

CONFORMAL FUNDAMENTAL FORMS AND THE ASYMPTOTICALLY POINCARÉ–EINSTEIN CONDITION

SAMUEL BLITZ^b, A. ROD GOVER[‡] & ANDREW WALDRON[‡]

ABSTRACT. An important problem is to determine under which circumstances a metric on a conformally compact manifold is conformal to a Poincaré–Einstein metric. Such conformal rescalings are in general obstructed by conformal invariants of the boundary hypersurface embedding, the first of which is the trace-free second fundamental form and then, at the next order, the trace-free Fialkow tensor. We show that these tensors are the lowest order examples in a sequence of conformally invariant higher fundamental forms determined by the data of a conformal hypersurface embedding. We give a construction of these canonical extrinsic curvatures. Our main result is that the vanishing of these fundamental forms is a necessary and sufficient condition for a conformally compact metric to be conformally related to an asymptotically Poincaré–Einstein metric. More generally, these higher fundamental forms are basic to the study of conformal hypersurface invariants. Because Einstein metrics necessarily have constant scalar curvature, our method employs asymptotic solutions of the singular Yamabe problem to select an asymptotically distinguished conformally compact metric. Our approach relies on conformal tractor calculus as this is key for an extension of the general theory of conformal hypersurface embeddings that we further develop here. In particular, we give in full detail tractor analogs of the classical Gauß Formula and Gauß Theorem for Riemannian hypersurface embeddings.

Keywords: Extrinsic conformal geometry, hypersurface embeddings, Poincaré–Einstein metrics, Yamabe problem

CONTENTS

1. Introduction	2
1.1. Riemannian Geometry	7
2. Tractor Calculus	8
2.1. Conformal Densities	8
2.2. Tractor Bundles	9
2.3. The Thomas- D Operator	11
2.4. Tractor Insertion	13
2.5. The W -Tractor	15
3. Conformally Embedded Hypersurfaces	17
3.1. Hypersurface Tractor Calculus	18
3.2. Proof of Theorem 1.3	20
3.3. Proof of Corollary 1.4	22
3.4. Normal Operators	25
3.5. Holography	27
4. Fundamental Forms	31
4.1. The Canonical Extension of $\mathring{\mathbb{I}}$	31
4.2. Canonical Transverse Differential Operators	32
4.3. Constructing Higher Fundamental Forms	34
4.4. Tractor Fundamental Forms	42
Acknowledgements	46
References	46

1. INTRODUCTION

Codimension one embedded submanifolds, or *hypersurfaces*, in Riemannian d -manifolds (M^d, g) have long been understood to be important because (for example) they arise naturally in foliations, and as the boundary of both domains and manifolds with boundary. In this setting the construction of local (and some global) invariants is well understood through the equations of Gauss, Codazzi and Ricci combined with Weyl's classical invariant theory. The goal of this article is the study of invariants of hypersurfaces embedded in a conformal manifold (M^d, \mathbf{c}) , where \mathbf{c} denotes an equivalence class of smoothly conformally related metrics. The main results include a sequence of conformal fundamental forms that should be viewed as higher order analogs of the trace-free second fundamental form. They are natural symmetric trace-free two tensors, on a hypersurface, that are extrinsic conformal invariants. The higher forms capture higher jet data of the interaction between the hypersurface and the ambient conformal structure. Therefore they are well-suited to the study of Poincaré–Einstein metrics.

A manifold $(M \setminus \Sigma, g^o)$ with boundary Σ is said to be conformally compact when $g = s^2 g^o$ extends as smoothly as a metric to the boundary for any choice of *defining function* s for Σ —meaning that Σ is the zero locus of $s \in C^\infty M$ and ds is nowhere zero along Σ . Moreover, when the interior metric g^o is Einstein, $(M \setminus \Sigma, g^o)$ is said to be *Poincaré–Einstein*. These structures have attracted intense scrutiny, partly because of their relation to the AdS/CFT correspondence of [41] and anomalies [36] in physics, as well as to basic geometric considerations [16, 33, 8, 2, 34, 1, 39, 43, 53, 17]. Given a conformally embedded hypersurface $\Sigma \hookrightarrow (M, \mathbf{c})$, we say a metric g^o on $M \setminus \Sigma$ is *asymptotically Poincaré–Einstein* if its trace-free Schouten \mathring{P}^{g^o} (or equivalently trace-free Ricci) tensor satisfies

$$(1.1) \quad \mathring{P}^{g^o} = \mathcal{O}(s^{d-3}),$$

where

$$g^o = s^{-2}g,$$

for some $g \in \mathcal{C}$ and s a defining function for Σ . Here $\mathcal{O}(s^k)$ denotes any smooth rank two tensor times the function s^k . We therefore ask the question: when is the metric on a conformally compact manifold conformal to an asymptotically Poincaré–Einstein metric? More generally we treat the following problem:

Problem 1.1. Given a conformally embedded hypersurface $\Sigma \hookrightarrow (M, \mathcal{C})$, under what conditions does $\mathcal{C}|_{M \setminus \Sigma}$ contain an asymptotically Poincaré–Einstein metric?

It is well known that the embedding $\Sigma \hookrightarrow (M, \mathcal{C})$ must be umbilic meaning that the trace-free second fundamental form $\mathring{\mathbb{H}}$ must vanish ($\forall p \in \Sigma$) [22, 38]. Thus, given a general conformal embedding $\Sigma \hookrightarrow (M, \mathcal{C})$, its trace-free second fundamental form oughxewed as an obstruction to the existence of a Poincaré–Einstein metric in the conformal class along $M \setminus \Sigma$; at the next order, an obstruction is the trace-free Fialkow tensor (of [18]; see also [52]). In Theorem 1.8 below we establish a necessary and sufficient condition for solving Problem 1.1 in terms of vanishing conditions for higher fundamental forms, which we now discuss.

For a hypersurface Σ , embedded in a dimension $d \geq 3$ Riemannian manifold (M^d, g) , the induced metric \bar{g} on the hypersurface is sometimes referred to as the first fundamental form. The second fundamental form \mathbb{H} is a rank two, natural, symmetric tensor along Σ that captures the failure of the conormal to be parallel. Such diffeomorphism invariant tensors built from the metric g , its derivatives, and the conormal are termed *natural*, see *e.g.* [4, 50, 25, 31]. For simplicity we do not consider parity odd tensors constructed from the Levi–Civita symbol/Hodge dual.

The part of \mathbb{H} that is trace-free with respect to the induced metric \bar{g} is denoted $\mathring{\mathbb{H}}$. Then

$$\mathring{\mathbb{H}} = \mathbb{H} - \bar{g}H,$$

defines the *mean curvature* H of $\Sigma \hookrightarrow (M, g)$. The *trace-free second fundamental form* $\mathring{\mathbb{H}}$ has the important property that when computed with respect to a conformally related metric $\Omega^2 g$, with $0 < \Omega \in C^\infty M$, it obeys

$$\mathring{\mathbb{H}}^{\Omega^2 g} = \Omega \mathring{\mathbb{H}}^g.$$

The *Fialkow tensor* of expression (3.5) below is another rank two tensor with a similar conformal transformation. Such *conformal hypersurface invariants* (see [28, Definition 4.1] for a formal definition) are important because they play a key rôle in geometric PDE boundary problems (see for example [12, 13, 25, 45]), in generalizing the scattering program of Melrose, Graham-Zworski and others [44, 34], and in the mathematics of the AdS/CFT program [41, 36]. Indeed, while the trace-free Schouten tensor of a metric $g^\circ = s^{-2}g$ on the interior of a conformally compact structure becomes singular as one approaches Σ , the quantity $s\mathring{P}^{g^\circ}$ is finite in this limit. Moreover this equals the trace-free part of the curvature-adjusted Hessian on $M \setminus \Sigma$

$$(1.2) \quad (\nabla^2 s + sP)_\circ.$$

Here ∇ is the Levi-Civita connection of g and P is the corresponding Schouten tensor. Importantly the above display extends smoothly to the boundary and there equals the trace-free second fundamental form $\mathring{\mathbb{H}}$. This suggests building higher fundamental forms from jets of the Hessian (1.2) transverse to Σ .

Later, we define a notion of the *transverse order* of a natural tensor along Σ by suitably counting transverse derivatives of the ambient metric g that are used in the formula for the given invariant along Σ —see Section (3.5) for details. The trace-free second fundamental form $\mathring{\mathbb{H}}$ has transverse order one and is the first example of what we define to be a conformal fundamental form. The trace-free part of the Fialkow tensor is a transverse order two example.

Definition 1.2. Let $n \in \mathbb{N}$. A *conformal fundamental form* is any natural trace-free section L of $\odot^2 T^* \Sigma$ with transverse order n that obeys

$$L^{\Omega^2 g} = \Omega^{2-n} L^g.$$

■

We call a conformal fundamental form of transverse order $n-1 > 0$ an *n*th conformal fundamental form, and for brevity, often drop the adjective conformal when $n > 2$. Because, by their very definition, conformal fundamental forms measure how the ambient metric extends off of the hypersurface, their leading transverse order terms are expressed in terms of ambient curvature quantities. Indeed, the first three non-trivial examples are made from the Weyl, Cotton, and Bach tensors, respectively.

In dimensions $d \neq 3$, a third fundamental form is

$$W_{\hat{n}ab\hat{n}},$$

where W is the Weyl tensor of (M, g) and \hat{n} is the unit conormal to Σ . In dimensions $d \neq 5$, a fourth fundamental form is

$$(1.3) \quad \frac{1}{d-5} \bar{\nabla}^c W_{c(ab)\hat{n}}^\top + C_{\hat{n}(ab)}^\top + HW_{\hat{n}ab\hat{n}},$$

where C is the Cotton tensor of (M, g) , \top denotes the projection of $TM|_\Sigma$ to $T\Sigma$, and $\bar{\nabla}$ is the Levi-Civita connection of (Σ, \bar{g}) . In dimensions $d \neq 3, 5, 7$, a fifth fundamental form is

$$\begin{aligned} & - \frac{2(d-4)}{d-7} (\bar{\Delta} \bar{F}_{ab} - \bar{\nabla}_{(a} \bar{\nabla} \cdot \bar{F}_{b)\circ} - (d-3) \bar{P}_{(a} \cdot \bar{F}_{b)\circ} - 2\bar{J} \bar{F}_{ab}) - \frac{d-4}{d-5} \bar{B}_{ab} \\ & + B_{(ab)\circ}^\top - 2(d-4) HC_{\hat{n}(ab)}^\top - (d-4) H^2 W_{\hat{n}ab\hat{n}} - \frac{d-4}{6(d-1)(d-2)} (\bar{\nabla}_{(a} \bar{\nabla}_{b)\circ} - 2\bar{P}_{(ab)\circ}) \bar{\Pi}^2 \\ & + \frac{d-4}{(d-2)(d-3)} \bar{\Pi}_{ab} (\bar{\nabla}_c \bar{\nabla}_d - (d-2) \bar{P}_{cd}) \bar{\Pi}^{cd} - \frac{d-4}{(d-2)^2} \bar{\nabla} \cdot \bar{\Pi}_{(a} \bar{\nabla} \cdot \bar{\Pi}_{b)\circ}, \end{aligned}$$

where B and \bar{B} are the Bach tensors of M and Σ , respectively (bars are universally used to denote hypersurface quantities and objects), see (2.10). Our conventions for tensors and for the Schouten tensor P and its trace J are given in Section 1.1. The above third and fourth fundamental forms vanish for generic embeddings in conformally flat spaces. It turns out that no fifth fundamental form exists with the same property.

The above tensor structures might at first seem arcane, but in fact they follow from a general theory of conformally invariant normal operators along hypersurfaces. The foundations of such were developed in [25], although important lacunæ in precisely this setting remain. We shall show that conformal fundamental forms are central to the theory of conformal hypersurface embeddings. They are atoms for the construction of *hypersurface invariants*, *i.e.*, natural tensors distinguished by the property that they are unchanged, or transform in a simple (“covariant”) way when the metric g is replaced by a conformally related metric $\hat{g} = e^{2\omega} g$, with $\omega \in C^\infty(M)$.

The identification and construction of conformal hypersurface invariants is complicated on all fronts. On one hand, the equations of Gauss, Codazzi, and Ricci are all badly behaved under conformal transformation, and on the other, even the construction of intrinsic conformal invariants is difficult and incomplete [7, 21]. There are two main tools which provide an effective route to resolving these difficulties. The first is the conformally invariant local calculus based around the conformal Cartan or tractor connection [6]. This enables a replacement of the Riemannian Gauss-Codazzi-Ricci hypersurface theory with an analogous conformally invariant set of basic equations and identities. While the basic elements of this are established in [6, 35, 46, 52, 29, 28, 31], gaps have remained and we treat these in Section 3; see also Theorem 1.3 and its Corollary 1.4 below. These results are crucial for the effective use of the second main tool which is *holography*.

The tractor approach to embedded hypersurfaces begins by extending the tangent bundle to M to a rank $d+2$ vector bundle $\mathcal{T}M$ equipped with a Cartan connection $\nabla^\mathcal{T}$ canonically determined by the data (M, \mathbf{c}) which preserves a tractor metric $h : \Gamma(\odot^2 \mathcal{T}M) \rightarrow C^\infty M$. These are respectively termed the *(standard) tractor bundle* and *tractor connection*; the precise definitions are given in Section 2.2. Because the conformal class of metrics \mathbf{c} induces a conformal class of metrics \mathbf{c}_Σ along Σ , the hypersurface also comes equipped with a tractor bundle $\mathcal{T}\Sigma$ and connection $\bar{\nabla}^\mathcal{T}$. It turns out that there is a simple relationship between $\mathcal{T}M$ and $\mathcal{T}\Sigma$ [22], and so it is natural to search for the conformal analogs of the Gauß formula and the Gauß equation for Riemannian hypersurface embeddings; early results for such were presented in [28, 31, 19]. Recall that the Gauß formula and Gauß equation relate ambient and hypersurface connections

and curvatures, respectively; see Equations (3.2) and (3.3). To obtain optimal conformal analogs of these, we use the Thomas- D operator which extends the tractor connection to a map from tractors to tractors. To understand this operator, one needs to enlarge the space of tractor objects to include sections of tensor products of tractor bundles as well as weighted tractor bundles (obtained by tensoring tractor fields with conformal densities), see Section 2.1. Denoting a weight w tractor bundle of tensor type Φ by $\mathcal{T}^\Phi M[w]$, the Thomas- D operator is a map

$$D : \Gamma(\mathcal{T}^\Phi M[w]) \rightarrow \Gamma(\mathcal{T}M \otimes \mathcal{T}^\Phi M[w-1]).$$

A quick way to understand the Thomas- D operator is that it is a conformally invariant triplet of differential operators in correspondence with the weight, gradient and Laplace operators. Detailed formulæ are given in Section 2.3. Because the Thomas- D operator is a powerful tool for proliferating conformal invariants [6], it is advantageous to have an analog of the Gauß formula relating the bulk and hypersurface Thomas- D operators:

Theorem 1.3 (Gauß–Thomas formula). *Let $w + \frac{d}{2} \neq 1, \frac{3}{2}, 2$. Acting on weight w tractors, the bulk tangential and hypersurface Thomas- D operators obey*

$$\hat{D}_A^T \stackrel{\Sigma}{=} \hat{D}_A + \Gamma_A^\sharp - \frac{X_A}{\bar{d} + 2w - 2} \left\{ 2\Gamma_B^\sharp \circ \hat{D}^B + \Gamma^{B\sharp} \circ \Gamma_B^\sharp + \frac{1}{\bar{d}(\bar{d}-1)} \left[(\hat{D}K) \wedge X \right]^\sharp - \frac{(3\bar{d}+2)wK}{2\bar{d}(\bar{d}-1)} \right\}.$$

The above explicit tractor result refines and simplifies the implicit one of [31]. The notation $\stackrel{\Sigma}{=}$ denotes equalities that hold along the hypersurface. The tangential Thomas- D operator \hat{D}^T is introduced in Proposition 3.2. Like the projection of the bulk Levi-Civita connection ∇^\top along Σ to $T\Sigma$, the operator \hat{D}^T obeys the tangentiality property

$$\hat{D}^T U \stackrel{\Sigma}{=} \hat{D}^T U',$$

for any pair of tractors U and U' satisfying $U \stackrel{\Sigma}{=} U'$. The tractor Γ_{ABC} is built from the second and third conformal fundamental forms; see Equation (3.6). Also, K denotes $\hat{\mathbb{H}}^2$ and is termed the *rigidity density*. Finally, the canonical tractor X is defined in Section 2.2. Theorem 1.3 is proved in Section 3.2.

The commutator of two Thomas- D operators includes a curvature tractor called the W -tractor—see Proposition 2.15—which unifies in a conformally covariant way, the Weyl, Cotton and Bach tensors [21, 24]. It obeys an analog of the Gauß theorem, which essentially follows as a corollary of the Gauß–Thomas formula:

Corollary 1.4 (Gauß–Thomas Equation). *Let $d > 5$. Then the bulk and hypersurface W -tractors are related by*

$$(1.4) \quad \begin{aligned} W_{ABCD}^\top|_\Sigma &= \bar{W}_{ABCD} - 2L_{A[C}L_{D]B} - 2\bar{h}_{A[C}F_{D]B} + 2\bar{h}_{B[C}F_{D]A} - \frac{2}{(d-1)(d-2)}\bar{h}_{A[C}\bar{h}_{D]B}K \\ &+ 2X_{[A}T_{B]CD} + 2X_{[C}T_{D]AB} - 2X_A X_{[C}V_{D]B} + 2X_B X_{[C}V_{D]A} \\ &+ \frac{1}{3(d-1)(d-2)}X_A X_{[C}\hat{D}_{D]}\hat{D}_B K - \frac{1}{3(d-1)(d-2)}X_B X_{[C}\hat{D}_{D]}\hat{D}_A K, \end{aligned}$$

where

$$T_{ABC} := 2\hat{D}_{[C}F_{B]A} + \frac{1}{(d-1)(d-2)}\bar{h}_{A[B}\hat{D}_{C]}K \in \Gamma(\mathcal{T}\Sigma \otimes \wedge^2 \mathcal{T}\Sigma[-3]),$$

and $V_{AB} \in \Gamma(\odot^2 \mathcal{T}\Sigma[-4])$ is a symmetric tractor built from curvatures such that $X^A V_{AB} = X_B V$ for some $V \in \Gamma(\mathcal{E}M[-4])$.

The tractors L_{AB} and F_{AB} are defined in Section 3.1. This corollary is proved in Section 3.3.

Our tractor hypersurface technology lays the groundwork for a forthcoming study of higher extrinsic Laplacian powers and generalized Willmore energies [9]. One such energy that emerges directly from our fundamental form study is given below.

Theorem 1.5. *Let $d = 5$ and $\Sigma \hookrightarrow (M, \mathbf{c})$ be a closed conformally embedded hypersurface. Then the integral defined for the metric \bar{g} induced by $g \in \mathbf{c}$ and given by*

$$(1.5) \quad \int_{\Sigma} d\text{Vol}(\bar{g}) \left(-\frac{1}{2} \bar{\mathbb{I}} \cdot \bar{\Delta} \bar{\mathbb{I}} + 6 \bar{\mathbb{I}} \cdot C_n^{\top} - 4 \bar{\mathbb{I}} \cdot \bar{P} \cdot \bar{\mathbb{I}} + \frac{7}{2} \bar{J}K + 6H \bar{\mathbb{I}}^3 + 6H \bar{\mathbb{I}} \cdot \bar{\mathbb{I}}\bar{\mathbb{I}} \right),$$

is independent of the choice of $g \in \mathbf{c}$.

Theorem 1.3 and its Corollary 1.4 concern the relationship between hypersurface objects and their bulk counterparts. It is also interesting to study how extensions of hypersurface quantities to the bulk vary in directions transverse to the hypersurface. For this one would like to have a theory of conformally invariant normal operators. This has been developed in [25]. For example, if $f \in C^{\infty}M$ is some extension of $\bar{f} \in C^{\infty}\Sigma$, then a conformally invariant second order normal operator is given by

$$\delta_R^{(2)} f = \bar{\Delta} \bar{f} - (d-3)(\hat{n}^a \hat{n}^b \nabla_a \nabla_b + H \nabla_{\hat{n}}) f|_{\Sigma} \in \Gamma(\mathcal{E}\Sigma[-2]).$$

In Lemma 3.11, in preparation for our study of conformal fundamental forms, we apply the theory of [25] to trace-free symmetric tensors.

We now turn to the second main tool: holography. Given an embedded hypersurface $\Sigma \hookrightarrow M$, the holographic approach is to set up a PDE problem on the ambient conformal manifold M whose solution is determined sufficiently accurately by the data $\Sigma \hookrightarrow M$ so that it encodes information about hypersurface invariants. Indeed, the idea is that the PDE solution determines the ambient geometry, at least to some order, so that ambient invariants determine hypersurface invariants.

For the construction of extrinsic conformal hypersurface invariants one begins with the data of a conformal embedding $\Sigma \hookrightarrow (M, \mathbf{c})$ and seeks a PDE problem that, given this data, determines a distinguished metric (again, to some order) in the conformal class \mathbf{c} . In [29, 28], it was shown that a suitable problem is the one of finding an asymptotic singular Yamabe metric. Uniqueness of solutions to the singular Yamabe problem on round [40] and asymptotically hyperbolic structures [5, 42, 3] is well understood and the appearance of hypersurface invariants as the obstruction to smooth solutions was first observed in [3]. Detailed definitions are given in Section 3.5. The key idea is to find a metric g^o on $M \setminus \Sigma$ whose scalar curvature obeys

$$(1.6) \quad S_{g^o} = -d(d+1) + \mathcal{O}(s^d),$$

where

$$g^o = s^{-2}g$$

for some $g \in \mathbf{c}$ and s a defining function for Σ . The pair (g, s) determines a distinguished conformal density $\sigma = [g; s] = [e^{2\omega}g; e^{\omega}s] \in \Gamma(\mathcal{E}M[1])$ (again see Section 2.1 for a detailed definition of a weight w conformal density bundle $\mathcal{E}M[w]$) from which conformal hypersurface invariants can be efficiently extracted using tractor calculus methods. This program is executed in the series of papers [29, 28, 31, 19].

Examining Equation (1.6), it is evident that the amount of extrinsic data encoded by an asymptotic singular Yamabe metric grows as the dimension increases. On the other hand, the complexity of the extrinsic curvature quantities determined this way escalates rapidly with increasing dimension. This can be simplified and captured using the conformally invariant higher order normal derivative operators discussed above.

In contrast to an asymptotic singular Yamabe metric, which always exists for generic embedding data, the existence of an asymptotic Poincaré–Einstein metric restricts the allowed conformal embeddings. This motivates our search for further conformally invariant obstructions to the existence of Poincaré–Einstein metrics. Indeed, given a defining function s for Σ , the standard approach, following [16], is to use s as a coordinate in a collar neighborhood $I \times \Sigma$, with $I \ni s$, of the boundary and then attempt to solve the asymptotic Poincaré–Einstein condition (1.1) via a “Fefferman–Graham” expansion

$$g = \bar{g} + sh_1 + \cdots + s^{d-2}h_{d-2} + \mathcal{O}(s^{d-1}).$$

Here, \bar{g} , h_1, \dots, h_{d-2} are symmetric rank two tensors on M that are in the kernel of $\mathcal{L}_{\frac{\partial}{\partial s}}$ and $\iota_{\frac{\partial}{\partial s}} h_k = 0$. The tensor $h_1|_{\Sigma}$ is the second fundamental form of the Riemannian embedding $\Sigma \hookrightarrow (M, g)$ and there always exists a choice of $g \in \mathbf{c}$ such that this embedding is minimal and $h_1|_{\Sigma}$ is the trace-free second fundamental form. A general conformal embedding is not umbilic but, as discussed above, does determine an asymptotic singular Yamabe metric $g^o = g/s^2$. Thus, asking whether the higher tensors h_k with $k \in \{2, \dots, d-2\}$ are compatible with the Poincaré–Einstein condition gives another perspective for why the trace-free second fundamental form $\hat{\mathbb{H}}$ is the first in a sequence of $d-2$ conformally invariant tensors characterizing the local obstruction to a conformal embedding being asymptotically Poincaré–Einstein. This sequence of fundamental forms is provided by the jets of the Hessian expression in Equation 1.2; see Section 4.1.

Our main result establishes that conformal fundamental forms are the obstruction to a conformal embedding admitting an asymptotically Poincaré–Einstein metric. However, the detailed picture here is both rich and subtle. For example, in $d = 5$ dimensions the fourth fundamental form displayed in Equation (1.3) is ill-defined. However, for umbilic embeddings, the hypersurface invariant

$$(1.7) \quad C_{\hat{n}(ab)}^\top + HW_{\hat{n}ab\hat{n}}$$

is both conformally invariant and has transverse order three. We call a transverse order $n-1$ hypersurface invariant a *conditional fundamental form* iff it is an n th conformal fundamental form on embeddings for which some lower transverse order fundamental form vanishes. This suggests two generalizations of umbilicity:

Definition 1.6. We say that a conformal hypersurface embedding $\Sigma \hookrightarrow (M^d, \mathbf{c})$ is *hyperumbilic* if, for each and every $n \in \{2, \dots, \lceil \frac{d+1}{2} \rceil\}$, an n th fundamental form vanishes. ■

Lemma 3.12) establishes that the above and following definitions are well-defined. As an application, it follows from Theorem 1.3 that $\hat{D}_A^T \stackrel{\Sigma}{=} \hat{D}_A$ for hyperumbilic embeddings in dimensions $d \geq 4$.

Definition 1.7. We say that a conformal hypersurface embedding $\Sigma \hookrightarrow (M^d, \mathbf{c})$ is *überumbilic* if the embedding is hyperumbilic and, for each and every $n \in \{\lceil \frac{d+3}{2} \rceil, \dots, d-1\}$, an n th conditional fundamental form vanishes. ■

The above definitions concern the second through $(d-1)$ th fundamental forms. At the weight where one would expect a d th fundamental form, an obstruction to the Poincaré–Einstein problem of a different nature is known to appear, namely the symmetric, trace-free, conformally invariant Fefferman–Graham tensor [16]. This tensor is intrinsic to the boundary conformal geometry $(\Sigma, \mathbf{c}_\Sigma)$ and must necessarily vanish for $\mathbf{c}|_{M \setminus \Sigma}$ to contain a smooth Poincaré–Einstein metric. The appearance of conditional fundamental forms for $n \geq \lceil \frac{d+3}{2} \rceil$ in Definition 1.7 is a feature of the theory of normal operators of [25], see in particular Lemma 4.7.

