" in substantiating the oddness of the Numicon strip [307], and they visually reject the even Numicon strip as signifying an odd number [329]. Jane visually mediates evenness by covering the "one more" of the odd Numicon strip to mediate evenness and uncovering it to mediate oddness [313]. Jane also uses nonspecific words and narratives— "one more" [313] and "one more on one side" [334]—for oddness and— "no extra bits" [331] and "same amount of dots on each one" [334]—for evenness. Hence, the students' engagement with the Numicon strip seems to have been successful in refocusing their attention to the generic structure of oddness and evenness.

There are a number of key features evident in this episode that I claim helped to create an interdiscursive environment conducive for students to transition from a numeric (specific) discourse about odd and even numbers to talking more generally about (nonspecific) oddness and evenness. Note, I am not claiming that the students have transitioned to a deductive discourse on odd and even numbers at this stage; I am only suggesting there is discursive evidence of some students moving in this direction. Having introduced the Numicon strips to the students, the Teacher-Researcher pushes the students to endorse them as "numbers" (and thereby specific signifiers) [298; 306; 320; 328] and, as such, she grounds the meaning of the strips within a (numeric) discourse in which the students are already familiar. This move assists these newcomers to nonspecific reasoning in taking their first steps across the commognitive conflict between an empirical discourse and a deductive discourse on

odd and even numbers. However, the Teacher-Researcher also intentionally steers the students towards seeing the general, rather than the particular, in these strips in several ways.

First, the Teacher-Researcher presses the students to identify and focus on generic structural features visible in the strips; that is, she asks the students what "*show*[s] … it's odd" and asks them to provide a verbal narrative for their visually mediated realizations of oddness [308; 312; 320]. Second, she pushes for a nonspecific, *visual* substantiation routine—which can be performed immediately—versus a specific, *counting* substantiation routine—which requires more time— in endorsing the strip as odd: she asks, "How did you know *straight away*?" [330]. Third, the Teacher-Researcher builds on the students' visually mediated substantiations of oddness and evenness, providing new, nonspecific words and narratives for the students to endorse and potentially adopt for themselves in future substantiations of oddness: she re-voices Jane's keywords "one more" [313] as an "extra one" within the narrative, "it's the extra one that makes it odd right?" [314].

Finally, the Teacher-Researcher actively redirects the students from specific to nonspecific ways of talking. This is evident when two complementary discourses emerge (numeric and generic) in substantiating a Numicon strip as odd ("nine" versus the "extra one" [309]). Here, having just pointed to the extra circle on the end of the odd strip [307] to substantiate endorsing it as being odd, Jane proceeds to check its oddness by counting the dots on the top of the fold to confirm that it is "nine" [309]. In other words, despite having used a visual substantiation to confirm the Numicon strip's oddness according to its asymmetrical structure, Jane seems to seek reassurance that this is the case with a numeric substantiation—number nine. Jane can be described as using hybrid-substantiating narratives here, appearing to come from two distinct discourses—generic narratives from a deductive discourse (in the first instance) and numeric/specific narratives from an empirical discourse (in the second instance). At this point, Jane needs a discursive push to abandon her numeric substantiation (i.e., a need to count) in favour of a generic-only substantiation routine. Accordingly, the Teacher-Researcher signals the "need to count" part of her routine as being unnecessary [310] and that visually substantiating

with “extra one” is legitimate now. The Teacher-Researcher thereby prompts the students to select generic talk as the leading discourse [310].

This redirecting teacher move is also evident when students return to numeric ways of talking [323]. Here, the Teacher-Researcher asked the students to consider what another (hypothetical) odd number would “look like” [320] with regard to “odd + odd”, hoping that the students will consider the generic structural composition of odds according to the odd Numicon strip they have just encountered. That is to say, she hopes that they will adopt the generic substantiation narratives that had just been offered. Instead, Danny proceeds to list particular odd numbers [323], expanding the scope of examples that he perceived Jane to have offered in her response [321] to this question. To redirect the students, the Teacher-Researcher reintroduces a nonspecific visual mediator (an even Numicon strip [328]), reminding them of less specific ways of signifying oddness and evenness and leading them back towards talk about nonspecific odd and even numbers.

6.4.2.1 Summary. The findings so far show that the task situation has made interdiscursive use of elements of the students’ familiar number-specific discourse to support them in taking initial steps in an emerging generic discourse. The introduction of the Numicon strip was intended to draw on precedents from a familiar number-specific discourse, but the Teacher-Researcher’s deliberate promotion of the generic properties of oddness and evenness was simultaneously intended to expand the students’ precedent-search-space so that students might interpret the concept cartoon task in a different way. Now, when students are considering the narrative “odd + odd = even,” the words “odd” and “even” may trigger generic realizations of oddness and evenness (i.e., asymmetrical and symmetrical structures) instead of (or as well as) labels for particular numbers. Expanding the students’ discourse about odd and even numbers in this way has laid the groundwork for students to use generic examples in explorative ways to generalise about the sums of odds and evens.

6.4.3 Explorative First Steps in a Deductive Discourse About the Sums of Odds and Evens

Having presented findings that show how Jane and Zara were steered towards nonspecific realizations and generic substantiations of oddness and evenness, I now

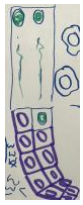
turn to whether, and how, the students use these with regards to universal statements about the sums of odds and evens. In the following episode, the Teacher-Researcher prompts Jane to use her previous narrative, regarding a second nonspecific odd having "one more" [Episode 12, 334], to substantiate the structure that would occur if a second odd number were added.

Episode 13

Jane and Zara's Generic Substantiations for "Odd + Odd = Even"

Turn	Speaker	What was said	What was done
335.	T.R.	So … She [Jane] said that with odd numbers there's one more and with another odd number is there going to be an extra one again?	
…	…	…	…
337.	JANE:	…if it was like three then we could put it there.	Picks up Numicon 3 tile and connects it with the long odd Numicon strip with the odd ends fitting together.
		And then it's not gonna be like that and there's gonna be one left no more.	Removes the Numicon 3 tile and runs her finger around the asymmetrical end of the odd Numicon strip.
338.	T.R.:	Ah! He says is it like a jigsaw puzzle so they kind of fit together?	Jerry (the puppet) whispers.
339.	JANE, ZARA, ROBERT together:	Yeah.	
…	…	…	…
341.	ZARA:	Yes, so if you have something like a square. If you have something like this.	Draws a rectangle.
342.	JANE:	A rectangle.	
343.	ZARA:	Yes, it's an oblong. So, if you have like two circles on each then it will be even.	Draws two circles in the rectangle.

		And just keep on going down.	Draws two lines going down from each of the circles.
		But if you added on an extra one here, then it wouldn't be even.	Draws the extra circle (bottom-right).
…	…	…	…
345.	T.R.:	I see. He [Jerry] says can we use another colour to show how another odd number would fit with that?	Jerry (the puppet) whispers.
346.	JANE:	Yes.	Picks up a pen to add onto Zara's drawing.
347.	ZARA:	So, then if you put like another one ["one" is taken here to mean another "odd"] there then it would be even.	Referring to her drawing.
…	…	…	…
354.	T.R.:	Yeah. Okay. So, Danny, see if you agree. Zara said, there's an even number here.	Danny looks up from his own writing. T.R. points to the rectangle
		And what makes it odd is this extra one there.	
		So, we don't know what number it is, but we can tell it's odd because it's got this extra one there. Is that right?	Points to the extra circle.
355.	ZARA, DANNY, JANE, together:	Yeah.	Danny returns to his own work.
356.	T.R.:	Jane was going to add onto that to show how when we add on another odd number, it doesn't matter what it is…	Jane draws one extra purple circle in a square to complete the rectangle.

357.	ZARA:	It's even.	Jane continues to add pairs of purple circles in squares to extend the figure. 
358.	T.R.	Could we just keep going on and on and on and on and on? [Referring to the pairs of circles Jane is drawing]	Jane continues extending the pairs of purple circles in the drawing.
359.	ZARA:	Yep, and it wouldn't matter because it's still even.	

In this episode, we see a number of changes have occurred in terms of the students' performances. First, both Zara and Jane now endorse the binary narrative "odd + odd = even", and there is no evidence of uncertainty about the narrative they endorse. This observation marks a substantial shift from Episode 11 where, with only a number-specific discourse at their disposal, Zara was undecided [284] and Jane endorsed Benny's bilateral narrative— "odd + odd = sometimes even and sometimes odd." Second, the Teacher-Researcher's deliberate elevation of the generic properties of oddness and evenness in Episode 12 seems to have successfully expanded Jane's and Zara's' precedent-search-space so that they have been able to reinterpret the task situation "odd + odd = even". In Episode 11, the students' task was to 'select a narrative' while drawing from a number-specific precedent search-space. Now, in Episode 12 their task is to provide a substantiation for their endorsement, and both Jane and Zara are seen to be drawing from an altered precedent-search-space which includes generic realizations of odd and even.

Note that the Teacher-Researcher still has a role to play in influencing the students' interpretation of the task situation here. Through the guise of Jerry, the puppet, she asks the students to draw "another odd *number*" [345] so that the two figures "fit together…like a jigsaw puzzle" [338]. Here, her use of the word "number" thus prompts an action (to perform a calculation) from the students' precedent-search-space drawn from a numeric discourse; yet the generic shape of the figure is nevertheless appropriate from the perspective of the emerging deductive discourse.

The Teacher-Researcher also provides a narrative for the students' actions and re-voices their own narratives [354; 356], emphasising key interdiscursive features for the students to endorse and hopefully adopt for themselves. For instance, when she says, "we don't know what number it is, but we can tell it's odd because it's got this extra one there," she establishes Zara's drawing as a number but also emphasises its genericity by establishing it as an unknown quantity.

With the support of the Teacher-Researcher, and now armed with new discursive tools to talk more generically, Zara and Jane have altered their procedure. They have tweaked a familiar routine (performing a calculation for "odd + odd") from their numerical discourse and, instead of using two number-specific visual mediators and numeric words, they perform a similar substantiation routine but using nonspecific visual mediators and generic words. Initially, Jane chooses a numeric example (Numicon 3 tile) to add to the odd Numicon strip, and, by running her finger around the exterior of the two visual mediators, she visually mediates the generic even (rectangular) shape created by the two odds. This gesture can be viewed as Jane divorcing from the counting-the-dots approach that was previously evident [Episode 12, 309]. The Teacher-Researcher draws attention to the infinite scope of Zara and Jane's drawing to elevate the genericity of this visual mediator, asking, "Could we just keep going on and on and on and on and on?" [358]. And Zara's reply [359], "Yep, and it wouldn't matter because it's still even", shows she endorses the genericity of the visually mediated substantiation for "odd + odd = even".

6.4.3.1 The Nature of the Learners' Participation. Using Lavie et al.'s (2019) characteristics in analysing Jane and Zara's performance (their performance post Episode 12), I will now point to several markers that suggest their participation in this new discourse is explorative.

Within Episode 13, we see the students' new, more generic, substantiation routine performed twice. Jane substantiates "odd + odd = even" using a Numicon 3 tile and an odd Numicon strip [337], and thereby combines visual mediators from two distinct discourses: number-specific from an empirical discourse (with respect to the Numicon 3 tile) and generic from a deductive discourse (with respect to the Numicon strip). Then, Zara and Jane collaboratively perform the substantiation routine using two generic "rectangular + 1" drawings. In each of these performances, the use of

"if…then…because" conditional statements in both Jane [337] and Zara's [341; 343; 347; 359] narratives (as they visually mediate the asymmetry of the odds they connect together and visually mediate the symmetry of the even result) shows that the output of each step is connected to the input of the next and towards the outcome. Thus, the "bondedness" of the steps within the girls' routine points towards explorative (outcome-oriented) participation. While the procedures are similar, they also vary to some degree too—they are "flexible". Remember, too, that Jane and Zara already realize that small number numeric examples (Episode 11) are consistent with "odd + odd = even". Hence, Jane and Zara are showing that they realize there is now more than one way to perform the same task and produce the same outcome, and the "flexibility" of their routine further suggests explorative participation.

Jane and Zara's participation can also be described as "agentic" in several ways. First, both Zara and Jane independently perform further variations of this routine during the course of the lesson focused on the "odd + odd = even" concept cartoon. Second, Zara's use of the generic words "square" [341] and "oblong" [343] and Jane's "rectangle" [342] are new words for the student group—they had not been previously offered by the Teacher-Researcher. Hence, Zara and Jane appear to have personalised the new nonspecific ways of talking offered so far, which represents an agentic move from a number-specific discourse. Third, the drawings that Zara and Jane collaborate on are also somewhat personalised versions (rather than direct replications) of the Numicon strip.

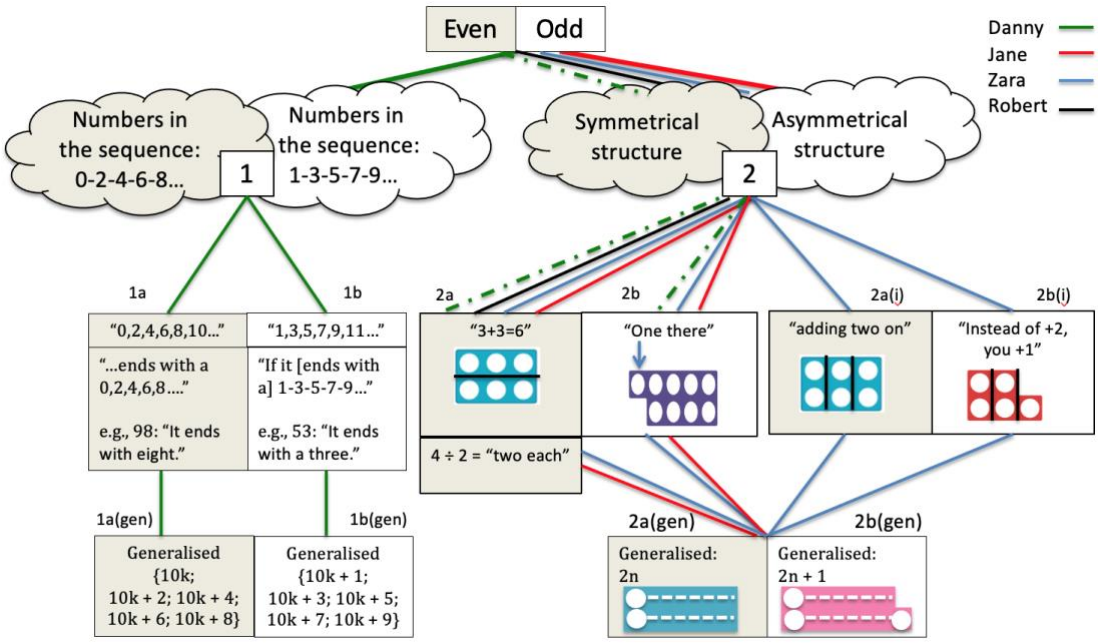
The girls' drawings also point to a degree of "objectification" of the discourse—replacing talk about processes with talk about (abstract) objects. Admittedly, their discourse is not fully objectified: both students continued to describe a process as being performed by an actor— "we," [337]; "you" [341; 343; 347]—and with concrete objects. However, while the words "odd" and "even" have not changed from signifying concrete objects to signifying abstract objects, the concrete objects themselves changed: from numbers to shapes. This is a key step: to be able to objectify odd and even as abstract entities requires a means to endorse all odd numbers and all even numbers as being conflated into one all-encompassing unit. Furthermore, by drawing lines from each of the dots that "just keep on going down" [343], Zara also implies that her drawings of even and odd each represent a number

continuing indefinitely. Where it may have been possible (although time-consuming) to count the pairs of dots on the Numicon strip to establish the particular number it signified, Zara's drawing offers an even more abstract realization of even. Zara moves towards highlighting the *multiple of two* structure of *any* even number (and how adding "an extra one" is *any* odd number). This may be interpreted as signalling her first independent steps in a deductive discourse on odd and even.

6.4.3.2 Summary. The growth in Jane and Zara's realizations of odd and even are shown visually in Figure 12. Their trees now have branches (red for Jane and blue for Zara) that extend to generalised realizations of odd and even (realization boxes 2a(gen) and 2b(gen)).

Figure 12

Janes and Zara's Meta-Level Learning in a Structure-Based Discourse



Although Jane and Zara's discourse has transformed, they cannot yet be considered to be participating fully in a deductive discourse on odd and even numbers. Jane and Zara's task is still to perform a calculation for "odd + odd," yet their routine is now deemed appropriate in the eyes of the teacher because, from the perspective of the emerging deductive discourse, their substantiation routine signifies "*any* odd + *any* odd = even." Jane's previous doubts about whether "odd + odd = even" holds for big numbers "we can't even count," are put to rest because the words "odd" and "even" are no longer just properties of particular numbers; they are asymmetric (odd) and

symmetric (even) structures, and it is now the generic structure (the symmetry of even and asymmetry of odd), rather than specific numbers, that is the object in their substantiating narratives. Admittedly, odd and even are not yet fully objectified—they are still concrete objects—but they are of a different, more generic form. Moreover, the process of objectification is expected to occur gradually over time (so Jane and Zara would not be expected to be talking in an objectified manner), meaning these signs of objectification at such an early stage in a new discourse may, in fact, be considered to be promising and surprising signs of explorative participation. Jane and Zara might be considered to be participating exploratively in a discourse on odd and even numbers that is neither fully empiric nor fully deductive. They are participating in a midway discourse consisting of generic words and visual mediators within which they can now account for *all* cases of odd and even numbers in endorsing "odd + odd = even" using a routine from a pre-existing discourse.