Our main result solves Problem 1.1 in terms of Definition 1.7:

Theorem 1.8. *A conformal class of metrics \mathbf{c} admits an asymptotic Poincaré–Einstein metric iff the conformal embedding $\Sigma \hookrightarrow (M^d, \mathbf{c})$ is überumbilic.*

1.1. Riemannian Geometry. In this section we outline our notations for Riemannian structures. When their meaning is obvious, these conventions will be extended to other bundles. Throughout the article, M will denote a d -manifold, which we equip either with a Riemannian metric g or a conformal class of metrics $\mathbf{c} = [g] = [\Omega^2 g]$ (where $0 < \Omega \in C^\infty M$). Unless otherwise indicated, all structures will be assumed smooth. Also, while we work exclusively in Riemannian signature, many results carry over to the pseudo-Riemannian setting upon obvious adjustments. To avoid confusion, the exterior derivative is denoted by d . For a given metric g , the Riemannian curvature tensor R for the associated Levi-Civita connection ∇ (we adorn ∇ and other metric

dependent operators/tensors with a superscript g when this dependence is not clear) is defined by

$$R(x, y)z = (\nabla_x \nabla_y - \nabla_y \nabla_x)z - \nabla_{[x, y]}z,$$

where $[\cdot, \cdot]$ is the Lie bracket and $x, y, z \in \Gamma(TM)$ are (smooth) vector fields.

We will use an abstract index notation in which a section of TM is denoted v^a and, for example $R(x, y)z$ becomes $x^c y^d R_{cd}{}^a{}_b z^b$. This has the advantage of directly yielding formula applicable to the case where local coordinate choices have been made. In this notation, the isomorphism between tangent and cotangent bundle TM and T^*M given by the metric tensor g_{ab} allows us to “raise and lower indices” and compute traces in the standard way. A Kronecker delta δ_b^a is used to denote the identity endomorphism of the tangent bundle. A dot will be used for abstract index contractions, so that $v^a w_a =: v \cdot w$, $\nabla_a v^a =: \nabla \cdot v$ and $v \cdot v =: v^2$, this notation will also be applied to higher rank symmetric tensors in an obvious way. Another succinct contraction notation that we use is to write, for example, $x^c y^d R_{cd}{}^a{}_b z^b = R_{xy}{}^a{}_z$ or even $x^{ab} u_a w_b = x_{uw}$ and $x_{uu} = (u \cdot)^2 x$. We employ bracket notations for groups of indices that are totally skew symmetric or symmetric. For example, $x^{ab} = x^{[ab]} + x^{(ab)}$ where $x^{[ab]} := \frac{1}{2}(x^{ab} - x^{ba})$ and $x^{(ab)} := \frac{1}{2}(x^{ab} + x^{ba})$. Also $u \wedge v$ denotes $\frac{1}{2}(u^a v^b - u^b v^a)$. A \circ will be used to denote the trace-free part of group of indices, so that $x_{(ab)\circ} := x_{(ab)} - \frac{1}{d} g_{ab} x^c{}_c$. We use standard \wedge and \odot notations for antisymmetric and symmetric tensor products of vector bundles and \odot_\circ for a further decomposition to the trace-free part. Given a tensor $v^{abc\dots e}$ we will use the shorthand $\mathcal{E}(v)$ to denote $v^{abc\dots e} t$ where t is an unspecified tensor or tensor-valued operator and a, b, c, \dots, e are any open indices. Also, given an endomorphism acting on sections of a vector bundle, we define its zeroth power to be the identity.

When specializing to conformal geometries, the decomposition

$$R_{abcd} = W_{abcd} + g_{ac} P_{bd} - g_{bc} P_{ad} + g_{bd} P_{ac} - g_{ad} P_{bc},$$

where W_{abcd} is the trace-free *Weyl tensor*, is particularly useful. In the above, P_{ab} is the symmetric *Schouten tensor*, which is related to the *Ricci tensor* $Ric_{bd} = R_{ab}{}^a{}_d$ in dimensions $d \geq 3$ via the trace adjustment

$$Ric_{ab} = (d-2)P_{ab} + g_{ab} J, \quad J = P_a^a.$$

We mostly deal with the case $d > 2$, but in two dimensions we define $J = Sc/2$ where $Sc := Ric_a^a$ is the scalar curvature in any dimension. The covariant curl of the Schouten tensor gives the *Cotton tensor*

$$C_{abc} = \nabla_a P_{bc} - \nabla_b P_{ac}.$$

2. TRACTOR CALCULUS

A conformal structure (M, \mathbf{c}) is the data of a smooth d -manifold M equipped with a conformal (equivalence) class \mathbf{c} of Riemannian metrics meaning that if \mathbf{c} is a non-empty set of Riemannian metrics on M , and if $g \in \mathbf{c}$, then $\hat{g} \in \mathbf{c}$ iff $\hat{g} = \Omega^2 g$ for some smooth positive function $\Omega \in C^\infty M$. Such (M, \mathbf{c}) have, in general, no distinguished connection on the tangent bundle but there is a canonical connection on a related bundle of rank $d+2$. This conformal *tractor connection* and surrounding tractor calculus provide the analog for conformal geometry of the Levi-Civita connection and its (“Ricci”) calculus on Riemannian manifolds [6, 15]. To treat this carefully we require the notion of a conformal density.

2.1. Conformal Densities. A weight w conformal density is a section of a certain line bundle $\mathcal{EM}[w]$ defined as follows: A conformal manifold (M, \mathbf{c}) may be viewed as a ray subbundle $\mathcal{G} \in \odot^2 T^*M$ with fiber at $P \in M$ given by all possible values of g_P for $g \in \mathbf{c}$. The bundle \mathcal{G} is a principal bundle with structure group \mathbb{R}_+ . The density bundle $\mathcal{EM}[w]$ is the line bundle associated to \mathcal{G} via the irreducible representation $\mathbb{R}_+ \in t \mapsto t^{-\frac{w}{2}} \in \text{End}(\mathbb{R})$, for each $w \in \mathbb{R}$. Then sections of $\Gamma(\mathcal{EM}[w])$ are equivalent to functions F of \mathcal{G} with the homogeneity property

$$F(P, \Omega^2 g) = \Omega^w F(P, g).$$

Alternately, one may view these as equivalence classes $F = [g; f] = [\Omega^2 g; \Omega^w f]$ where $f \in C^\infty M$. Note that the section space of the bundle $\mathcal{E}M[0]$ is isomorphic to the space of smooth functions $C^\infty M$; we often make this identification without comment. Importantly, a nowhere vanishing weight $w = 1$ conformal density $\tau = [g; t]$ canonically determines a metric $g_\tau \in \mathbf{c}$ by choosing an equivalence class representative $\tau = [g_\tau; 1]$. We refer to such a density as a *true scale* and the corresponding metric in \mathbf{c} as a choice of scale. We will also employ the term *scale* for weight $w = 1$ conformal densities whose zero locus is a hypersurface Σ , since these determine a metric on $M \setminus \Sigma$. As a point of notation, given a vector bundle \mathcal{B} , we use $\mathcal{B}[w]$ as shorthand for $\mathcal{B} \otimes \mathcal{E}M[w]$. Note that $\mathbf{g} := \tau^2 g_\tau \in \odot^2 T^*M[2]$ is independent of the choice of true scale τ and is called the *conformal metric*, which will be used for index raising and lowering in the obvious way. Note that on densities we get a connection on densities of weight w by acting with the operator $\tau^w \circ d \circ \tau^{-w}$. In fact this is the Levi-Civita connection (of the bundle $\mathcal{E}M[w]$) for the metric g^τ determined by the scale τ .

A conformal structure also determines log-density bundles, $\mathcal{F}M[w]$; see [27] for details. A weight- w *log-density* $\lambda \in \Gamma(\mathcal{F}M[w])$ is an equivalence class of (metric, function) pairs $\lambda = [g; \ell] = [\Omega^2 g; \ell + w \log \Omega]$. If $\varphi = [g; f]$ is a positive, weight w density $0 < \varphi \in \Gamma(\mathcal{E}M[w])$, then $\log \varphi = [g; \log f]$ is a weight- w log density. Also if λ is a weight w log density, then for any $r \in \mathbb{R}$, the product $r\lambda$ is a weight rw log density; in many contexts it thus suffices to focus on weight 1 log densities.

2.2. Tractor Bundles. As noted above, on a general conformal manifold (M, \mathbf{c}) with $d \geq 3$, there is a canonical connection on a closely related natural vector bundle of rank $d + 2$. This bundle $\mathcal{T}M$ is termed the *standard tractor bundle*. A choice of metric $g \in \mathbf{c}$ determines an isomorphism

$$(2.1) \quad \mathcal{T}M \stackrel{g}{\cong} \mathcal{E}M[1] \oplus TM[-1] \oplus \mathcal{E}M[-1].$$

This use of the metric to provide this isomorphism is called a *choice of splitting*. An abstract index notation (using capital latin letters) is often used for sections of the tractor bundle so, for example, $T^A \in \Gamma(\mathcal{T}M)$ denotes a standard tractor. Given $g \in \mathbf{c}$ and the corresponding splitting as in the above display, we can write $T^A \stackrel{g}{=} (t^+, t^a, t^-)$; we often employ a column vector notation for this and will be careful to display the metric dependence unless the context makes it clear. Choosing a conformally related metric $\Omega^2 g$, with Ω a positive smooth function, the isomorphism changes according to

$$T^A \stackrel{\Omega^2 g}{=} (t^+, t^a + \Upsilon^a t^+, t^- - \Upsilon \cdot t - \frac{1}{2} |\Upsilon|_g^2 t^+),$$

where $\Upsilon = \Omega^{-1} d\Omega$. General tensor products of $\mathcal{T}M$ with itself are often denoted $\mathcal{T}^\Phi M$, where Φ labels the particular tensor product under consideration. Further tensoring with a weight w conformal density bundle gives (weighted) tractor bundles $\mathcal{T}^\Phi M[w] := \mathcal{T}^\Phi M \otimes \mathcal{E}M[w]$. For brevity we often also call these *tractor bundles*. The first non-zero component, as determined by the isomorphism (2.1) above, of a tractor $T^A \stackrel{g}{=} (t^+, t^a, t^-)$ is called its *projecting part*, because it is necessarily conformally invariant; this is denoted $q^*(T)$. The same notion extends to general tractor tensors [6] (although the projecting part may no longer be an irreducible element of the tensor bundle of M). We term q^* the *extraction map*.

The tractor bundles $\mathcal{T}M[1]$ and $\odot^2 \mathcal{T}^*M$ are endowed with distinguished canonical sections X^A and h_{AB} , respectively termed the *canonical tractor* and *tractor metric*. In any choice of splitting these are given by, respectively,

$$X^A \stackrel{g}{=} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad h_{AB} \stackrel{g}{=} \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{g}_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and the inverse tractor metric is denoted by h^{AB} . The tractor metric provides an isomorphism between $\mathcal{T}M$ and its dual $\mathcal{T}^*M \stackrel{g}{=} \mathcal{E}M[1] \oplus T^*M[1] \oplus \mathcal{E}M[-1]$. Note that the canonical tractor is null, *i.e.*, $X^2 := h(X, X) = 0$.

The canonical tractor gives the complex

$$\mathcal{E}M[-1] \xrightarrow{X^A} \mathcal{T}M \xrightarrow{X_A} \mathcal{E}M[1].$$

In the above, the first map is multiplication by the canonical tractor. The second map denotes contraction by $X_A := h_{AB}X^B$. Here and henceforth, we employ the tractor metric to raise and lower tractor indices and to identify the tractor bundle with its dual. The above sequence of maps underlies the isomorphism of Equation (2.1). For later use, note that we will employ a tilde notation $\tilde{V}^A := [V^A] = [V^A + X^A U]$, where $U \in \Gamma(\mathcal{E}M[-1])$ for elements of the cokernel of the map X^A above. Note that, because the skew product of two canonical tractors is zero, then $X^{[A}\tilde{V}^{B]} := X^{[A}V^{B]}$ defines a section of $\wedge^2\mathcal{T}M[1]$. We will employ a tilde notation for higher tensor analogs of this cokernel. We will even use analogs of the notation $X^{[A}\tilde{V}^{B]}$ involving sums of tilded terms when it can be proved that such combinations define tractor tensors.

The tractor bundle has a canonical *tractor connection*

$$\nabla^{\mathcal{T}} : \Gamma(\mathcal{T}M) \rightarrow \Gamma(\mathcal{T}M \otimes T^*M)$$

that may be viewed as the conformal analog of the Levi-Civita connection. In the choice of splitting determined by $g \in \mathfrak{c}$, this acts according to

$$(2.2) \quad \nabla_a^{\mathcal{T}} T^B \stackrel{g}{=} \begin{pmatrix} \nabla_a t^+ - t_a \\ \nabla_a t^b + \delta_a^b t^- + P_a^b t^+ \\ \nabla_a t^- - P_{ab} t^b \end{pmatrix}.$$

Recall from above that a true scale $\tau \in \Gamma(\mathcal{E}M[1])$ canonically defines a metric $g_\tau \in \mathfrak{c}$. Combining τ , the canonical tractor, and connection, we may then form the (one-form)-valued standard tractor

$$Z_a^A := \tau \nabla_a^{\mathcal{T}} (\tau^{-1} X^B) \stackrel{g_\tau}{=} \begin{pmatrix} 0 \\ \delta_a^b \\ 0 \end{pmatrix}.$$

In terms of the tractor connection coupled to the Levi-Civita connection (of the metric g^τ)—which we henceforth denote simply ∇ —the above display reads $Z = \nabla X$.

Together with the tractor metric, for each true scale τ , we then can uniquely define the weight -1 tractor Y^A by the decomposition

$$h^{AB} = X^A Y^B + g^{ab} Z_a^A Z_b^B + X^B Y^A.$$

We refer to the triplet (X, Z, Y) above as *injecting operators* (or *injectors*) because given a standard tractor in a choice of scale $T^A \stackrel{g_\tau}{=} (t^+, t^a, t^-)$, we may write

$$T^A = t^+ Y^A + t^a Z_a^A + t^- X^A.$$

For later use, let us record how the tractor connection acts on the injectors

$$\begin{aligned} \nabla_b X^A &= Z_b^A, \\ \nabla_b Z_a^A &= -P_{ab} X^A - g_{ab} Y^A, \\ \nabla_b Y^A &= P_b^a Z_a^A. \end{aligned}$$

This is simply a rewriting of Equation (2.2); in the above the tensors $P := P^{g^\tau}$ is determined by the scale τ .

By way of notation, given a tractor such as $T^{ABC\dots}_{EFG\dots}$, then a quantity such as $T^{AbC\dots}_{EFg\dots}$ denotes $Z_B^b Z_g^G T^{ABC\dots}_{EFG\dots}$ so that in the running example above $T^a = t^a$. Moreover quantities like $X_A T^{ABC\dots}$ will be denoted X^{+BC} and $T_{EFG} Y^E$ as T^-_{FG} . The next section develops machinery for efficiently passing between Riemannian quantities and their tractor counterparts.

2.3. The Thomas- D Operator. The Levi-Civita-coupled tractor connection ∇ can be used to define an invariant, second order operator on tractors that plays the *rôle* of a conformal gradient. This is termed the *Thomas- D operator* which is the map

$$\Gamma(\mathcal{T}^\Phi M[w]) \rightarrow \Gamma(\mathcal{T}M \otimes \mathcal{T}^\Phi M[w-1])$$

defined acting on $T \in \Gamma(\mathcal{T}^\Phi M[w])$ by, in a scale τ ,

$$(2.3) \quad D^A T := (d+2w-2)wY^A T + (d+2w-2)\mathbf{g}^{ab}Z_a^A \nabla_b T - X^A(\Delta + wJ^{g^\tau})T.$$

In the above $\Delta := \mathbf{g}^{ab}\nabla_a \nabla_b$ and the injectors (Y, Z) are determined in terms of τ as explained above.

The Levi-Civita connection can also act on log densities: for $\lambda \in \Gamma(\mathcal{F}M[w])$, we define, in the scale τ ,

$$\nabla \lambda := d(\lambda - w \log \tau) \in \Gamma(T^*M).$$

Here we used that the combination $\lambda - w \log \tau$ defines an element of $C^\infty M$ in the obvious way. Note that $\nabla^{\tau'} \lambda$ in a scale $\tau' = \Omega \tau$ equals $\nabla^\tau \lambda - w \Upsilon$, where $\Upsilon = d \log \Omega$.

Acting on weight w log densities we define

$$D : \Gamma(\mathcal{F}[w]) \rightarrow \Gamma(\mathcal{T}M[-1])$$

acting on $\lambda \in \Gamma(\mathcal{F}[w])$ by (in the scale τ)

$$(2.4) \quad D^A \lambda := (d-2)wY^A \lambda + (d-2)\mathbf{g}^{ab}Z_a^A \nabla_b \lambda - X^A(\Delta \lambda + wJ^{g^\tau}).$$

Lemma 2.1. $D^A T$ and $D^A \lambda$, as respectively given in Equations (2.3) and (2.4), are defined independently of the choice of true scale τ .

Proof. The proof of this lemma is contained in [24, Section 2] for the case $D^A T$ where T is a tractor; the log-density statement can be established using the same proof *mutatis mutandis*. \square

Remark 2.2. Given the above lemma, we will sometimes write abbreviated formulæ such as

$$D^A T = \begin{pmatrix} w(d+2w-2)T \\ (d+2w-2)\nabla T \\ -(\Delta + wJ)T \end{pmatrix}$$

for the action of the Thomas- D operator on weighted tractors. We also omit the decoration τ when the dependence on it drops out. \blacksquare

A remarkable property of the Thomas- D operator is that it is null [20], meaning

$$D^A \circ D_A = 0.$$

Because the Thomas- D operator is second order, it does not obey a Leibniz rule. However, upon a slight modification, the failure of the Leibniz property is rather mild. For that we first make a definition:

Definition 2.3. Suppose that $w \neq 1 - \frac{d}{2}$. The *hatted- D operator*

$$\hat{D}^A : \Gamma(\mathcal{T}^\Phi M[w]) \rightarrow \Gamma(\mathcal{T}M \otimes \mathcal{T}^\Phi M[w-1])$$

is defined acting on $T \in \Gamma(\mathcal{T}^\Phi M[w])$ by

$$\hat{D}^A T := \frac{1}{d+2w-2} D^A T.$$

\blacksquare

The following result of [37] is proved in [28].

Proposition 2.4. Let $T_i \in \Gamma(\mathcal{E}^\Phi M[w_i])$ for $i = 1, 2$, and $h_i := d+2w_i$, $h_{12} := d+2w_1+2w_2-2$ with $h_i \neq 0 \neq h_{12}$. Then,

$$\hat{D}^A(T_1 T_2) - (\hat{D}^A T_1) T_2 - T_1 (\hat{D}^A T_2) = -\frac{2}{h_{12}} X^A (\hat{D}_B T_1) (\hat{D}^B T_2).$$

Another important identity is

$$(2.5) \quad \hat{D}^A X^B = h^{AB}.$$

Using that

$$X^A D_A T = w(d + 2w - 2)T,$$

for any weight w tractor, Equation (2.5) is then an easy corollary of Proposition 2.4.

It is useful to introduce the *weight operator* \underline{w} defined on sections $T \in \Gamma(\mathcal{E}M[w])$ by $\underline{w}T = wT$. It is easy to check that \underline{w} is a derivation acting on tractors. Acting on log-densities $\lambda \in \Gamma(\mathcal{F}M[w])$, we define (see [27])

$$\underline{w}\lambda = w.$$

Moreover, for $T \in \Gamma(\mathcal{T}^\Phi M[w])$ and $\lambda \in \Gamma(\mathcal{F}M[w'])$, we extend \underline{w} (and similarly ∇) to act on the product $\lambda T \in \Gamma(\mathcal{T}^\Phi M[w] \otimes \mathcal{F}M[w'])$ by the Leibniz property:

$$\underline{w}(\lambda T) = \lambda \underline{w}T + T \underline{w}\lambda = w\lambda T + w'T \in \Gamma(\mathcal{T}^\Phi M[w] \oplus \mathcal{T}^\Phi M[w] \otimes \mathcal{F}M[w']).$$

We can now write a universal formula for the Thomas- D operator on any tensor product bundle of density and log-density bundles:

$$D^A := Y^A(d + 2\underline{w} - 2)\underline{w} + \mathbf{g}^{ab} Z_a^A \nabla_b(d + 2\underline{w} - 2) - X^A(\Delta + J\underline{w}).$$

Using these notations, acting on weight $w \neq 1 - \frac{d}{2}$ tractors, the operator \hat{D} can be written in terms of D via $\hat{D} = D \circ \frac{1}{d + 2\underline{w} - 2}$, where we define the operator $\frac{1}{d + 2\underline{w} - k}$ on a weight w' log-density λ (for $k \neq d$) by

$$(2.6) \quad \frac{1}{d + 2\underline{w} - k} \lambda := \frac{\lambda}{d - k} - \frac{2w'}{(d - k)^2} \in \Gamma(\mathcal{F}M[w'] \oplus \mathcal{E}M[0]),$$

and

$$(2.7) \quad \frac{1}{d + 2\underline{w} - k} (\lambda T) := T \frac{1}{d + 2\underline{w} + 2w - k} \lambda \in \Gamma(\mathcal{T}^\Phi M[w] \oplus \mathcal{T}^\Phi M[w] \otimes \mathcal{F}M[w']).$$

Remark 2.5. It is even possible to define an operator $\frac{1}{\alpha\underline{w} + \beta}$ acting on sections of $\otimes^k \mathcal{F}M[w']$ by appealing to the formal power series expression

$$\frac{1}{\alpha\underline{w} + \beta} = \frac{1}{\beta} - \frac{\alpha}{\beta^2} \underline{w} + \frac{\alpha^2}{\beta^3} \underline{w}^2 + \dots,$$

since only finitely many terms are ever needed. ■

In general we prefer to avoid working with sections of Whitney sum bundles such as those arising when the inverse weight operator above acts on log-density bundles. Happily, the following result shows that composing the Thomas- D operator with the action of $\frac{1}{d + 2\underline{w} - 2}$ on log densities results in a tractor.

Lemma 2.6. *For any $0 \neq \beta \in \mathbb{R}$,*

$$\left(D \circ \frac{1}{\alpha\underline{w} + \beta} \right) \lambda = \frac{1}{\beta} D\lambda \in \Gamma(\mathcal{T}M[-1]).$$

Proof. This follows from a straightforward computation in a choice of scale, or otherwise can be seen as a direct consequence of Equation (2.6). □

There is an analog of Proposition (2.4) for the algebra of Thomas- D operators and log densities; this is another case where the need for Whitney bundles is obviated.

Lemma 2.7. *Let λ be any log density and let $w \neq \frac{2-d}{2}$. Then,*

$$\hat{D} \circ \lambda - \lambda \circ \hat{D} : \Gamma(\mathcal{T}^\Phi M[w]) \rightarrow \Gamma(\mathcal{T}M \otimes \mathcal{T}^\Phi M[w - 1]),$$

and moreover

$$\hat{D} \circ \lambda - \lambda \circ \hat{D} = (\hat{D}\lambda) - \frac{2}{d + 2\underline{w}} X(\hat{D}\lambda) \cdot \hat{D}.$$

Proof. This is an easy computation in a choice of scale. □

A particularly useful application of the Thomas- D operator is the construction of tractors from Riemannian tensors.

2.4. Tractor Insertion. Given a conformal density-valued, weight $w + r$, trace-free, rank r Riemannian tensor $t_{abc\dots}$, there exists a canonical map (see for example [21]) to insert $t_{abc\dots}$ into a tractor $T^{ABC\dots}$ with the same symmetries as $t_{abc\dots}$. This map is denoted by q ; there are three particular instances relevant for our computations:

Lemma 2.8. *Let $g \in \mathbf{c}$.*

(i) *Given $v_a \in \Gamma(T^*M[w + 1])$ where $w \neq 1 - d$, then*

$$q(v_a) =: V^A \in \Gamma(\mathcal{T}M[w])$$

$$\stackrel{g}{=} \begin{pmatrix} 0 \\ v^a \\ -\frac{\nabla \cdot v}{d+w-1} \end{pmatrix},$$

where

$$D_A V^A = X_A V^A = 0.$$

(ii) *Given $t_{ab} \in \Gamma(\odot^2 T^*M[w + 2])$ where $w \neq -d, 1 - d$, then*

$$q(t_{ab}) =: T^{AB} \in \Gamma(\odot^2 \mathcal{T}M[w])$$

$$\stackrel{g}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & t^{ab} & -\frac{\nabla \cdot t^a}{d+w} \\ 0 & -\frac{\nabla \cdot t^b}{d+w} & \frac{\nabla \cdot \nabla \cdot t + (d+w)P_{ab}t^{ab}}{(d+w)(d+w-1)} \end{pmatrix},$$

where

$$D_A T^{AB} = 0 = X_A T^{AB}.$$

(iii) *Given $t_{abcd} \in \Gamma(\otimes^4 T^*M[w + 4])$, where $w \neq 1 - d, 2 - d$, such that t has the algebraic symmetries of the Riemann tensor and is trace-free, then*

$$q(t_{abcd}) =: T^{ABCD} \in \Gamma(\otimes^4 \mathcal{T}M[w])$$

where

$$T^{abcd} \stackrel{g}{=} t^{abcd}$$

$$T^{abc-} \stackrel{g}{=} -\frac{\nabla_d t^{dabc}}{d+w-1}$$

$$T^{a-b-} \stackrel{g}{=} \frac{\nabla_a \nabla_c t^{abcd} + (d+w-1)P_{ac}t^{abcd}}{(d+w-1)(d+w-2)},$$

where T also has the algebraic symmetries of the Riemann tensor and

$$D_A T^{ABCD} = X_A T^{ABCD} = 0 = h_{AC} T^{ABCD}.$$

Proof. The proofs of the above three results can either be given in a much more general setting (see [21]), or by explicit computation whose intricacy increases in concordance with tensor rank. Here we give the lowest rank example.

The ‘‘bottom slot’’ V^- of $q(v_a) := V^A \stackrel{g}{=} (v^+, v^a, v^-)$ can be computed by writing out the constraint $D_A V^A \stackrel{g}{=} 0$ for some $g \in \mathbf{c}$:

$$0 = (d + 2w - 2)(wv^- + \nabla \cdot v + Jv^+ + dv^-) - (\Delta + (w - 1)J)v^+ + 2\nabla \cdot v + dv^-$$

$$= (d + 2w)(\nabla \cdot v + (d + w - 1)v^-),$$

where the second equality comes from the requirement that $X_A V^A = 0$. When $w \neq -\frac{d}{2}$, this yields the quoted result. If $w = -\frac{d}{2}$, we need to verify that the result for V^A given for a choice of $g \in \mathbf{c}$ defines a section of $\Gamma(\mathcal{T}M[w])$. This is easily established by transforming the quoted result to a conformally related metric. \square

Note that the insertion operator q is a right-inverse of the extraction map

$$q^* \circ q = \text{Id},$$

but q is *not* a left-inverse of the extraction map,

$$q \circ q^* \neq \text{Id}.$$

The above operator, acting on rank-2 tractors, is computed in the following lemma for use later.

Lemma 2.9. *Let the tractor $T \in \Gamma(\odot^2 \mathcal{T}M[w])$, where $w \neq 1 - \frac{d}{2}, -\frac{d}{2}, -d, 1 - d$, obey*

$$X_A T^{AB} = 0 \quad \text{and} \quad q^*(T) \in \Gamma(\odot^2 T^* M[w+2]).$$

Then

$$(q \circ q^*)(T) = \tilde{T},$$

where

$$\tilde{T}_{AB} := T_{AB} - \frac{2}{(d+w)(d+2w)} X_{(A} D^C T_{B)C} + \frac{1}{(d+w)(d+w-1)(d+2w)} X_A X_B D^C \hat{D}^D T_{CD}.$$

Proof. We will establish that there is a unique \tilde{T} that satisfies

$$\hat{D}^A \tilde{T}_{AB} = X^A \tilde{T}_{AB} = 0 = h^{AB} \tilde{T}_{AB},$$

and obeys $q^*(\tilde{T}) = q^*(T)$ whenever $q^*(T) \in \Gamma(\odot^2 T^* M[w+2])$. This ensures that $(q \circ q^*)(T) = \tilde{T}$. For that, we use the operator version of Proposition 2.4, valid acting on tractors of weight $w \neq 1 - d/2, -d/2$:

$$(2.8) \quad \hat{D}^A \circ X^B = X^B \hat{D}^A + h^{AB} - \frac{2}{d+2w} X^A \hat{D}^B.$$

We first verify that $X \cdot \tilde{T} = 0$. Because $X^2 = 0 = X^A T_{AB}$, we simply need to check that $X^A \hat{D}^B T_{AB}$ vanishes. Applying Equation (2.8) we have that

$$\begin{aligned} X^A \hat{D}^B T_{AB} &= \left(\hat{D}^B X^A - h^{AB} + \frac{2}{d+2w} X^B \hat{D}^A \right) T_{AB} \\ &= \frac{2}{d+2w} X^B \hat{D}^A T_{AB}, \end{aligned}$$

where we have used that $h^{AB} T_{AB} = 0 = X^A T_{AB}$. Because T is symmetric, we are left with the identity

$$\frac{d+2w-2}{d+2w} X^A \hat{D}^B T_{AB} = 0.$$

Thus, thanks to the weight assumptions, it follows that $X^A \hat{D}^B T_{AB} = 0$, and hence $X^A \tilde{T}_{AB} = 0$. Similarly, we have that $h^{AB} \tilde{T}_{AB} = 0$.

Finally, we check that $\hat{D}^A \tilde{T}_{AB} = 0$. We do this in stages. First, we evaluate $\hat{D}^A (X_A \hat{D}^C T_{BC})$:

$$\begin{aligned} \hat{D}^A (X_A \hat{D}^C T_{BC}) &= \left[(w-1) + d + 2 - \frac{2(w-1)}{d+2(w-1)} \right] \hat{D}^C T_{BC} \\ &= \frac{(d+w-1)(d+2w)}{d+2w-2} \hat{D}^C T_{BC}. \end{aligned}$$

Next, we evaluate the term $\hat{D}^A (X_B \hat{D}^C T_{AC})$:

$$\begin{aligned} \hat{D}^A (X_B \hat{D}^C T_{AC}) &= X_B \hat{D}^A \hat{D}^C T_{AC} + \hat{D}^C T_{BC} - \frac{2}{d+2(w-1)} X^A \hat{D}_B \hat{D}^C T_{AC} \\ &= X_B \hat{D}^A \hat{D}^C T_{AC} + \hat{D}^C T_{BC} \\ &\quad - \frac{2}{d+2(w-1)} \left[\hat{D}_B (X^A \hat{D}^C T_{AC}) - h_B^A \hat{D}^C T_{AC} + \frac{2}{d+2(w-1)} X_B \hat{D}^A \hat{D}^C T_{AC} \right] \\ &= \frac{(d+2w)(d+2w-4)}{(d+2w-2)^2} X_B \hat{D}^A \hat{D}^C T_{AC} + \frac{d+2w}{d+2w-2} \hat{D}^C T_{BC}. \end{aligned}$$

Last, we evaluate the term $\hat{D}^A (X_A X_B \hat{D}^C \hat{D}^D T_{CD})$:

$$\hat{D}^A (X_A X_B \hat{D}^C \hat{D}^D T_{CD}) = \frac{(d+w-1)(d+2w)}{d+2w-2} X_B \hat{D}^C \hat{D}^D T_{CD}.$$

Combining these terms, we find that $\hat{D}^A \tilde{T}_{AB} = 0$, thus completing the proof. \square

Remark 2.10. Note that if a weight $w \neq 1 - \frac{d}{2}, -\frac{d}{2}$ tractor $\tilde{T}^{AB\dots}$ obeys

$$X_A \tilde{T}^{AB\dots} = 0 = \hat{D}_A \tilde{T}^{AB\dots},$$

then it follows directly from Equation (2.8) that

$$X_A \hat{D}^C \tilde{T}^{AB\dots} = -\tilde{T}^{CB\dots}.$$

■

Often it is useful to change the projecting part of tractor; the operator in the following lemma is an instance of this.

Lemma 2.11. *Let $V^A \in \Gamma(\mathcal{T}M[w])$ and $T \in \Gamma(\odot_0^2 M[w])$. Then if $w \neq -1, -1 - \frac{d}{2}$,*

$$r(V^A) := V^A - \frac{1}{w+1} \hat{D}^A (X_B V^B)$$

obeys

$$X_A r(V^A) = 0,$$

while if $w \neq 0, -1, -\frac{d}{2}, -1 - \frac{d}{2}, -2 - \frac{d}{2}$,

$$\begin{aligned} r(T^{AB\dots}) := & T^{(AB)\circ} - \frac{2}{w} \hat{D}^A (X_C T^{C|B)\circ}) + \frac{1}{w(w+1)} \hat{D}^A \hat{D}^B \circ (X_C X_D T^{CD}) \\ & - \frac{8}{wd(d+2w+2)} X^{(A} \hat{D}^B) \circ (\hat{D}_C (X_D T^{CD})), \end{aligned}$$

obeys

$$X_A r(T^{AB}) = 0 = h_{AB} r(T^{AB}).$$

Proof. The proof is an elementary application of the identity

$$X_A \hat{D}^A T = wT,$$

valid for any weight $w \neq 1 - \frac{d}{2}$ tractor T , the fact that X and D are null, and Equation (2.8). □

Remark 2.12. The map r of the above lemma can be generalized, modulo distinguished weights, to tractors T of arbitrary tensor type so that $r(T)$ is both tractor trace-free and in the kernel of contraction by the canonical tractor X . ■

The algebra of multiple Thomas- D operators, and in particular commutators of these, will be of particular importance. This leads to a tractor tensor that generalizes the Weyl tensor.

2.5. The W -Tractor. To begin with, we consider the curvature of the tractor connection. The commutator of tractor connections gives the *tractor curvature* $\Omega_{ab}{}^{AB} \in \Gamma(\wedge^2 T^*M \otimes \wedge^2 \mathcal{T}M)$ which acts on a standard tractor V^A according to

$$[\nabla_a, \nabla_b] V^A = \Omega_{ab}{}^A{}_B V^B =: \Omega_{ab}{}^\sharp V^A \in \Gamma(\wedge^2 T^*M \otimes \mathcal{T}M).$$

The operator $\Omega_{ab}{}^\sharp$ extends by linearity to act on general tractor tensors. In general, if Ω^{AB} is any skew-symmetric tractor, then Ω^\sharp denotes the operator on tractors $V^{AB\dots}$ defined by $\Omega^\sharp V^{AB\dots} := \Omega^A{}_C V^{CB\dots} + \Omega^B{}_C V^{AC\dots} + \dots$. For a choice of metric $g \in \mathfrak{c}$, the tractor curvature is given in an obvious matrix notation by

$$\Omega_{ab}{}^C{}_D = \begin{pmatrix} 0 & 0 & 0 \\ C_{ab}{}^c & W_{ab}{}^c{}_d & 0 \\ 0 & -C_{abd} & 0 \end{pmatrix}.$$

The projecting part of the tractor curvature is the Weyl tensor, which is indeed conformally invariant. In three dimensions, the Weyl tensor vanishes, and the Cotton tensor is then the projecting part and hence invariant in this dimension.

At this juncture, to build its tractor analog, we could either insert the Weyl tensor into a tractor or instead view the Thomas- D operator as the tractor analog of the gradient operator

and study commutators of Thomas- D operators. Both routes will lead us to the so-called W -tractor. Let us begin with the first route. In dimension $d > 4$, using Proposition 2.8, we can define the weight -2 , rank 4, W -tractor by

$$W^{ABCD} := q(W_{abcd}),$$

where W_{abcd} is the Weyl tensor. By construction, W has the symmetries of the Weyl tensor

$$W^{[ABC]D} = W^{ABCD} + W^{BACD} = W^{ABCD} - W^{CDAB} = 0 = W^A{}_{A'}{}^{CD}.$$

We have used the same letter W for both the Weyl curvature and W -tractor because the projecting part of the latter, W^{abcd} is—by construction—the Weyl tensor. The remaining tensor content of the W -tractor is given below.

Lemma 2.13. *Let $d > 4$. Then in any scale $g \in \mathfrak{c}$, the W -tractor is given by*

$$(2.9) \quad \begin{aligned} W^{abc-} &= C^{abc}, \\ W^{a-b-} &= \frac{\Delta P^{ab} - \nabla_c \nabla^b P^{ca} + P_{cd} W^{adbc}}{d-4}. \end{aligned}$$

All other components are either zero or determined by the symmetry of the W -tractor.

Proof. This lemma follows from Proposition 2.8 and the curvature identities

$$C^{abc} = \frac{\nabla_d W^{dcab}}{d-3}, \quad \nabla_a \nabla_c W^{abcd} = (d-3)(\Delta P^{bd} - \nabla_c \nabla^b P^{cd}).$$

□

Remark 2.14. In $d = 4$ dimensions, the W -tractor is not well-defined. Instead we may consider the equivalence class of tractors defined by the relation $\tilde{W}^{ABCD} \sim \tilde{W}^{ABCD} + X^{[A} V^{B][D} X^C]$ for V^{BD} any rank 2, symmetric, trace-free, weight -4 tractor. In any choice of scale $g \in \mathfrak{c}$ there is always a representative $\tilde{W}^{abcd} = W^{abcd}$, $\tilde{W}^{abc-} = C^{abc}$ and $\tilde{W}^{a-b-} = 0$.

Note that the residue of the $d - 4$ pole in Equation (2.9) continued to four dimensions is precisely the four dimensional Bach tensor

$$(2.10) \quad B^{ab} = \Delta P^{ab} - \nabla_c \nabla^b P^{ca} + P_{cd} W^{adbc}.$$

For that reason, the W -tractor is often defined with an overall factor of $d - 4$ so that it is well-defined in four dimensions; this has the disadvantage that in higher dimensions, the projecting part only equals the Weyl tensor up to an overall non-zero multiplicative factor. Also note that we use the above tensor expression to define a Bach tensor $B^{ab} \stackrel{g}{=} (d-4)W^{a-b-}$ in dimensions higher than four. ■

Turning to the other route, the commutator of Thomas- D operators is given below:

Proposition 2.15. *In dimension $d \neq 4$, the commutator of Thomas- D operators obeys*

$$[D_A, D_B] = (d+2w-4)(d+2w-2)W_{AB}^\sharp + 4X_{[A} W_{B]C}^\sharp \circ D^C.$$

In $d = 4$, the commutator of two Thomas- D operators is

$$[D_A, D_B] = 4w(w+1)\tilde{W}_{AB}^\sharp + 4X_{[A}\tilde{W}_{B]C}^\sharp \circ D^C,$$

and the sum of equivalence classes on the right hand side is a tractor-valued operator.

Proof. This result was first given in [26] using an ambient construction. Alternatively one can—at the cost of quite some computation—verify the first displayed equation using Equation (2.3) and the injectors introduced above. The same applies for the second display, but we must first verify that its right hand side is defined as a tractor operator independently of the choice of equivalence class representative chosen for \tilde{W} . It is sufficient to check that the right-hand side

of the second display, acting on a tractor $V^C \in \Gamma(\mathcal{T}M[w])$ and contracted with $Y^A Y_C$, does not depend on the choice of metric representative.

$$\begin{aligned}
& Y_A Y_C \left[4w(w+1) \tilde{W}^{ABCE} V_E + 4X^{[A} \tilde{W}^{B]ECF} D_E V_F \right] \\
&= 4w(w+1) \tilde{W}^{-B-E} V_E + 2\tilde{W}^{BE-F} D_E V_F - 2X^B \tilde{W}^{-E-F} D_E V_F \\
&= 4w(w+1) \tilde{W}^{-B-E} V_E + 2w(d+2w-2) \tilde{W}^{B--F} V_F \\
&\quad + 2(d+2w-2) \tilde{W}^{Be-F} \nabla_e V_F - 2X^B \tilde{W}^{-E-F} D_E V_F \\
&= 4(w+1) \tilde{W}^{Be-F} \nabla_e V_F - 2X^B \tilde{W}^{-E-F} D_E V_F \\
&= 4(w+1) \left[X^B \tilde{W}^{-e-F} + Z_b^B \tilde{W}^{be-F} \right] \nabla_e V_F - 2(d+2w-2) X^B \tilde{W}^{-e-F} \nabla_e V^F \\
&= 4(w+1) Z_b^B \tilde{W}^{be-F} \nabla_e V_F.
\end{aligned}$$

In the above computation, we used that $\tilde{W}^{+ABC} = 0$ and that the Thomas- D can be expanded in terms of the injectors according to (2.3). Importantly, in the second and fourth equalities, we rely on cancellations peculiar to $d = 4$. Because $\tilde{W}^{be-F} \sim \tilde{W}^{be-F} + X^{[b} V^{e][F} X^{-]} = \tilde{W}^{be-F}$, it follows that the above display defines a tractor. \square

3. CONFORMALLY EMBEDDED HYPERSURFACES

We now develop the invariant theory for a hypersurface Σ smoothly embedded in a conformal manifold (M, \mathbf{c}) . Here we mean that Σ is a smoothly embedded codimension-1 submanifold of (M, \mathbf{c}) , and denote this by $\Sigma \hookrightarrow (M, \mathbf{c})$. It is simplifying, although not strictly necessary for the study of local invariant theory, to assume that Σ is closed and orientable, and that its embedding in M is separating. In this case $M = M^+ \sqcup \Sigma \sqcup M^-$. Given a metric $g \in \mathbf{c}$, we denote the unit conormal by \hat{n} , and take as convention that it points in the direction of M^+ . The first fundamental form $I_{ab} = (g_{ab} - \hat{n}_a \hat{n}_b)|_\Sigma$ equals the induced metric \bar{g}_{ab} of Σ . The second fundamental form is then

$$(3.1) \quad \Pi_{ab} = I_a^c \nabla_c \hat{n}_b^\flat|_\Sigma = \nabla_a^\top \hat{n}_b^\flat|_\Sigma \in \Gamma(\odot^2 T^* \Sigma),$$

where \hat{n}^\flat on the right hand side of the above display is any smooth extension of \hat{n} to M . Note that the image of the projection of the tangent bundle TM along the hypersurface by the endomorphism $I_b^a = g^{ac} I_{bc}$ is isomorphic to $T\Sigma$. We therefore identify these spaces and use the same abstract index notation for hypersurface tensors as for ambient ones. We use a \top notation for projection to $T\Sigma$, and also for more general tensors. For example if ω is a one-form in $\Gamma(T^*M)|_\Sigma$, then $\top(\omega_a) \equiv \omega_a^\top := I_a^b \omega_b \in \Gamma(T^*\Sigma)$. Contraction by the unit conormal, followed by projection is denoted, for example for a two-form, by $\omega_{\hat{n}b}^\top := (\hat{n}^a \omega_{ac}) I_b^c$. When $\omega \in \Gamma(T^*M)$ we use ω^\top to denote $(\omega|_\Sigma)^\top$. A ring over the projection symbol \top denotes an additional projection to the (hypersurface) trace-free part. In these terms, the Gauß formula then reads

$$(3.2) \quad \bar{\nabla}_a \bar{v}^b = \nabla_a^\top v^b|_\Sigma + \hat{n}^b \Pi_{ac} \bar{v}^c,$$

where in the first term on the right hand side, v^a denotes any smooth extension of a hypersurface vector $\bar{v}^a \in \Gamma(T\Sigma)$ to M . Also, $\bar{\nabla}$ is the Levi-Civita connection of the induced metric \bar{g} . Quite generally, we use a bar notation on objects to indicate when they belong to the hypersurface Σ . For example, curvature quantities intrinsic to the hypersurface will be denoted by bars as will be the hypersurface dimension, so $d-1 = \bar{d}$. The Gauß equation is then given by

$$(3.3) \quad R_{abcd}^\top = \bar{R}_{abcd} - \Pi_{ac} \Pi_{bd} + \Pi_{ad} \Pi_{bc}.$$

We also record the trace-free and traced Codazzi–Mainardi equations:

$$(3.4) \quad \begin{aligned} \bar{\nabla}_{[a} \bar{\Pi}_{b]c} - \frac{1}{\bar{d}-2} \bar{g}_{c[a} \bar{\nabla} \cdot \bar{\Pi}_{b]} &= \frac{1}{2} W_{abc\hat{n}}^\top, \\ \bar{\nabla}_a H - \frac{1}{\bar{d}-2} \bar{\nabla} \cdot \bar{\Pi}_a &= -P_{a\hat{n}}^\top. \end{aligned}$$

3.1. Hypersurface Tractor Calculus. Tractor calculus is a key technical tool for studying conformal embeddings. A basic observation [6] is that the unit conormal \hat{n}^g and mean curvature H^g of an embedded hypersurface depend on the conformal representative $g \in \mathfrak{c}$ as

$$(\hat{n}^{\Omega^2 g}, H^{\Omega^2 g}) = (\Omega \hat{n}^g, \Omega^{-1}(H^g + \hat{n}^g \cdot d \log \Omega)),$$

where $0 < \Omega \in C^\infty M$ and its exterior derivative appearing above are restricted to Σ . Henceforth we reuse the notation \hat{n}^a and H for the corresponding density-valued objects. So, given a metric $g \in \mathfrak{c}$, the triple

$$N^A \stackrel{g}{=} \begin{pmatrix} 0 \\ \hat{n}^a \\ -H \end{pmatrix} \stackrel{\Sigma}{=} \hat{n}^a Z_a^A - H Y^A$$

labels a section of the tractor bundle $\mathcal{T}M|_\Sigma$. The section N^A is termed the *normal tractor* and obeys

$$h(N, N) = 1.$$

Note that the projecting part of N is the unit normal, which indeed defines a vector-valued weight $w = -1$ conformal density. The normal tractor mimics the *rôle* of the unit conormal in Riemannian hypersurface theory. In particular, projection of the tractor bundle along Σ by $I_B^A := \delta_B^A - N^A N_B$ gives a bundle isomorphic to the tractor bundle $\mathcal{T}\Sigma$ of $(\Sigma, \bar{\mathfrak{c}})$, where $\bar{\mathfrak{c}}$ is the induced conformal class of metrics [22]. The tractor $I_{AB} := h_{AB}|_\Sigma - N_A N_B$ is the *tractor first fundamental form*. Hence, we shall identify these spaces and use the same abstract index notation for ambient and hypersurface tractors. In particular the hypersurface tractor metric is given by $\bar{h}_{AB} = I_{AB}$. Moreover, together the normal tractor and tractor metric give an isomorphism

$$\mathcal{T}M|_\Sigma \cong \mathcal{N}\Sigma \oplus \mathcal{T}\Sigma,$$

where $\mathcal{N}\Sigma$ is the tractor normal bundle (defined by orthogonal decomposition completely analogously to the normal bundle for Riemannian hypersurface embeddings). Note that for the canonical tractor X^A , we have $I_B^A X^B = X^A$, whose image under the bundle isomorphism is the hypersurface canonical tractor \bar{X}^A ; thus we often drop the bar and denote this by X^A .

Recall that the trace-free combination

$$\mathring{\Pi} := \Pi - H \bar{g},$$

is conformally invariant and defines a section of $\Gamma(\odot_\circ^2 T^* \Sigma[1])$. In dimensions $d > 3$, it can be inserted into a symmetric, trace-free hypersurface tractor

$$L^{AB} := \bar{q}(\mathring{\Pi}_{ab}) \in \Gamma(\odot_\circ^2 \mathcal{T}\Sigma[-1])$$

termed the *tractor second fundamental form* [35, 46]. In the above, we have applied the insertion map of Lemma 2.8 on the conformal manifold $(\Sigma, \bar{\mathfrak{c}})$. Note that in a choice of scale g ,

$$L^{AB} \stackrel{g}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathring{\Pi}^{ab} & -\frac{1}{d-2} \bar{\nabla} \cdot \mathring{\Pi}^a \\ 0 & -\frac{1}{d-2} \bar{\nabla} \cdot \mathring{\Pi}^b & \frac{\bar{\nabla} \cdot \bar{\nabla} \cdot \mathring{\Pi} + (d-2) \bar{P}^{ab} \mathring{\Pi}_{ab}}{(d-2)(d-3)} \end{pmatrix}.$$

The hypersurface conformal curvature quantity $L^2 = \mathring{\Pi}^2 =: K \in \Gamma(\mathcal{E}\Sigma[-2])$ was termed the rigidity density because it was employed to describe rigid strings in [47].

Remark 3.1. In the theory of surfaces in \mathbb{R}^3 , the square of the second fundamental form is sometimes termed the third fundamental form $\mathbb{I}\mathbb{I}$. If $\Sigma \hookrightarrow \mathbb{R}^3$ is a surface expressed as the level set of a unit defining function $s : \mathbb{R}^3 \rightarrow \mathbb{R}$, then the second fundamental form \mathbb{I} is the Hessian of s , so that $\mathbb{I} = \nabla n$ where $n = \nabla s$. The third fundamental form then obeys

$$\mathbb{I}\mathbb{I}_{ab} = -\nabla_n \nabla_a n_b|_\Sigma = \mathbb{I}_{ab}^2.$$

Indeed one might even define higher fundamental forms by taking successive normal derivatives, because for $k \in \mathbb{Z}_{\geq 0}$,

$$\mathbb{I}_{ab}^{k+1} = \frac{(-1)^k}{k!} \nabla_n^k \nabla_a n_b \Big|_{\Sigma}.$$

In Section 3.5 we give a definition of the *transverse order* of an hypersurface invariant by viewing formulæ such as that above as functionals of a general curved ambient metric and then counting normal/transverse derivatives on this metric. The third fundamental form for a hypersurface embedded in a Riemannian manifold defined by $\mathbb{I}_{ab} := -\nabla_n \nabla_a n_b \Big|_{\Sigma} = \mathbb{I}_{ab}^2 - R_{\hat{n}ab\hat{n}}$ then has transverse order two. A main aim of this article is to develop a theory of conformally invariant higher fundamental forms. \blacksquare

In dimensions $d > 3$, manipulating the trace of the Gauß equation (3.3) produces a conformally invariant third fundamental form, termed the Fialkow tensor $F \in \Gamma(\odot^2 T^* \Sigma[0])$ and defined by [18] (see also [52]),

$$(3.5) \quad \left(\mathring{\mathbb{I}}_{ab}^2 - \frac{1}{2(d-2)} \mathring{\mathbb{I}}^2 \bar{g}_{ab} - W_{\hat{n}ab\hat{n}} \right) = (d-3) (P_{ab}^{\top} - \bar{P}_{ab} + H \mathring{\mathbb{I}}_{ab} + \frac{1}{2} H^2 \bar{g}_{ab}) =: (d-3) F_{ab}.$$

We shall call the above relation the *Fialkow–Gauß equation*.

The trace-free Fialkow tensor, denoted \mathring{F} , is also of special interest. In particular, in dimensions $d > 4$, it too can be inserted (using Lemma 2.8) into a symmetric, trace-free tractor

$$F^{AB} := q(\mathring{F}_{ab}) \in \Gamma(\odot_{\circ}^2 \mathcal{T}\Sigma[-2])$$

termed the *Fialkow tractor*. In a choice of scale g ,

$$F^{AB} \stackrel{g}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathring{F}^{ab} & -\frac{1}{d-3} \bar{\nabla} \cdot \mathring{F}^a \\ 0 & -\frac{1}{d-3} \bar{\nabla} \cdot \mathring{F}^b & \frac{\bar{\nabla} \cdot \bar{\nabla} \cdot \mathring{F} + (d-3) \bar{P}^{ab} \mathring{F}_{ab}}{(d-3)(d-4)} \end{pmatrix}.$$

It is also useful to define a weight -1 , rank three tractor that combines the Fialkow and trace-free second fundamental forms,

$$(3.6) \quad \Gamma_{ABC} := 2N_{[C} L_{B]A} + 2X_{[C} F_{B]A} + \frac{K}{d(d-1)} X_{[C} \bar{h}_{B]A}.$$

From its Definition (3.5), we see that Fialkow tensor measures the difference between the bulk and hypersurface tractor connections through their respective Schouten tensors. This leads to an analog of Equation (3.2) (see [31]), that we term the Fialkow–Gauß formula

$$(3.7) \quad \bar{\nabla}_a \bar{V}^B = \nabla_a^{\top} V^B \Big|_{\Sigma} - \Gamma_a^B{}^C \bar{V}^C.$$

Here $\bar{V} \in \Gamma(\mathcal{T}\Sigma)$ is any standard hypersurface tractor and we have used the isomorphism discussed above between the hypersurface tractor bundle and the projection of the bulk tractor bundle restricted to Σ , to extend this to $V \in \Gamma(\mathcal{T}M)$. The displayed formula does not depend on the choice of such an extension.