6.5 Discussion

Transitioning from accepting and using empirical reasoning to using deductive proofs for universal narratives about the sums of odds and evens involves meta-level learning, where learners come to view and use words like "odd" and "even" as abstract objects in their own right rather than as labels for particular numbers. The commognitive stance has traditionally considered that at meta-level junctions like this, learners may have little intrinsic motivation to adopt the new rules and so only ritualized (socially motivated and process-oriented) initiation into the new discourse is possible. However, taking inspiration from Cooper and Lavie's (2020) notion of interdiscursivity and their principles of exploration-enabling tasks, I designed a task to support learners in taking explorative (intrinsically motivated and outcome-oriented) first steps in a deductive discourse on odd and even numbers. And where Cooper and Lavie (2020) provided theoretical accounts of how discursive transformations might transpire with the implementation of exploration-enabling tasks, I have complemented the theory with an empirical account of how two students engaged with the task, which shows how interdiscursivity supported their explorative first steps in a new discourse.

For the task situation to do its interdiscursive job in enabling explorative participation, I showed the importance of the "sweet spot". In order to take

explorative first steps in a new discourse, students need to be intrinsically motivated in their initiation. We saw how a bilateral narrative triggered Jane's dissatisfaction with small-number, numeric examples. It was this need to account for numbers beyond ones that are countable, along with the group's lack of discursive apparatus to resolve the commognitive conflict created, which opened the door from an empirical discourse towards transitioning towards new and more generic ways of talking. Had the cartoon simply included binary narratives, a numerical example that endorsed one narrative would imply the rejection of the other, and the student group may have remained in agreement and seen no need for new ways of talking. Hence, rather than hoping that students may intuitively and spontaneously question the certainty of binary narratives, the inclusion of bilateral narratives in tasks involving endorsing and substantiating universal narratives may help to facilitate doubt within a learner (or group of learners) and elevate the possibility of generating an intrinsic motivation (rather than social motivation) to consider new ways of talking. This learning environment "sweet spot" helps interdiscursive mediation to take place.

We also saw the usefulness of the Numicon strips, which were designed to be open to two interpretations—a particular number and generic evenness (or generic oddness)—with the goal of supporting learners in reinterpreting a task situation to elicit a different action. But the introduction of a new visual mediator like this cannot be considered a panacea that will magically transform students from numeric to generic ways of talking. In accord with other researchers' stances (Cooper & Lavie, 2020; Ben-Zvi & Sfard, 2007; Nachlieli & Tabach, 2019), the empirical account clearly showed the crucial role that the teacher plays in facilitating students' discursive transformations. The Teacher-Researcher deliberately grounded the mediator within a discourse with which the students were already familiar: by referring to the strips as "numbers", the students were able to draw on precedents from a familiar *number-specific* discourse and use their *perform-a-calculation* routine. But she also highlighted the genericity of the strips and prompted the students to select generic ways of talking as the leading discourse by: (i) pressing the students to identify and focus on generic structural features visible in the strips; (ii) pushing for a nonspecific, *visual* substantiation routine; (iii) providing new, nonspecific words and narratives for the students to endorse and potentially adopt for themselves; and (iv) redirecting the students from specific to nonspecific ways of talking.

These teacher moves served to expand the students' precedent-search-space so that when the students subsequently considered the narrative "odd + odd = even," the words "odd" and "even" triggered generic realizations of oddness and evenness (i.e., asymmetrical and symmetrical structures) instead of (and as well as) labels for particular numbers. Without the guidance of an expert, the interdiscursive potential of tools like the Numicon strips would likely be lost. The possibility of the strips signifying generic oddness and generic evenness would unlikely be realized because, for performers in an empirical discourse, they would only signify particular numbers. It should not go unnoticed that the role of the teacher described here contrasts with the traditional commognitivist view, where modelling a new discourse for their students to adopt is considered critical. Of course, this is because, in this study, the goal of teaching was not to transform rituals into explorations—it was to enable explorations from the very beginning.

Through the interdiscursive use of these new visual mediators, the students were encouraged to interpret the task situation in a way that would elicit an action from their precedent-search-space (drawn from their familiar numeric discourse) that was also appropriate from the perspective of the emerging deductive discourse. Indeed, within the space of the three episodes, Zara and Jane's performances changed quite substantially. They modified a familiar routine (performing a calculation for "odd + odd") from their numerical discourse and performed it using nonspecific visual mediators and generic words, which meant they could account for numbers beyond those that were countable.

There were several markers suggesting the students' performances of this modified routine were more explorative than ritualistic. The steps in the students' routines were "bonded" to each other and towards the outcome of endorsing "odd + odd = even." The slight variations in their routines also marked flexibility. Furthermore, I pointed to Zara's "personalisation" of generic words and visual mediators—a characteristic that I claim is a possible sub-category of Lavie et al.'s (2019) "performer's agency"—as signalling their exploratory participation. And although there was no evidence of the students alienating the procedure (Jane and Zara continued to describe the procedure as being performed by actors) or reifying the words "odd" and "even" (the use of these words did not change from signifying

concrete objects to signifying abstract objects), the concrete objects themselves changed from numbers to shapes—and thereby they changed from many discrete objects to one all-encompassing object. Hence, the task appeared to have fulfilled its goal of enabling these students' explorations in a new discourse.

However, consistent with Cooper and Lavie's (2020) disclaimer, it should be stressed that I am not suggesting Jane and Zara were participating exploratively in a fully deductive discourse. Rather, from their perspective, the students were participating in a familiar discourse with new discursive tools and, from the perspective of the Teacher-Researcher, they were participating exploratively in a midway discourse, somewhere between empirical and deductive. The students were drawing on familiar precedents of performing a numeric calculation but were performing the routine with generic visual mediators and thus were able to talk in more general terms. Although the students were not aware of it, these new ways of talking could at some point come to have new meaning: the students could, with further support, come to learn that proving universal statements deductively requires talk about abstract objects.

It is important to note that, in order to fulfil my aim of showing how interdiscursivity could support learners' discursive transformations, I intentionally selected data coming from two students whose meta-learning journey was both successful and relatively concise. Yet a similar story could not be told about all the students involved in the lesson. Indeed, the reader may be somewhat satisfied by the markers indicating Jane and Zara's explorative participation yet may be left wondering about the other two group members' participation and learning. With regards to Robert, little can be claimed about his learning since, throughout the lesson, he remained largely quiet and passive. Of course, his lack of verbal contributions does not necessarily imply he did not learn—self-communication (thinking) and interpersonal communication are indeed perceived as united within the commognition framework. However, within a commognitive framework, research on human development is the study of the development of discourses (Sfard, 2008, p. 124), and the unavailability of his interpersonal discourse to see or use makes it difficult (near impossible) to determine the extent of Robert's development.

Danny, on the other hand, was certainly an active participant in this lesson, yet the implementation of this task and interdiscursive efforts of the Teacher-Researcher were admittedly not so successful in ushering him into generic ways of talking. An absence of discursive mechanisms to engage Danny dialogically help explain the shortcomings of the lesson with regard to Danny, but this story is reserved for Chapter 7 of the thesis. Nevertheless, this acknowledgement provides a cautionary note with regard to falsely assuming that exploration-enabling tasks result in meta-level learning for all. Some students' meta-level learning journeys may be longer and more complicated, while some may not make such journeys at all. Moreover, even when students like Zara and Jane could be seen exploratively participating in a midway discourse, there was often evidence to show them fluxing between this and their more familiar empirical discourse. Hence, while exploration-enabling tasks may support learner's explorative initiation into a new discourse, one should be mindful that these are indeed only first steps: learners will undoubtedly require further guidance in taking further steps towards fully embracing new ways of talking and leaving old ways behind.

Nonetheless, while previous research has shown meta-level learning to commence with ritual performance and the process of de-ritualisation to be often gradual and slow (e.g., Sfard & Lavie, 2005; Lavie & Sfard, 2016), the findings of this study provide an alternate take on the learning journey. Meta-level learning, it appears, can commence in more exploratory ways, which may result in more meaningful learning experiences, and the process may involve both *pro-explorisation*—becoming increasingly exploratory via more exploration-enabling opportunities to learn—as well as de-ritualisation. The findings suggest that the interdiscursive design and implementation of tasks can be critical in opening exploration-enabling opportunities to learn and in moving from a ritualised initiation into a new discourse. Whether such a pathway turns out to be a shorter cut to meta-level learning than the ritual-to-exploration route remains to be seen.

CHAPTER 7: Research Study 4:
How a Discursive Connection Facilitated Danny's Dialogical Engagement.

7.1 Introduction

The belief that maths talk somehow facilitates mathematics learning is well established and seems to be gaining in popularity in terms of research, developments in classroom practice, and educational policy (e.g., J. Hunter, R. Hunter, & Restani, 2020; Leftsein & Snell, 2014; Mercer, Hennessey & Warwick, 2017; NCTM, 2000; O'Connor & Michaels, 2019; Topping & Trickey, 2015). Dialogic education or discourse-based pedagogies are described by Leftsein and Snell (2014) as learning that is: *collective*—teachers and children address learning tasks together; *reciprocal*—they share ideas and consider alternative viewpoints; *supportive*—they help each other reach common understandings; *cumulative*—they build on their own and each other's ideas; and *purposeful*—the talk has specific educational goals (pp. 23–24). That is to say, multiple voices, authored by individual students, are expressed and listened to with the goal of advancing collective thinking and reaching consensus. The benefits of dialogical approaches to learning have been described as, "non-hierarchical, democratic, pluralistic, ethically and culturally sensitive, and inherently egalitarian" (Kennedy, 2009, p. 71) and as supporting the emergence of "deep connected mathematical understandings" (J. Hunter, R. Hunter, & Restani, 2020, p. 178). However, Sfard (2019) observed that the connection between communication and learning is often assumed and oversimplified. Sfard (2019) was critical of a general "unshaken confidence" (p. 89) in the benefits of classroom talk, and she pointed to her own empirical accounts (i.e., Sfard & Kieran, 2001; Chan & Sfard, 2020,) as evidence of *dis*engaged interaction and *un*productive communication occurring in classrooms. Hence, a need for the advancement of theory to underpin dialogic pedagogies was signalled.

Indeed, as was acknowledged in the story presented in the previous two chapters, a discursive division between two students (Jane and Zara) and a third student (Danny) remained stubborn despite much communication occurring. Danny was committed to his own discursive ways. Sfard (2019) offered the notion of *dialogic*

engagement as a means of providing more detail about *what kinds* of communication are helpful and *when* as well as theoretical insights into *how* it supports learning. Accordingly, the study in this chapter addresses two aims of this thesis: it investigates *what helped Danny's initiation into a structure-based deductive discourse on odd and even numbers* and seeks *to explore whether his first steps in this discourse could be described as explorative.*

7.2 The Commognitive Framework

7.2.1 Dialogic engagement

Sfard's (2019) dialogic engagement provides a lens through which to look at the relationship between learning and communicating. At the heart of this perspective is the rejection of monologic ideas of "truths" to be learnt in favour of viewing learning as being the process of constructing knowledge via active participation. Such a view is inspired by Bakhtin's (1986) call for multi-vocality and dialogic (persuasive) voices over monologic (authoritative) voices. For Bakhtin, engaging dialogically meant allowing one voice to shape another. Sfard also acknowledged Buber's (1923/1958) *I-it* and *I-thou* distinction in the origins of her notion of dialogic engagement: the former assumes an external "objective" or "true" perspective, while the latter assumes at least two perspectives where one's mind can be changed as a result of seeing something through the eyes of another. Similarly, Wegerif (2013) contrasted monologic theories of educational progress that assume theories become replaced and superseded by new ones with dialogic notions: he argued that progress in a dialogue does not tend to occur in a linear "from A to B" manner but usually through augmentation "from A to both A and B" (p. 29).

Building on these ideas of dialogicalism, Sfard (2019) defined dialogic engagement as being when a learner: (i) is aware of, and remains alert to, the incommensurability between their own discourse and the new one (especially in cases of persistent disagreements); (ii) tries to figure out the rationale of the other discourse and reasons for favouring it (beginning with emulating the new discourse); and (iii) strives to adopt the new discourse for themselves (learns to think according to the rules of the new discourse) (p. 95). However, Sfard suggested that dialogic engagement might be vital for some types of learning but not for others: her approach separates learning into two categories and explains how different types of

learning call for different kinds of interaction. The following section provides an overview of the process and types of learning accounted for within the commognitive framework and describes how Sfard's (2019) definition of dialogic engagement might be used as a means to examine the interaction of participants and explain how and why learning might occur or fail to occur.

7.2.2 The Process of Learning

The process of learning mathematics is seen commognitively as routinization or individualising historically developed routines (Lavie et al., 2019). Having observed others commonly performing routines in certain places and at certain times, learners gradually become "independent, agentive implementers of those routines" (Lavie et al., p. 156). In particular task situations, particular precedents—past situations considered sufficiently similar to the present one—will be triggered for a learner that will help them with determining what routine to perform. Different participants facing the same task situation may respond differently because they may be using different precedent identifiers (features of the present task considered sufficiently similar to a previous one to justify acting in a similar way). Hence, Lavie et al. (2019) defined a routine as a participant's interpretation of the task situation together with the procedure performed in response.

Learning can be at the object-level or metal level. Since object-level learning is a matter of simply growing realizations for mathematical objects, Sfard (2019) claimed that "in principle" this type of learning could be achieved via intrapersonal communication (p. 92)—that is to say, interpersonal conversation is optional rather than necessary. In contrast, Sfard (2019) viewed interpersonal talk as being "indispensable" for meta-level learning to occur (p. 92) since a learner adopting the new incommensurable discourse would need to revise previously endorsed narratives, and the *way* in which they would come to endorse narratives would also need to change. Commognitive conflicts—communicative interaction whereby participants use the same signifiers (words or symbols) in different ways or perform the same mathematical tasks according to different rules— (Sfard, 2008, p. 162) are seen as playing an essential role in learning, as they provide space for participants to consider new ways of talking. Resolving a commognitive conflict involves one of the participants gradually accepting and adopting the incommensurable discourse and

abandoning his or her own, which occurs through the learner's exposure to, and participation in, the discourse he or she is supposed to individualise and the support he or she receives from other participants.

According to Sfard (2019), in order to attempt meta-level learning (and come to endorse and adopt the incommensurable discourse), the learner must engage dialogically. Hence, in analysing apparent learning successes or failures, one should be asking whether participants were dialogically engaged. Note that the implication here is that dialogical engagement might *not* be necessary for object-level learning. Furthermore, while Sfard provided an operational definition of dialogic engagement and suggested that it is vital for meta-level learning, *how* such engagement might be fostered is unclear. Sfard (2019) suggested that choosing to engage dialogically may "seem to be a matter of moral commitment" (p. 95); however, she admitted that more work was needed to theoretically and empirically account for how dialogic engagement might commence and be sustained. Correspondingly, in this chapter, I focus on one student, Danny, participating within a group context, as he engaged in proving activity regarding the sums of odd and even numbers. I use Sfard's (2008) commognitive framework to explain the type of learning that Danny underwent, and put Sfard's notion of dialogic engagement to work empirically as a lens through which to explain *when* and *how* learning occurred or failed to occur.

7.3 Conduct of the Study

In this section, I describe the story so far in terms of Danny's discourse compared with Jane and Zara's. Details about the task situations and their implementation are omitted since these have been provided in previous chapters. I conclude the section by articulating two research questions that guide this study and providing a brief overview of the data analysis.

7.3.1 The Story So Far

In Chapter 5, I used the Grove of Realizations tool to expose two sources of commognitive conflict within the student group: object-level and meta-level. While the majority of the students within the group (Jane, Zara, and Robert) held structure-based realizations of odd and even numbers based on numbers' asymmetry (odd) or symmetry (even), Danny held sequence-based realizations of odd and even.

Consequently, Danny's sequence-based substantiation routine for endorsing numbers' oddness and evenness conflicted with the ones of other students. The content-specific communicational conflict was shown here to exist at the object-level. However, a communicational conflict between Danny and the other students was also shown to exist at the meta-level (a commognitive conflict). Zara and Jane were shown to be lacking discursive tools to talk about all cases of even and all cases of odd, but Danny was equipped to substantiate "odd + odd = even" and "even + odd = odd". Danny had encapsulated infinite cases of odd and even numbers into two finite sets of numbers—numbers that ended in 1,3,5,7,9 (for odd) and 0,2,4,6,8 (for even)—and he performed a kind of "proof by exhaustion" with these digits. Hence, his realizations of odd and even were generalised.

Subsequently, in Chapter 6, I showed how Jane and Zara's need for new discursive tools to resolve the commognitive conflict opened the door for shifting away from an empirical discourse; they had an intrinsic motivation to consider another discourse with new rules, as their current discourse was limited in its productiveness to settle their disagreement. Accordingly, the interdiscursive learning environment facilitated by the Teacher-Researcher was successful in ushering them into new and more generic ways of talking—it supported Zara and Jane to engage in meta-level learning. Nevertheless, in Section 6.5, I also acknowledged that while the interdiscursive learning environment had been successful in ushering Jane and Zara into new ways of talking, it had been a different story for Danny.

This section of findings now picks up from where the reader was left at the end of Chapter 5 in terms of the story of Danny's discursive development. In particular, it aims to examine why, if learning at the object-level is considered easier than that at the meta-level, Danny's participation in structure-based talk on odd and even numbers was neither easy nor straightforward? My analysis is led by the following research questions:

> *(i) To what extent, and for what reasons, does Danny participate in structure-based talk on odd and even numbers?*
>
> *(ii) To what extent is Danny dialogically engaged?*

7.3.2 Data Analysis

As with the previous studies, I present episodes of a dialogue and utilise Sfard's (2008) commognitive framework to examine and describe Danny's performances. To address the first question and analyse the extent of Danny's participation in structure-based talk, I look for (i) growth in Danny's realizations that extend to structural realizations of odd and even numbers (which I present in Grove of Realizations visuals using green branches for Danny (red for Jane and blue for Zara)) and (ii) evidence to suggest whether his participation is more ritualistic or explorative. While the reasons I suggest for his participation in a structure-based talk are somewhat speculative, I look for indicators of Danny's task (his interpretation of the task situation) to support my suggestions. To assess how dialogically engaged Danny is, I use Sfard's (2019) definition of dialogic engagement and ask:

- Is Danny aware of and alert to the communicational gap between his talk and the other students'?
- Is he trying to figure out the rationale of the other mode of talk and reasons for favouring it?
- Is he striving to adopt the new mode of talk for himself?

7.4 Findings

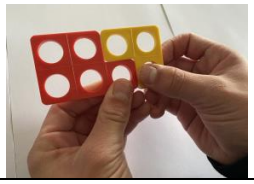
I provide three sections of findings, which form three parts to the story of Danny's participation in structure-based talk and the degree to which he engages dialogically changes. The first section analyses episodes of dialogue taken from the group's initial encounter with the first concept cartoon ("odd + odd"), which show Danny participating briefly in structure-based talk in a manner that I describe as "ritual-ish." The second section examines episodes of dialogue following Jane and Zara's initiation into a deductive discourse on odd and even numbers that builds on their structure-based realizations. Here, Danny's participation in structure-based talk ceases temporarily and I examine the change in his task to account for his reluctance to participate in structure-based ways of talking about odd and even. Then, in the third section, episodes of Danny's dialogue are used to show *when*—for what reasons—Danny participates in structure-based talk and *how*—to what extent—his participation changes.

7.4.1 Part 1. Danny's Initial and "Ritual-Ish" Participation in Structure-Based Talk

7.4.1.1 Danny's Reasons for Engaging in Structure-Based Talk. Episode 14 provides dialogue that is useful in helping identify Danny's task when he participates in structure-based talk on odd and even numbers for the first time. Previously, during the classification task when Numicon 5 was turned over, Danny had said, "I know something. Two odds make an even." [110]. Now, in Episode 14, when the "odd + odd" concept cartoon task situation is introduced, Danny immediately endorses "odd + odd = even" [239], before the Teacher-Researcher even has chance to read the three narratives, and he reminds the group that he has already provided this narrative independently [242; 244; 246].

Episode 14

Danny's Surprising Structure-Based Substantiation

Turn	Speaker	What was said	What was done
238.	T.R.:	All right. Jerry has some friends in Livraria. He's got a friend called Hanna, and Manu…	Points to first character on sheet.
239.	DANNY:	Hanna's right ["odd + odd = even"].	Interrupting.
240.	T.R.:	… and Benny. And they've all got some different ideas…This is what Hanna said.	Points to Hanna's speech bubble.
241.	JANE:	Yeah, that's right. [Referring to Hana's narrative "odd + odd = even"]	
242.	DANNY:	I was right, it's right.	
243.	T.R.:	What about Manu? ["odd + odd = odd"]	
244.	DANNY:	'Cos I already did it for you guys.	
245.	JANE:	It's wrong. ["odd + odd = odd"].	
246.	DANNY:	I already did it for you guys.	
247.	JANE:	Odd plus odd equals sometimes even and sometimes odd.	Reads Benny's conjecture in the cartoon.
248.	DANNY:	No wrong. See watch this.	Refers to Benny's narrative. Holds up two Numicon 3 tiles and fits them together.
249.	JANE:	I sometimes think that's right, because, um it might be- 'cos three and three -but what if there's another number and we add them together?	Referring to Danny's demonstration of "odd + odd = even".
250.	DANNY:	OK watch this.	Takes one of the Numicon 3 tiles and rotates it to fit with Numicon 5. 

253.	JANE:	So, because an odd and odd, like a three and three, equals odd [probably meant even] but if you put it together, they're a small number... But some odd numbers are like really big—we don't, we can't even count them—they might not, they might be odd but can't go even."	Danny shakes his head and quietly says "No".
…	…	…	…
255.	DANNY:	[Interrupting the T.R.] I learnt this since I was six!	

Upon first inspection, Danny's substantiations to endorse "odd + odd = even" and reject the bilateral narrative seem surprising: they appear to be consistent with the other students' structural-based talk and seem incongruous with Danny's previous dominantly sequence-based talk. By arranging two Numicon tiles in a symmetric 2 x 3 array [248], and then exchanging one of the Numicon 3 tiles for a Numicon 5 tile to make a symmetric 2 x 4 array [250], Danny twice offers a numeric example to reject the statement "odd plus odd equals sometimes even and sometimes odd." Although, Danny's numeric examples provide insufficient grounds to reject Benny's narrative (they only prove the "sometimeness" of "odd + odd = even" and are silent with respect to possibilities where "odd + odd" turns out to be "odd") and fail to convince Jane to reject it (as evidenced in by her response [253]), they are consistent: both examples provide numeric and visual substantiations that appear to attend to the even (or symmetrical) composition resulting from adding two odd (asymmetrical) tiles.

Note, though, that Danny's visual substantiations show how the two asymmetric tiles fit to make a symmetric array but are devoid of any accompanying spoken narrative [248; 250], except for him saying "watch this". A corresponding verbal narrative might have addressed the oddness of the two numbers selected, the evenness of the resulting sum, and possibly how or why these examples substantiated endorsing "odd + odd = even" and thus rejected the bilateral narrative. The absence of such a narrative coupled with the speed of his endorsement of "odd + odd" [239] and rejection of the bilateral narrative [244; 248], suggests that Danny considers the even outcome of adding two odds to be so obvious that no explanation is required. Indeed,

Danny's narratives and performance so far provide evidence to suggest that he has entered the concept cartoon task situation from a different place to Jane and, correspondingly, his task differs to hers (and the other students in the group). Jane has intuitive hunches about the narratives presented [241; 245] but has not yet decided which should be endorsed; hence, her primary task seems to be to decide which narrative should be endorsed. However, for Danny, the statement "odd + odd = even" is old news [255]. So, for him, the task appears not to be about determining which narrative should be endorsed, but about convincing the others that he is right [242; 244; 246]; that they, too, should endorse "odd + odd = even".

If convincing the others that he is right (i.e., "odd + odd = even") is indeed Danny's task, his choice of routine makes more sense. It was evident during the classification task that the other three students agreed on structure-based substantiations to endorse a number's evenness or oddness (this was illustrated by multiple branches stemming from branch 2 on the grove of realizations in Chapter 5, Figure 8—leaving him largely alone in his sequence-based talk. Hence, Danny may well be aware of his peers' preference for talking "structure" (not "sequence"), and so his structure-based substantiation may represent a compromise on his talk of choice, which he is willing to make in order to relate to them. At this point, I remind the reader of the previous activity of classifying numbers as odd or even described in Chapter 5, in which Danny initially rejected the other students' structure-based talk and then attuned slightly to this model of talk whilst simultaneously positioning it as inferior to his own. For example, he said "I know, I know but that's a way you can check" [Episode 15: 120] with regard to Jane's "eleven and eleven" substantiation for why 22 was even. In other words, as illustrated in the grove of realizations for the group (Chapter 5, Figure 9 by the dashed green lines, there is evidence that Danny *could* engage in structure-based talk, but he chose not to because he viewed this mode of talk as inferior and as offering him nothing mathematically. The reason behind Danny's surprising structure-like substantiation in this episode, therefore, may actually reflect his task: he may be attempting to talk the talk of his peers in order to convince them of something he is already sure of.

7.4.1.2 The Nature of Danny's Participation in Structure-Based Talk. In Episode 15, the Teacher-Researcher invites Danny to repeat his previous visual

demonstration, one that she had perceived as Danny's *structure-based* substantiation [270]. She predicts that Danny will show how the visual mediators' (two Numicon 3 tiles) odd structures connect to make an even shape, symmetrical in structure (a 2 x 3 array) and hopes that he might provide structure-based substantiating narratives to accompany his visual demonstration this time. The further hope is that this will resonate with the other students (since their substantiating narratives for oddness and evenness have so far been structure-based). But although Danny uses the Numicon tiles to provide a visual substantiation, his spoken substantiation does not pertain to structure or shape at all: Danny is substantiating that his endorsement of "odd plus odd" (three plus three) is "even" because the resulting sum (six) is in the "[even] sequence, two, four, six" [271]. He reconfirms that the sequence is "actually the most important thing" [272], not, as the teacher was hoping he would say, the symmetry of the sum's structural composition.

Episode 15

Danny's "Part-Structure-Part-Sequence-Based" Routine

Turn	Speaker	What was said	What was done
267.	T.R.:	Ok. So, Danny, you did something with those pieces before to kind of show that odd and odd made even. Could you show us that again?	
268.	DANNY:		Grabs the two Numicon pieces of three and puts them together to make an array of 2 x 3.
269.	T.R.:	Can you see what Danny's doing [addressing the other group members]? So, you've taken two odd numbers there, how has that proven that it's even?	
270.	DANNY:	'Cos, erm three plus three is six and six is even. Cos, it goes in a sequence two, four, six.	
271.	JANE:	What's a sequence?	
272.	DANNY:	Yeah, 'cos it's actually the most important thing.	

This routine of Danny's [268; 270] can be described as "part-structure-part-sequence-based": his actions (connecting two odd Numicon tiles) signify the structural composition of odd and even, but his narratives are still very much sequence-based. However, the steps in this substantiation routine do not appear to be "bonded" to one another—the output from his actions does not feed into the latter steps—rather, his initial act of connecting the two Numicon tiles is superfluous in achieving the final outcome (i.e., endorsing "six is even" [270] because it is in the sequence of even numbers). He also maintains a level of hierarchy between the sequence-based parts of this routine and the structural-based parts by restating that *sequence* is "actually the most important thing" [273] (not structure): "it's always the sequence" [259]; "you just need the digits" [266]; "The digits are like the most important" [266]. Hence, his participation in structure-based talk is ritualistic and might be described as him paying "lip-service" to the other students' structure-based talk. Danny makes a concession in enacting a part-structure-based routine simply to support his task of convincing the others rather than having any intrinsic motivation to adopt it for himself.

At this point, I posit that his performance of this part-structure-part-sequence routine might be described as being *ritual-ish*. In ritual performances, participants are concerned with *how* to proceed rather than the result and are *socially motivated* to adhere to goals set by others. In Danny's case, his performance is consistent with rituals in that his routine consists of unbonded steps—so it is *process*-oriented—and he desires to talk the same talk as his peers—therefore, he is *socially* motivated. Paradoxically though, he can actually be described as being *intrinsically* motivated and *outcome*-oriented to perform this ritual—his task is to convince the others they should agree with him and endorse "odd + odd = even", and performing the ritual is a means to this end.

7.4.1.3 How Dialogically Engaged is Danny? Danny's willingness to perform a structure-based routine in his interaction with other students shows that he is aware of communicational gap (sequence-based versus structure-based) between his talk and theirs. However, the ritual-ish nature of his performance suggests he is not yet striving to figure out the logic of the other mode of talk, and there is no evidence to suggest he sees reasons to favour it. Rather than striving to adopt the others' mode of

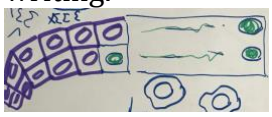
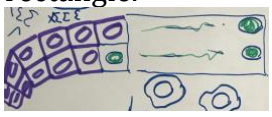
talk as a discourse for himself, Danny only appears to be motivated to adopt it for the sake of the others.

7.4.2 Part 2. Danny's Resistance to Structure-Based Talk

The story now shifts to when the Teacher-Researcher has introduced the Numicon strips to the student group and has facilitated Jane and Zara's use of these nonspecific (but structure-based) visual mediators in their numeric routines (see Chapter 6, Section 6.4.3). During this part of the group session, there were very few instances where Danny actively participated at all. Instead, Danny mainly worked silently, independently, and according to his own pre-existing sequence-based talk. In the following episode, Zara and Jane provide structure-based substantiations to endorse "odd + odd = even", and the Teacher-Researcher makes several deliberate efforts to dialogically engage Danny in the discussion involving new generic ways of talking. She asks Danny if he agreed with the structure-based substantiations [354], if they made sense to him [365], and if he would revoice the substantiations [367]. But Danny resists these invitations by briefly indicating agreement before removing himself from the discussion and returning to his own work [355] and re-directing the discussion to "more complicated" [358] yet "simpler" numeric ways of talking [368], In doing so, he maintains an elevated positioning of his talk over the one being offered to him.

Episode 16

Danny's Resistance to Structure-Based Talk

Turn	Speaker	What was said	What was done
354.	T.R.:	Yeah. Ok. So, Danny, see if you agree. Zara said, there's an even number here and what makes it odd is this extra one there. So, we don't know what number it is, but we can tell it's odd because it's got this extra one there. Is that right?	Danny looks up from his own writing.  T.R. draws the blue rectangle with two blue circles and lines going down (as Zara had done in the previous drawing). T.R. draws the connecting blue circle in the blue square (left).
355.	ZARA, DANNY, JANE:	Yeah	Danny returns to his own work.
355.	T.R.:	And then… hold on, hold on… and then Jane was going to add onto that to show how when we add on another odd number, it doesn't matter what it is…	Jane draws one extra circle in a square to complete the rectangle. 
356.	ZARA:	It's even	
357.	T.R.:	It's going to be even. Could we just keep going on with that?	Danny looks towards T.R. and Jane's work. Jane continues to add pairs of purple circles in squares to extend the figure.
358.	DANNY:	I was right. I did it this way. I just did the complicated way.	Holding up his written work.
…	…	…	…
365.	T.R.:	Alright let's stop there. So, before we go onto yours Danny, does that make sense what Zara and Jane have just said?	
366.	DANNY:	Yeah.	
367.	T.R.:	How are they proving that an odd and odd will always make even?	
368.	DANNY:	Because always like... you don't have to use the ones. There's actually a quicker way.	

This episode shows the meta-level learning that Jane and Zara had made (in Chapter 6) (as signalled by the new branches 2a(gen) and 2b(gen) for Jane and Zara on the Grove of Realization tool, Chapter 6, Figure 12). As Chapter 6 showed, Jane and Zara's task was to produce a new endorsed narrative regarding the sum of two odds. Therefore, they had an intrinsic and outcome-oriented motivation to adopt the Teacher-Researcher's interdiscursive tools because these new tools (i) accounted for all cases of odd and even (where their discursive tools had not) and (ii) aligned with their structure-based realizations of odd and even. In contrast, Episode 16 shows the Teacher-Researcher fails to dialogically engage Danny's participation in a deductive discourse that builds on structure-based realizations of odd and even numbers. I posit that this episode reveals a change in Danny's task. Initially, Danny aimed to convince the other students (Jane, Zara, and Robert) that he was right— "odd + odd = even"—and his ritual-ish participation in structure-based talk was a means to this end. Now, Danny, Jane, and Zara endorse the same narrative ("odd + odd = even"), and they each have discursive tools (albeit different ones) to substantiate this. Danny recognises this situation [358] and, accordingly, his previous reason for participating in structure-based talk disappears—he no longer needs to convince his peers that "odd + odd = even", so this motivation for 'talking their talk' is redundant. Danny's participation suggests he has no desire to be ushered into new ways of talking—his talk works just fine! Seemingly, his task now is to position his sequence-based ways of talking as being superior (i.e., more "complicated" [358] and "quicker" [368]).