Formulæ like the right hand side of Equation (3.7) that are expressed in terms of an operator acting on an extension of an object defined only along the hypersurface play an important *rôle* in the study of extrinsic geometry. In general, we call an operator O acting on sections a of a vector bundle over the bulk manifold *tangential* if $(Oa)|_{\Sigma}$ only depends on the restriction $\bar{a} = a|_{\Sigma}$ (*i.e.*, it is independent of the choice of extension a of \bar{a}). This defines an operator O_{Σ} along Σ by $O_{\Sigma} \bar{a} := (Oa)|_{\Sigma}$. Note that the right hand side is a holographic formula for the quantity $O_{\Sigma} \bar{a}$; this notion is developed in detail in Section 3.5. The operator ∇^{\top} is tangential. Our next task is to develop a tangential analog of the Thomas- D operator in order to relate the bulk Thomas- D operator to its hypersurface counterpart.

Proposition 3.2. *Let $w + \frac{d}{2} \neq 1, \frac{3}{2}, 2$ and N^e be any extension of the normal tractor. Then, the operator*

$$\hat{D}^T : \Gamma(\mathcal{T}^{\Phi} M[w]) \rightarrow \Gamma(\mathcal{T}M \otimes \mathcal{T}^{\Phi} M[w-1]),$$

given by

$$\hat{D}_A^T := \hat{D}_A - N_A^e N_e^B \hat{D}_B + \frac{X_A}{d+2w-3} \left(N_e^B N_e^C \hat{D}_B \hat{D}_C + \frac{wK}{d-2} \right),$$

is tangential. Moreover, if O^A is any operator acting on tractors of weight $\frac{1-d}{2}$ that obeys

$$O^A \circ X_A = 0,$$

then the operator

$$O \cdot \tilde{D}^T := O^A \circ (\hat{D}_A - N_A^e N_e^B \hat{D}_B)$$

is also tangential.

The tangential Thomas- D operator was first introduced in [23]; the proof of the above result is given in [31], and is simplified using holographic ideas. Note that, unlike the Levi-Civita connection, which becomes tangential upon projection by the first fundamental form, projection by the tractor first fundamental form alone does not render the Thomas- D tangential. Also note that the last term proportional to K in the definition of \hat{D}^T is separately tangential, but has been added because the operator in parentheses will play a special rôle.

An immediate application of the *tangential Thomas- D operator* is the tractor analog of the defining equation for the second fundamental form (3.1):

Lemma 3.3. *Let $d > 3$ and N^e be any extension of the normal tractor. Then the tractor second fundamental form obeys*

$$L_{AB} = \hat{D}_{(A}^T N_{B)}^e \Big|_{\Sigma} - \frac{1}{d-3} (X_{(A} N_{B)} K + X_A X_B M),$$

with $K = L^2 = (\hat{D} N^e)^2 \Big|_{\Sigma}$ and $M = L_{AB} F^{AB} = F^{AB} (\hat{D}_A N_B^e) \Big|_{\Sigma}$.

The proof of this lemma is given in Subsection 4.1 and relies on holographic ideas.

As promised in the introduction, the following subsections provide proofs of Theorem 1.3 and Corollary 1.4.

3.2. Proof of Theorem 1.3. We prove the Gauss-Thomas formula 1.3 acting on a tractor vector of arbitrary weight and then generalize. That is, we look to prove the following lemma.

Lemma 3.4. *Let $V \in \Gamma(\mathcal{T}\Sigma[w])$. Then, the bulk tangential and hypersurface Thomas- D operators obey*

$$\begin{aligned} (\hat{D}_A^T)_{\Sigma} V^B &= \hat{D}_A V^B + \Gamma_A^{\#} V^B \\ &\quad - \frac{X_A}{d+2w-2} \left\{ 2\Gamma_C^{\#} \circ \hat{D}^C V^B + \Gamma_C^{\#} \circ \Gamma_C^{\#} V^B \right. \\ &\quad \left. + \frac{1}{d(d-1)} [(\hat{D}K) \wedge X]^{\#} V^B - \frac{(3\bar{d}+2)wK}{2d(d-1)} V^B \right\}. \end{aligned}$$

Proof. We rely heavily on [31, Lemma 4.9], which states that

$$(3.8) \quad (\hat{D}^{TA})_{\Sigma} = \begin{pmatrix} w \\ \nabla^{\top a} \\ -\frac{\Delta^{\top} + w\bar{J}}{d+2w-2} + \frac{wK}{2(d-1)(d+2w-2)} \end{pmatrix}$$

We check the lemma by contracting both sides with the injectors and checking that it holds for all three. First note that $w = \bar{w}$ and $X_A \Gamma^A{}_{BC} = 0$. Thus, the lemma holds upon contraction with X^A .

Next, we check the lemma upon contraction with \bar{Z}_a^A . Note that, according to Equation (3.8), $\bar{Z}_a^A \hat{D}^T{}_A = (\nabla_a^{\top})_{\Sigma}$. Because $X_A \bar{Z}_a^A = 0$, we have that

$$(\nabla_a^{\top})_{\Sigma} V^B = \bar{\nabla}_a V^B + \Gamma_a{}^B{}_C \bar{V}^C,$$

which is the Fialkow-Gauss equation. Thus, the lemma holds upon contraction with \bar{Z}_A^A .

Finally, we check that the identity holds upon contraction with \bar{Y}^A . From Equation. (3.8), we have that

$$\begin{aligned}\bar{Y}^A \left[\hat{D}_A^T - \hat{D}_A \right] &= \left(-\frac{\Delta^\top + w\bar{J}}{\bar{d} + 2w - 2} + \frac{wK}{2(\bar{d} - 1)(\bar{d} + 2w - 2)} \right) + \frac{1}{\bar{d} + 2w - 2} (\bar{\Delta} + w\bar{J}) \\ &= -\frac{1}{\bar{d} + 2w - 2} \left(\Delta^\top - \bar{\Delta} - \frac{wK}{2(\bar{d} - 1)} \right).\end{aligned}$$

We explicitly compute the tractor Laplacian difference $\Delta^\top - \bar{\Delta}$.

From the definition of the tractor Laplacian and defining $\Gamma_\perp^{ABC} := \Gamma^{ABC} - 2N^{[C}L^{B]A} \in \Gamma(\mathcal{T}^3\Sigma[-1])$, we have

$$\begin{aligned}\Delta^\top V^B &= \nabla_a^\top (\bar{\nabla}^a V^B + \Gamma^{aBC} V_C) \\ &= \nabla_a^\top (\bar{\nabla}^a V^B + \Gamma_\perp^{aBC} V_C + 2N^{[C}L^{B]a} V_C) \\ &= \bar{\Delta} V^B + \Gamma^{aB}{}_C \bar{\nabla}_a V^C + \bar{\nabla}_a (\Gamma_\perp^{aBC} V_C) + \Gamma_a{}^B{}_E \Gamma_\perp^{aEC} V_C + \nabla_a^\top (2N^{[C}L^{B]a} V_C) \\ &= \bar{\Delta} V^B + \Gamma^{aB}{}_C \bar{\nabla}_a V^C + (\bar{\nabla}_a \Gamma_\perp^{aBC}) V_C + (\Gamma^{aBC} - 2N^{[C}L^{B]a}) \bar{\nabla}_a V_C \\ &\quad + \Gamma_a{}^B{}_E \Gamma^{aEC} V_C - 2\Gamma_a{}^B{}_E N^{[C}L^{E]a} V_C + 2 \left(\nabla_a^\top (N^{[C}L^{B]a}) \right) V_C + 2N^{[C}L^{B]a} \nabla_a^\top V_C \\ &= \bar{\Delta} V^B + 2\Gamma^{aB}{}_C \bar{\nabla}_a V^C + (\bar{\nabla}_a \Gamma_\perp^{aBC}) V_C \\ &\quad + \Gamma_a{}^B{}_E \Gamma^{aEC} V_C - 2\Gamma_a{}^B{}_E N^{[C}L^{E]a} V_C + 2 \left(\nabla_a^\top (N^{[C}L^{B]a}) \right) V_C + 2N^{[C}L^{B]a} \Gamma_{aC}{}^E V_E.\end{aligned}$$

Writing $\Gamma = \Gamma_\perp + 2NL$ allowed us to use the Fialkow–Gauß equation (3.7) for the above simplifications.

We now break up this calculation into smaller parts.

$$\begin{aligned}\bar{\nabla}_a \Gamma_\perp^{aBC} &= \bar{\nabla}_a (\bar{Z}_A^a \Gamma_\perp^{ABC}) \\ &= \bar{Z}_A^a \bar{\nabla}_a \Gamma_\perp^{ABC} + \Gamma_\perp^{ABC} (\bar{J} \bar{X}_A - \bar{d} \bar{Y}_A) \\ &= \bar{Z}_A^a \bar{\nabla}_a \Gamma_\perp^{ABC} - \bar{d} \Gamma_\perp^{-BC}.\end{aligned}$$

Here, the second equality comes from the fact that $\bar{\nabla}_a \bar{Z}_A^a = -\bar{P}_a^b \bar{X}_A - \bar{g}_a^b \bar{Y}_A$ and the last equality holds because $X_A \Gamma^{ABC} = 0$. Similarly,

$$\begin{aligned}\nabla_a^\top N^{[C}L^{B]a} &= \left(\nabla_a^\top N^{[C} \right) L^{B]a} + N^{[C} \left[\bar{\nabla}_a (L^{B]A} \bar{Z}_A^a) + \Gamma_a{}^B{}_E L^{Ea} \right] \\ &= \left(\nabla_a^\top N^{[C} \right) L^{B]a} + N^{[C} \left[(\bar{\nabla}_a L^{B]A}) \bar{Z}_A^a - \bar{d} L^{B]-} + \Gamma_a{}^B{}_E L^{Ea} \right]\end{aligned}$$

In order to simplify the above two displays, we need results that follow from Equation (2.2):

$$\begin{aligned}\bar{Z}_A^a \bar{\nabla}_a L^{AC} &= 2L^{-C}, \\ \bar{Z}_A^a \bar{\nabla}_a F^{AC} &= 3F^{-C}, \\ \nabla_a^\top N^B &= L_a^B.\end{aligned}$$

Using the above identities, we can write

$$\begin{aligned}\bar{\nabla}_a \Gamma_\perp^{aBC} &= 2 \left(\bar{\nabla}_a X^{[C} \right) F^{B]a} + 6X^{[C} F^{B]-} + \frac{1}{\bar{d}(\bar{d} - 1)} (\nabla_a K) X^{[C} \bar{h}^{B]a} - \bar{d} \Gamma_\perp^{-BC} \\ &= 4X^{[C} F^{B]-} + \frac{1}{\bar{d}(\bar{d} - 1)} \left(X^{[C} \hat{D}^{B]} K + 2K X^{[C} \bar{h}^{B]-} \right) - \bar{d} \Gamma_\perp^{-BC} \\ &= (2 - \bar{d}) \Gamma_\perp^{-BC} + \frac{1}{\bar{d}(\bar{d} - 1)} X^{[C} \hat{D}^{B]} K\end{aligned}$$

and

$$2\nabla_a^\top N^{[C}L^{B]a} = 2(2-d)N^{[C}L^{B]-} + N^C\Gamma^{ABE}L_{AE} - N^B\Gamma^{ACE}L_{AE}.$$

We can now use these formulas to write

$$\begin{aligned} \Delta^\top \bar{V}^B &= \bar{\Delta}V^B + 2\Gamma^{aBC}\bar{\nabla}_a V_C + (2-\bar{d})\Gamma_\perp^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{D}^{B]}K + \Gamma_a{}^B{}_E\Gamma^{aEC}V_C \\ &\quad - 2\Gamma_a{}^B{}_E N^{[C}L^{E]a}V_C + 2N^{[C}L^{B]a}\Gamma_{aC}{}^E V_E \\ &\quad + \left(2(2-d)N^{[C}L^{B]-} + N^C\Gamma^{ABE}L_{AE} - N^B\Gamma^{ACE}L_{AE}\right)V_C \\ &= \bar{\Delta}\bar{V}^B + 2\Gamma^{aBC}\bar{\nabla}_a V_C + (2-\bar{d})\Gamma^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{D}^{B]}K + \Gamma_a{}^B{}_E\Gamma^{aEC}V_C \\ &\quad - 2\Gamma_a{}^B{}_E N^{[C}L^{E]a}V_C + 2N^{[C}L^{B]a}\Gamma_{aC}{}^E V_E + (N^C\Gamma^{ABE}L_{AE} - N^B\Gamma^{ACE}L_{AE})V_C \\ &= \bar{\Delta}V^B + 2\Gamma^{aBC}\bar{\nabla}_a V_C + (2-\bar{d})\Gamma^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{D}^{B]}K + \Gamma_a{}^B{}_E\Gamma^{aEC}V_C \\ &= \bar{\Delta}V^B + 2\Gamma^{ABC}\hat{D}_A V_C - (\bar{d}+2w-2)\Gamma^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{D}^{B]}K + \Gamma_A{}^B{}_E\Gamma^{AEC}V_C \\ &= \bar{\Delta}V^B + 2\Gamma^A{}_A{}^C\hat{D}_C V^B - \frac{(\bar{d}+1)wK}{\bar{d}(\bar{d}-1)}V^B + 2\Gamma^{ABC}\hat{D}_A V_C - (\bar{d}+2w-2)\Gamma^{-BC}V_C \\ &\quad + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{D}^{B]}K + \Gamma_A{}^B{}_E\Gamma^{AEC}V_C \\ &= \bar{\Delta}V^B + 2\Gamma^{A\sharp}\circ\hat{D}_A V^B - (\bar{d}+2w-2)\Gamma^{-BC}V_C + \frac{V_C}{\bar{d}(\bar{d}-1)}X^{[C}\hat{D}^{B]}K \\ &\quad + \Gamma^{A\sharp}\circ\Gamma_A{}^\sharp V^B - \frac{(\bar{d}+1)wK}{\bar{d}(\bar{d}-1)}V^B. \end{aligned}$$

In the display above, the second equality comes the definition of Γ_\perp . The third equality is a result of the last four terms canceling, and the fourth equality comes from the fact that $\Gamma \in \ker X_\perp$ (where X_\perp denotes contraction by X). The last inequality follows from $\Gamma_A{}^A{}_E\Gamma^{EBC}V_C = 0$.

But,

$$\left(\hat{D}_A^T - \hat{D}_A\right)V^B = \bar{Z}_a^A\Gamma_A{}^B{}_C V^C - \frac{X_A}{\bar{d}+2w-2}\left(\Delta^\top - \bar{\Delta} - \frac{wK}{2(\bar{d}-1)}\right)V^B,$$

so the proof is completed by combining terms involving wK . \square

Remark 3.5. Applying the same techniques in the proof above but accounting for the possible linear actions on multiple indices, the proof of the general Gauß–Thomas formula follows easily. \blacksquare

3.3. Proof of Corollary 1.4. In this section, we provide a proof of Corollary 1.4 as well as an additional corollary.

Proof of Corollary 1.4. Recall that the Gauß equation is a corollary of the Gauß formula, in the sense that it is obtained by applying the latter to $[\nabla_a^\top, \nabla_b^\top]v_c$ where v_c is an extension of a hypersurface tangent vector. Similarly, the present proof could be completed by applying the Gauß–Thomas formula to $[\hat{D}_A^T, \hat{D}_B^T]V_C$. But, because \hat{D} is not a derivation, that computation is rather involved. Instead, we approach the proof via equality of all possible contractions (in some scale $g \in \mathfrak{c}$) by hypersurface injectors $(X^A, \bar{Z}_a^A, \bar{Y}^A)$ on both sides of the lemma’s result. Note that it is unnecessary to check contractions with more than one \bar{Y} —this only probes V_{AB} . Also, without loss of generality, we may choose g to be a scale in which the mean curvature H^g of the embedding $\Sigma \hookrightarrow (M, g)$ vanishes.

We begin by contracting with a single X . For that, we first use Proposition 2.4 and the Fialkow tractor identities $\hat{D}^A F_{AB} = 0 = X^A F_{AB}$, $0 = F_A{}^A$ as well as the ansatz $X^A V_{AB} = X_B V$, to obtain

$$\begin{aligned} X^A T_{ABC} &= \frac{1}{(d-1)(d-2)} X_{[B} \hat{D}_{C]} K, \\ X^C T_{ABC} &= -F_{AB} - \frac{K}{(d-1)(d-2)} \bar{h}_{AB} - \frac{1}{2(d-1)(d-2)} X_A \hat{D}_B K, \\ X^A X_{[B} V_{C]A} &= 0. \end{aligned}$$

Now $X^A W_{ABCD}^\top = 0$, so we need to show contraction of the right-hand side of Equation (1.4) with X^A vanishes. Clearly $X^A \bar{W}_{ABCD} = 0$ and the contraction of X with the second term also vanishes because $X^A L_{AB} = 0$. Using $X^A F_{AB} = 0$ along with the identities of the above display, the remaining terms are

$$\begin{aligned} & -2X_{[C} F_{D]B} + \frac{2}{(d-1)(d-2)} X_{[D} \bar{h}_{C]B} K \\ & + 2X_{[C} F_{D]B} - \frac{2}{(d-1)(d-2)} X_{[D} \bar{h}_{C]B} K - \frac{1}{(d-1)(d-2)} X_B X_{[C} \hat{D}_{D]} K \\ & \quad + \frac{1}{(d-1)(d-2)} X_B X_{[C} \hat{D}_{D]} K = 0. \end{aligned}$$

Because the W -tractor has Weyl curvatures symmetries this establishes consistency of the identity when any index is contracted with a canonical tractor.

Next, note that $\bar{Z}_a^A \bar{Z}_b^B \bar{Z}_c^C \bar{Z}_d^D W_{ABCD}^\top = W_{abcd}^\top$ and that the trace-free Gauß equation says

$$W_{abcd}^\top = \bar{W}_{abcd} - 2\hat{\Pi}_{a[c} \hat{\Pi}_{d]b} - 2\bar{g}_{a[c} \hat{F}_{d]b} + 2\bar{g}_{b[c} \hat{F}_{d]a} - \frac{2}{(d-1)(d-2)} \bar{g}_{a[c} \bar{g}_{d]b} K.$$

It is easy to check, using $X_A \bar{Z}_a^A = 0$, that this is the right hand side of Equation (1.4) when contracted with this combination of injectors.

The last case to check is contraction of Equation (1.4) by $\bar{Z}_a^A \bar{Z}_b^B \bar{Z}_c^C \bar{Y}^D$. By directly applying the definitions of L_{AB} , F_{AB} , \bar{W}_{ABCD} , and the hatted hypersurface Thomas- D operator, after some computation, we find for the right-hand side

$$\begin{aligned} & \bar{Z}_a^A \bar{Z}_b^B \bar{Z}_c^C \bar{Y}^D \left[\bar{W}_{ABCD} - 2L_{A[C} L_{D]B} - 2\bar{h}_{A[C} F_{D]B} + 2\bar{h}_{B[C} F_{D]A} - \frac{2}{(d-1)(d-2)} \bar{h}_{A[C} \bar{h}_{D]B} K \right. \\ & \quad \left. + 2X_{[A} T_{B]CD} + 2X_{[C} T_{D]AB} \right] \\ & = \bar{C}_{abc} + \frac{2}{d-2} \hat{\Pi}_{c[a} \bar{\nabla} \cdot \hat{\Pi}_{b]} + 2\bar{\nabla}_{[a} \hat{F}_{b]c} - \frac{1}{(d-1)(d-2)} \bar{g}_{c[a} \bar{\nabla}_{b]} K. \end{aligned}$$

We must then contract with the left-hand side with the same injector product. Because we use a scale where $H^g = 0$,

$$\bar{Z}_a^A \bar{Z}_b^B \bar{Z}_c^C \bar{Y}^D W_{ABCD}^\top = C_{abc}^\top|_\Sigma.$$

Showing that this contraction yields equality in Equation (1.4) is now equivalent to showing that, when $H^g = 0$, the projected Cotton tensor is related to the hypersurface Cotton tensor by

$$C_{abc}^\top|_\Sigma = \bar{C}_{abc} + 2\bar{\nabla}_{[a} \hat{F}_{b]c} + \frac{2}{d-2} \hat{\Pi}_{c[a} \bar{\nabla} \cdot \hat{\Pi}_{b]} - \frac{1}{(d-1)(d-2)} \bar{g}_{c[a} \bar{\nabla}_{b]} K.$$

For that, first observe that the projected covariant derivative of the first fundamental form obeys

$$\nabla_a^\top I_{bc}|_\Sigma = -\Pi_{ab} \hat{n}_c - \Pi_{ac} \hat{n}_b \stackrel{H^g=0}{=} -\hat{\Pi}_{ab} \hat{n}_c - \hat{\Pi}_{ac} \hat{n}_b.$$

Applying this identity, the trace-free Fialkow-Gauss Equation (3.5), and the traced Codazzi-Mainardi equation, the projected Cotton tensor can be written in terms of the hypersurface

Cotton tensor:

$$\begin{aligned}
C_{abc}^\top|_\Sigma &= (\nabla_a P_{bc})^\top - (a \leftrightarrow b) \\
&= \nabla_a^\top P_{bc}^\top + \mathring{\Pi}_{ab} P_{\hat{n}c}^\top + \mathring{\Pi}_{ac} P_{\hat{n}b}^\top + \hat{n}_b \mathring{\Pi}_a^d P_{dc}^\top + \hat{n}_c \mathring{\Pi}_a^d P_{bd}^\top - (a \leftrightarrow b) \\
&= \nabla_a^\top P_{bc}^\top + \hat{n}_b \mathring{\Pi}_a^d P_{dc}^\top + \hat{n}_c \mathring{\Pi}_a^d P_{bd}^\top - (a \leftrightarrow b) + \frac{2}{d-2} \mathring{\Pi}_{c[a} \bar{\nabla} \cdot \mathring{\Pi}_{b]} \\
&= \bar{\nabla}_a P_{bc}^\top - (a \leftrightarrow b) + \frac{2}{d-2} \mathring{\Pi}_{c[a} \bar{\nabla} \cdot \mathring{\Pi}_{b]} \\
&= \bar{\nabla}_a \bar{P}_{bc} + \bar{\nabla}_a \mathring{F}_{bc} + \frac{\bar{g}_{bc}}{2(d-1)(d-2)} \bar{\nabla}_a K - (a \leftrightarrow b) + \frac{2}{d-2} \mathring{\Pi}_{c[a} \bar{\nabla} \cdot \mathring{\Pi}_{b]} \\
&= \bar{C}_{abc} + 2\bar{\nabla}_{[a} \mathring{F}_{b]c} + \frac{2}{d-2} \mathring{\Pi}_{c[a} \bar{\nabla} \cdot \mathring{\Pi}_{b]} - \frac{1}{(d-1)(d-2)} \bar{g}_{c[a} \bar{\nabla}_{b]} K.
\end{aligned}$$

The second above relies on previous display, while the third relies the trace of the Codazzi–Mainardi equation. The second last line uses the Fialkow–Gauß equation. This completes the proof. \square

Remark 3.6. The corollary does not contain an explicit formula for the tractor V_{AB} for reasons of brevity only. It measures the difference between hypersurface and bulk Bach tensors. While explicit knowledge of the tensor content of V_{AB} is unnecessary for the computations that follow, it is nonetheless interesting. A computer-aided computation gives

$$V_{AB} = \bar{q}(U_{ab}) + \frac{1}{(d-1)(d-4)(d-5)} \bar{h}_{AB} U,$$

where, for $d \neq 7$,

$$\begin{aligned}
(3.9) \quad \Gamma(\mathcal{E}\Sigma[-4]) \ni U &= \frac{d-3}{d-1} K^2 + 2\mathring{\Pi}^{ad} \mathring{\Pi}^{bc} \bar{W}_{abcd} - 2(d-3) \mathring{\Pi} \cdot \mathring{F} \cdot \mathring{\Pi} + (d-3)(d-5) \mathring{F}^2 \\
&+ \frac{1}{d-7} (\bar{D}_A L_{BC}) \left(\hat{D}^A L^{BC} \right) - L^{BC} N^A N^D \delta_R W_{ABCD},
\end{aligned}$$

$$\begin{aligned}
\Gamma(\odot^2 T_\circ^* \Sigma[-2]) \ni U_{ab} &= \frac{1}{d-5} \bar{B}_{ab} - \frac{1}{d-4} B_{(ab)\circ}^\top + \frac{2}{d-7} E \mathring{F}_{ab} \\
&+ \frac{1}{6(d-1)(d-2)} \bar{\nabla}_{(a} \bar{\nabla}_{b)\circ} K - \frac{1}{(d-2)(d-3)} \mathring{\Pi}_{ab} \bar{\nabla} \cdot \bar{\nabla} \cdot \mathring{\Pi} + \frac{1}{(d-2)^2} \bar{\nabla} \cdot \mathring{\Pi}_{(a} \bar{\nabla} \cdot \mathring{\Pi}_{b)\circ} \\
&+ 2H C_{\hat{n}(ab)}^\top + H^2 W_{\hat{n}ab\hat{n}} - \frac{1}{3(d-1)(d-2)} K \bar{P}_{(ab)\circ} - \frac{1}{d-3} \mathring{\Pi}_{ab} \mathring{\Pi} \cdot \bar{P}.
\end{aligned}$$

Here the operator $E \in \text{End}(\Gamma(\odot_\circ^2 T^* \Sigma))$ is defined by

$$E \mathring{X}_{ab} := \bar{\Delta} \mathring{X}_{ab} - \bar{\nabla}_{(a} \nabla \cdot \mathring{X}_{b)\circ} - (d-3) \bar{P}_{c(a} \mathring{X}_{b)\circ}^c - 2\bar{J} \mathring{X}_{ab}.$$

When $\bar{d} = 6$, the operator E defines a conformally invariant map $\Gamma(\odot_\circ^2 T^* \Sigma) \rightarrow \Gamma(\odot_\circ^2 T^* \Sigma[-2])$. Note that this calculation recovers the fifth fundamental form described in the introduction. \blacksquare

A further corollary of the Gauß–Thomas equation in Theorem 1.3 characterizes the Fialkow tractor in terms of the W -tractor and tractor second fundamental form and generalizes Equation (3.5).