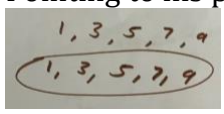
7.4.2.1 How Dialogically Engaged is Danny? In terms of his dialogical engagement at this point in time, Danny is aware of the communicational gap between the two modes of talking, but I suggest he is resistant to endorsing and adopting structure-based ways of talking for a couple of reasons. Danny is not motivated in the same way as Jane and Zara were because: (i) the generic tools offered by the Teacher-Researcher do not align with his realizations of even and odd so adopting them would require him to divorce from his own sequence-based talk; and (ii) he has no need for structure-based talk, as he already had discursive tools to talk about odd and even generally. This was signalled by the solid green branches *1a(gen)* and *1b(gen)* for Danny extending to generalised realizations of odd and even, (Chapter 6, Figure 12). In short, Danny has only *social* reasons—and no *mathematical* reasons—to engage dialogically, so he chooses to continue in his own discursive ways.

7.4.3 Part 3. Danny's Explorative Participation in Structure-Based Talk

This final section of findings provides evidence of Danny endorsing and adopting structure-based substantiations for "odd + odd = even" for the first time. To explain the change in his participation, I look at how the situation and his task have changed again. I also examine Danny's participation for signs of him participating in a structure-based routine exploratively.

Episode 17

Jane Builds on Danny's Proof For "Odd + Odd = Even"

Turn	Speaker	What was said	What was done
472.	T.R.:	Was there an argument we used before that helped us prove for any number, that it didn't matter which one [which pair of odd numbers] it was?	
473.	DANNY:	The last digit. One, three, five, seven, nine plus one, three, five, seven, nine. Any [pair] of these numbers equals even.	Pointing to his paper. 
…	…	…	…
479.	JANE:	What we're trying to say is like… This is still odd. Because if you actually put them together, they will actually be even, because there's no, nothing pointing out and nothing uneven. Just, they're even and you can share it.	Holds up a Numicon 9 tile. Takes two Numicon 9 tiles and shows how they slot together.
480.	DANNY:	That's how you can prove it.	
481.	T.R.	That's how you can prove it right because those bits kind of go together?	
482.	ZARA:	Yeah.	Jane and Danny nod.
…	…	…	…
488.	JANE:	So, if there's two bits there that are actually even but if you put one, it's uneven, but if you put another, the same piece together it'll be even,	Holding a Numicon 9 tile. Points to the extra circle on the bottom-right of the tile.

		there's no bits there, you can share it, evenly.	Connects the Numicon 9 tiles to make a 2 x 9 array (as before).
…	…	…	…
493.	DANNY:	No, I just need to explain this. So, it's like divided by two … but…	Takes two Numicon 9 tiles to make a 2 x 9 array.
		if you make another 5 it's same, it's fourteen, divided by two, it's seven. It's always, and then it's	Replaces one of the Numicon 9 tiles with a Numicon 5 tile and forms a 2 x 7 array.
		even one, even three.	Replaces the Numicon 5 tile with Numicon 1 tile and then Numicon 3 tile.
494.	JANE:	Even this.	Takes a Numicon 7 tile and connects it with a Numicon 9 tile.
…	…	…	…
496.	DANNY:	[Interjecting] Any number.	

7.4.3.1 Danny's Reasons for Engaging in Structure-Based Talk. In this episode, Danny repeats his "last digit" (proof-by-exhaustion) substantiation for endorsing "odd + odd = even". However, when Jane performs a substantiation using two Numicon 9 tiles to form an even shape with "nothing pointing out and nothing uneven" [481], Danny [480] now endorses Jane's structure-based narratives. Danny then begins to systemically demonstrate that Jane's procedure (involving a number "nine") can be performed with each and every odd number in his list (9+9; 9+5; 9+1; 9+3) [493], and Jane completes the demonstration (9+7) [494]. I now describe how Danny has made an apparent U-turn regarding his previous rejection of and resistance to structure-based talk by claiming Jane's procedure has provided Danny with what I term a *discursive connection*: a discursive opportunity for linking the two realizations— structure-based and last-digit (or sequence-based)—together:

(i) Danny's last-digit substantiation reduces all cases of odd to two sets of five digits (1,3,5,7,9) [473].

(ii) Importantly, Jane's selection of two Numicon 9 tiles represents a selection of two numbers with the last digit "9" from Danny's lists.

(iii) Jane shows that selecting two such numbers from Danny's list form a shape with, "nothing pointing out and nothing uneven… and you can share it" [479].

Previously for Danny, adopting his peers' structure-based substantiations required him to exchange his talk for theirs, and their discursive ways appeared to provide no mathematical reason for him to do so. But now, Danny is able to show that any combination of odds from his "last-digit" checklist (1,3,5,7,9) not only produces a sum with an even last digit but *also* are all divisible by two [493]. In other words, Danny can perform a structure-based substantiation routine *without divorcing* from his sequence-based routine; rather, the new routine *utilises* his sequence-based realizations. The discursive connection creates a space where Danny coalesces the structure-based routine with his own sequence-based one.

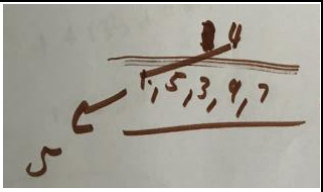
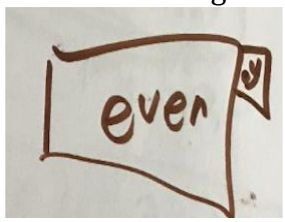
Furthermore, in incorporating this structure-based step in his substantiation routine, Danny has the discursive tools to simplify his proof-by-exhaustion. Previously, Danny had reduced infinite cases of odd to five realizations—$10k+1$, $10k+3$, $10k+5$, $10k+7$, $10k+9$—and was thereby able to reduce his substantiation routine for infinite cases of "odd + odd" to just 15 possible cases ([1+1], [1+3], [1+5], [1+7], [1+9], [3+3], [3+5], [3+7], [3+9], [5+5], [5+7], [5+9], [7+7], [7+9], [9+9]). Now, he realizes a connection between his sequence-based realizations of odd and even and the others' structure-based realizations and, by encapsulating these realizations of odd (from $10k+1$; $10k+3$, $10k+5$, $10k+7$, $10k+9$) to $2k+1$, Danny can complete his routine in one step. In terms of his dialogical engagement, where Danny previously only had social reasons to endorse and adopt structure-based talk, he now has a mathematical reason to favour this alternative and appears to be striving to adopt it for himself.

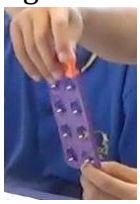
7.4.3.2 The Extent of Danny's Participation in Structure-Based Talk on Odd and Even Numbers. I now examine Danny's performances in both Episodes 17 and 18 for further evidence of Danny coalescing the two modes of talking and endeavouring to turn the alternative talk into a discourse for himself as well as for features that point

to his explorative participation in this new deductive discourse. Accordingly, Episode 5 provides instances of Danny substantiating his endorsement of "even + odd = odd".

Episode 18

Danny's Hybrid "Structure-and-Sequence-Based" Routine for "Even + Odd = Odd"

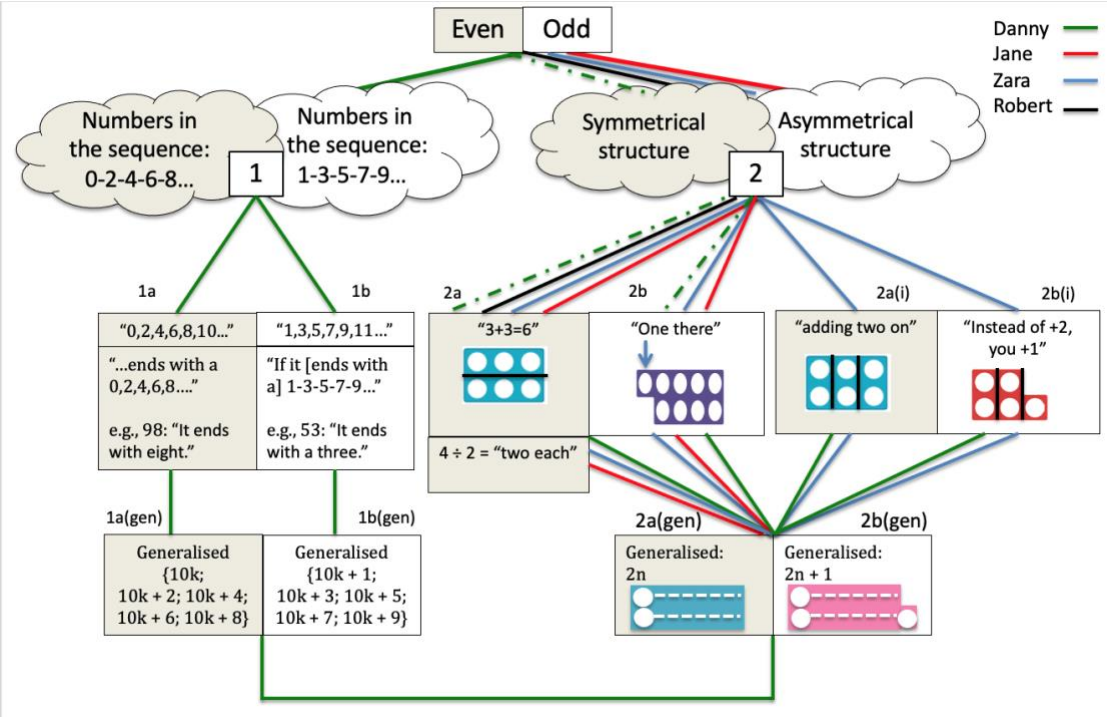
Turn	Speaker	What was said	What was done
598.	DANNY:	Because, um, if you try um four plus one is five, you can try all of these numbers	 Writes 1, 5, 3, 9, 7, and then writes 4 above this list. Connects the 4 and 1 with a line and draws an arrow to the result "5."
		but it is just the one that matters. If there's the one sticking out. There's always like, there's always a gap.	Places Numicon 8 and 7 tiles together.
608.	DANNY:	You have to like; you have to put a picture in mind. Like this.	Draws a rectangle. 
		This is like even. This is a odd.	Labels the rectangle "Even". Draws an extra one to show odd. Labels the extra one "*o*" (for odd) and "*y*".
620.	DANNY:	'Cos it's always sticking out. I know something. If the picture is in your mind. So, if there is like, this looks like a machine gun shooter.	Holds up a Numicon 9. Points to the top extra one on Numicon 9.

		It always is odd. But if it's like covered it looks like, like just like a rectangle. It's always even. And it looks like that. It looks like this. It has a one here and it looks like a gun, it is an odd, plus an even.	Holds and points it like a gun. Takes a Numicon 1 and adds it to the Numicon 9 to for a rectangle.  Shows the Numicon 9 like a gun. Connects Numicon 10 to Numicon 9.
647. … 649.	DANNY: … DANNY:	Ah, I got the best idea… … Technically there's always a one sticking out. So, if you put a even …it's like two, two, two, two, two, two, two: it goes always forever; it goes like long.	Smiles broadly. … Holds up one finger as he says "one". Writes '2,2,2,2,2' Points to the long even Numicon strip.
… 651.	… DANNY:	… Yeah, and then and then you put an odd, how…Who, yeah… whoever cares how much this is like, this will always be odd because this is a odd. And this is, this part doesn't make a difference.	… Points to the extra one. Points to long part of the odd Numicon strip (the rows of two).

Episode 18 shows expansions in Danny's realizations for odd and even. Accordingly, there is now evidence to show a connecting branch between Danny's generalised sequence-based realizations and generalised structure-based realizations in the grove of realizations (shown by a solid green branch from 1a and b(gen) to 2a and b(gen) in Figure 14). Danny endorses sharing or "divided by two" [493] and "multiple of two", substantiations of even "*like two, two, two, two, two, two, two*" [649], and a rectangle-like structure to visually mediate evenness [608; 620]. So, Danny's realizations of even relate to both forms of 2*n* and *n*2 (shown by green branches linking 2a(gen) with realizations of even numbers at 2a and 2a(i)). Similarly, Danny's "one sticking out" and "always a gap" [598] substantiations of odd, and the figures "like a gun [620] he uses to visually mediate odd, show his realizations of odd are of the form 2*n*+1 [620] and *n*2+1 [649] (shown by green branches linking 2b(gen) with realizations of odd numbers at 2b and 2b(i)). Danny has encapsulated his "last-digit" realizations of even (10*k*; 10*k*+2; 10*k*+4; 10*k*+6; 10*k*+8) and odd (10*k*+1; 10*k*+3; 10*k*+5; 10*k*+7; 10*k*+9) as all being in the form 2*n* and 2*n*+1. The growth in Danny's tree of realizations (the additional green branches within the Grove of Realizations, Figure 13) mark learning for Danny at the object-level.

Figure 13

Danny's Expansions in Structure-Based, Generalised Realizations for Even and Odd



7.4.3.3 The Nature of Danny's Participation in Structure-Based Talk. In contrast to Danny's ritual-ish performance of the part-structure-part-sequence-based routine at the beginning of the session (Episode 14), I claim there is now evidence that Danny is participating in a deductive *and* structure-based discourse on odd and even numbers exploratively. First, where the steps were unbonded in his routine before, the ways in which he uses the Numicon tiles [620] and the Numicon strip [649] are geared towards the *outcome* of proving the generalizable outcomes and the structure-and-sequence steps in his routine are "bonded". Second, Danny expresses an urgency to explain how the result of adding any two odd Numicon tiles will make it possible to divide by two [493] quickly. He proceeds to claim this (divisibility of two) is the case for *any* pair of odd numbers [496]; this represents an "agentic" move (he does not require prompting from others to perform this routine) from his previous claim that the tiles formed a *number* in the "sequence" of evens [270] and that the sequence was "the most important thing" [272]. Agentivity in his routines is further evidenced in Episode 18 by the stretches of his routine (involving structure-based steps) becoming longer, and he no longer waits for invitations from the other group members to perform this routine.

Third, Danny shows "applicability" in his performance of this new routine: having previously performed the routine (in Episode 17) to substantiate "odd + odd = even", he also performs it several times (in Episode 18) to substantiate "even + odd + odd." Fourth, he refers to his previous proof by exhaustion— "you can try all of these numbers" [598]—but he now also uses Numicon tiles to demonstrate how the combinations of these numbers result in "one sticking out". Danny is now realizing that the, hitherto unrelated, structure-based proving routine can be used to perform the same task: he is developing "flexibility" in his routine ways of substantiating his endorsement of universal narratives involving the sums of odds and evens. Fifth, Danny's visual mediation of the structure-based (Numicon-like) even [608] is also more abstract (and thus more "objectified") than any visuals that have been offered by other group members. Zara had previously drawn something similar—her drawing included two dots with lines extending from each one to represent the "2*n*" of even—but Danny's version is simply a rectangle. Furthermore, Danny can be seen adopting a structure-based narrative—rectangle—for even that had previously been offered by Jane, but also by giving odd "gun-like" properties, he adds a novel take on

the structure of odd. Hence, his performance here displays characteristics of "personalisation" of the discourse (something I claimed in Chapter 6, Section 6.5 as being a sub-category of Lavie et al.'s (2019) performer's agency).

7.4.3.4 How Dialogically Engaged is Danny? In Sections 7.4.1 and 7.4.2, Danny was considered to be dialogically disengaged. He was aware of the communicational gap between his discourse and that of the other students, but there was no evidence to suggest he trying to figure out the rationale of their structure-based talk. Although there were some small signs of Danny using such talk, he appeared to be motivated to do this for the sake of the other students so that they might endorse his routine—he was not striving to adopt it for himself. Now, however, because Danny can see how the structure-based talk connects with his own, he appears to be striving to adopt it for himself. He has mathematical reasons to favour the new mode of talk. I posit that, at this point, Danny is engaging dialogically.

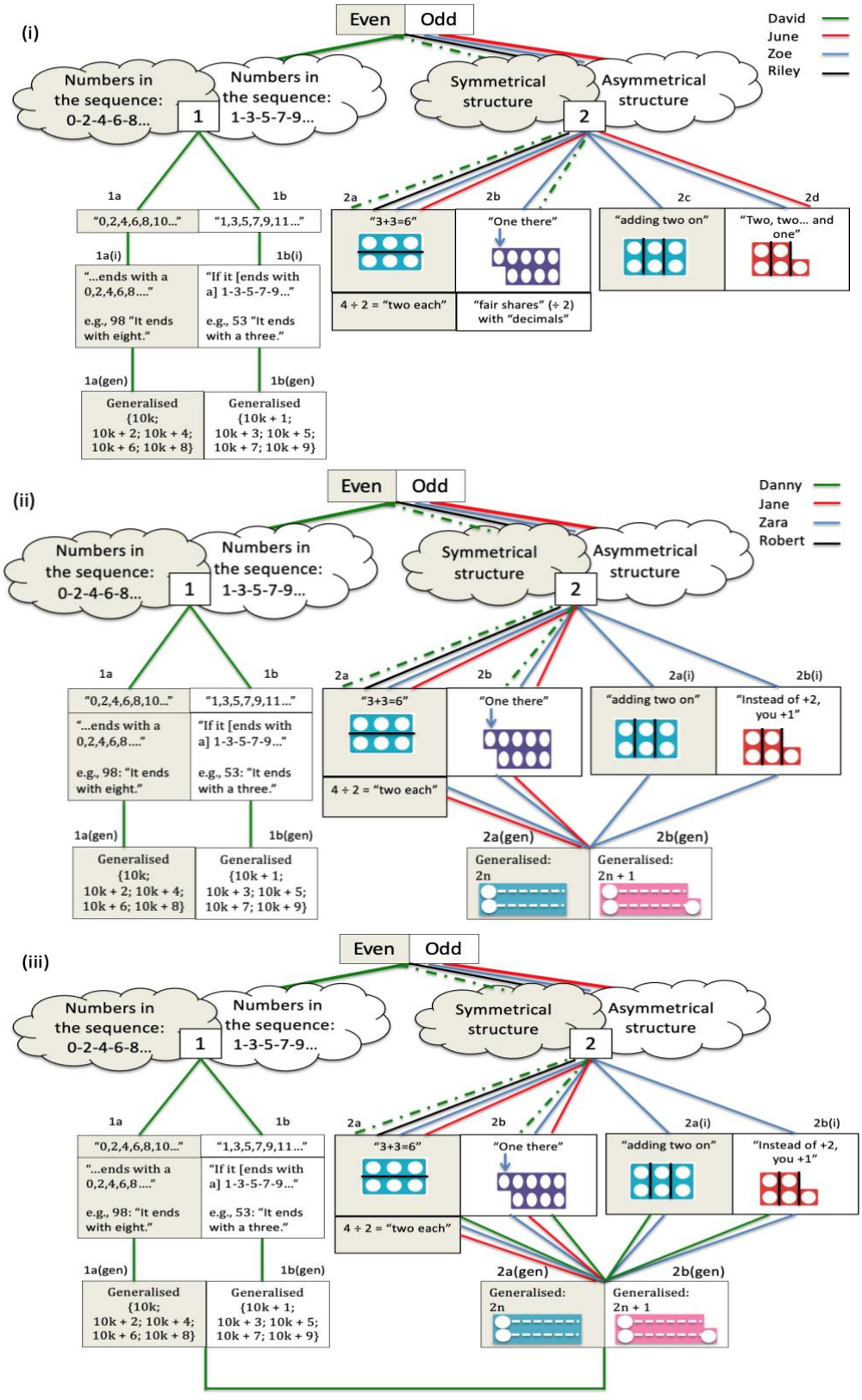
7.4.4 Summary of Findings

I now summarise Danny's participation in a structure-based, deductive discourse showing how a change in task situation triggered new tasks for Danny and influenced his motivations for engaging dialogically. This three-part summary of Danny's course of dialogical development is presented in Table 6. I also provide visual accompaniment of each part of Danny' story with the growth of the group's Grove of in Figure 14.

As Figure 14 shows, the initial grove (i) exposed the source of conflict between Danny and the other students. The use of words, narratives and substantiation routines within the student group were initially incommensurable because they were grounded in, and developed from, different realizations of odd and even (as were established in Chapter 5, Section 5.5). That is to say, because the students had realized the mathematical objects odd and even in different ways, the manner in which they operated (their routines) with these mathematical objects was different. Initially, Danny's realizations were distinct from those of his peers' at both the object-level and the meta-level. At the object-level, Danny's realizations were sequence-based while those of the other students were structure-based. At the meta-level, Danny had discursive tools to account for cases of odd and even, while the other students lacked tools to talk about odd and even as generalised mathematical objects.

Figure 14

The Growth of the Grove of Realizations



Hence, the branches of Danny's realization tree are disconnected from the other students, but more established. As Table 6 shows, while the first two parts of Danny's story showed he was certainly cognizant of the communicational gap between his sequence-based talk and the structure-based talk, Danny did not favour the alternative mode of talk, nor did he endeavour to adopt it for himself. In this part of the story, Danny's task appeared to be to convince the others he was right—that "odd + odd = even"—and he adopted structure-based talk in a ritual-ish manner in a bid to "talk their talk".

However, when Jane and Zara made meta-level learning (shown in Figure 14 (ii) by red and blue branches to 2a(gen) and 2b(gen)) they also had discursive tools to talk about all odd and even numbers, so Danny's reason for talking their talk disappeared and he resisted invitations to participate in structure-based talk. At this point, Danny's task appeared to shift to maintaining incommensurability between their modes of talking and positioning his sequence-based one as superior.During both parts of Danny's story: (i) the alternative mode of talk did not align with his realizations of odd and even; (ii) adopting his peers' talk required Danny to divorce from his own; and (iii) his discourse did the same job—it accounted for all cases of odd and even—but in a different way. Hence, Danny had only social reasons to consider adopting his peers' discursive ways—there were no mathematical benefits for him.

It wasn't until a discursive connection was made between the two modes of talk (in the third part of Danny's story) that Danny was able to perform a new structure-based routine that: (i) *utilised* his sequence-based one—he did not have to abandon his familiar discursive ways; and (ii) *simplified* his procedure—his proof-by-exhaustion could be encapsulated into a one-step procedure. Danny now had social *and* mathematical reasons to favour and adopt the other students' discursive ways. Only at this point was he described as engaging dialogically.

The discursive connection that occurred in part 3 of Danny's story is signalled in the Grove of Realizations (Figure 14 (iii) by the green branch between 1a and b(gen) and 2a and b(gen)) and shows Danny grew structure-based realizations of even (i.e., in the form of *2n*) and odd (i.e., in the form *2n+1*) which were *connected* to his sequence-based ones—they were *not superseded* by them. The connection also shows

how the students attuned their discursive ways and progressed towards resolving the commognitive conflict. Namely, the comparison of the before and after groves shows that while Jane and Zara made meta-level learning, Danny's learning was at the object-level, and both learning journeys were necessary for the students to meet at a point of consensus (shown in Figure 14(iii) at 2a(gen) and 2b(gen)). At this point in the story, Danny and Jane were able to talk with similar discursive tools and according to the same rules. They had reached a point of commensurability.

When describing Danny's performance of this new structure-based routine, I pointed to several explorative signs: flexibility, bondedness, applicability, performer's agentivity, and objectification of the discourse. Note, his performance did not follow a typical ritual-to-exploration transition (commognitively-speaking). Danny was already performing exploratively with his sequence-based discursive tools and resisted performing ritually with new structure-based ones. Thus, because he was able to coalesce the two routines into one, he continued to perform exploratively (he did not need to abandon his own tools) using a combination of sequence-and-structure-based discursive tools.

Table 6

The Course of Danny's Dialogical Engagement

Part	Situation	Danny's task	The nature of Danny's participation in a structure-based discourse	Danny's reasons for his participation or non-participation	The nature of Danny's reasons to engage dialogically	Is Danny dialogically engaged?
1	Two discourses (structure-based and sequence-based) are distinct at the object level and meta-level.	Convincing the others he is right ("odd + odd = even")	'Ritual-ish': Danny performs it to 'talk-the talk' of the others.	The structure-based talk does not align with Danny's realizations of odd and even. Adopting the new mode of talk requires Danny to divorce from his own. Danny's discourse is more objectified than theirs—he already has discursive tools to account for all cases of odd and even.	Social	No
2	The other students have made meta-level learning. The two discourses are distinct at the object-level only.	Positioning his discourse to be superior.	Danny does not participate: he maintains incommensurability between his (sequence-based) discourse and the other (structure-based) one.	The structure-based talk does not align with Danny's realizations of odd and even. Adopting the new mode of talk requires Danny to divorce from his own. Danny's discourse does the same job as the other discourse (it accounts for all cases of odd and even). Danny considers his discourse to be superior.	Social	No
3	A discursive connection is made between the structure-based discourse and Danny's sequence-based discourse.	Modifying and simplifying his substantiation of "even + odd = odd"	Exploratory.	The new discourse utilises and simplifies Danny's procedure. Danny is not required to divorce from his routine.	Social and mathematical	Yes

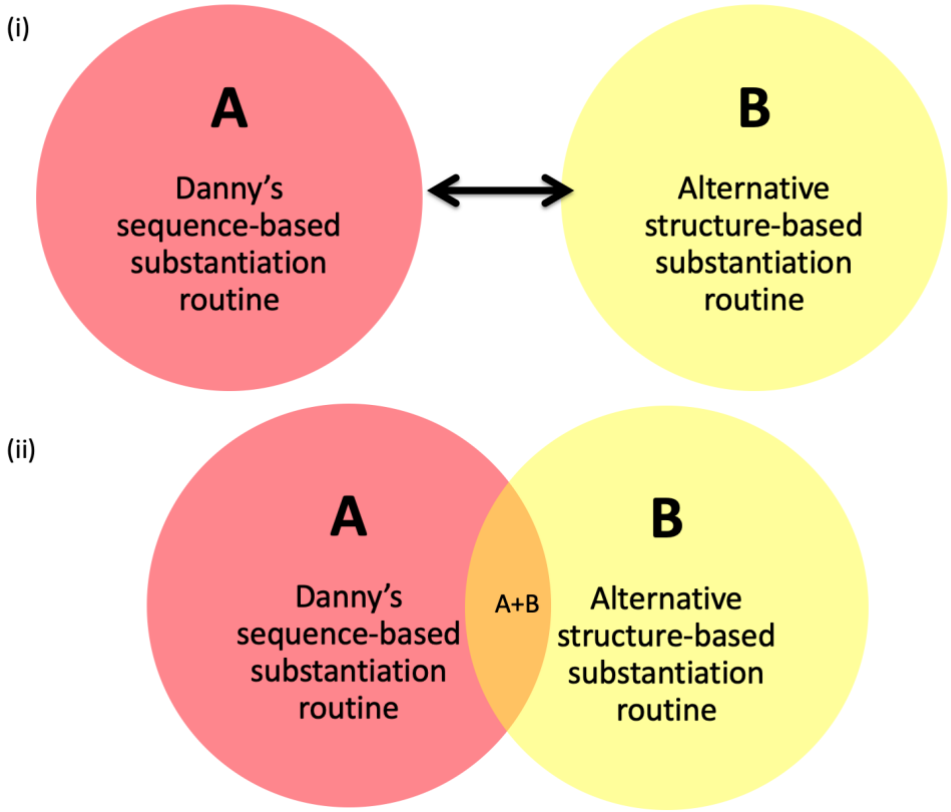
7.5 Discussion

In line with Sfard's (2019) concern that maths *talk* is often too readily conflated with maths *learning,* this chapter aimed to contribute to some missing pieces of the puzzle (both theoretical and empirical) about when and how communication supports learning in mathematics. Sfard (2019) distinguished between two types of learning—object-level and meta-level—and suggested dialogic engagement was required for the latter, more demanding, type of learning to occur. Sfard also operationalised dialogic engagement in terms of three defining conditions and suggested that a person opting for dialogical engagement might do so out of good conscience. However, in describing how such engagement might be fostered, Sfard acknowledged that more work was needed to theoretically and empirically account for how dialogic engagement might commence and be sustained.

The story described in this chapter has shown that even with much group discussion and many efforts to engage Danny in his peers' talk, learning at the object-level was not a simple process. For Danny, expanding his tree of realizations to include structure-based realizations proved difficult because he had already objectified odd and even numbers according to sequence-based realizations. Here, just as a commognitive conflict is required for meta-level learning to take place, the communicational conflict within the group had an indispensable role to play in object-level learning. Danny's exposure to the mode of talk he was supposed to individualise and the support he received from other participants was vital. However, where resolving a commognitive conflict at the meta-level involves one of the participants gradually accepting and adopting the incommensurable discourse and abandoning his or her own (Sfard 2008 p. 162), at the object-level, it was important to establish a discursive connection between Danny's discourse and the new one.

Figure 15

The Relation of Two Substantiation Routines: From (i) Discrete Discourses to (ii) a Discursively Connected Space Where Danny Coalesces a Routine From Two Discourses



For Danny, adopting his peers' structure-based substantiations initially required him to exchange his talk for theirs, and their discursive ways appeared to provide no mathematical reason for him to do so. This situation is shown visually in Figure 15 (i): choosing substantiation *B* required leaving *A*. But, as is shown by the overlapping region in Figure 15 (ii), a discursive connection was established which created a space where Danny was able to coalesce the structure-based routine with his own sequence-based one (A+B). Danny was presented with a way to perform his structure-based substantiation routine without divorcing from his sequence-based routine—the new routine utilised his sequence-based realizations. This implication is in keeping with Wegerif's (2013) sentiment that progress in learning is not a supersession, from A to B; rather, it "occurs through augmentation…from A to both A and B" (p. 29). Hence, the findings of this study suggest that, in some situations, interpersonal communication may actually play a necessary role in supporting the occurrence of object-level learning. However, the findings also point to a special kind

of communication being necessary—one that provides a discursive connection between the two discourses.

The findings within this chapter support Sfard and Kieran's (2001) warning that "the merits of learning-by-talking cannot be taken for granted" (p. 70). Indeed, there was no shortage of interpersonal dialogue within the group, and many efforts were made to engage Danny in structure-based talk, yet, for much of the group session, Danny's discursive ways remained unchanged. It was not until Danny engaged dialogically that learning took place. However, where Sfard suggested a person might engage dialogically out of a "moral commitment" (2019, p. 95), that did not seem to be the case for Danny. While a *social* motivation—to talk the same talk as his peers—may have been provided for Danny to engage dialogically, he remained disengaged until a *discursive connection* between his talk and his peers was made. Admittedly, these findings are focused on the learning and engagement of just one student yet examining micro-phenomena within this one case has proven enlightening with regard to fostering dialogical engagement. It has shown that, for some students and in some situations, social motivation alone may be insufficient grounds for engaging dialogically—mathematical incentives may also be required.

CHAPTER 8: Conclusions

This final chapter is divided into three sections. First, I provide an overview of the rationale for decisions made about the aims, theory, and methodology of the thesis overall. Here I also summarise the main ideas, key findings, arguments, and contributions of the research stories, including a review of the Grove of Realizations tool. In the second section, I discuss the implications of the research and its findings for further research into proof and proving at the primary school level, its theoretical implications for further research in commognition (and the potential of the Grove of Realizations tool), and its practical implications for educators and educational practice—particularly in primary schools. Finally, I acknowledge and address the limitations of the studies within the thesis.

8.1 Summary of Research

The literature review (Chapter 2) provided an overview of research literature in the area of proof and proving, with a particular focus on the primary school sector. The review highlighted the difficulties that students have with transitioning from endorsing and using empirical arguments to proving deductively. While it was somewhat clear that young students could produce generic proofs and their arguments could be considered as deductive, it was unclear how to judge whether an author of an argument was actually seeing and indicating specificity or generality in the particular examples they used. It also remained unclear how students transitioned from empirical to deductive reasoning, and, therefore, how their transition might be aided. To shed light on these unknowns, this thesis used the commognitive framework to characterise primary school students' use of examples within the empirical-generic space to help with assessing whether students were using examples generically rather than empirically in their reasoning. The thesis also used the commognitive lens and its analytical tools to examine sources of communicational breaches in students' transition from empirical to deductive discourses and examine their first steps in a deductive discourse. The aims here were to explore whether their first steps could be exploratory and to investigate what helped or hindered their initiation.

In the theoretical framework and methodology chapter (Chapter 3), I provided an overview of Sfard's (2008) commognitive framework and justified it as an appropriate theoretical approach for addressing the research questions. Situated within a participationist paradigm, where learning is conceptualised as participating in discourse, previous use of the commognitive framework has shed light on the development of thinking and the process at other critical transitions in mathematical learning (Ben-Zvi & Sfard, 2007; Caspi & Sfard, 2011; Lavie & Sfard, 2009; Sfard 2007). Caspi and Sfard (2011) described algebra as a meta-discourse and were thus able to show arithmetic-to-algebraic difficulties as meta-level changes in discourse. In a similar way, I described empirical and deductive reasoning as two incommensurable discourses: each has different uses of words, uses different visual mediators in different ways, endorses different narratives, and performs tasks according to different rules. The transition from empirical to deductive reasoning contains elements of abstraction and requires discourse-on-discourse substantiations rather than direct identification ones. Accordingly, deductive discourses on mathematical objects were defined as meta-discourses of empirical ones. The difficulties with transitioning from empirical to deductive discourses could thus be perceived as difficulties in learning at the meta-level. The context of the research and the research procedure was also described in this chapter. The commognitive framework guided the four research studies that made up the thesis as a whole. Within each study, the commognitive framework provided the means by which to conduct a fine-grained discursive analysis on the data. It supplied discursive tools with which to detect commonalities and differences in the ways that learners communicated as well as when and how their discourses changed.

I described the first research study—How can we tell if learners are referring to the general in the particular?: Discursive markers of generic example-use—in Chapter 4. Here I used my entire data set (from all 28 students involved in the research) and utilised Sfard's (2008) commognitive framework to study students' use of examples when endorsing universal narratives about the sums of odds and evens. Whereas some previous studies have pointed to the value of generic examples for learners, others have flagged the difficulty in determining whether learners were seeing the general in the particular example or were attending to the specific example in focus. To address this gap in the research, my goal in this first study was to identify

discursive markers, for researchers' and educators' use, that might signal when examples are being used generically rather than empirically.