Corollary 3.7. *Let $7 \neq d > 5$. Then the Fialkow tractor obeys*

$$(d-3)F_{AB} = \left(L_A^C L_{CB} - \frac{1}{d-1} K \bar{h}_{AB} - W_{NABN} \right) - \frac{1}{d-1} X_{(A} \hat{D}_{B)} K - \frac{1}{(d-4)(d-5)} X_A X_B U,$$

where $U \in \Gamma(\mathcal{E}\Sigma[-4])$ is the density built from curvatures given in Equation (3.9).

Proof. The proof amounts to tracing Equation (1.4) with the hypersurface tractor metric. Note that U is given in the previous remark. \square

3.4. Normal Operators. Conformally invariant operators that take derivatives in directions normal to the hypersurface are important, especially for the construction of higher fundamental forms. The first of these is obtained by asking how the Thomas- D operator acts in normal directions; indeed there exists a canonical conformal Robin operator produced this way [22]:

Definition 3.8. The operator

$$\delta_R : \Gamma(\mathcal{T}^\Phi M[w]) \rightarrow \Gamma(\mathcal{T}^\Phi M[w-1])|_\Sigma$$

defined, for $w \neq 1 - \frac{d}{2}$, by

$$\delta_R T := N^A \hat{D}_A T|_\Sigma,$$

and, for $w = 1 - \frac{d}{2}$, by

$$(3.10) \quad \delta_R T \stackrel{g}{=} (\nabla_{\hat{n}} - (1 - \frac{d}{2})H^g)T|_\Sigma,$$

for any $g \in \mathfrak{c}$, is termed the *tractor Robin operator*. \blacksquare

Remark 3.9. In fact, for any weight w tractor T and $g \in \mathfrak{c}$, the tractor Robin operator obeys

$$\delta_R T \stackrel{g}{=} (\nabla_{\hat{n}} T - wH^g T)|_\Sigma,$$

which shows, by a dimensional continuation argument, that Equation (3.10) defines a tractor in the stated codomain. Also, in the special case that T is a conformal density, the above is the conformally invariant Robin combination of Neumann and Dirichlet operators first constructed by Cherrier [14]. \blacksquare

Higher order conformally invariant analogs of the Robin operator acting on a weight w tractor T , can be defined as follows [25]

$$\Gamma(\mathcal{T}^\Phi M[w-k])|_\Sigma \ni \delta_R^{(k)} T := N^{A_1} \dots N^{A_{k-1}} \delta_R D_{A_1} \dots D_{A_{k-1}} T.$$

It is not difficult to verify that the operator $\delta_R^{(k)}$ has transverse order at most k (see Section 3.5). This bound is saturated at generic weights. One such example of the above family of operators is given below.

Lemma 3.10. *Let $\tau \in \Gamma(\mathcal{E}M[w])$. Then, given $g \in \mathfrak{c}$ where $\tau = [g; t]$,*

$$\begin{aligned} \delta_R^{(2)} t \stackrel{g}{=} & (d+2w-3) \left[\hat{n}^a \hat{n}^b \nabla_a \nabla_b t - (2w-1)H \nabla_{\hat{n}} t + w(P_{\hat{n}\hat{n}} + \frac{1}{2}(2w-1)H^2)t \right] \\ & - \left[\bar{\Delta} + w \left(\bar{J} + \frac{1}{2(d-2)}K \right) \right] t. \end{aligned}$$

When $w = \frac{3-d}{2}$,

$$\delta_R^{(2)} = -\square_Y - \frac{d-3}{4(d-2)}K,$$

where $\square_Y := \bar{\Delta} + \frac{3-d}{2}\bar{J}$ is the Yamabe operator of (Σ, \bar{c}) .

Proof. The proof can be found in [25]. \square

Observe that we can extend $\delta_R^{(k)}$ to a map $\Gamma(T^\varphi M[w]) \rightarrow \Gamma(T^\varphi \Sigma[w-k])$ acting on trace-free sections of $T^\varphi M[w]$, by the composition of maps $\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\top} \circ \delta_R^{(k)} \circ q$ whenever this is defined at weight w , and will denote this also by $\delta_R^{(k)}$ with $\delta_R^{(1)} \equiv \delta_R$. We use the same notation for the operator generalizing this to a slightly larger class of weights.

Lemma 3.11. *Let $t \in \Gamma(\odot_\circ^2 T^* M[w])$. Then, given $g \in \mathfrak{c}$ and when $w \neq 3$,*

$$\delta_R t_{ab} \stackrel{g}{=} \overset{\circ}{\top} \left[(\nabla_{\hat{n}} - (w-2)H) t_{ab} + \frac{2}{w-3} \bar{\nabla}_{(a} t_{\hat{n}b)}^\top \right],$$

and when $w \neq 4, -d, 2-d$,

$$\begin{aligned} \delta_R^{(2)} t_{ab} &\stackrel{g}{=} \overset{\circ}{\top} \left\{ (d+2w-7) \left[\left(\hat{n}^c \hat{n}^d \nabla_c \nabla_d t_{ab} \right) - (2w-5) H (\nabla_{\hat{n}} t_{ab}) + \frac{4}{w-4} \bar{\nabla}_{(a} (\hat{n} \cdot \nabla_{\hat{n}} t)_{b)}^\top \right. \right. \\ &\quad \left. \left. + (w-2) \left(P_{\hat{n}\hat{n}} + \frac{1}{2} (2w-5) H^2 \right) t_{ab} - \frac{4(w-2)}{w-4} H \bar{\nabla}_{(a} t_{\hat{n}b)}^\top - \frac{2w}{w-4} (\bar{\nabla}_{(a} H) t_{\hat{n}b)} \right] \right. \\ &\quad - \frac{4}{d+w-2} \overset{\circ}{\Pi}_{ab} \bar{g}^{cd} (\nabla_{\hat{n}} t_{cd}) \\ &\quad - (\bar{\Delta} + (w-2) \bar{J}) t_{ab}^\top - \frac{4}{w-4} \bar{\nabla}_{(a} \bar{\nabla}_{b)} t_{\hat{n}}^\top + 4 \bar{P}_{(a} \cdot t_{b)} + 4 \hat{F}_{(a} \cdot t_{b)} \\ &\quad + \frac{4}{d+w-2} \overset{\circ}{\Pi}_{ab} \bar{\nabla} \cdot t_{\hat{n}}^\top - 4 \overset{\circ}{\Pi}_{c(a} \bar{\nabla}^c t_{\hat{n}b)}^\top - \frac{4}{w-4} \bar{\nabla}_{(a} (\overset{\circ}{\Pi}_{b)} t_{\hat{n}c)}^\top - \frac{4(d+w-5)}{d-2} (\bar{\nabla} \cdot \overset{\circ}{\Pi})_{(a} t_{\hat{n}b)} \\ &\quad + \frac{2(d+2w-6)}{(w-3)(w-4)} \bar{\nabla}_a \bar{\nabla}_b t_{\hat{n}\hat{n}} - \frac{2(d+2w-6)}{w-3} \bar{P}_{ab} t_{\hat{n}\hat{n}} - \frac{4(w-2)}{d+w-2} H \overset{\circ}{\Pi}_{ab} t_{\hat{n}\hat{n}} \\ &\quad \left. + 2 \overset{\circ}{\Pi}_{(a}^2 \cdot t_{b)} - \frac{4}{d+w-2} \overset{\circ}{\Pi}_{ab} \overset{\circ}{\Pi} \cdot t - \left(\frac{2}{d-1} + \frac{w-6}{2(d-2)} \right) K t_{ab} - 2 \overset{\circ}{\Pi}_{ab}^2 t_{\hat{n}\hat{n}} \right\}. \end{aligned}$$

Proof. We proceed by working in a generic dimension d and with generic weight w rank-2 trace-free symmetric tensors $t \in \Gamma(\odot_{\circ}^2 T^* M[w])$. For generic weights and dimensions, we may compute the composition of maps $\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\top} \circ \delta_R^{(k)} \circ q$. In both the cases $k=1, 2$, we must compute $\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\top}$, so we first compute this operator on a general rank-2 tractor $T \in \Gamma(\odot_{\circ}^2 \mathcal{T}M[w])$. Note that $\overset{\circ}{\top}$ maps $T \mapsto \overset{\circ}{T}$ where $\overset{\circ}{T} := (I_A^{A'} I_{B'}^B - \frac{1}{d+1} I_{AB} I^{A'B'}) T_{A'B'} |_{\Sigma}$. Because \bar{r} achieves $X \cdot \bar{r}(\overset{\circ}{T}) = 0$, we have that $(\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\top})(T)_{ab} = \bar{Z}_{Aa} \bar{Z}_{Bb} \bar{r}(\overset{\circ}{T})^{AB}$. Thus, for generic dimensions and weights, we have that

$$\begin{aligned} (\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\top})(T)_{ab} &= \bar{Z}_{Aa} \bar{Z}_{Bb} \overset{\circ}{T}^{AB} \\ &\quad - \frac{2}{w} \bar{Z}_{B(a} \bar{\nabla}_{b)} (X_C \overset{\circ}{T}^{CB}) + \frac{2}{w(d+1)} \bar{g}_{ab} \hat{D}_C (X_D \overset{\circ}{T}^{CD}) \\ &\quad + \frac{1}{w(w+1)} \bar{Z}_{Bb} \bar{\nabla}_a \hat{D}^B (X_C X_D \overset{\circ}{T}^{CD}) \\ &\quad + \frac{8 \bar{g}_{ab}}{(d-1)(d+1)(d+2w+1)} \hat{D}_C (X_D \overset{\circ}{T}^{CD}). \end{aligned}$$

We now compute each of the terms above. First, using Equation (2.2), we obtain

$$\bar{Z}_{Bb} \bar{\nabla}_a (X_C \overset{\circ}{T}^{CB}) = \bar{\nabla}_a \overset{\circ}{T}_b^+ + \bar{g}_{ab} \overset{\circ}{T}^{+-} + \bar{P}_{ab} \overset{\circ}{T}^{++},$$

$$\bar{Z}_{Bb} \bar{\nabla}_a \hat{D}^B (X_C X_D \overset{\circ}{T}^{CD}) = [\bar{\nabla}_a \bar{\nabla}_b - \frac{\bar{g}_{ab}}{d+2w+1} (\bar{\Delta} + (w+2) \bar{J}) + (w+2) \bar{P}_{ab}] \overset{\circ}{T}^{++}.$$

Next (see for example the Appendix B of [49]), we have

$$\hat{D}_C (X_D \overset{\circ}{T}^{CD}) = \frac{1}{d+2w-1} (-[\bar{\Delta} - (d+w-1) \bar{J}] \overset{\circ}{T}^{++} + (d+2w+1) \bar{\nabla}_a \overset{\circ}{T}^{+a} + (d+w-1)(d+2w+1) \overset{\circ}{T}^{+-}).$$

Finally, because $\overset{\circ}{T}^{AB}$ is hypersurface tractor-trace-free, we have that $0 = \overset{\circ}{T}_a^a + 2 \overset{\circ}{T}^{+-}$. Thus,

$$\bar{Z}_{Aa} \bar{Z}_{Bb} \overset{\circ}{T}^{AB} + \frac{2 \bar{g}_{ab}}{d-1} \overset{\circ}{T}^{+-} = \overset{\circ}{T}_{ab} - \frac{1}{d-1} \bar{g}_{ab} \overset{\circ}{T}_c^c =: \overset{\circ}{T}_{(ab)\circ}.$$

Substituting these identities into the above display for $(\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\top})(T)$ gives

$$(\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\top})(T)_{ab} = \frac{1}{w(w+1)} \bar{\nabla}_{(a} \bar{\nabla}_{b)\circ} \overset{\circ}{T}^{++} - \frac{2}{w} \bar{\nabla}_{(a} \overset{\circ}{T}_{b)\circ}^+ - \frac{1}{w+1} \bar{P}_{(ab)\circ} \overset{\circ}{T}^{++} + \overset{\circ}{T}_{(ab)\circ}.$$

Proving the lemma now amounts to computing the components of $\overset{\circ}{T}$ when $T = \delta_R \circ q(t)$ or $T = \delta_R^{(2)} \circ q(t)$. Note that by construction,

$$\begin{aligned} \overset{\circ}{T}^{++} &\stackrel{\Sigma}{=} T^{++}, \\ \overset{\circ}{T}_b^+ &\stackrel{\Sigma}{=} \bar{g}_b^c T_c^+, \\ \overset{\circ}{T}_{(ab)\circ} &\stackrel{\Sigma}{=} \bar{g}_a^c \bar{g}_b^d T_{cd} + \frac{1}{d-1} \bar{g}_{ab} (2T^{+-} + T_{\hat{n}\hat{n}}). \end{aligned}$$

Thus, we can simplify our calculations by only computing the components of T appearing on the right hand side above.

We begin with $T = \delta_{\mathbb{R}} \circ q(t)$. We can use Equation (2.8) and Definition (3.8) to show that

$$\begin{aligned} X_A T^{AB} &= X_A \delta_{\mathbb{R}} q(t)^{AB} = \delta_{\mathbb{R}} X_A q(t)^{AB} - N_A q(t)^{AB} \\ &= -N_A q(t)^{AB}, \end{aligned}$$

where the second equality holds because $X \cdot q(t_{ab}) = 0$ by definition. Thus, using (2.8), we have

$$T^{++} = 0, \quad T_a^+ = -t_{\hat{n}a}, \quad T^{+-} = \frac{\hat{n} \cdot \nabla \cdot t}{d + w - 2}.$$

Using again (2.8) as well as Equation (2.2), Remark (3.9), and the fact that $q(t)$ has weight $w - 2$, we have that

$$T_{ab} = \nabla_{\hat{n}} t_{ab} - (w - 2) H t_{ab} - \frac{2 \hat{n}_{(a} \nabla \cdot t_{b)}}{d + w - 2},$$

so that (in the scale g)

$$\begin{aligned} \overset{\circ}{T}_{(ab)\circ} &\stackrel{\Sigma}{=} \top[\nabla_{\hat{n}} t_{ab} - (w - 2) H t_{ab}] + \frac{1}{d-1} \bar{g}_{ab} (\hat{n}^a \hat{n}^b \nabla_{\hat{n}} t_{ab} - (w - 2) H t_{\hat{n}\hat{n}}) \\ &\stackrel{\Sigma}{=} \overset{\circ}{\top}[\nabla_{\hat{n}} t_{ab} - (w - 2) H t_{ab}]. \end{aligned}$$

Combining the above and noting that T has weight $w - 3$, we have

$$\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\top} \circ \delta_{\mathbb{R}} \circ q(t) = \overset{\circ}{\top} \left[\nabla_{\hat{n}} t_{ab} - (w - 2) H t_{ab} + \frac{2}{w-3} \bar{\nabla}_{(a} t_{\hat{n}b)\circ} \right].$$

We finish the proof by handling $T = \delta_{\mathbb{R}}^{(2)} \circ q(t)$. The tractor computations in this case become unwieldy but not difficult. To manage these, we used the computer algebra system FORM [51]; this computation is documented in [10]. The lemma results from this computation. \square

To uncover further invariants of the embedding $\Sigma \hookrightarrow (M, \mathbf{c})$ and associated operators, we introduce more powerful holographic machinery.

3.5. Holography. A useful way to treat hypersurfaces is in terms of defining functions: recall that s is a *defining function* for Σ if $s \in C^\infty M$, $\Sigma = \{P \in M \mid s(P) = 0\}$, and $ds|_\Sigma$ is nowhere vanishing. We shall also assume that $s > 0$ on M^+ . A weight $w = 1$ conformal density given by $\sigma = [g; s]$, where s is a defining function and $g \in \mathbf{c}$, is called a *defining density*. Defining functions can be used to analyze hypersurface invariants; for example, the second fundamental form can be written as $\Pi_{ab} = \mathcal{P}_{ab}|_\Sigma$ where

$$\mathcal{P}_{ab} = \left(\nabla_a - |ds|_g^{-1} (\nabla_a s) \nabla_{|ds|_g^{-1} \text{grad } s} \right) \frac{\nabla_b s}{|ds|_g}.$$

The quantity \mathcal{P}_{ab} is an example of a *preinvariant*, namely a smooth (and suitably polynomial) function of generic metrics g and defining function s , whose restriction to Σ is independent of the choice of defining function s , and therefore defines some hypersurface invariant, see [28] for the detailed definition. Let us choose local coordinates (s, y^i) in a neighborhood of Σ where $i = 1, \dots, d-1$ such that along Σ the vector fields $\partial/\partial y^i$ are tangent to Σ . Then the *transverse order* of a preinvariant \mathcal{P} is defined by writing \mathcal{P} as a function of the coordinate components of g and their derivatives, and computing the minimum, over all such coordinate representations of \mathcal{P} , of the highest order of $\partial/\partial s$ derivatives of g upon restriction to Σ . The transverse order is an invariant of the corresponding hypersurface invariant: for example, the second fundamental form has transverse order one. We will also say that an operator O has transverse order $k \in \mathbb{Z}_{\geq 0}$ when there exists v in the domain of O such that $O(s^k v)|_\Sigma \neq 0$, but $O(s^{k+1} v')|_\Sigma = 0$, for all v' in the domain of O .

We can use this notion of preinvariants to show that Definitions 1.6 and 1.7 are well-posed, as established by the following result.

Lemma 3.12. *Suppose $2 \leq n \leq d-1$. Then if the conformal embedding $\Sigma \hookrightarrow (M, \mathbf{c})$ is such that at least one ℓ th fundamental form vanishes for every $2 \leq \ell < n$, then up to an overall non-zero coefficient, there is a unique n th fundamental form.*

Proof. We begin by considering the leading transverse order term in the preinvariant expression for an n th fundamental form. Using coordinates $\{s, y^i\}$ in a collar neighborhood $I \times \Sigma \subset M$, we can express any preinvariant in terms of a defining function s , partial derivatives ∂_s and $\partial_i = \partial/\partial y^i$, the metric components g_{ab} , and the components of its inverse g^{ab} . Because fundamental forms are conformally invariant, it is useful to view part of the preinvariant alphabet—the metric g and the defining function s —as representatives of weighted densities: the metric is a weight 2 representative of the conformal class of metrics \mathbf{c} and the defining function is a representative of a defining density $\sigma = [g; s]$ with weight 1. We next determine the leading transverse order term in the preinvariant expression for an n th fundamental form using Definition 1.2.

From the definition of transverse order, the leading derivative term in the preinvariant for the n th fundamental form must be of the form

$$O_{(ab)}{}^{cd} \partial_s^{n-1} g_{cd}|_{\Sigma},$$

where, as an operator, $O_{(ab)}{}^{cd}$ has transverse order 0. Moreover the above is annihilated by the normal and hypersurface trace. Note that the weight of an n th fundamental form is $3 - n$ and its transverse order is $n - 1$. By considering only conformal transformations by a constant we may still analyze expressions such as that displayed above in terms of weights. Because the weight of the operator ∂_s is -1 , the weight of the above display is $3 - n + w_O$ where the operator $O_{(ab)}{}^{cd}$ has weight w_O . Hence we must have that $w_O = 0$. By an elementary weight argument, one sees that this operator is algebraic and therefore made only from the metric, its inverse, and a preinvariant for the conormal. Together with elementary $O(d)$ and $O(d - 1)$ representation theory, this implies that (along Σ) this operator must be a non-zero multiple of the trace-free hypersurface projector, and hence proportional to

$$\overset{\circ}{\mathbb{T}}_e(\partial_s^{n-1} g_{ab})|_{\Sigma},$$

where $\overset{\circ}{\mathbb{T}}_e$ is any preinvariant expression for the operator $\overset{\circ}{\mathbb{T}}$.

Now suppose that $L^{(n)}$ and $L^{(n)'}$ are two n th fundamental forms with the same coefficient for the above-displayed term and the conformal embedding $\Sigma \hookrightarrow (M, \mathbf{c})$ is such that an ℓ th fundamental form vanishes for every $\ell < n$. We then seek to show that $L^{(n)} - L^{(n)'} \stackrel{\Sigma}{\equiv} 0$. Clearly, because $L^{(n)}$ and $L^{(n)'}$ have the same leading term, their difference must have transverse order at most $n - 2$. Put another way,

$$L_{ab}^{(n)} - L_{ab}^{(n)'} = P_{(ab)}{}^{cd} \partial_s^{n-2} g_{cd}|_{\Sigma} + \text{lower-order terms},$$

where $P_{(ab)}{}^{cd}$ is a preinvariant operator with weight -1 and transverse operator order 0. But then $\partial_s^{n-2} g_{ab}|_{\Sigma}$ can be rewritten as an $(n - 1)$ th fundamental form plus lower-order terms. Thus, a proof via induction only requires that we check that the second fundamental form is unique up to an overall non-zero coefficient, which is again easily verified by an elementary weight and representation theoretic argument. \square

Given a hypersurface embedding $\Sigma \hookrightarrow (M, \mathcal{S})$, where M is a smooth manifold endowed with any structure \mathcal{S} , and a hypersurface invariant $\overline{\mathcal{P}}$, we call a smooth extension \mathcal{P} of $\overline{\mathcal{P}}$ to M , a *holographic formula* when \mathcal{P} is canonically determined by the structure \mathcal{S} . In many cases, we use this terminology also when \mathcal{P} (or $\mathcal{P}(\mathcal{S})$) is only uniquely determined up to some asymptotic order in a defining function. The notion of a holographic formula extends to geometric differential operators in a straightforward way.

The possibility to define hypersurface invariants in terms of preinvariants that do not depend on any particular choice of defining function can be leveraged by a clever choice of defining function; such a choice leads to holographic formulæ. In fact, a key result [40, 5, 42, 3] is that there is a unique metric g^o on the d -dimensional manifold M^+ with boundary $\partial M^+ = \Sigma$, whose scalar curvature obeys

$$(3.11) \quad S c^{g^o} = -d(d - 1),$$

and such that in M^+

$$g^o = s^{-2}g,$$

for some suitably smooth defining function s and $g \in \mathbf{c}$. The problem of finding s such that the pair (g, s) give such a metric g^o is a general case of the Loewner–Nirenberg problem [40]. The metric g^o is called the *singular Yamabe metric*. Observe that if (g, s) determines the singular Yamabe metric g^o , then so does $(\Omega^2 g, \Omega s)$. In other words, g^o is determined by the conformal density $\sigma := [g; s] \in \Gamma(\mathcal{EM}[1])$.

An important insight is that, because σ is uniquely defined on M^+ by the data $\Sigma \hookrightarrow (M, \mathbf{c})$, the jets of σ can be used to efficiently study the conformal hypersurface embedding. Recasting Equation (3.11) as a tractor equation is extremely helpful for this. To understand why, first recall that formulæ for Riemannian hypersurface invariants simplify when given a *unit defining function* s , *viz.* a defining function that satisfies

$$g(n, n) = |ds|_g^2 = 1,$$

where $n := ds$. (It suffices for many applications to find s obeying the above condition in a neighborhood of Σ .) For example, the second fundamental form is given by the Hessian of s restricted to Σ . There is a neat conformal analog of this picture: A *unit conformal defining density* is a weight one conformal density σ subject to the condition

$$(3.12) \quad h(I_\sigma, I_\sigma) = 1,$$

where $I_\sigma := \hat{D}\sigma$. In general, we call a tractor obtained by acting with the hatted Thomas- D operator on a scale, a *scale tractor*, and drop the subscript σ when it is clear to do so (the same convention will be applied to other objects whose dependence of σ is denoted this way).

Now, note that given $g \in \mathbf{c}$ where $\sigma = [g; s]$,

$$(3.13) \quad I^A = \hat{D}\sigma \stackrel{g}{=} \begin{pmatrix} s \\ ds \\ -\frac{1}{d}(\Delta^g s + J^g s) \end{pmatrix},$$

so that

$$I^2 := h(I, I) \stackrel{g}{=} |ds|_g^2 - \frac{2}{d}s(\Delta^g s + J^g s).$$

Away from Σ , written in terms of the metric $g^o = s^{-2}g$, the right hand side of the above display equals $-2J^{g^o}/d = -Sc^{g^o}/(d(d-1))$. Hence the unit conformal Condition (3.12) is equivalent to the singular Yamabe Equation (3.11).

Given the data $\Sigma \hookrightarrow (M, \mathbf{c})$, the one-sided solution g^o to the singular Yamabe problem of Equation (3.11) in M^+ depends on global information of M^+ and \mathbf{c} . However, locally determined metrics g^o on $M^+ \subset M^d$ that are smooth up to the boundary and obey

$$Sc^{g^o} = -d(d-1) + \mathcal{O}(s^d)$$

in an open neighborhood of Σ , where s is any defining function for Σ , always exist [3] and are easily constructed [28]. We term such a metric g^o an *asymptotic singular Yamabe metric*. Here the notation $\mathcal{O}(s^k)$ denotes s^k times any smooth function, and we will use a similar notation involving powers of densities in an obvious way. In terms of defining densities σ and their corresponding scale tractor, the above-displayed equation stipulates that

$$(3.14) \quad I_\sigma^2 = 1 + \sigma^d \mathcal{B},$$

where $\mathcal{B} \in \Gamma(\mathcal{EM}[-d])$. Solutions σ to the above equation are termed *asymptotic unit defining densities*. While these are not unique, the quantity

$$(3.15) \quad B_\Sigma := \mathcal{B}|_\Sigma \in \Gamma(\mathcal{E}\Sigma[-d])$$

is, and is termed the *obstruction density*. The obstruction density B_Σ is a local invariant of the conformal embedding $\Sigma \hookrightarrow (M, \mathbf{c})$, and is the obstruction to solving $I_\sigma^2 = 1$ smoothly on $\overline{M^+}$ [3, 28]. Moreover, B_Σ is variational, meaning that it is the functional gradient with

respect to variations of the embedding $\Sigma \hookrightarrow M$ of a Willmore-type energy functional [32] (see also [30]).