My findings showed the occurrence of example-use to be more nuanced and multi-layered than an empiric-generic dichotomy. I identified four categories of example-use according to whether the example revealed numeric or generic realizations of odd and even and whether the substantiations associated with the realizations were inductive or deductive. These categories were found to be blurred and overlapping. It was quite possible for students' example-use to fall somewhere between two categories or to flux between categories. Within each category, I provided a range of subtle discursive markers that implicitly pointed towards students using examples generically, rather than empirically.

In the second study—Using a Grove of Realizations to reveal communicational breache*s*— (Chapter 5), the focus shifted to an analysis of the communication among a group of four students. Here I analysed how these students classified numbers as odd and even and how they substantiated universal narratives about the sums of odd and even numbers. The analysis established a discursive baseline, which pertained to realizations that each student brought to the lesson. This baseline was necessary to trace students' discursive developments. Furthermore, this analysis exposed two types of communicational breaches within the group: object-level and meta-level. At the object-level, the findings showed students' substantiation procedures were grounded in their realizations of odd and even. Because the students attended to different features of oddness and evenness, they agreed on *which* numbers are odd and even but disagreed about *why*. And their realizations also differed in terms of objectification: one student had discursive tools to account for all cases of even and odd numbers, while the others did not.

The findings suggested that students' awareness of the breach in communication may not be sufficient to engender meaningful engagement with each other, even when pedagogical moves are made in this direction. Furthermore, these findings showed how the content-specific nature of a learner's talk (whether their realizations are structure-based or sequence-based), combined with the sophistication of that discourse (whether the learner has objectified numbers or not), affects their readiness to engage in deductive reasoning and shapes the nature of their

substantiations. These findings are significant for research and practice, as they show that the content-specific nature of a learners' realizations of the mathematical objects and the degree of objectification are pivotal to their readiness to engage in deductive reasoning.

In the third study—From numeric talk to generic talk: How and interdiscursive learning environment supports students' explorative entry into an unfamiliar discourse— (Chapter 6), I focused on two students (Jane and Zara) from the group. I examined their discursive transition from using empirical reasoning to using deductive reasoning for universal narratives about the sums of odds and evens. From the commognitive perspective, it was previously assumed that learners would only be able to take ritualised—not explorative—first steps in the emerging discourse requiring abstract generalisation: they would be socially, not intrinsically motivated to participate and they would focus on the procedure rather than the outcome. However, taking inspiration from Cooper and Lavie's (2020) notion of interdiscursivity and their principles of exploration-enabling tasks, I designed a task to support learners in taking explorative (intrinsically motivated and outcome-oriented) first steps in a deductive discourse on odd and even numbers. Where Cooper and Lavie (2020) provided theoretical accounts of interdiscursivity, I complemented this with an empirical account of how two students were supported to take explorative first steps in a new discourse.

In the final study—How a discursive connection facilitated Danny's dialogical engagement— (Chapter 7), the focus switched to just one student from the group, Danny. In this study, I used Sfard's (2019) notion of dialogical engagement as a lens through which to explain why Danny's participation in structure-based talk was neither easy nor straightforward. Danny's discursive journey began with him vehemently rejecting structure-based talk. He then endorsed it but maintained incommensurability between this mode of talk and his own, which I interpreted as a concession Danny briefly made to "talk the talk" of the others to convince the others he was right (i.e., "odd + odd = even"). It was only when Danny saw what I termed a "discursive connection"—an opportunity for linking two modes of talk together that he finally engaged dialogically. The case of Danny showed that learners might need

more than a social reason to engage dialogically and adopt a discourse for themselves—they might require a mathematical reason to do so.

In these analyses, I introduced a new methodological tool called the Grove of Realizations to map the students' realizations of odd and even numbers in one visual model for the group. Initially, in the second study (Chapter 5), the tool mapped the extent of students' pre-existing realizations and helped reveal object-level and meta-level differences in realizations within the group. In doing so, the tool helped clarify the effectiveness of the group's interpersonal communication and expose the sources of communicational breaches. The group's Grove of Realizations then provided a discursive baseline for tracking the developments of the same students' discourses in Studies 3 and 4 (Chapters 6 and 7). The tool was an effective means for the researcher to map the participants' realizations regarding the mathematical objects in question—odd and even—and then to contrast their realizations within the group's grove to expose the sources of communicational breaches within the group. It assisted me, the researcher, in communicating findings effectively and efficiently, and it provided a visual metaphor for the reader to make sense of the analysis.

8.2 Implications of the Research

In this section, I summarise the implications of this research and its findings for further research into proof and proving at the primary school level (including practical implications for conducting empirical research), its theoretical implications for further research in commognition, and its practical implications for mathematics education.

8.2.1 Implications for Further Research into Proof and Proving at the Primary School Level

8.2.1.1 Opening the Space Between Empirical and Generic Example-Use.

Traditionally, example-use was perceived as an empiric-generic dichotomy and determining whether learners were seeing particularity or generality in the example they used in their reasoning was deemed problematic (Mason & Pimm, 1984; Reid & Vallejo Vargas, 2018; Yopp & Ely, 2016). However, the study in Chapter 4 opened up the space between empirical and generic example-use: it showed example-use to be a more nuanced and multi-layered space within an empiric-generic spectrum. I

identified discursive markers of generality in learners' use of examples, which suggested that determining intent of genericity of examples when reasoning might not be an impossible task. Given that the focus of this research was only on students' example-use regarding universal statements on odd and even numbers, further exploration into this empiric-generic space is needed to examine the ways these discursive markers are present in learners' reasoning in different contexts (e.g., different task situations, learners of different levels of schooling, different mathematics foci) and how they point to learners demonstrating and reasoning generality on a wider scale.

8.2.1.2 The Commognitive Framework as a Theoretical Approach and an Analytical Tool. The commognitive framework was instrumental in unearthing each student's realizations about odd and even numbers and highlighted distinctions in the students' reasoning that may otherwise have been hidden. The framework helped expose sources of communicational breaches at the object-level (content-specific talk) and meta-level (empirical and deductive discourses) and pinpointed when and how they began to be resolved. Adopting the commognitive stance shed light on why students resist or struggle to make sense of one another's thinking, why they struggle with transitioning from empirical reasoning to deductive reasoning, and why these difficulties are common with learners around the world and at different levels of education. It may be natural to be puzzled about why, when students can classify numbers as odd and even according to differences in their structure, they struggle to use the structure of odds and even to prove universal statements. However, the commognitivist appreciates that this difficulty arises because of incommensurability in discourses at the empirical-deductive junction. In empirical discourses, odd and even numbers feature as specific numbers and the words odd and even appear as adjectives. Conversely, in deductive discourses, the same words are used as nouns denoting abstract encapsulations of the sets of infinite odd and even numbers. The two discourses produce conflicting narratives. Learners have no common criteria to resolve the contradictions—giving up previously endorsed, seemingly self-evident and entrenched stories about mathematical objects is necessary for them to resolve the conflict, but it is not easy.

Rather than being puzzled, then, by the difficulties that students face with

transitioning to proving deductively, the commognitivist expects the transition to be a slow process and that learners will flux between their old empirical discourse and the new deductive one. To use an analogy:

> *Casting out a favourite pair of old, well-worn, and comfortable shoes in favour of new ones that do not seem to fit yet may be neither a quick nor easy thing to do. One may prefer to chop and change between the old and the new, to begin with. There might need to be a good reason to select the new pair over the old, and the new pair may only seem appropriate for certain occasions, to begin with. Only when the new ones begin to fit more easily and are seen as fit for purpose, may the owner be inclined to discard the old ones for good.*

The present research and its findings thus suggest that further research may find it useful to adopt the commognitive framework as a theoretical approach and an analytical tool to examine these critical transitions in other proof-related learning contexts that students appear to find challenging.

8.2.1.3 Practical Implications for Conducting Empirical Research. With regards to supporting learners in making empirical-deductive transitions, this thesis also has practical implications for conducting empirical research. First, as a means of addressing a limitation of commognitive research, with regard to its reliance on observable discourses, the use of a puppet in this research may be considered as somewhat innovative. In playing the role of the teacher in the group sessions, I would have been seen both as a figure of authority but also an unfamiliar one, so there was a good chance that students would be less confident than they might usually be in their groupwork, and therefore less vocal. However, including the puppet as a group member would likely have reduced the formality of the setting and reduced student anxiety so that the students may have interacted more readily. Additionally, the puppet played the role of a novice and a sceptic with regards the "odd + odd" and "even + odd" concept cartoon narratives, and thus he provided a space for the students to work harder to explain their reasoning and to play the "experts" (Simon et al., 2008). Of course, as was demonstrated with the passivity of one student from the focus group of this research, the use of puppets does not necessarily mean that *all* students will interact readily. Nevertheless, puppet-use might be considered as a means to encourage young students' discourse, particularly in contexts such as the

studies described in this thesis where the Teacher-Researcher is unknown to the group.

Second, the task situations in these studies involved the students considering narratives presented in a concept cartoon. Concept cartoons (Keogh & Naylor, 1993) have been previously used in science education contexts and shown to be useful in stimulating and provoking discussion and in pre-empting disagreements (including faulty arguments). In the present research, they were found to be particularly useful also in the mathematical context. The rationale was that the awareness of alternative ideas within the cartoon would prompt students to re-evaluate their ideas. In Study 3's focus group (Chapter 6), the bilateral narrative (e.g., "odd + odd = sometimes even and sometimes odd") was shown to trigger doubt for one student regarding the sufficiency of small number examples and rich group debate ensued. It was the students' need to account for numbers beyond ones that are countable, along with the group's lack of discursive apparatus to resolve the commognitive conflict created, which opened the door from an empirical discourse to new and more generic ways of talking. Had the cartoon simply included binary narratives, a numerical example that endorsed one narrative would imply the rejection of the other, and the student group may have remained in agreement and seen no need for new ways of talking. Hence, the study suggested that bilateral narratives might play a helpful part in facilitating doubt within a learner (or group of learners), and in generating an intrinsic motivation to consider new ways of talking. Further research would help establish the usefulness of the inclusion of bilateral narratives in tasks (including concept cartoon tasks) involving endorsing and substantiating universal narratives.

Third, Study 3 also showed that the interdiscursive learning environment—the coalescing of discursive elements from two discourses—was successful in supporting two students' transition from a numeric discourse on odd and even numbers towards generic ways of talking. The introduction of new visual mediators—Numicon strips, which were designed to be open to two interpretations (a particular odd or even number *and* generic oddness or evenness)—was seen as particularly useful here. Through the interdiscursive use of these strips, the students were encouraged to interpret the task situation in a way that would elicit an action from their familiar numeric discourse that was also appropriate from the perspective of the emerging

deductive discourse. The findings suggested that the interdiscursive design and implementation of tasks could be critical in supporting learners' transitioning to a new discourse, and thus could be critical in supporting learners to traverse the empirical-deductive divide. This study provided the first empirical account of Cooper and Lavie's (2020) theory of interdiscursivity *in action*. Therefore, further studies into how interdiscursivity may support learners at other discursive junctions, like the empirical-deductive one, is needed.

8.2.2 Implications for Further Research in Commognition

8.2.2.1 Theoretical Implications.

Object-Level Learning. Sfard (2019) claimed that the simpler and more straightforward kind of learning—object-level learning—could be achieved via intrapersonal communication; interpersonal conversation is optional. However, the findings of the studies in Chapters 5 and 7 showed object-level learning to be neither simple nor straightforward, even *with* interpersonal talk. The object-level division between the students' talk was stubborn: the students were unwilling to adjust their discursive ways and resisted treating one another's articulated realizations of even and odd as the same thing.

This phenomenon (i.e., a persistent object-level division between students' discourses) was considered to have occurred because the conflict was not about the *outcome* of classifying numbers as odd or even (they agreed on this); it was about *how* the students substantiate oddness and evenness. Hence, the task presented no good reason for the students to modify their ways of talking. In addition to this, one student's discourse was shown to be distinct both at the object-level (his was sequence-based and theirs was structure-based) *and* at the meta-level (his was more objectified than the other students). He dismissed his peer's discourse not only because it was different at the object-level but because: his worked and produced the same outcome as that of his peers; and, his was more efficient than the alternatives—he already had appropriate discursive machinery to account for all cases of "odd + odd" in his substantiations of "odd + odd = even" (and all cases of "even + odd" in substantiations of "even + odd = odd"). In terms of implications for the commognitive theory, the findings suggest that in some instances, object-level learning is not so

easy, and learners might need a mathematical, not just a social reason to endorse and adopt a new discourse for themselves.

Meta-Level Learning. The commognitive stance traditionally maintains that meta-level learning can only commence with ritual, not explorative, participation, and the process of de-ritualization is often likely to be gradual and slow (e.g., Sfard & Lavie, 2005; Lavie & Sfard, 2019). The rationale for this is that the productiveness of new meta-rules cannot be appreciated by a newcomer, so their participation is expected to be socially motivated to conform with a teacher's instructions or examples, and the newcomer is liable to focus on the steps of the procedure, rather than the outcome. However, in Study 3 (Chapter 6), the findings from the implementation of a task I designed, using principles of Cooper and Lavie's (2020) exploration-enabling task design, showed how interdiscursivity supported two students to take *explorative* first steps in a deductive discourse on even and odd numbers. The findings provide empirical support for the alternate view of the learning journey proposed by Cooper and Lavie (2020): they showed that learning could commence in more exploratory ways.

If exploratory participation is the goal of teaching (Lavie et al., 2019), facilitating interdiscursive learning environments could potentially bypass learners' ritualised participation in a new discourse. Facilitating interdiscursive learning environments might also result in more meaningful learning experiences since the learner is likely to be more outcome-oriented and intrinsically motivated when taking their first steps in a new discourse than they would be when taking ritualised first steps. Hence, the process of learning might be described as involving both (what I termed) "pro-explorisation"—becoming increasingly more exploratory via more exploration-enabling opportunities to learn—as well as de-ritualisation. Therefore, the implication is that commognitive research should include the interdiscursive design and implementation of other tasks and examine if, when, and how other exploration-enabling opportunities to learn facilitate learners' explorative initiation into new discourses in different learning contexts (i.e., ones that involve meta-level learning). Of course, the mathematics involved, and the individual discourses of particular students involved in each context will be key considerations in creating alternate and exploratory pathways. Whether such a pathway turns out to be a shorter cut to meta-

level learning than the ritual-to-exploration route, though, also needs further investigation.

8.2.2.2 The Grove of Realizations Tool. In terms of its affordances for further research, the Grove of Realizations tool as it stands has the potential to be used and developed for similar purposes *within* groups in different contexts. It also has the means to show differences *between* groups who engage in similar tasks engaging in the same task by visually representing which groups are experiencing commognitive conflict (where realizations boxes are not shared) and which groups are talking the same talk (where all group members have shared realization boxes on the tool). Nevertheless, while the tool has the potential to be used in future commognitive research, more work is needed (including deciding what should constitute root notes and constructing realization boxes) when developing the tool for different tasks and in different contexts

8.2.3 Practical Implications for Mathematics Education—Particularly in the Primary School Sector

The thesis supports a body of research advocating for the inclusion of mathematical proof and proving at the primary school level and therefore has implications for those in charge of designing and reviewing educational policies, curricula and curriculum materials, and teachers. Studying the focus group of students showed that, in the absence of pressing students for proof, the group consensus about an answer (in this case about which numbers are even and odd) might have prematurely signalled shared reasoning. Hence, the findings demonstrated that students engaged in rich and fruitful discussion as a result of engaging in proving activity and highlighted the role that proving can play in signalling differences in reasoning within a group of students that may otherwise have remained hidden. The findings also suggested that an awareness of the generality of example-use in reasoning per se might not be so challenging and rare (although discerning the subtle and implicit discursive markers in students' reasoning admittedly requires dedicated time and effort). Thus, to those who play a role in education, the findings of this thesis imply that proving activity and mathematical proof are both achievable and important goals for young students and that they deserve a place in policy, curricula, educational materials, and in the classroom.

At a practical level for teachers and designers of curriculum materials (as with my comments in Section 8.2.1 (iii)), the use of concept cartoons may be a simple yet promising way to engage students in proving activity. Rather than simply hoping that arguments that are useful for advancing students' thinking and reasoning will surface within group discussions (which may include faulty arguments or bilateral narratives), they can be pre-empted and included in the characters' narratives. Of course, the teachers themselves could present such arguments during discussions in the event that they do not arise within group discussion, but the advantage of having them in the cartoon is that all narratives have equal status. For teachers of young students, the use of puppets may provide another simple yet promising way to both encourage students to participate in proving activity and motivate students to work harder with their reasoning.

8.3 Limitations of the Studies Within the Thesis

This thesis has provided novel theoretical contributions regarding how students' arguments unfold and develop during proving activity, and what might be required to support their development. To that end, it makes an original contribution to international research on proof and proving, and it is the first research of its kind in New Zealand. However, the research should be considered as an initial exploration into what primary school students' first steps in a deductive discourse might entail and more studies (possibly similar to this one) are needed to provide a clearer and more comprehensive picture. Given the limited amount of research that has been conducted on students' first steps in a deductive discourse, though, the research and its findings might prompt further research. Despite endeavouring to fulfil the research aims in the best possible way, several limitations of the studies must be acknowledged. Namely, the research was limited in terms of the context-bound nature of the study, the role I played as Teacher-Researcher, the commognitive lens used with which to view my data, and the pitfalls of commognitive research. I also acknowledge the limitations of the Grove of Realizations tool.

8.3.1 The Context-Bound Nature of the Study

The research findings from the four studies that make up this thesis are bounded by the context in which the studies were conducted. The research examined a case—the development of primary school students' participation in a deductive discourse—

which was bounded by being a discourse on odd and even numbers and was examined by gathering data from groups of primary school students who were presented with specific task situations. As such, the findings were not intended to be generalisable; however, they raise questions for further exploration and inquiry in different contexts (i.e., the same case but with different task situations, different primary school participants, whole class as well as group contexts, different mathematical foci).

Indeed, it was argued in Chapter 3 (Section 3.1.2.1) that primary school students draw from a different (less established) precedent-search-space than mathematicians and, as has been demonstrated from studying just one group of students in this thesis, the precedent-search-spaces of individual students are diverse too. Hence, one should be mindful that the ways in which students interpret the same task situations and what they consider to be an appropriate way to respond are expected to differ. For instance, how students' individual discourses unfold in the same task situations as students who have been explicitly taught what canonical proofs entail is likely to be significantly different to what was described in the four studies in this thesis.

It should also not be ignored that the learning described in Studies 3 and 4 (Chapters 6 and 7) occurred in a group setting of four students working with one teacher. Had the same group of students been working together in a whole class setting, it is unlikely that the way in which these students' discourses unfolded and developed would have been the same. Therefore, one recommended direction for further research would be to investigate whether and how students' first steps in a deductive discourse occur in different contexts such as these, and to determine the nature of their first steps (i.e., ritualistic or exploratory).

8.3.2 Limitations of the Teacher-Researcher Role

Commognitive researchers must remain cognisant of the fact that they are participants in their research (Sfard, 2008, p. 278). This is even true when they are merely observing the activity of others, as observed participants may interpret anything the researcher does—be it verbal, gestural, or even when remaining silent—as value-laden. Hence, in playing the role of teacher and researcher, my

participation would have influenced the responses of those I worked with and observed to an even greater extent than a researcher playing the role of an observer.

Despite this risk, playing the role of teacher myself was a more appropriate way to answer my research questions. I particularly wanted to refrain from explicitly teaching the students conventional ways of proving in order to provide opportunities for the students to demonstrate their inchoate ways of substantiating numbers as odd and even and substantiating their endorsement of universal narratives and to take explorative steps in a new discourse. However, as noted in the literature review (Chapter 2, Section 2.2), proof does not feature in the New Zealand curriculum and primary school teachers are not necessarily well-versed with proving practices (Knuth, 2002; G. J. Stylianides et al., 2007). It was also noted that teachers have been shown to over-direct the learning (Sullivan, 2011). Hence, conducting research that involved students working with their own teacher would have entailed both considerable vetting and training of teachers and produced different research limitations. Furthermore, the risks of students picking up on my reactions to certain student responses by way of my body language and thus "learning" what was seen to be a favourable or unfavourable response was just as likely to have occurred with any person playing the role of teacher.

8.3.3 The Pitfalls of Commognitive Research

Within a commognitive framework, research on human development is the study of the development of discourses (Sfard, 2008, p. 124). While undertaking commognitive research and viewing the data collected for this research through a commognitive lens afforded new insights into the empirical-deductive space, it is likely to have obscured other data, types of analysis, and findings from view. Furthermore, as is the nature of using the commognitive framework for analysis, the researcher is highly dependent on observable communication, which means that some parts of an individual's discourse may remain undetected. Only when a participant's discourse is made public can it be commented upon. With regard to the findings of the four studies, it is quite possible that some students in the group held realizations for odd and even that were not articulated. Indeed, although Robert was a member of the group focused on in Chapters 6–8, his passivity and lack of interpersonal communication meant that little could be said about his discourse on

odd and even numbers at all. Even when students made their reasoning public, it was not always comprehensible, so it was sometimes difficult to be sure what an individual intended. An example of this was the assumption that was made about Jane's realizations of odd in Study 2 (Chapter 5), regarding her "two, two and one" substantiation narrative with regard to endorsing five as even: where it was assumed to be a realization in the form of "a double plus one" ($2n+1$), it could also have signalled a realization in the form of "repetitions of two (2+2+2+2…) plus one". In situations like this, looking across the whole transcript for clues rather than focusing on single utterances transformed such assumptions, at the very least, from wild guesses into best guesses.

8.3.4 The Limitations of the Grove of Realizations Tool

The affordances of the Grove of Realizations tool have already been described in this chapter (8.1 and 8.2.2). However, the tool also has some limitations that are worth noting. First, the Grove of Realizations tool was only been developed for the one task and with the one group described in Studies 3, 4, and 5 (Chapters 5, 6, and 7). Its development and usefulness for other tasks remain to be demonstrated. Second, while the tool neatly represented the realizations of the four students who comprised the focus group, its capacity to represent larger groups is constrained. That is, at a certain point the number of different coloured branches for individual students would become too confusing. Third, certain aspects of the groupwork were lost in the analysis. For example, I collapsed several similar realizations made observable in the discursive data into one box, meaning that the number of realizations of the same ilk was lost. Thus, a realization that was made observable for many different numbers (e.g., 3+3; 5+5; 6+6) would appear no different in the tool than one that had been made for only one number. Fourth, even though the tool allowed for a variety or realizations of the same ilk to be shown (e.g., it showed "3 + 3 = 6" and "4 ÷ 2 = two each" in the same realization box), it did not allow for the representation of who articulated which version of the realization, nor how frequently each student articulated each realization—only one branch for a given student was shown connecting to a box on the tool. Hence, the tool does not necessarily have the capacity to distinguish between a student who was very talkative and articulated a realization

many times and in many different ways, from one who was largely passive and only articulated the same realization just once.

Furthermore, the Grove of Realizations tool only provides a means of visually showing which realizations were made observable in the data and by whom; establishing *how* realizations were made observable and *why* learning occurred would require examining the discursive data to pinpoint what changed for the participant for this expansion of realizations to occur. Hence, the tool alone may not be particularly meaningful—it needs to be accompanied by detailed discursive analysis. These features and limitations were concessions I made for simplicity of the tool and in meeting my intended goals of examining consistencies and inconsistencies within students' discourses in order to examine the source of a commognitive conflict and to trace students' discursive growth. However, they may be useful considerations for future and different uses of the Grove of Realizations tool.

8.4 Concluding Remarks

The impetus for the research conducted and presented in this thesis was to produce an original contribution to research about proof in the primary school sector. Towards these goals, I hope my thesis lends itself towards encouraging increasing numbers of researchers, educators, and policymakers to recognise that mathematical proof and the process of proving do not just belong with the work of mathematicians, nor occur solely at higher levels of education; they have a place in all levels of mathematical learning. Just as the greatest penned works of an acclaimed author and the scribbles of an infant may be described by one word—"writing"—and the *process of writing* is just as important for each person, the word "proof" should apply equally to work of young students having figured out why two odds make an even and that of the most accomplished mathematicians—and, here too, the *process of proving* is just as important at each end of the spectrum. Furthermore, the considerable effort and time that is required for children to transform their scribbles into legible, grammatically correct, and coherent writing might also be compared with the struggles that students have with transforming their rudimentary ways of proving into more conventional ones. The struggle should be anticipated—it is unavoidable, but it is also necessary.

I exit with wordplay, summarising my work:

I have proof that mathematical proof and proving are not childproof!

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Appendices

Appendix A: Permission to Use Published Material

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SET: mathematical proof paper
3 messages

Jo Knox <jkno020@aucklanduni.ac.nz> 9 September 2020 at 15:24
To: john.huria@nzcer.org.nz

Hello John
I am about 6 months off submitting my PhD thesis and would like to use my paper I submitted for SET within my rationale/motivation section.
https://www.nzcer.org.nz/nzcerpress/set/articles/proving-proof-has-place-primary-classroom
This will be modified slightly but will closely resemble the version accepted by NZCER/SET.
Please can you advise whether this is possible, and if so how I obtain permission to do so from the journal?
Kind regards,
Jo Knox

John Huria <john.huria@nzcer.org.nz> 9 September 2020 at 17:18
To: Jo Knox <jkno020@aucklanduni.ac.nz>

Kia ora Jo
We're happy for you to use your Set article in your PhD thesis, and please accept this email as granting NZCER's permission to do so.
Please see our policy on self-archiving for further information.
Best wishes for the final run home!
Ngā manaakitanga

John Huria Senior Editor–Kaietita Matua, NZCER Press
New Zealand Council for Educational Research | DDI: 04 802 1438 | www.nzcer.org.nz
Level 10, West Block, Education House, 178-182 Willis St, PO Box 3237, Wellington

From: Jo Knox <jkno020@aucklanduni.ac.nz>
Sent: Wednesday, 9 September 2020 3:24 PM
To: John Huria <john.huria@nzcer.org.nz>
Subject: SET: mathematical proof paper

[Quoted text hidden]

NZCER self-archiving policy.pdf
132K

Jo Knox <jkno020@aucklanduni.ac.nz> 9 September 2020 at 17:36
To: John Huria <john.huria@nzcer.org.nz>

Many thanks for your email and permission John.
Kind regards,
Jo
[Quoted text hidden]

Appendix B: Participation Information Sheets

SCIENCE
DEPARTMENT OF MATHEMATICS

Addressee
Address

July 21, 2021

Department of Mathematics
Level 4, Building 303,
38 Princes Street
Auckland, New Zealand
T +64 9 373 7599 ext. xxxxx
E xxxxxxx@auckland.ac.nz
W www.math.auckland.ac.nz

The University of Auckland
Private Bag 92019
Auckland 1142
New Zealand

PARTICIPANT INFORMATION SHEET

(Parent)

Project title: Mathematical tasks that promote mathematical processes in the primary classroom.

Name of researcher: Jo Knox (PhD student)

Name of research supervisors: Caroline Yoon

Name of research co-supervisor: Fiona Ell, Igor Kontorovich

Researcher Introduction:

I am Jo Knox, and I am a PhD student in The University of Auckland, School of Science, Mathematics, and Technology Education. My supervisors are Caroline Yoon, Fiona Ell and Igor Kontorovich.

This Project:

The reason I am doing this research is to find out about students' mathematical thinking and understanding as they work on mathematical activities.

The research goal is to design mathematical tasks that can be used in primary classrooms to help students develop deep conceptual understanding in mathematics.

Invitation to Participate:

Your child is invited to participate in this research because s/he meets the age requirements for this research. To find potential participants like your child, I

have contacted primary/intermediate schools in Auckland that have existing relationships with the University of Auckland. Your child's school has given permission and I have selected three classes to work with.

Your child's participation is voluntary. Even if you consent to your child being involved in this research, he/she will also be read a participant information sheet and be asked to sign an assent form. The Principal has given assurance that your child's learning, academic standing, assessment and academic relationships with the school or members of staff will not be affected by either refusal or agreement to participate and will not be affected by their performance on the mathematics tasks.

Project Procedures:

If your child chooses to participate, he/she will be observed, as he/she works on a mathematical task together with other students in a small group of two-four. The researcher (myself) will introduce and explain the task. The purpose of the sessions is to see how students respond to the task: how they think, how they explain and justify their answers.

He/she may be asked to work on the mathematical tasks on up to four different occasions (school days), with each session taking approximately one hour. The maximum time commitment for students is therefore four hours.

Data Storage, Retention, destruction and Future Use

How: I will collect the data by taking student samples of work, taking field notes and video-recording the sessions. Data will not be used for any other purpose outside of this project, and only research personnel will have access to them. I will transcribe the videotapes and audiotapes. The un-identified transcripts of these tapes will be kept indefinitely for research purposes and publication. Information about your child's age, gender, ethnicity and achievement level will be obtained from their teacher.

Data (i.e., videotapes, audiotapes, and your child's work on the tasks) will be securely stored at the University of Auckland for six years. Electronic data will be kept on two University of Auckland password-protected computers. After 6 years has elapsed, the data will be securely destroyed using reputable destruction services.

Right to Withdraw from Participation

Your child has the right to withdraw from the interview/focus group at any time without giving a reason. However, it will be impossible to remove any of his/her own data that they have already given if he/she wishes to withdraw my participation after the project has started.

Anonymity and Confidentiality

The preservation of confidentiality is paramount. The information your child shares within these sessions will remain confidential to the researcher and her supervisors.

Pseudonyms will be used to label the samples of student written work and videotapes, as well as in the transcripts and any publications.

If the information that your child provides is reported/published, this will be done in a way that does not identify your child as its source.

If you and your child consent, his/her snapshot may be published in reports and academic journals. However, this part is optional – you may choose not to allow video images of your child to be published or shown in professional presentations, while still being involved in the other aspects of this project. Should you agree to their images being shown at professional presentations and research publications, their anonymity, along with their teacher's, parent's and school's, cannot be guaranteed as they may be recognized from the image.

If you choose not to allow your child's image to be published or shown in professional presentations, any information they provide will be reported/published in a way that does not identify the students as its source.

A copy of the research findings will be made available to you if you wish.

If you have any questions, please feel free to contact my supervisors or myself. If you are willing to participate in this research, please complete the enclosed Consent Form and return it to me at the address below.

CONTACT DETAILS AND APPROVAL

Student Researcher name and contact details	**Supervisor name and contact details**	**Head of Department/School name and contact details**
Jo Knox School of Science, Mathematics, and Technology Education, The University of Auckland, Private Bag 92019, Auckland j.knox@auckland.ac.nz	Associate Professor: Dr. Caroline Yoon School of Science, Mathematics, and Technology Education, The University of Auckland, Private Bag 92019, Auckland Ph. +64 9 923 8740 c.yoon@auckland.ac.nz	Prof. Steven Galbraith School of Science, Mathematics, and Technology Education, The University of Auckland, Private Bag 92019, Auckland Ph. +64 9 923 8778 s.galbraith@auckland.ac.nz

For any queries regarding ethical concerns you may contact the Chair, The University of Auckland Human Participants Ethics Committee, The University of Auckland, Office of the Vice Chancellor, Private Bag 92019, Auckland 1142. Telephone 09 373-7599 extn. 83711.

Email: ro-ethics@auckland.ac.nz

APPROVED BY THE UNIVERSITY OF AUCKLAND HUMAN PARTICIPANTS ETHICS COMMITTEE ON 1st April 2019 (expires on 9th October 2020). Reference Number **019961**

THE UNIVERSITY OF AUCKLAND
Te Whare Wānanga o Tāmaki Makaurau
NEW ZEALAND

SCIENCE
DEPARTMENT OF MATHEMATICS

Addressee
Address

July 21, 2021

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Level 4, Building 303,
38 Princes Street
Auckland, New Zealand
T +64 9 373 7599 ext. xxxxx
E xxxxxxx@auckland.ac.nz
W www.math.auckland.ac.nz

The University of Auckland
Private Bag 92019
Auckland 1142
New Zealand

PARTICIPANT INFORMATION SHEET

(Principal/BOT)

Project title: Mathematical tasks that promote mathematical processes in the primary classroom.

Name of researcher: Jo Knox (PhD student)

Names of research supervisor: Caroline Yoon

Name of research co-supervisors: Fiona Ell, Igor Kontorovich

Researcher Introduction:

I am Jo Knox, and I am a PhD student in The University of Auckland, School of Science, Mathematics, and Technology Education. My supervisors are Caroline Yoon, Fiona Ell and Igor Kontorovich.

This Project:

The reason I am doing this research is to find out about students' mathematical thinking and understanding as they work on mathematical activities.

The research goal is to design mathematical tasks that can be used in primary classrooms to help students develop deep conceptual understanding in mathematics.

This project will continue for two years, however, the commitment of time required from participants for this part of the study is a maximum of four hours (1 hour sessions over a maximum 4 separate occasions/school days).

I expect that the results from this project will help support teachers with improving students' mathematical learning and understanding. The only risk I have identified associated with this research is the time required to participate in the research. Despite my utmost efforts to minimize risks, I can't prevent unforeseen circumstances where they might become upset. To manage this

risk, in such cases, students will be referred to their class teacher or school student counselor.

Only myself (the researcher) and my supervisors will have access to the data collected.

Invitation to Participate:

Your school is invited to participate in this research because some of your students (Year 3-8) meet the age requirements for this research.