The key point now is that the first d jets of σ are uniquely defined by solving Equation (3.14), and therefore can be used to efficiently construct invariants of the conformal embedding $\Sigma \hookrightarrow (M, \mathbf{c})$. These are given as holographic formulæ. For example, the first two jets are required for the scale tractor $I_\sigma|_\Sigma$ and this gives a holographic formula for the normal tractor.

Lemma 3.13 (Gover [22]). *Let σ be a defining density for Σ that obeys*

$$I_\sigma^2 = 1 + \mathcal{O}(\sigma^2),$$

where $I_\sigma = \hat{D}\sigma$. Then

$$I_\sigma|_\Sigma = N.$$

Recall that the normal tractor N encodes both the unit normal and mean curvature (in any scale). The trace-free part of the second fundamental form can be obtained holographically in several ways, the first example is given below.

Lemma 3.14. *Let σ be a defining density for Σ that obeys*

$$I_\sigma^2 = 1 + \mathcal{O}(\sigma^2),$$

where $I_\sigma = \hat{D}\sigma$. Then the projecting part of $\nabla I_\sigma|_\Sigma$ equals the trace-free second fundamental form.

Proof. Let $g \in \mathbf{c}$ and $\sigma = [g; s]$. First note that $I_\sigma^2 = 1 + \mathcal{O}(\sigma^2)$ implies that

$$|n|_g^2 = 1 + \frac{2s}{d}(\nabla \cdot n + Js) + \mathcal{O}(s^2).$$

Then using the scale tractor Equation (3.13) and tractor connection Equation (2.2), it follows that the projecting part of $\nabla I_\sigma|_\Sigma$ is

$$\left(\nabla n - \frac{1}{d}g\nabla \cdot n \right) \Big|_\Sigma = \left(\nabla^\top n + \hat{n}\nabla_n n - gH \right) \Big|_\Sigma.$$

In the above $n := ds$ and, using the first display of this proof, it follows that $n|_\Sigma = \hat{n}$. In the above we have also used that Lemma 3.13 implies that $\frac{1}{d}\nabla \cdot n|_\Sigma = H$. Moreover $\nabla_n n = \frac{1}{2}d|n|_g^2 = \frac{1}{d}n\nabla \cdot n + \mathcal{O}(s)$ which equals $\hat{n}H$ along Σ . Using $\bar{g} = g|_\Sigma - \hat{n} \odot \hat{n}$, and $\nabla^\top n|_\Sigma = \nabla^\top (n/|n|_g)|_\Sigma$, the result now follows. \square

Thanks to Lemma 3.13 we now have a canonical extension I_σ of the normal tractor N , and thus can construct a canonical extension of the Robin operator of Definition (3.8).

Lemma 3.15. *The operator*

$$I \cdot D : \Gamma(\mathcal{T}^\Phi M[w]) \rightarrow \Gamma(\mathcal{T}^\Phi M[w-1]),$$

where $I \cdot DT := I^A D_A T$, obeys

$$I \cdot DT \Big|_\Sigma = (d + 2w - 2) \delta_R T.$$

Proof. The proof of the above lemma is given in [22] (see also [11]) and uses that $I \cdot DT$, in a choice of scale $g \in \mathbf{c}$ for which $\sigma = [g; s]$, is given by

$$I \cdot DT = (d + 2w - 2)(\nabla_n + w\rho_s)T - s(\Delta + wJ^g)T,$$

where $\rho_s := -\frac{1}{d}(\Delta^g s + sJ^g)$. \square

The above lemma presages a useful algebraic relationship between several objects present in the calculus so far.

Proposition 3.16 ([27]). *Suppose $\sigma \in \Gamma(\mathcal{E}M[1])$ obeys $I_\sigma^2 \neq 0$, and denote by $\mathfrak{h} : \Gamma(\mathcal{T}^\Phi M[w]) \rightarrow \Gamma(\mathcal{T}^\Phi M[w])$ the operator defined by $\mathfrak{h}f = (d + 2w)f$. Then, viewing $x := \sigma : \Gamma(\mathcal{T}^\Phi M[w]) \rightarrow \Gamma(\mathcal{T}^\Phi M[w+1])$ as a multiplicative operator and $y := -\frac{1}{I^2}I \cdot D : \Gamma(\mathcal{T}^\Phi M[w]) \rightarrow \Gamma(\mathcal{T}^\Phi M[w-1])$ as a differential operator, commutators of the operators (x, \mathfrak{h}, y) satisfy the $\mathfrak{sl}(2)$ defining relations,*

$$[\mathfrak{h}, x] = 2x, \quad [x, y] = \mathfrak{h}, \quad [\mathfrak{h}, y] = -2y.$$

Remark 3.17. In many cases, we assume that σ solves the singular Yamabe problem to some order $k \geq 2$. In these situations, we may neglect the $\frac{1}{I^2}$ coefficient when using the above-displayed relations because it equals unity to sufficiently high order for the problem at hand. \blacksquare

The next lemma gives a holographic formula for the tangential Thomas- D operator of Proposition 3.2.

Lemma 3.18. *Let $w + \frac{d}{2} \neq 1, \frac{3}{2}, 2$ and σ be a unit conformal defining density for Σ that obeys $I_\sigma^2 = 1 + \mathcal{O}(\sigma^2)$. Then, along Σ ,*

$$(3.16) \quad \hat{D}_A^T = \hat{D}_A - I_A I \cdot \hat{D} + \frac{X_A}{d + 2w - 3} I \cdot \hat{D}^2.$$

Proof. Using $I^2 = 1 + \mathcal{O}(\sigma^2)$ and Proposition 2.4,

$$(I \cdot \hat{D})^2|_\Sigma = \left[I_A I \cdot \hat{D} \hat{D}^A + \frac{X_A K}{d - 2} \hat{D}^A \right] \Big|_\Sigma = N_A N_B \hat{D}^B \hat{D}^A + \frac{wK}{d - 2}.$$

Substituting this identity into the definition of \hat{D}^T in Proposition 3.2 completes the proof. \square

4. FUNDAMENTAL FORMS

The singular Yamabe problem provides a canonical extension of $\mathring{\mathbb{I}}$. Hence, in the spirit of holography, canonical fundamental forms can be constructed by applying high transverse order operators to this extension.

4.1. The Canonical Extension of $\mathring{\mathbb{I}}$. As noted in Lemma 3.3, the tractor analog L of $\mathring{\mathbb{I}}$ is (essentially) given by the tangential Thomas operator \hat{D}^T acting on an extension N^e of N . The scale tractor I_σ of a unit conformal defining density σ gives a canonical extension of the normal tractor N and, in turn, of the tractor second fundamental form L . The projecting part of the latter gives a conformally-invariant canonical extension of $\mathring{\mathbb{I}}$. We will compute this extension explicitly, but doing so requires the promised proof of Lemma 3.3.

Proof of Lemma 3.3. Let σ be an asymptotic unit conformal defining density for $\Sigma \hookrightarrow (M, \mathfrak{c})$. When not stated explicitly, in what follows P^{AB} denotes $\hat{D}^A I_\sigma^B$ for σ an asymptotic unit defining density. Because $d > 3$ we have that $I^2 = 1 + \mathcal{O}(\sigma^4)$. We will apply Lemma 3.18 and that the normal tractor obeys $N = I|_\Sigma$. Evaluating $\hat{D}_A I^2$ yields, after some massaging using Proposition 2.4, the identity

$$(4.1) \quad I \cdot \hat{D} I_A = \frac{(\hat{D} I)^2 X_A}{d - 2} + \mathcal{O}(\sigma^2).$$

Defining the extension $K^e := (\hat{D} I_\sigma)^2$ of the rigidity density K , it follows using the above identity that

$$\delta_R K^e|_\Sigma = -2(d - 3)M,$$

see [31]. Thus, noting that $I \cdot \hat{D} X = I$ (see Equation (2.5)), and employing the holographic formula for \hat{D}^T in Lemma 3.18, we then have the following identity:

$$\hat{D}_{(A}^T I_{B)}|_\Sigma = \hat{D}_{(A} I_{B)}|_\Sigma - \frac{d-4}{(d-2)(d-3)} I_{(A} X_{B)} K|_\Sigma - \frac{2}{d-2} X_A X_B M.$$

A holographic formula for the tractor second fundamental form is given in [31]:

$$(4.2) \quad \hat{D}_A I_B|_\Sigma = L_{AB} + \frac{2}{d-2} I_{(A} X_{B)} K|_\Sigma + \frac{3d-8}{(d-2)(d-3)} X_A X_B M.$$

This formula applied to the second display above gives

$$\hat{D}_{(A}^T I_{B)} \Big|_{\Sigma} = L_{AB} + \frac{1}{d-3} I_{(A} X_{B)} K \Big|_{\Sigma} + \frac{1}{d-3} X_A X_B M.$$

This completes the proof. \square

Remark 4.1. In the proof above, the tractor $P := \hat{D}I$ played a key rôle generating an extension of the tractor second fundamental form. The same tractor (and higher transverse order counterparts) can also be used to construct tractor higher fundamental forms as well as the rigidity density K (and normal derivatives thereof). Note that we use the notation P for the tractor $\hat{D}I$ because, away from Σ , and computed in the scale σ , one has that $q^*(P)$ equals the Schouten tensor of the singular metric g^o . In what follows we will see that the tractor P provides an interpolation between the Schouten tensor of the singular metric g^o and trace-free second fundamental form. \blacksquare

Note that $X \cdot P = 0$ because $X \cdot \hat{D}T = wT$ for any weight w tractor T , so $q^*(P_{AB}) \in \Gamma(\odot_{\circ}^2 T^* M[1])$. Hence, writing $\sigma = [g; s]$, the canonical extension of $\hat{\Pi}$ is given by

$$(4.3) \quad \hat{\Pi}_{ab}^e := q^*(P_{AB}) \stackrel{g}{=} \nabla_a n_b + s P_{ab} + \rho g_{ab},$$

where $\rho = -\frac{1}{d}(\Delta^g s + J^g s)$. This result was computed by directly computing $\hat{D}I$. Note that the above may be written $(\nabla_{(a} \nabla_{b)} \circ + P_{(ab)\circ})s$, where the second order, conformally invariant operator in parentheses is termed the *almost Einstein operator*.

4.2. Canonical Transverse Differential Operators. One could construct higher fundamental forms by extending Lemma 3.11 to $\delta_{\mathbb{R}}^{(k)}$ for $k > 2$ and applying these operators to $\hat{\Pi}^e$, but this is inefficient. Fortunately, holography provides a canonical extension $I \cdot D$ of the Robin operator. Indeed, applying powers of the operator ID_{σ} , generically given by

$$(4.4) \quad \text{ID}_{\sigma} = q^* \circ r \circ I \cdot D \circ q,$$

to $\hat{\Pi}^e$ is the key step for producing formulæ for higher fundamental forms. To compute ID (dropping the subscript σ for brevity), we first need the following result.

Lemma 4.2. *Let $\theta = [g; t] \in \Gamma(\otimes^r T^* M[w])$ and $\sigma = [g; s] \in \Gamma(\mathcal{E}M[1])$. Then*

$$\nabla^{\sigma} \theta := [g; s \nabla t - (w - r) \text{d}s \otimes t + (\text{d}s \otimes g)^{\sharp} t] \in \Gamma(\otimes^{r+1} T^* M[w + 1]),$$

where for a general covector ω , we denote $(\omega \otimes g)_{a}^{\sharp} t_b := \omega_b t_a - g_{ba} \omega^c t_c$ (when $r = 1$) and extends, in the standard Leibniz way, to higher rank r tensors.

Proof. First note that, using the notation provided in the lemma, the Levi-Civita connection acting on $\theta := [g, t] \in \Gamma(\otimes^r T^* M[w])$ obeys

$$\nabla^{\Omega^2 g} t^{\Omega^2 g} = \nabla^{\Omega^2 g} (\Omega^w t^g) = \Omega^w (\nabla^g t^g + (w - r) \Upsilon \otimes t^g - (\Upsilon \otimes g)^{\sharp} t^g).$$

Here $\Upsilon := \text{d} \log \Omega$. Further, denoting $n := \text{d}s$, $n^{\Omega^2 g} = \text{d}(\Omega s) = \Omega(n^g + \sigma \Upsilon)$. Therefore, $n^{\Omega^2 g} \otimes t^{\Omega^2 g} = \Omega^{w+1} (n^g + \sigma \Upsilon) \otimes t^g$ and $(n^{\Omega^2 g} \otimes \Omega^2 g)^{\sharp} = \Omega([n^g + s \Upsilon] \otimes g)^{\sharp}$. Combining these conformal transformations with the above display completes the proof. \square

Note that if σ is a defining density for a hypersurface Σ subject to $I_{\sigma}^2 = 1 + \mathcal{O}(\sigma)$ then, along Σ , the operator ∇^{σ} is tensor multiplication by the unit conormal \hat{n} .

The operator ID of Equation (4.4) is not defined for many weights, so we instead make the following definition.

Definition 4.3. Let $\sigma \in \Gamma(\mathcal{E}M[1])$ be any weight one density, $I_{\sigma} = \hat{D}\sigma$, and $\tau_{ab} \in \Gamma(\odot_{\circ}^2 T^* M[w])$ be any rank two, trace-free density-valued symmetric tensor where $w \neq 3, 2 - d$. Also let $\widehat{M} := M \setminus \mathcal{Z}(\sigma)$. Then we define the map

$$\text{ID}_{\sigma} : \Gamma(\odot_{\circ}^2 T^* \widehat{M}[w]) \rightarrow \Gamma(\odot_{\circ}^2 T^* \widehat{M}[w - 1])$$

by the following formula:

$$\begin{aligned} \sigma \mathbb{ID}_\sigma \tau_{ab} := & -\mathbf{g}^{cd} \left(\nabla_c^\sigma \nabla_d^\sigma \tau_{ab} + \frac{2d}{(w-3)(d+w-2)} \nabla_{(a}^\sigma \nabla_{|c}^\sigma \tau_{d|b)\circ} - \frac{4}{d} [\nabla_{(a}^\sigma, \nabla_{|c}^\sigma] \tau_{d|b)\circ} \right) \\ & - \frac{4}{d} \sigma^2 W^c{}_{ab}{}^d \tau_{cd} + \left[\left(w - 2 + \frac{d-1}{2} \right)^2 - \left(\frac{d-1}{2} \right)^2 + 2 \right] I_\sigma^2 \tau_{ab}. \end{aligned}$$

■

The combination of terms in the above display is distinguished because they in fact allow the operator \mathbb{ID}_σ to be defined along $\mathcal{Z}(\sigma)$. This result is described in the following lemma.

Lemma 4.4. *Let $\sigma \in \Gamma(\mathcal{EM}[1])$ be any weight one density and $\tau_{ab} \in \Gamma(\odot_\circ^2 T^*M[w])$ any rank two, trace-free density-valued symmetric tensor where $w \neq 3, 2-d$. Let $g \in \mathbf{c}$ for which $\sigma = [g; s]$, $\tau_{ab} = [g; t_{ab}]$, and $\mathbb{ID}_\sigma \tau_{ab} = [g; \mathbb{ID}_s t_{ab}]$. Then*

$$\begin{aligned} \mathbb{ID}_s t_{ab} = & (d+2w-6) \left([\nabla_n + (w-2)\rho] t_{ab} - \frac{2(w-2)}{(w-3)(d+w-2)} n_{(a} \nabla \cdot t_{b)\circ} + \frac{2}{w-3} \left[n_c \nabla_{(a} t_{b)\circ}^c + (\nabla n)_{(a} \cdot t_{b)\circ} \right] \right) \\ & - s \left(\Delta t_{ab} + (w-2) J t_{ab} + \frac{2d}{(w-3)(d+w-2)} \nabla_{(a} \nabla \cdot t_{b)\circ} - 4P_{(a} \cdot t_{b)\circ} \right), \end{aligned}$$

with $n := ds$ and $\rho = -\frac{1}{d}(\Delta s + Js)$, and \mathbb{ID}_σ is a well-defined map

$$\mathbb{ID}_\sigma : \Gamma(\odot_\circ^2 T^*M[w]) \rightarrow \Gamma(\odot_\circ^2 T^*M[w-1]).$$

Proof. The proof amounts to a computation of $\mathbb{ID}_\sigma \tau_{ab}$ in a choice of scale away from $\mathcal{Z}(\sigma)$, applying Lemma 4.2 twice, and the identity $I_\sigma^2 \stackrel{g}{=} n^2 + 2s\rho$. The resulting tensor is proportional to s and hence the apparent singularity of \mathbb{ID}_σ along $\mathcal{Z}(\sigma)$ in Definition 4.3 is removable. \square

For weights at which the composition of maps $q^* \circ r \circ I \cdot D \circ q$ acting on trace-free symmetric rank-2 tensors is defined, it is not difficult (but somewhat tedious) to show that Equation (4.4) holds. In this sense, the operator \mathbb{ID}_σ is the action of $I \cdot D$ on rank-two symmetric tensors. Also we define the analog of the operator $I \cdot \hat{D}$ when $w \neq 3 - \frac{d}{2}$ by

$$\hat{\mathbb{ID}}_\sigma := \frac{1}{d+2w-6} \mathbb{ID}_\sigma.$$

For weights at which the appropriate composition of maps is defined, we also have that $\hat{\mathbb{ID}}_\sigma = q^* \circ r \circ I \cdot \hat{D} \circ q$.

Note that, given a defining density σ subject to $I_\sigma^2|_\Sigma = 1$, when $w \neq 3, 3 - \frac{d}{2}$, the operator δ_R acting on tensors is expressed in terms of $\hat{\mathbb{ID}}_\sigma$ according to

$$\delta_R = \mathring{\dagger} \circ \hat{\mathbb{ID}}_\sigma - \frac{2}{w-3} \mathring{\Pi} (\hat{n} \cdot)^2.$$

Remark 4.5. Proposition 3.16 shows that the operator $I \cdot D$ obeys an $\mathfrak{sl}(2)$ Lie algebra. Moreover, it is not hard to check that

$$[\nabla^\sigma, \sigma] = 0.$$

Hence we may expect that \mathbb{ID}_σ and σ obey a similar algebra. Indeed suppose $\sigma \in \Gamma(\mathcal{EM}[1])$ obeys $I_\sigma^2 \neq 0$, and call $\mathfrak{h} : \Gamma(\odot_\circ^2 T^*M[w]) \rightarrow \Gamma(\odot_\circ^2 T^*M[w])$ the operator defined by $\mathfrak{h}\tau = (d+2w-4)\tau$ and view $\sigma : \Gamma(\odot_\circ^2 T^*M[w]) \rightarrow \Gamma(\odot_\circ^2 T^*M[w+1])$ as a multiplicative operator. Then, acting on the subspace of sections τ_{ab} subject to the divergence condition

$$\mathbf{g}^{ab} \nabla_a^\sigma \tau_{bc} = 0,$$

straightforward algebra shows that commutators of the operators $(\sigma, \mathfrak{h}, -\frac{1}{I_\sigma^2} \mathbb{ID})$ satisfy the $\mathfrak{sl}(2)$ defining relations,

$$[\mathfrak{h}, \sigma] = 2\sigma, \quad [\sigma, -\frac{1}{I_\sigma^2} \mathbb{ID}] = \mathfrak{h}, \quad [\mathfrak{h}, -\frac{1}{I_\sigma^2} \mathbb{ID}] = -2 \left(-\frac{1}{I_\sigma^2} \mathbb{ID} \right).$$

Away from Σ and in the $g^\circ = \mathbf{g}/\sigma^2$ scale, the operator \mathbb{ID} acting on a trace- and divergence-free symmetric tensor τ_{ab} is given by

$$\mathbb{ID} \tau_{ab} \stackrel{g^\circ}{=} - \left(\Delta - \frac{2J}{d} \left[\left(w - 2 + \frac{d-1}{2} \right)^2 - \left(\frac{d-1}{2} \right)^2 - 2 \right] \right) \tau_{ab} + 4P_{(a} \cdot \tau_{b)\circ}.$$

The above algebra is likely a key ingredient for constructing a calculus for the study of tensor wave equations (see [23] for the analogous problem for differential forms). \blacksquare

4.3. Constructing Higher Fundamental Forms. One might think that having constructed the canonical conformally-invariant operator \mathbb{ID}_σ with transverse order 1, as well as a canonical extension of $\mathring{\mathbb{I}}$, that we could now directly compute higher fundamental forms by applying powers of \mathbb{ID}_σ to $\mathring{\mathbb{I}}^e$. However, as could be predicted by [25], the k th power of \mathbb{ID}_σ does not always produce an operator of transverse order k . More precisely, the candidate $(k-2)$ th fundamental form

$$\mathbb{ID}_\sigma^k \mathring{\mathbb{I}}^e|_\Sigma,$$

has transverse order strictly less than $k+1$ when $k \geq \frac{d-1}{2}$. In this section, we first prove that fundamental forms with transverse order $n \geq \frac{d+1}{2}$ can indeed not be constructed by applying powers of \mathbb{ID} to $\mathring{\mathbb{I}}^e$. We will then explicitly construct canonical fundamental forms with transverse order $n < \frac{d+1}{2}$. Finally we construct canonical conditional fundamental forms with transverse order $n \geq \frac{d+1}{2}$ and use these ingredients to prove Theorem 1.8.

We begin by defining operators that often have the appropriate transverse order.

Definition 4.6. Let $\Sigma \hookrightarrow (M^d, \mathbf{c})$ be a conformal embedding with asymptotic unit defining density σ . Then, when $k=0$, define $\delta_{d,w}^{(0)} := \mathring{\top} : \Gamma(\odot_\sigma^2 T^* M[w]) \rightarrow \Gamma(\odot_\sigma^2 T^* \Sigma[w])$. For $k=1$ and $w \neq 3$, call

$$\delta_{d,w}^{(1)} := \delta_R : \Gamma(\odot_\sigma^2 T^* M[w]) \rightarrow \Gamma(\odot_\sigma^2 T^* \Sigma[w-1]),$$

while for $k \in \mathbb{Z}_{\geq 2}$ and $w \notin \{2-d, \dots, k-d\} \cup \{3, \dots, k+2\}$, let

$$\delta_{d,w}^{(k)} := \delta_R \circ \mathbb{ID}_\sigma^{k-1} : \Gamma(\odot_\sigma^2 T^* M[w]) \rightarrow \Gamma(\odot_\sigma^2 T^* \Sigma[w-k]).$$

\blacksquare

Acting on sections of $\odot_\sigma^2 T^* M[w]$ for $w \neq 3$, the transverse order of $\delta_{d,w}^{(1)}$ is 1. When $w \notin \mathbb{Z}$, the operator $\delta_{d,w}^{(k)}$ is always defined. Moreover, when $2w$ is not an integer the operators $\delta_{d,w}^{(k)}$ for $k \geq 2$ have transverse order k (see Equation (4.5) below). However, when $k \in \mathbb{Z}_{\geq 2}$ and $2w \in \mathbb{Z}$, the operator $\delta_{d,w}^{(k)}$ may not be defined or it could fail to have transverse order k . In particular, for w an integer and $k \geq 2$, the operators $\delta_{d,w}^{(k)}$ are only defined in the three regions where w obeys $w < 2-d$, $k-d < w < 3$, or $k+2 < w$ (the second of these could be empty). The following lemma characterizes the transverse order of $\delta_{d,w}^{(k)}$ in these cases:

Lemma 4.7. Fix $d \geq 3$ and $k \geq 2$, and let $w \in \mathbb{Z}$ be such that

$$w < 2-d, \quad k-d < w < 3, \quad \text{or} \quad k+2 < w.$$

Then, the transverse order of $\delta_{d,w}^{(k)}$ is strictly less than k if and only if

$$\frac{7-d}{2} \leq w < 3 \quad \text{and} \quad \frac{d+2w-3}{2} \leq k \leq d+2w-5.$$

Proof. To evaluate the transverse order of $\delta_{d,w}^{(k)}$, we compute the coefficient of ∇_n^k . For that we first examine the leading derivative structure of the operator \mathbb{ID}_σ . From Lemma 4.4, acting on a weight w tensor $t_{ab} \in \Gamma(\odot_\sigma^2 T^* M[w])$ the only terms with non-zero transverse order are $\nabla_n t_{ab}$, $n_{(a} \nabla \cdot t_{b)\circ}$, $n_c \nabla_{(a} t_{b)\circ}^c$, $s \Delta t_{ab}$, and $s \nabla_{(a} \nabla \cdot t_{b)\circ}$ (here $\sigma = [g; s]$ and we work in the scale g). In a choice of coordinates (s, y^1, \dots, y^{n-1}) , we can write

$$\nabla_a = n_a \partial_s + \text{ltots},$$

where $n = ds$ and ltots denotes terms of lower transverse order. Thus, we have that

$$n_c \nabla_{(a} t_{b)\circ}^c = n_c n_{(a} \partial_s t_{b)\circ}^c + \text{ltots} \quad \text{and} \quad s \nabla_{(a} \nabla \cdot t_{b)\circ} = s n_{(a} \partial_s \nabla \cdot t_{b)\circ} + \text{ltots}.$$

Because σ is an asymptotic unit defining density, we have that $\nabla_n \circ n \stackrel{\Sigma}{=} n(\nabla_n + H)$ and so the operator $\delta_{\mathbb{R}}$ composed with the conormal n_a has transverse order zero. Therefore, only the terms ∇_n and $s\Delta$ in \mathbb{ID}_{σ} can contribute to the leading transverse derivatives in the operator $\delta_{d,w}^{(k)}$.

From the above, and again consulting Lemma 4.4, we conclude that

$$\delta_{d,w}^{(k)} = \mathring{\top} \circ \nabla_n \circ \prod_{i=0}^{k-2} [(d+2w-2i-6)\nabla_n - s\Delta] + \text{ltots}.$$

Because $\Delta = \nabla_n^2 + \text{ltots}$, we have that

$$\delta_{d,w}^{(2)} \stackrel{\Sigma}{=} \mathring{\top} \circ (d+2w-7)\nabla_n^2 + \text{ltots}.$$

We now proceed inductively to find the general coefficient of the leading transverse derivative term. Suppose that for $k \geq 3$,

$$\delta_{d,w}^{(k-1)} = \mathring{\top} \circ \left[\prod_{i=1}^{k-2} (d+2w-k-i-3) \right] \nabla_n^{k-1} + \text{ltots}.$$

Then, because

$$\delta_{d,w}^{(k)} = \delta_{d,w-1}^{(k-1)} \circ \mathbb{ID}_{\sigma},$$

we have that

$$\delta_{d,w}^{(k)} = \mathring{\top} \circ \left[\prod_{i=1}^{k-2} (d+2w-k-i-5) \right] \nabla_n^{k-1} \circ [(d+2w-6)\nabla_n - s\Delta] + \text{ltots}.$$

Hence, as required, we find

$$(4.5) \quad \delta_{d,w}^{(k)} = \mathring{\top} \circ \left[\prod_{i=1}^{k-1} (d+2w-k-i-4) \right] \nabla_n^k + \text{ltots}.$$

To find when $\delta_{d,w}^{(k)}$ has transverse order strictly less than k , we study when the leading coefficient in the above display vanishes. That is, we wish to find k and w that obey

$$(4.6) \quad d+2w-2k-3 \leq 0 \leq d+2w-k-5.$$

Because $k \geq 2$, we find that the right inequality implies that $w \geq \frac{7-d}{2}$. Further, the inequality above can be rewritten in terms of k to obtain $\frac{d+2w-3}{2} \leq k \leq d+2w-5$. Equation 4.6 has no solutions for w in the range $w < 2-d$ since that would require $d < -3$. Therefore, when $w < 2-d$, the transverse order of $\delta_{d,w}^{(k)}$ is k . The left inequality of 4.6 rules out $w > k+2$ so the only remaining case is $k-d < w < 3$, which, in combination with Equation 4.6, gives the ranges of k and w quoted in the Lemma. □

An asymptotic unit defining density σ only allows the construction of local invariants with transverse order at most $d-1$, so the highest fundamental form we could determine from it is the d th fundamental form. In view of the above lemma, we next focus on fundamental forms built from σ of transverse order less than $\frac{d+1}{2}$.

Definition 4.8. Let $d \geq 3$ and let $2 \leq n < \frac{d+3}{2}$. The *canonical n th fundamental form* $\mathring{\mathbb{H}}$ is defined by

$$\mathring{\mathbb{H}} := \delta_{d,1}^{(n-2)} \mathring{\mathbb{H}}^e.$$

■

Corollary 4.9. *The canonical n th fundamental form is a fundamental form.*

Proof. From Equation 4.3, in a choice of scale $\sigma = [g; s]$, we have $\mathring{\mathbb{I}}_{ab}^e = \nabla_{(a}n_{b)\circ} + s\mathring{P}_{ab}$. Therefore, we have that $\nabla_n^k \mathring{\mathbb{I}}_{ab}^e \stackrel{\Sigma}{=} \nabla_n^k \nabla_{(a}n_{b)\circ} + k\nabla_n^{k-1} \mathring{P}_{ab}$ for any positive integer k . There exists a scale g for which s is a unit defining function for Σ (see, for example [33]); thus in what follows, we can assume that $|ds|^2 = 1$, so that $\nabla_n n_a = 0$. Therefore, we can write that

$$\begin{aligned} \nabla_n \nabla_a n_b &= R_{nabn} - (\nabla_a n^c)(\nabla_c n_b) \\ &= W_{nabn} - P_{ab} + 2n_{(a}P_{b)n} - g_{ab}P_{nn} + \text{ltots}. \end{aligned}$$

Applying the above display to $\nabla_n^k \mathring{\mathbb{I}}_{ab}^e$, we have

$$\nabla_n^k \mathring{\mathbb{I}}_{ab}^e \stackrel{\Sigma}{=} \nabla_n^{k-1} W_{nabn} + (k-1)\nabla_n^{k-1} \mathring{P}_{ab} + 2n_{(a} \nabla_n^{k-1} P_{b)n} + \text{ltots}.$$

To verify that the first two transverse order $k+1$ terms in the above do not cancel in general, suppose that (M, \mathbf{c}) is conformally flat: For generic Σ , the metric $g \in \mathbf{c}$ for which $|ds|^2 = 1$ has $P \neq 0$ which has transverse order 2. Because the Weyl tensor here vanishes, the combination $\nabla_n^{k-1} W_{nabn} + (k-1)\nabla_n^{k-1} \mathring{P}_{ab}$ has transverse order $k+1$; this must hold for general conformally-curved manifolds.

Using Lemma 4.7, if $n \in \mathbb{Z}$ satisfies $2 \leq n < \frac{d+3}{2}$, then

$$\delta_{d,1}^{(n-2)} = \alpha \mathring{\mathbb{T}} \circ \nabla_n^{n-2} + \text{ltots}$$

for some non-zero coefficient α . Thus, in a choice of scale where $|ds|^2 = 1$, we have

$$\mathring{\mathbb{I}}_{ab} = \alpha \mathring{\mathbb{T}} \circ \nabla_n^{n-2} \mathring{\mathbb{I}}_{ab}^e + \text{ltots} = \alpha \mathring{\mathbb{T}} \circ \nabla_n^{n-3} (W_{nabn} + (n-3)\mathring{P}_{ab}) + \text{ltots}$$

and so $\mathring{\mathbb{I}}$ has transverse order $n-1$. Further, by construction, $\mathring{\mathbb{I}}$ is a conformal tensor density of weight $3-n$. The corollary follows. \square

Remark 4.10. In dimensions $d=3,4$, the canonical n th fundamental form is defined for all $n \leq d-1$. In $d=4$ dimensions a well-defined fourth fundamental form also exists and was given in Equation (1.3). When $d > 3$, the operator $\delta_{d,1}^{(1)}$ can be used to compute

$$\mathring{\mathbb{I}}_{ab} := \delta_{d,1}^{(1)} \mathring{\mathbb{I}}_{ab}^e = -\mathring{\mathbb{I}}_{(ab)\circ}^2 + W_{\hat{n}ab\hat{n}}.$$

For $d > 5$, applying the operator $\delta_{d,1}^{(2)}$ gives

$$\begin{aligned} \mathring{\mathbb{I}}_{ab} := \delta_{d,1}^{(2)} \mathring{\mathbb{I}}_{ab}^e &= -(d-4)(d-5)C_{n(ab)}^\top - (d-4)(d-5)HW_{\hat{n}ab\hat{n}} - (d-4)\bar{\nabla}^c W_{c(ab)\hat{n}}^\top \\ &\quad + 2W_{c\hat{n}(a}\mathring{\mathbb{I}}_{b)\circ}^c + (d^2 - 7d + 18)\mathring{F}_{(a}\mathring{\mathbb{I}}_{b)\circ} + (d-6)\bar{W}_{ab}^c \mathring{\mathbb{I}}_{cd} \\ &\quad + \frac{d^3 - 10d^2 + 25d - 10}{(d-1)(d-2)} K \mathring{\mathbb{I}}_{ab}. \end{aligned}$$

The above computation was performed using the symbolic manipulation program FORM [51]. Documentation of our FORM code can be found in [10]. The above canonical fundamental forms have leading transverse derivative terms that are identical to the fundamental forms displayed in the introduction up to an overall non-zero coefficient. Also note that, in dimensions $d > 3$, the canonical third fundamental form $\mathring{\mathbb{I}}$ recovers the trace-free Fialkow tensor:

$$(4.7) \quad \mathring{\mathbb{I}} = -(d-3)\mathring{F}.$$

■

We now turn our attention to conditional fundamental forms for $d \geq 5$. This relies on hyperumbilicity. Corollary 4.9 lets us reframe hyperumbilicity in terms of canonical fundamental forms. In particular, a conformal embedding is hyperumbilic iff $\mathring{\mathbb{I}} = \dots = \mathring{\mathbb{I}}_{\underline{k}} = 0$, where $k = \lceil \frac{d+1}{2} \rceil$. We now study consequences of hyperumbilicity.

Our next task is to construct canonical criteria for conditional fundamental forms. We begin with technical lemmata.

Lemma 4.11. *Let $d \geq 3$, $k = \lceil \frac{d+1}{2} \rceil$, and $\Sigma \hookrightarrow (M^d, \mathbf{c})$ be a hyperumbilic conformal embedding with corresponding asymptotic unit defining density σ . Then, for all $0 \leq \ell \leq k-2$,*

$$\nabla_n^\ell \hat{\Pi}^e \stackrel{\Sigma}{=} 0.$$

Proof. We proceed by induction on ℓ . The $\ell = 0$ case is clear since hyperumbilicity implies that $\hat{\Pi} = 0$. Next, suppose that $\nabla_n^{\ell-1} \hat{\Pi}^e \stackrel{\Sigma}{=} 0$ for all $1 \leq \ell \leq k-2$. We then compute directly:

$$\begin{aligned} \nabla_n^\ell \hat{\Pi}_{ab}^e &\stackrel{\Sigma}{=} (\bar{g}_a^c + n_a n^c)(\bar{g}_b^d + n_b n^d) \nabla_n^\ell \hat{\Pi}_{cd}^e \\ &\stackrel{\Sigma}{=} \hat{\top} \circ \nabla_n^\ell \hat{\Pi}_{ab}^e + \frac{\bar{g}_{ab}}{d-1} \bar{g}^{cd} \nabla_n^\ell \hat{\Pi}_{cd}^e + 2n_{(a} (n \cdot \nabla_n^\ell \hat{\Pi}_{b)}^e)^\top + n_a n_b n \cdot (\nabla_n^\ell \hat{\Pi}^e) \cdot n \\ &\stackrel{\Sigma}{=} \hat{\top} \circ \nabla_n^\ell \hat{\Pi}_{ab}^e + 2n_{(a} (n \cdot \nabla_n^\ell \hat{\Pi}_{b)}^e)^\top + (n_a n_b - \frac{\bar{g}_{ab}}{d-1}) n \cdot (\nabla_n^\ell \hat{\Pi}^e) \cdot n, \end{aligned}$$

where the third line holds because $g^{ab} \hat{\Pi}_{ab}^e = 0$. By hyperumbilicity, we have that $\frac{\bar{g}_{ab}}{d-1} = \delta_{a,1}^{(\ell)} \hat{\Pi}^e = 0$ for $0 \leq \ell \leq k-2$. Consulting Equation 4.5, we see that $\hat{\top} \circ \nabla_n^\ell \hat{\Pi}^e|_\Sigma = 0$ for all $0 \leq \ell \leq k-2$ (the lower order terms in Equation 4.5 also vanish by hyperumbilicity). Thus, it only remains to show that $n^a \nabla_n^\ell \hat{\Pi}_{ab}^e \stackrel{\Sigma}{=} 0$.

By the inductive assumption and Leibniz rule, we also have that

$$n^a \nabla_n^\ell \hat{\Pi}_{ab}^e \stackrel{\Sigma}{=} \nabla_n^\ell (n^a \hat{\Pi}_{ab}^e).$$

Using the tractor identity $I^A P_{AB} = \frac{K X_B}{d-2}$ and that $P = q(\hat{\Pi}^e)$, we have $n^a \hat{\Pi}_{ab}^e = \frac{s \nabla \cdot \hat{\Pi}^e}{d-1}$ where σ is s in the scale g . Applying this identity to the above display, we have that

$$\begin{aligned} n^a \nabla_n^\ell \hat{\Pi}_{ab}^e &\stackrel{\Sigma}{=} \frac{1}{d-1} \nabla_n^\ell (s \nabla^a \hat{\Pi}_{ab}^e) \\ &\stackrel{\Sigma}{=} \frac{\ell}{d-1} \nabla_n^{\ell-1} \nabla^a \hat{\Pi}_{ab}^e \\ &\stackrel{\Sigma}{=} \frac{\ell}{d-1} \nabla^a \nabla_n^{\ell-1} \hat{\Pi}_{ab}^e \\ &\stackrel{\Sigma}{=} \frac{\ell}{d-1} (\nabla^\top{}^a \nabla_n^{\ell-1} \hat{\Pi}_{ab}^e + n^a \nabla_n^\ell \hat{\Pi}_{ab}^e) \\ &\stackrel{\Sigma}{=} \frac{\ell}{d-1} n^a \nabla_n^\ell \hat{\Pi}_{ab}^e, \end{aligned}$$

where the second, third, and fifth lines follow from the inductive assumption. Thus, we have that $n^a \nabla_n^\ell \hat{\Pi}_{ab}^e \stackrel{\Sigma}{=} 0$ so long as $\ell \neq d-1$, but $\ell \leq k-2 < d-1$, so the lemma follows. \square

Lemma 4.12. *Let $d \geq 5$, $k = \lceil \frac{d+1}{2} \rceil$, and $\Sigma \hookrightarrow (M^d, \mathbf{c})$ be a hyperumbilic conformal embedding with corresponding asymptotic unit defining density σ . Then,*

$$P_{AB} = \sigma^{k-1} Q_{AB} + \sigma^{k-2} X_{(A} T_{B)} + \sigma^{k-3} X_A X_B U,$$

where $Q \in \Gamma(\odot^2 \mathcal{T}M[-k]) \cap \ker X$, $T \in \Gamma(\mathcal{T}M[-k]) \cap \ker X$, and $U \in \Gamma(\mathcal{E}M[-k])$.

Proof. As usual P_{AB} denotes $\hat{D}_A \hat{D}_B \sigma$. From Lemma 4.11 we have that $\nabla_n^m \hat{\Pi}^e|_\Sigma = 0$ for all $0 \leq m \leq k-2$, so that $\hat{\Pi}^e = \sigma^{k-1} Q_{ab}$ for some $Q \in \Gamma(\odot^2 T^* M[2-k])$. Because $P = q(\hat{\Pi}^e) = q(\sigma^{k-1} Q)$, of interest is the difference $q(\sigma^{k-1} Q) - \sigma^{k-1} q(Q) = [q, \sigma^{k-1}] Q$. Note, because $k \leq \frac{d}{2} + 1$, $q(Q)$ is defined for $d \geq 5$, and from Lemma 2.8) it follows that, for any integer $\ell \geq 2$,

$$(4.8) \quad q(\sigma^\ell Q)_{AB} - \sigma^\ell q(Q)_{AB} = \sigma^{\ell-1} X_{(A} Z_{B)}^b t_b + X_A X_B \mathcal{O}(\sigma^{\ell-2}),$$

for some $t_b \in \Gamma(T^* M[1-k])$. Clearly $\sigma^{\ell-1} Z^b t_b = \sigma^{\ell-1} q(t) + X \mathcal{O}(\sigma^{\ell-1})$, so setting $\ell = k-1$, Equation 4.8 becomes

$$q(\sigma^{k-1} Q)_{AB} - \sigma^{k-1} q(Q)_{AB} = \sigma^{k-2} X_{(A} q(t)_{B)} + \sigma^{k-3} X_A X_B U,$$

for some $U \in \Gamma(\mathcal{E}M[-k])$. Defining $T_A := q(t)_A \in \Gamma(\mathcal{T}M[-k])$ and $Q_{AB} := q(Q)_{AB} \in \Gamma(\odot^2 \mathcal{T}M[-k])$, the lemma follows. \square

Remark 4.13. For hyperumbilic embeddings, $\mathring{\nabla}(\nabla_n^m \mathring{\Pi}^e) = 0$ for all $0 \leq m \leq k - 2$ (where $k = \lceil \frac{d+1}{2} \rceil$). Also, as shown in the proof of Lemma 4.9, the quantity $\mathring{\nabla}(\nabla_n^{k-1} \mathring{\Pi}^e)$ has transverse order k , so it follows from the above proof that the transverse order of $Q_{ab}|_\Sigma$ is k . ■

Armed with the above technical lemma, as well as Lemma 2.7, we can construct a set of canonical conditional fundamental forms.

Definition 4.14. Let $d \geq 5$ and $\Sigma \hookrightarrow (M^d, \mathbf{c})$ be a conformal embedding with corresponding asymptotic unit defining density σ . Further, let $\tau \in \mathcal{EM}[1]$ be any true scale. Then, for $\frac{d+3}{2} \leq n \leq d - 1$, define the n th canonical conditional fundamental form by

$$\mathring{\underline{\Pi}} := \bar{q}^* \circ \bar{r} \circ \mathring{\nabla} \circ (I \cdot D^{n-2}(P \log \tau) - \log \tau I \cdot D^{n-2}P) \in \Gamma(\odot_\circ^2 T^* \Sigma [3 - n]).$$

■

Remark 4.15. It follows that the expression $I \cdot D^{n-2}P \log \tau - \log \tau I \cdot D^{n-2}P$ in the above definition is a tractor by repeated application of Lemma 2.7. ■

Proposition 4.16. Let $\Sigma \hookrightarrow (M^d, \mathbf{c})$ with $d \geq 5$ be a hyperumbilic conformal embedding and let $k := \lceil \frac{d+1}{2} \rceil$. Then, for all $k + 1 \leq n \leq d - 1$, the n th canonical conditional fundamental form is a fundamental form.

Proof. We begin by showing that the transverse order of the n th canonical conditional fundamental form is $n - 1$. Recycling the computation in the proof of Lemma 4.7, but promoting the weight w to an operator (as necessary to act on log densities), we find that

$$I \cdot D^{n-2} \stackrel{\Sigma}{\cong} \nabla_n^{n-2} \circ \left[\prod_{i=1}^{n-2} (d + 2\underline{w} - n - i + 2) \right] + \text{ltots},$$

and in turn

$$I \cdot D^{n-2} \circ P_{AB} \stackrel{\Sigma}{\cong} \nabla_n^{n-2} \circ P_{AB} \circ \left[\prod_{i=1}^{n-2} (d + 2\underline{w} - n - i) \right] + \text{ltots} \circ P_{AB}.$$

Because $k + 1 \leq n \leq d - 1$, we have that $d - n - (n - 2) = d - 2n + 2 \leq 0$ and $d - n - 1 \geq 0$. Therefore the product above has \underline{w} as one of its factors, so we have that

$$\prod_{i=1}^{n-2} (d + 2\underline{w} - n - i) = \alpha \underline{w} + \mathcal{O}(\underline{w}^2),$$

for some $\alpha \neq 0$. So, remembering that $\mathring{\nabla}$ includes restriction to Σ and $\underline{w} \log \tau = 1$ while $\underline{w}^2 \log \tau = 0$, we can write

$$\mathring{\nabla} \circ I \cdot D^{n-2}(P_{AB} \log \tau) = \alpha \mathring{\nabla}(\nabla_n^{n-2} P_{AB}) + \text{ltots}(P_{AB} \log \tau).$$

Further, again using $\underline{w} 1 = 0$, we have that $\log \tau I \cdot D^{n-2}P_{AB} = \log \tau \text{ltots}(P_{AB})$. Note that $P = q(\mathring{\Pi}^e)$ and, from the proof of Corollary 4.9, we have $\mathring{\nabla}(\nabla_n^m \mathring{\Pi}^e)$ has transverse order $m + 1$. Hence, because $Z_a^A Z_b^B (\mathring{\nabla} \circ I \cdot D^{n-2})(P_{AB})$ has transverse degree $n - 1$ and the hypersurface-intrinsic operator $\bar{q}^* \circ \bar{r}$ acting on $(\mathring{\nabla} \circ I \cdot D^{n-2})(P_{AB})$ cannot change its transverse order, we conclude that the n th canonical conditional fundamental form has transverse order $n - 1$.

Finally we need to show that the n th canonical conditional fundamental form is independent of τ . Suppose that Σ is embedded hyperumbilically and let $\ell := n - k - 1$. Note that $0 \leq \ell \leq d - k - 2$. Then, from Lemma 4.12, we have that

$$\mathring{\underline{\Pi}} := (\bar{q}^* \circ \bar{r} \circ \mathring{\nabla})(\Pi_n),$$

where

$$\begin{aligned} \Pi_n := & I \cdot D^{\ell+(k-1)} \left([\sigma^{k-1} Q_{AB} + \sigma^{k-2} X_{(A} T_{B)} + \sigma^{k-3} X_A X_B U] \log \tau \right) \\ & - \log \tau I \cdot D^{\ell+(k-1)} \left(\sigma^{k-1} Q_{AB} + \sigma^{k-2} X_{(A} T_{B)} + \sigma^{k-3} X_A X_B U \right). \end{aligned}$$

Employing a quadratic Casimir of the $\mathfrak{sl}(2)$ algebra,

$$4yx + 2\mathfrak{h} + \mathfrak{h}^2 = 4xy - 2\mathfrak{h} + \mathfrak{h}^2 \stackrel{\Sigma}{=} \mathfrak{h}(\mathfrak{h} - 2),$$

we find the enveloping algebra recursion relation

$$y^{\ell+m+1}x^{m+1} \stackrel{\Sigma}{=} -y^{\ell+m}x^m(\ell+m+1)(\mathfrak{h}+m-\ell),$$

which can be solved to yield (for any non-negative integer m)

$$(4.9) \quad y^{\ell+m}x^m \stackrel{\Sigma}{=} (-1)^m y^\ell \prod_{i=1}^m (\ell+i)(\mathfrak{h}-\ell+i-1).$$

Note that when $m = 0$ in the above display, our convention is to define the product to be 1. In the $\mathfrak{sl}(2)$ notations of Lemma 3.16 we now have

$$\begin{aligned} \Pi_n = & -(-1)^n \log \tau \left[y^{\ell+(k-1)}x^{k-1}Q_{AB} \right. \\ & \left. + y^{\ell+1+(k-2)}x^{k-2}(X_{(A}T_B) + y^{\ell+2+(k-3)}x^{k-3}(X_A X_B U) \right] \\ & + (-1)^n \left[y^{\ell+(k-1)}x^{k-1}(Q_{AB} \log \tau) \right. \\ & \left. + y^{\ell+1+(k-2)}x^{k-2}(X_{(A}T_B) \log \tau) + y^{\ell+2+(k-3)}x^{k-3}(X_A X_B U \log \tau) \right]. \end{aligned}$$

We define the polynomials

$$(4.10) \quad F_{\ell,w,i}(u) := (\ell+i)(u+2w-\ell+i-1),$$

which obey $F_{\ell+1,w+1,i}(u) = F_{\ell,w,i+1}(u)$. Then, using Equation 4.9 and the fact that Q , $X \odot T$ and $X^2 U$ have weights $-k$, $1-k$, and $2-k$, respectively, we have that:

$$\begin{aligned} y^{\ell+(k-1)}x^{k-1} \circ Q_{AB} & \stackrel{\Sigma}{=} (-1)^{k-1} y^\ell \circ Q_{AB} \circ \prod_{i=1}^{k-1} F_{\ell,-k,i}(\mathfrak{h}), \\ y^{\ell+1+(k-2)}x^{k-2} \circ X_{(A}T_B) & \stackrel{\Sigma}{=} (-1)^{k-2} y^{\ell+1} \circ X_{(A}T_B) \circ \prod_{i=1}^{k-2} F_{\ell+1,1-k,i}(\mathfrak{h}) \\ & \stackrel{\Sigma}{=} (-1)^{k-2} y^{\ell+1} \circ X_{(A}T_B) \circ \prod_{i=2}^{k-1} F_{\ell,-k,i}(\mathfrak{h}), \\ y^{\ell+2+(k-3)}x^{k-3} \circ X_A X_B U & \stackrel{\Sigma}{=} (-1)^{k-3} y^{\ell+2} \circ X_A X_B U \circ \prod_{i=1}^{k-3} F_{\ell+2,2-k,i}(\mathfrak{h}) \\ & \stackrel{\Sigma}{=} (-1)^{k-1} y^{\ell+2} \circ X_A X_B U \circ \prod_{i=3}^{k-1} F_{\ell,-k,i}(\mathfrak{h}). \end{aligned}$$

Defining the polynomial

$$f_j(u) := \prod_{i=j}^{k-1} (\ell+i)(u-2k-\ell+i-1) = \prod_{i=j}^{k-1} F_{\ell,-k,i}(u),$$

and remembering that $\mathfrak{h}(1) = d$, we may rewrite Π_n :

$$\begin{aligned}
(4.11) \quad \Pi_n = & -(-1)^{n+k-1} \log \tau \left\{ f_1(d) y^\ell Q_{AB} \right. \\
& \left. - f_2(d) y^{\ell+1} (X_{(A} T_B) + f_3(d) y^{\ell+2} (X_A X_B U) \right\} \\
& + (-1)^{n+k-1} \left\{ y^\ell [Q_{AB} f_1(\mathbf{h})(\log \tau)] \right. \\
& \left. - y^{\ell+1} [X_{(A} T_B) f_2(\mathbf{h})(\log \tau)] + y^{\ell+2} [X_A X_B U f_3(\mathbf{h})(\log \tau)] \right\}.
\end{aligned}$$

Note that

$$(4.12) \quad f_2(u) = F_{\ell, -k, 2}(u) f_3(u) \quad \text{and} \quad f_1(u) = F_{\ell, -k, 1}(u) F_{\ell, -k, 2}(u) f_3(u).$$

Now, for any polynomial f for which d is a root, the operator $f(\mathbf{h}) = f(d + 2\underline{w})$ obeys

$$f(\mathbf{h}) = 2f'(d)\underline{w} + \mathcal{O}(\underline{w}^2),$$

and so

$$f(\mathbf{h}) \log \tau = 2f'(d).$$

Thus, if $f_3(d) = 0$ then $f_1(d) = 0 = f_2(d)$ and moreover $f_1(\mathbf{h}) \log \tau$, $f_2(\mathbf{h}) \log \tau$, and $f_3(\mathbf{h}) \log \tau$ are independent of τ and hence Π_n would be independent of τ .

To establish τ independence, we analyze three cases. In the first two cases, we determine for which choices of n we have that $f_3(d) = 0$, the result of which depends on dimension parity. In the third case, we handle all of the choices of n excluded from the analysis in the first two cases. Note that, because $d \geq 5$, $k := \lceil \frac{d+1}{2} \rceil$, and the lemma is only concerned with n satisfying $k+1 \leq n \leq d-1$, we have that $k \geq 3$ so $n \geq 4$. Thus, when $k = 3$ we must have $d = 5$ and $n = 4$. We begin with the first, even dimension parity, case.