To find potential schools, I am contacting primary/intermediate schools in Auckland that have existing relationships with the University of Auckland. If your school expresses an interest in participating in this project, I would like to work with up to three classes from your school. If you are willing for your school to participate in this research, please complete the enclosed Consent Form and return it to me at the address below.

Should your school be selected, I would like permission to approach students in Years 3-8 to participate. Student participation is voluntary. Even if you give consent to your school participating in this project, the final decision will be that of the students (including parental consent) who will be given details and asked to sign a consent form (parents of students) and an assent form (students).

The school may decline this invitation to participate without penalty. Should you agree to participate, I would appreciate your assurance that teacher participation or non-participation in this research will have no effect on their standing in the school. I would also appreciate your assurance that students' learning, academic standing, assessment and academic relationships with the school or members of staff will not be affected by either refusal or agreement to participate and will not be affected by their performance on the mathematics tasks.

During video or audio recording, students who agree to participate in the research cannot ask for the recorder to be turned off during the sessions, but can choose to leave the room. Since students will be video and audio recorded together in a group, if a student decides to withdraw from participation after the project has started, it will be impossible to remove any of his/her own data that he/she has already given.

Project Procedures:

If your school chooses to participate, students will be observed as they work on mathematical tasks together with other students in small groups of two-four.

Students in each class will be asked to work on a mathematical task on up to four different occasions (school days). Each of these sessions will take one hour each. The total time commitment from participants is therefore a maximum of four hours. The times of these periods will be negotiated with you and the teachers of the students involved, fitting in with your timetable. Teachers will be required to allocate students to groups of 2-4 who will be removed from the class to work on the task in a quiet space. Teachers will not be required to do any planning or instruction for these withdrawal group sessions. Only those students who have given their assent, and have parental consent, will be directly observed by myself and will have their work photocopied for research use.

Data Storage, Retention, destruction and Future Use

I will collect the data by taking student samples of work, taking field notes and video-recording the sessions with students. Data will not be used for any other purpose outside of this project, and only research personnel will have access to them. I will transcribe the videotapes. Information about students' age, gender, ethnicity and achievement levels will be obtained from their teacher.

Data (i.e., field notes, videotapes, and students' written work) will be securely stored at the University of Auckland. Electronic data will be kept on two University of Auckland password-protected computers. The un-identified transcripts of these tapes will be kept indefinitely for research purposes and publication. All other data will be stored for 6 years. After 6 years has elapsed, the data will be securely destroyed using reputable destruction services.

The information from the study (including student work) may be published but the students' names and the name of your school will not be used.

Right to Withdraw from Participation

The school and students have the right to withdraw at any time without giving a reason. However, since these are group tasks, it will be impossible for a student to remove any of his/her own data that he/she has already given if he/she wishes to withdraw their participation after the project has started.

If your school withdraws within two weeks of signing the consent form or by June 22 2019, any data supplied by the students will not be used in the research. After this time it may be too late to withdraw data already collected.

Anonymity and Confidentiality

The preservation of confidentiality is paramount. The information your students share within these sessions will remain confidential to the researcher and her supervisors. Pseudonyms will be used to label the student written work and videotapes, as well as in the transcripts and any publications. If the information that students/teachers provide is reported/published, this will be done in a way that does not identify your students, your teachers or your school as its source.

If you and your students (and their parents/guardians) consent, student snapshots may be published in reports and academic journals. However, this part is optional – you may choose not to allow video images of your students to be published or shown in professional presentations, while still being involved in the other aspects of this project. Should you and your students (and parent/guardians) agree to their images being shown at professional presentations and research publications, their anonymity, along with their teacher's, parent's and school's, cannot be guaranteed as they may be recognized from the image. *Please indicate whether you consent to students (and their parents/guardians) being asked for their consent to use their image on the attached consent form.*

A copy of the research findings will be made available to you if you wish.

CONTACT DETAILS AND APPROVAL

A copy of the research findings will be made available to you if you wish, by indicating on the consent form.

If you have any questions, please feel free to contact my supervisors or myself. If you are willing to participate in this research, please complete the enclosed Consent Form and return it to me at the address below.

Student Researcher name and contact details	**Supervisor name and contact details**	**Head of Department/School name and contact details**
Jo Knox School of Science, Mathematics, and Technology Education, The University of Auckland, Private Bag 92019, Auckland j.knox@auckland.ac.nz	Associate Professor: Dr. Caroline Yoon School of Science, Mathematics, and Technology Education, The University of Auckland, Private Bag 92019, Auckland Ph. +64 9 923 8740 c.yoon@auckland.ac.nz	Prof. Steven Galbraith School of Science, Mathematics, and Technology Education, The University of Auckland, Private Bag 92019, Auckland Ph. +64 9 923 8778 s.galbraith@auckland.ac.nz

For any queries regarding ethical concerns you may contact the Chair, The University of Auckland Human Participants Ethics Committee, The University of Auckland, Office of the Vice Chancellor, Private Bag 92019, Auckland 1142. Telephone 09 373-7599 extn. 83711.

Email: ro-ethics@auckland.ac.nz

APPROVED BY THE UNIVERSITY OF AUCKLAND HUMAN PARTICIPANTS ETHICS COMMITTEE ON 1st April 2019 (expires on 9th October 2020). Reference Number **019961**

THE UNIVERSITY OF AUCKLAND
Te Whare Wānanga o Tāmaki Makaurau
NEW ZEALAND

SCIENCE
DEPARTMENT OF MATHEMATICS

Addressee
Address

July 21, 2021

Department of Mathematics
Level 4, Building 303,
38 Princes Street
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T +64 9 373 7599 ext. xxxxx
E xxxxxxx@auckland.ac.nz
W www.math.auckland.ac.nz

The University of Auckland
Private Bag 92019
Auckland 1142
New Zealand

PARTICIPANT INFORMATION SHEET
(Student/Class)

Project title: Mathematical tasks that promote mathematical processes in the primary classroom.
Name of researcher: Jo Knox (PhD student)
Name of research supervisor: Caroline Yoon
Name of research co-supervisor: Fiona Ell, Igor Kontorovich

Who am I?

I am Jo Knox, and I am a PhD student in The University of Auckland, School of Science, Mathematics, and Technology Education. My supervisor is Caroline Yoon.

What am I doing?

I am trying to design tasks that can help students think and better understand mathematics.

Why am I here?

Your school and teacher have allowed me to work in your class.

Your parents have also given permission for you to take part in my work if you want to.

What if I say yes? What if I say no?

You can say 'Yes' or 'No' to taking part in this study.

Your principal and teacher have both promised that your decision to take part or not take part will not affect your schoolwork or grades.

If you say **"yes",** you are agreeing to help me with my work.

- Please complete the Assent Form and return it to me at the end of this talk.
- I will watch you and you will be video recorded as you work on mathematical tasks together with other students in small groups of two-four in a room at

your school. The room will not be your classroom and your teacher will not be there.

- You may be asked to work on mathematical tasks during 1-hour sessions on up to four separate school days.
- You can change your mind about taking part at any time without giving me a reason and you can leave the room and return to your class at any time during your work with me.
- If you change your mind and/or leave the room after we begin working on the task, it will be impossible for you to remove any of your own work from the group work.
- I will not use your name or the school's name (I will use made-up names) if I use your written work, or what you have said during the group sessions in my research.
- If you agree to take part in this study, you can say 'Yes' or 'No' to having your image (picture/video of you working on the task) used in my research materials/publications. Saying 'Yes' will mean that other people may recognize you and your school.

Student Researcher name and contact details	**Supervisor name and contact details**	**Head of Department/School name and contact details**
Jo Knox School of Science, Mathematics, and Technology Education, The University of Auckland, Private Bag 92019, Auckland j.knox@auckland.ac.nz	Associate Professor: Dr. Caroline Yoon School of Science, Mathematics, and Technology Education, The University of Auckland, Private Bag 92019, Auckland Ph. +64 9 923 8740 c.yoon@auckland.ac.nz	Prof. Steven Galbraith School of Science, Mathematics, and Technology Education, The University of Auckland, Private Bag 92019, Auckland Ph. +64 9 923 8778 s.galbraith@auckland.ac.nz

For any queries regarding ethical concerns you may contact the Chair, The University of Auckland Human Participants Ethics Committee, The University of Auckland, Office of the Vice Chancellor, Private Bag 92019, Auckland 1142. Telephone 09 373-7599 extn. 83711.

Email: ro-ethics@auckland.ac.nz

APPROVED BY THE UNIVERSITY OF AUCKLAND HUMAN PARTICIPANTS ETHICS COMMITTEE ON 1st April 2019 (expires on 9th October 2020). Reference Number **019961**

Appendix C: Consent Forms

SCIENCE
DEPARTMENT OF MATHEMATICS

Addressee
Address

Department of Mathematics
Level 4, Building 303,
38 Princes Street
Auckland, New Zealand
T +64 9 373 7599 ext. xxxxx
E xxxxxxx@auckland.ac.nz
W www.math.auckland.ac.nz

The University of Auckland
Private Bag 92019
Auckland 1142
New Zealand

July 21, 2021

CONSENT FORM (Principal)

THIS FORM WILL BE HELD FOR A PERIOD OF 6 YEARS

Project title: **Mathematical tasks that promote mathematical processes in the primary classroom.**
Name of researcher: PhD student: Jo Knox
Name of research supervisor: Caroline Yoon
Name of research co-supervisor: Fiona Ell, Igor Kontorovich
Contact email address for researchers: j.knox@auckland.ac.nz; c.yoon@auckland.ac.nz; f.ell@auckland.ac.nz; i.kontorovich@auckland.ac.nz

I have read the Participant Information Sheet and I have understood the nature of the research. I have had the opportunity to ask questions and have them answered to my satisfaction.

- I agree to give permission for the researchers to approach students (and their parents/guardians) in Years 3, 4, 5 and 6 of the school, inviting them to participate in the study.
- I understand that students who consent to participate in this research will be observed and video recorded as they work on mathematical tasks (for up to a maximum of one-hour) in a small group of two-four students, working in a withdrawal room from their class. They will also be asked to answer some questions regarding the tasks. The expected time commitment from students will be up to a maximum of 1 hour on four different occasions.
- I understand that students cannot ask for the recorder to be turned off during the sessions, but they can choose to leave the room.
- I understand that the researcher will transcribe the recordings.
- I understand that data will only be accessible to the researchers above and electronic data will be kept for 6 years on two University of Auckland password-protected

computers, after which they will be securely destroyed using reputable destruction services.

- I understand that permission from the parent/guardian will be sought for students involved in this study to be videotaped. In this case, the parent, student and teacher will be advised that if they give permission for images of the student to be shown at professional presentations and research publications, the student's anonymity (and consequently the parent's and school's) cannot be guaranteed. I understand this is optional – the students may still participate even if they (or their parents, or principal) do not allow their video images to be shown. In the latter case, any information they provide will be un-identifiable in any presentations/publications.

- As the principal, I give / do not give (please delete as appropriate) permission for video images to be shown at professional presentstion and research publication, should the parent/guardian and teacher also give their consent).

- I understand that should consent to use video images not be given, any information, collected from the recordings, will be reported or published in a way that does not identify the source. The researchers will change any identifying details, by blurring out images, to make sure these cannot be linked to the school or individual students or teachers.

- I understand that students are free to withdraw participation at any time.

- I understand that because students will be video recorded together, it will be impossible to remove any of an individual student's own data that he/she has already given if he/she wishes to withdraw his/her participation after the project has started.

- I give assurance that a student's participation or non-participation will not affect their grades, learning, standing or assessment in the department .

- I understand that I am free to withdraw the school's participation at any time without reason.

- I understand that if I withdraw the school's participation within two weeks of signing the consent form or by 22 June 2019 any data supplied by the students will not be used in the research, and that after this time it may be too late to withdraw data already collected.

- I wish / do not wish to receive the summary of findings, which can be provided to me at this email/postal address:_____.

Name: _____

Signature _____ Date _____

APPROVED BY THE UNIVERSITY OF AUCKLAND HUMAN PARTICIPANTS ETHICS COMMITTEE ON 1st April 2019 (expires on 9th October 2020). Reference Number **019961.**

SCIENCE
DEPARTMENT OF MATHEMATICS

Addressee
Address

Department of Mathematics
Level 4, Building 303,
38 Princes Street
Auckland, New Zealand
T +64 9 373 7599 ext. xxxxx
E xxxxxxx@auckland.ac.nz
W www.math.auckland.ac.nz

The University of Auckland
Private Bag 92019
Auckland 1142
New Zealand

July 21, 2021

CONSENT FORM
(Parent/Guardian of Primary School Student)
THIS FORM WILL BE HELD FOR A PERIOD OF 6 YEARS

Project title: **Mathematical tasks that promote mathematical processes in the primary classroom.**
Name of researcher: PhD student: Jo Knox
Name of research supervisor: Caroline Yoon
Name of research co-supervisor: Fiona Ell, Igor Kontorovich
Contact email address for researchers: j.knox@auckland.ac.nz; c.yoon@auckland.ac.nz; f.ell@auckland.ac.nz; i.kontorovich@auckland.ac.nz

I have read the Participant Information Sheet and have understood the nature of the research and why I have been selected. I have had the opportunity to ask questions and have them answered to my satisfaction.

- I agree for my child to take part in this research.
- I understand that his/her participation is voluntary.
- I understand that the Principal has given an assurance that my child's learning, standing, assessment and academic relationships with the school or members of staff will not be affected by either refusal or agreement to participate.
- I understand that my child will be observed and video recorded as s/he works on mathematical tasks (approximately one-hour sessions on up to four different occasions) with other students during class time at a location within my child's school that will be inaccessible to his/her teacher.

- I understand that s/he cannot ask for the recorder to be turned off during the sessions, but s/he can choose to leave the room.

- I agree for my child to be video-recorded and that a pseudonym will be used instead of his/her real name to label the transcripts of the video-recordings.

- I understand that the researcher will transcribe the recordings.

- I understand that s/he is free to withdraw participation at any time.

- I understand that because s/he will be video recorded together with other students, it will be impossible to remove any of his/her own data that s/he has already given if s/he wishes to withdraw my participation after the project has started.

- I agree / do not agree to my child's video/image being shown at professional presentations and research publications. (I understand that if I agree, my child's anonymity, along with and my own identity and that of the school's cannot be guaranteed.)

- I understand that electronic data will be kept for 6 years on two University of Auckland password-protected computers, after which they will be securely destroyed using reputable destruction services.

- I wish / do not wish to receive a summary of findings, which can be emailed to me at this email address: _____.

Name _____

Signature _____ Date _____

APPROVED BY THE UNIVERSITY OF AUCKLAND HUMAN PARTICIPANTS ETHICS COMMITTEE ON 1st April 2019 (expires on 9th October 2020). Reference Number **019961**

Appendix D: Assent Form

SCIENCE
DEPARTMENT OF MATHEMATICS

Addressee
Address

July 21, 2021

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38 Princes Street
Auckland, New Zealand
T +64 9 373 7599 ext. xxxxx
E xxxxxxx@auckland.ac.nz
W www.math.auckland.ac.nz

The University of Auckland
Private Bag 92019
Auckland 1142
New Zealand

ASSENT FORM
(Primary School Student)
THIS FORM WILL BE HELD FOR A PERIOD OF 6 YEARS

Project title: **Mathematical tasks that promote mathematical processes in the primary classroom.**
Name of researcher: PhD student: Jo Knox
Name of research supervisor: Caroline Yoon
Name of research co-supervisor: Fiona Ell
Contact email address for researchers: j.knox@auckland.ac.nz; c.yoon@auckland.ac.nz; f.ell@auckland.ac.nz; i.kontorovich@auckland.ac.nz

- I have read the Participant Information Sheet and I have understood what the work is about and why I have been selected.
- I have been able to ask questions and have them answered to my satisfaction.
- I understand that I can say 'Yes' or 'No' to taking part in this work.
- I understand I will be video recorded as I work on mathematical tasks together with other students in small groups of two-four in a room at my school.
- I understand the room will not be my classroom and my teacher will not be there.
- I understand that I can change my mind about taking part at any time without giving a reason.
- I understand that I cannot ask for the video recorder to be turned off during the sessions, but I can choose to leave the room and go back to my class at any time.

- I understand that if I change my mind and/or leave the room after we begin working on the task, it will be impossible to remove any of my own work from the group work.

- I understand I may be asked to work on a task for up to 1 hour, and on 4 different days.

- I understand that my principal and teacher have both promised that my decision to take part or not take part will not affect my schoolwork or grades.

- I understand that information from this work may be shared and published but my name and the name of my school will not be used.

I want to take part in this work. **Yes No**

I am happy to be video recorded. **Yes No**

I am happy for my video/image to be shown at professional presentations and research publications. (I understand that Saying 'Yes' will mean that other people may recognize me and my school). **Yes No**

Name _____

Signature _____ Date _____

APPROVED BY THE UNIVERSITY OF AUCKLAND HUMAN PARTICIPANTS ETHICS COMMITTEE ON 1st April 2019 (expires on 9th October 2020). Reference Number **019961**