Case 1: If $k \geq 4$ and $d \geq 6$ is even, then \underline{n} is independent of τ when $k+1 \leq n \leq d-1$:

To see this first observe that the polynomial f_3 satisfies $f_3(d) = 0$ when

$$d - 2k - \ell + 2 \leq 0 \leq d - k - \ell - 2.$$

Using that $\ell := n - k - 1$, the above display implies that $d - k + 3 \leq n$. From the hypothesis, we only consider $n \leq d - 1$, so we have that $k \geq 4$. Rewriting the inequality above as a condition on n , we have

$$(4.13) \quad d - k + 3 \leq n \leq d - 1.$$

Using that d is even we have $d = 2k - 2$, and so Equation 4.13 becomes

$$k + 1 \leq n \leq d - 1,$$

as required.

Case 2: If $k \geq 4$ and $d \geq 7$ is odd, then \underline{n} is independent of τ when $k+2 \leq n \leq d-1$:

Because d is odd, $d = 2k - 1$. Then, Equation 4.13 becomes

$$k + 2 \leq n \leq d - 1,$$

as required.

So far we have dealt neither with the case $k = 3$ nor the case $n = k + 1 \geq 5$ and d odd. These are both encompassed by studying $n = k + 1 \geq 4$ and d odd.

Case 3: If $n = k + 1$ and d is odd, then \underline{n} is independent of τ :

Because d is odd, we have that $d = 2k - 1$. Because $n = k + 1$, we have that $\ell = 0$. Moreover (see Equation (4.10))

$$F_{0, -k, 2}(d) = 0 \text{ and } F_{0, -k, 2}(d + 2\underline{w}) = 4\underline{w}.$$

Thus, from Equations 4.11 and 4.12 and remembering that $\underline{w} \log \tau = 1$, we have that the only terms in Π_n that depend on τ are the terms proportional to $y^2 X_A X_B U$. Because $y^2 \propto I \cdot \hat{D}^2$, we can employ Equation (2.8) and Proposition 2.4 to see that

$$y^2 \circ X_A X_B \stackrel{\Sigma}{=} \mathcal{E}(X) + \mathcal{E}(I) + \mathcal{E}(h).$$

But, because $\overset{\circ}{\Pi} = \bar{q}^* \circ \bar{r} \circ \overset{\circ}{\Gamma}(\Pi_n)$ and

$$(\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\Gamma})(t_{(ab)} Z_A^a Z_B^b + \alpha X_{(A} V_{B)} + \beta I_{(A} V'_{B)} + \gamma h_{AB} S) = (\bar{q}^* \circ \bar{r} \circ \overset{\circ}{\Gamma})(t_{(ab)} Z_A^a Z_B^b),$$

for any non-vanishing $t_{ab} \in \Gamma(\odot^2 T^* M[w+2])$, $V \in \Gamma(\mathcal{T}M[w-1])$, $V' \in \Gamma(\mathcal{T}M[w])$, and $S \in \Gamma(\mathcal{E}M[w])$ for generic w , as well as any coefficients α, β, γ , we have that $\overset{\circ}{\Pi}$ is independent of τ when $n = k+1$ and d odd.

The above three cases prove that, for all $d \geq 5$ and n satisfying $k+1 \leq n \leq d-1$, the n th canonical conditional fundamental form $\overset{\circ}{\Pi}$ is independent of τ . \square

We have already seen that a conformal embedding $\Sigma \hookrightarrow (M^d, \mathbf{e})$ is hyperumbilic when all its canonical fundamental forms vanish. The above proposition in fact establishes that a hyperumbilic conformal embedding $\Sigma \hookrightarrow (M^d, \mathbf{e})$ is überumbilic when all its canonical conditional fundamental forms vanish.

Example 4.17. We can use Proposition 4.16 to compute $\overset{\circ}{IV}$ in $d=5$ for hyperumbilic embeddings (so $\overset{\circ}{II} = \overset{\circ}{III} = 0$) and find

$$\overset{\circ}{IV}_{ab} \stackrel{\Sigma_{\text{hyp}}}{=} 2C_{\hat{n}(ab)}^{\top}.$$

This computation relies on the expression for $I \cdot D^2$ in terms of the tractor connection and weight operators, which can be found in [31]. \blacksquare

Remark 4.18. Notice that the above fourth canonical conditional fundamental form and the fourth conditional fundamental form provided in Equation (1.7) of the introduction differ because they have differing invariance criteria; the latter requires only umbilicity. \blacksquare

We require a corollary of Lemma 4.11 before we prove our main result:

Corollary 4.19. *Let $d \geq 3$ and $\Sigma \hookrightarrow (M^d, \mathbf{e})$ be an überumbilic conformal embedding with corresponding asymptotic unit defining density σ . Then, for all $0 \leq \ell \leq d-3$,*

$$\nabla_n^\ell \overset{\circ}{\Pi}^e \stackrel{\Sigma}{=} 0.$$

Proof. In $d=3$ dimensions, überumbilicity is equivalent to umbilicity and thus the corollary holds trivially. When $d=4$, überumbilicity is equivalent to hyperumbilicity, and thus the result follows from Lemma 4.11. In dimensions $d \geq 5$, the proof follows that of Lemma 4.11 *mutatis mutandis*. \square

We can now prove our main result.

Proof of Theorem 1.8. In the interior of M , the trace-free Schouten tensor of the singular metric $g^\circ = s^{-2}g$, determined by the conformal density $\sigma = [g; s]$, obeys [6]

$$\sigma \overset{\circ}{P}^{g^\circ} = q^*(\nabla I_\sigma),$$

and this condition extends to the boundary Σ . Thus the asymptotic Poincaré–Einstein Condition (1.1) holds exactly when

$$q^*(\nabla I_\sigma) = \mathcal{O}(\sigma^{d-2}).$$

Because $\overset{\circ}{\Pi}^e = q^*(\nabla I)$, it only remains to establish that Σ is überumbilic iff $\overset{\circ}{\Pi}^e = \sigma^{d-2}T$ for some $T \in \Gamma(\odot^2 T^* M[3-d])$. Clearly if $\overset{\circ}{\Pi}^e = \sigma^{d-2}T$, then Σ is überumbilic. The converse follows from Corollary 4.19. \square

Remark 4.18 provides an example of a conditional higher fundamental forms that exists when the hyperumbilicity condition is weakened. Moreover, both the fourth and fifth fundamental forms given in the introduction exist in all even dimensions $d \geq 4$ without the supposition of any conditions whatsoever. The dimension parity dichotomy between Cases 1 and 2 in the proof of Proposition 4.16 also suggests that even dimensions may be more amenable for constructing invariant fundamental forms. Rather than searching for higher fundamental forms (with no invariance conditions) in even dimensions, one can consider weakened conditions in any dimension. In fact, when $d = 5$, a conditional fourth fundamental form exists if the embedding is umbilic, while when $d = 7$, a conditional fifth fundamental form exists if both $\mathring{\mathbb{I}}$ and $\mathring{\mathbb{III}}$ vanish. Existence results for higher conditional forms can be established for weakened analogs of the hyperumbilicity condition so long as the following hypothesis holds:

Hypothesis 4.20. *Let $\Sigma \hookrightarrow (M^d, \mathbf{c})$ be a conformal embedding with corresponding asymptotic unit defining density σ such that*

$$P_{AB} = \sigma^{k-1} Q_{AB},$$

where $Q \in \Gamma(\odot^2 \mathcal{T}M[-k]) \cap \ker X$ and the integer k obeys $1 \leq k \leq d-1$.

Proposition 4.21. *Let $d \geq 5$ and $\Sigma \hookrightarrow (M^d, \mathbf{c})$ be a conformal embedding subject to Hypothesis 4.20 for some k . Then for all $d-k+1 \leq n \leq d-1$, the n th canonical conditional fundamental form is a fundamental form.*

Proof. The proof follows that of Proposition 4.16 *mutatis mutandis*. \square

Remark 4.22. Hypothesis 4.20 is a significant strengthening of Lemma 4.12 but, for $k \leq 3$, is provably true by direct computation so long as $\mathring{\mathbb{I}} = \dots = \mathring{\mathbb{K}} = 0$. \blacksquare

From the statement of this proposition, when $d = 5$, we see that to construct $\mathring{\mathbb{IV}}$, we only need $k = 2$ and in turn need only require that $\mathring{\mathbb{I}} = 0$, as stated above. Similarly, for $d = 7$, we see that to construct $\mathring{\mathbb{V}}$, we need only that $k = 3$ and so we only require that $\mathring{\mathbb{I}} = \mathring{\mathbb{III}} = 0$.

4.4. Tractor Fundamental Forms. In this section, we give various identities relating tractor fundamental forms and bulk tractors. These are particularly useful for any holographic computation involving extrinsic conformal embedding data.

A canonical tractor n th fundamental form is given by $\bar{q}(\mathring{\mathbb{N}})$ in dimensions $d > n+1$. The tractor second fundamental form $L := \bar{q}(\mathring{\mathbb{II}})$ is defined for $d > 3$, and its holographic formula was given in Equation (4.2). A holographic formula for the canonical tractor third fundamental form, valid when $7 \neq d > 5$, is

$$(4.14) \quad \bar{q}(\mathring{\mathbb{III}}) = \mathring{P}_{AB}^t - \frac{2}{(d-3)(d-5)} X_{(A} \bar{D} \cdot \mathring{P}_{B)}^t + \frac{1}{(d-3)(d-4)(d-5)} X_A X_B \bar{D} \cdot \hat{D} \cdot \mathring{P}^t.$$

Here $\mathring{P}_{AB} := I \cdot \hat{D} P_{AB}$ and $\mathring{P}_{AB}^t := \bar{r} \circ \mathring{\Upsilon}(P_{AB})$; the above result is a direct application of Lemma 2.9 and the definition of $\mathring{\mathbb{III}}$ in Definition 4.8. Just as the Fialkow tensor is related to the canonical third fundamental form by a factor $-(d-3)$, see Equation (4.7), the Fialkow tractor and the canonical tractor third fundamental form obey

$$\bar{q}(\mathring{\mathbb{III}})_{AB} = -(d-3)F_{AB}.$$

The above relationship between F and $\bar{q}(\mathring{\mathbb{III}})$ and Equation (4.14) yield a corollary to Corollary 3.7 that gives an analog of the Fialkow–Gauß equation (3.5):

Corollary 4.23 (Fialkow–Gauß–Thomas Equation). *Let $7 \neq d > 5$ and σ be an asymptotic unit conformal defining density for Σ . Then,*

$$\begin{aligned} & \left(L_A^C L_{CB} - \frac{1}{d-1} K \bar{h}_{AB} - W_{NABN} \right) - \frac{1}{d-1} X_{(A} \hat{D}_{B)} K - \frac{1}{(d-4)(d-5)} X_A X_B U \\ & = -\mathring{P}_{AB}^t + \frac{2}{(d-3)(d-5)} X_{(A} \bar{D} \cdot \mathring{P}_{B)}^t - \frac{1}{(d-3)(d-4)(d-5)} X_A X_B \bar{D} \cdot \hat{D} \cdot \mathring{P}^t = (d-3)F_{AB}, \end{aligned}$$

where $U \in \Gamma(\mathcal{E}\Sigma[-4])$ is the density given in Equation (3.9).

The rigidity density $K = \mathring{\Pi}^2$ is an important conformal hypersurface invariant that measures the difference between ambient and hypersurface scalar curvatures:

$$K \stackrel{\Sigma}{=} 2(d-2)(J - P(\hat{n}, \hat{n}) - \bar{J} + \frac{d-1}{2}H^2).$$

The above follows by tracing the Fialkow–Gauß equation (3.5). Since we have canonical extension $\mathring{\Pi}^e$ of $\mathring{\Pi}$, the same follows for K . There are many situations in which normal derivatives of K^e are needed.

Definition 4.24. Let $K^e := P_{AB}P^{AB}$ as in the proof of Lemma 3.3, and

$$\begin{aligned} \dot{K} &:= \delta_R K^e \in \Gamma(\mathcal{E}\Sigma[-3]), \\ \ddot{K} &:= \delta_R I \cdot \hat{D} K^e \in \Gamma(\mathcal{E}\Sigma[-4]), \quad d \neq 6, \\ \ddot{\ddot{K}} &:= \delta_R I \cdot \hat{D}^2 K^e \in \Gamma(\mathcal{E}\Sigma[-5]), \quad d \neq 6, 8, \\ &\vdots \end{aligned}$$

■

In particular, because $P_{AB}P^{AB} = (\mathring{\Pi}^e)^2$, there are (rather useful) formulæ for \dot{K} , \ddot{K} , and $\ddot{\ddot{K}}$ in terms of the fundamental forms $\mathring{\Pi}$, $\mathring{\mathbb{I}}\mathring{\mathbb{I}}$, $\mathring{\mathbb{I}}\mathring{\mathbb{V}}$, $\mathring{\mathbb{V}}$, and hypersurface derivatives thereof. For this we introduce the following notational device.

Definition 4.25. Let

$$\Pi_{(2)} := \mathring{\mathbb{T}} \circ q(\mathring{\Pi}^e) \in \Gamma(\odot_{\circ}^2 \mathcal{T}\Sigma[-1]),$$

and, for $3 \leq m < d$ such that $m \notin \{\frac{d+1}{2}, \frac{d+3}{2}, \frac{d+5}{2}\}$, let

$$\Pi_{(m)} := (\bar{r} \circ \mathring{\mathbb{T}} \circ \delta_R \circ q) \circ \mathbb{D}_{\sigma}^{m-3}(\mathring{\Pi}^e) \in \Gamma(\odot_{\circ}^2 \mathcal{T}\Sigma[1-m]).$$

When $3 \leq m < d$ we define

$$\tilde{\Pi}_{(m)} := \bar{q}(\underline{\mathring{m}}).$$

■

Remark 4.26. The values $\{\frac{d+1}{2}, \frac{d+3}{2}, \frac{d+5}{2}\}$ are treated on a separate footing in Definition 4.25 for reasons of definedness of the operator \bar{r} ; see Lemma 2.11. Also, in dimensions d such that $m \notin \{\frac{d+1}{2}, \frac{d+3}{2}, \frac{d+5}{2}\}$, by construction we have that $\tilde{\Pi}_{(m)} = \Pi_{(m)} + \mathcal{E}(X)$, since $\tilde{\Pi}_{(m)} = (\bar{q} \circ \bar{q}^*)(\Pi_{(m)})$ and Lemma 2.9 says $\bar{q} \circ \bar{q}^* = \text{Id} + \mathcal{E}(X)$. So, when $m \in \{\frac{d+1}{2}, \frac{d+3}{2}, \frac{d+5}{2}\}$, if $T \in \Gamma(\odot^2 \mathcal{T}M[w]) \cap \ker X_{\perp}$, we may define $\Pi_{(m)}^{AB} T_{AB} := \tilde{\Pi}_{(m)}^{AB} T_{AB}$. ■

By construction, the rank two, trace-free hypersurface tractors $\Pi_{(m)}$ produce the corresponding fundamental form $\underline{\mathring{m}}$ upon acting by the extraction map \bar{q}^* . In general, if $\bar{q}^*(T^{AB}) = t_{ab}$ and $\bar{q}^*(U^{AB}) = u_{ab}$, we have that $t_{ab}u^{ab} = T_{AB}U^{AB}$. For this reason, the tractors $\Pi_{(m)}$ can be used to compute holographic formulæ for scalars built from contractions of fundamental forms. These formulæ are simpler than those for their constituent fundamental forms and are therefore particularly useful for computations of scalar densities, such as integrands for Willmore-like energies (see for example [31, 48]). We now give two such results.

Lemma 4.27. *Let $d > 4$, then the square of the third fundamental form has a holographic formula given by*

$$\mathring{\mathbb{I}}\mathring{\mathbb{I}}^2 = \dot{P}_{AB}\dot{P}^{AB}|_{\Sigma} - \frac{3d-2}{(d-1)(d-2)^2}K^2.$$

Proof. It follows from Definition 4.25 that $\mathring{\mathbb{I}}\mathring{\mathbb{I}}^2 = \Pi_{(3)AB}\Pi_{(3)}^{AB}$. Because the only appearance of $\Pi_{(3)}$ in this proof is when it is squared, any instances where $\tilde{\Pi}_{(3)}$ would be required can be replaced with $\Pi_{(3)}$. So, the proof amounts to relating \dot{P} to $\Pi_{(3)}$. As previously noted, $q(\mathring{\Pi}^e) = P$,

so $\Pi_{(3)}^{AB} = \bar{r} \circ \hat{\top}(\dot{P})$. In order to relate $\Pi_{(3)}$ to $\dot{P}|_{\Sigma}$ explicitly, following Lemma 2.11 specialized to hypersurface tractors, we need to rewrite $X_A \dot{P}^{AB}$:

$$\begin{aligned}
X_A \dot{P}^{AB} &= I \cdot \hat{D} X_A P^{AB} - I_A P^{AB} + \frac{2\sigma}{d-2} \hat{D}_A P^{AB} \\
&= -I_A \hat{D}_B I^A \\
(4.15) \quad &= -\left(\frac{1}{2} \hat{D}_B I^2 + \frac{X_B}{d-2} (\hat{D}I)^2\right) \\
&= -\frac{K X_B}{d-2} + \mathcal{O}(\sigma^{d-2}),
\end{aligned}$$

where the first and third lines are results of Proposition 2.4, the second line results from the properties of P , and the last line uses the definition of K . Further, observe that via Proposition 2.4, we have

$$(4.16) \quad I \cdot \dot{P}^B = \frac{1}{d-2} (\dot{K} X^B + K I^B) - \frac{2\sigma}{d-4} \left(\frac{1}{d-2} \hat{D}^B K - P^{AC} \hat{D}_C P_A^B \right) + \mathcal{O}(\sigma^{d-3}) \stackrel{\Sigma}{=} \frac{1}{d-2} (\dot{K} X + K N).$$

Using the above identities, the definition of \bar{r} and $\hat{\top}$, as well as the standard operator identity for $\hat{D} \circ X$, a tedious calculation along Σ yields

$$\Pi_{(3)AB} \stackrel{\Sigma}{=} \dot{P}_{AB} - \frac{d}{(d-1)(d-2)} X_{(A} \hat{D}_{B)} K - \frac{2}{d-2} \dot{K} I_{(A} X_{B)} - \frac{1}{d-2} K I_A I_B - \frac{1}{(d-1)(d-2)} K I_{AB}.$$

Squaring this identity gives the quoted result. \square

Lemma 4.28. *Let $d > 6$, then the product of the second and fourth fundamental forms has a holographic formula given by*

$$\hat{\Pi} \cdot \hat{\mathbb{V}} \stackrel{\Sigma}{=} (d-4) \left(P_{AB} \dot{P}^{AB} + \frac{4}{(d-2)^2} K^2 \right).$$

Proof. First, because $d > 5$, we see that $\hat{\mathbb{V}}$ is a canonical fundamental form (so not conditional). As in the previous lemma, we note that $\hat{\Pi} \cdot \hat{\mathbb{V}} = \Pi_{(2)AB} \Pi_{(4)}^{AB}$. Moreover, Equation (4.1) implies that $N^A P_{AB} = \frac{1}{d-2} K X_B$ so

$$\Pi_{(2)} = P|_{\Sigma} - \frac{2K}{d-2} N \odot X.$$

Thus we are tasked with computing $P_{AB} \Pi_{(4)}^{AB}$ along Σ . Remembering that $X \cdot P = 0$, it is sufficient to compute $\Pi_{(4)}^{AB}$ modulo terms proportional to X . Using Definition 4.25, we compute $\Pi_{(4)}$ in steps. Recall, from Equation (4.4), that

$$\Pi_{(4)} = (d-4) (\bar{r} \circ \hat{\top} \circ \delta_R \circ q \circ q^* \circ r \circ I \cdot \hat{D} \circ q) (\hat{\Pi}^e).$$

We outline this calculation proceeding from right to left in this sequence of operators.

First, as shown previously, $q(\hat{\Pi}^e) = P$, so $I \cdot \hat{D} q(\hat{\Pi}^e) = \dot{P}$. Therefore, using Lemma 2.11, we next compute $r(\dot{P})$:

$$r(\dot{P})_{AB} = \dot{P}_{AB} - \frac{d+2}{4d(d-2)} X_{(A} \hat{D}_{B)} K - \frac{2}{d(d-2)} h_{AB} K + \mathcal{O}(\sigma^{d-2}) + \mathcal{O}(\sigma^{d-4}) X_{(A} T_{B)},$$

for some tractor T_B . Here we used Equation (4.15) and the oft-used operator identity for $\hat{D} \circ X$ given in Equation (2.8).

Next, we need to compute $(q \circ q^* \circ r)(\dot{P})$. Before we continue, we consider the operators that come next: We are only interested in the $\bar{Z}_A \bar{Z}_B$ component of the tractor $\Pi_{(4)}$, so we can ignore terms proportional to X in $(\hat{\top} \circ \delta_R \circ q \circ q^* \circ r)(\dot{P})$ when doing this computation. Further, we can ignore terms proportional to I_A when computing $(\delta_R \circ q \circ q^* \circ r)(\dot{P})$ because these terms are projected out by $\hat{\top}$. The various projections, therefore, amount to ignoring terms proportional to X or I when computing $(q \circ q^* \circ r)(\dot{P})$, because $\delta_R \circ I \stackrel{\Sigma}{=} \mathcal{E}(X) + \mathcal{E}(I)$ and $\delta_R \circ X \stackrel{\Sigma}{=} \mathcal{E}(X) + \mathcal{E}(I)$. From Lemma 2.9, $(q \circ q^* \circ r)(\dot{P}) - r(\dot{P}) = \mathcal{E}(X)$, so

$$(q \circ q^* \circ r)(\dot{P}) = \dot{P} - \frac{2}{d(d-2)} K h + \mathcal{E}(X) + \mathcal{E}(I) + \mathcal{O}(\sigma^{d-2}).$$

Next, note that $(\delta_R \circ q \circ q^* \circ r)(\dot{P}) = \ddot{P} - \frac{2}{d(d-2)} \dot{K}h + \mathcal{E}(X) + \mathcal{E}(I) + \mathcal{O}(\sigma^{d-3})$, and we can apply the operator $\overset{\circ}{\mathbb{T}}$ to obtain

$$(\overset{\circ}{\mathbb{T}} \circ \delta_R \circ q \circ q^* \circ r)(\dot{P}) = \overset{\circ}{\mathbb{T}}(\ddot{P}) + \mathcal{E}(X).$$

Next, it is useful to note that $\overset{\circ}{\mathbb{T}}(\ddot{P}_{AB}) \overset{\Sigma}{\cong} I_A^{A'} I_B^{B'} \ddot{P}_{A'B'} + I_{AB}U$ for some $U \in \Gamma(\mathcal{E}\Sigma[-3])$, and also that $I_{AB}P^{AB} \overset{\Sigma}{\cong} 0$. Thus, finishing the calculation amounts to computing

$$(4.17) \quad P^{AB} \bar{r}(I_A^{A'} I_B^{B'} \ddot{P}_{A'B'} + I_{AB}U).$$

For this, we need the identity

$$X^A \ddot{P}_{AB} \overset{\Sigma}{\cong} -\frac{2}{d-2} (\dot{K}X_B + KN_B),$$

which is derived from Proposition 2.4, Equation (2.8), and Equation (4.16). Because we are contracting on P , any terms proportional to X or the tractor first fundamental form produced by \bar{r} in Equation 4.17 can be discarded. Hence, again consulting Lemma 2.11, we find that

$$\begin{aligned} \Pi_{(2)AB} \Pi_{(4)}^{AB} &\overset{\Sigma}{\cong} (d-4) P_{AB} I_A^A I_B^B \ddot{P}_{A'B'} \\ &\overset{\Sigma}{\cong} (d-4) (P_{AB} - \frac{2}{d-2} KN_{(A} X_{B)}) \ddot{P}^{AB} \\ &\overset{\Sigma}{\cong} (d-4) (P_{AB} \ddot{P}^{AB} + \frac{4}{(d-2)^2} K^2), \end{aligned}$$

where the first equality is a result of the identity $I_{AB}P^{AB} \overset{\Sigma}{\cong} 0$, the second equality is an application of Equation (4.15) to yield an identity for $I \cdot P$, and the last equality is a consequence of the display above expressing $X \cdot \ddot{P}$. \square

One more technical lemma is necessary in order to produce formulæ for \dot{K} and \ddot{K} in terms of the canonical fundamental forms.

Lemma 4.29. *Let $d > 5$. Then,*

$$\begin{aligned} (\hat{D}P)^2 &\overset{\Sigma}{\cong} (\hat{D}L)^2 + \hat{\mathbb{I}}\hat{\mathbb{I}}^2 + \frac{2}{(d-4)(d-5)} \hat{\mathbb{I}} \cdot \hat{\mathbb{I}}\hat{\mathbb{V}} - \frac{4(d-7)}{(d-3)(d-5)} \hat{\mathbb{I}} \cdot \hat{\mathbb{I}}\hat{\mathbb{I}} \cdot \hat{\mathbb{I}} + \frac{2(d-7)}{d-5} \hat{\mathbb{I}}^4 \\ &\quad - \frac{2(3d^3 - 34d^2 + 100d - 73)}{(d-1)(d-2)^2(d-5)} K^2. \end{aligned}$$

Proof. The proof is a tedious but straightforward application of Equation 4.2, Lemmas 1.3, 4.27, 4.28 and standard reorderings of tractor operators based on Proposition 2.4. \square

Employing these lemmas, we have formulæ for \dot{K} and \ddot{K} .

Proposition 4.30. *Let $d > 4$. Then,*

$$(4.18) \quad \dot{K} = 2\hat{\mathbb{I}} \cdot \hat{\mathbb{I}}\hat{\mathbb{I}}.$$

If $d > 6$,

$$(4.19) \quad \begin{aligned} \ddot{K} &\overset{\Sigma}{\cong} -\frac{2}{d-6} (\hat{D}L)^2 + \frac{2(d-7)}{d-6} \hat{\mathbb{I}}\hat{\mathbb{I}}^2 + \frac{2(d-7)}{(d-5)(d-6)} \hat{\mathbb{I}} \cdot \hat{\mathbb{I}}\hat{\mathbb{V}} + \frac{8(d-7)}{(d-3)(d-5)(d-6)} \hat{\mathbb{I}} \cdot \hat{\mathbb{I}}\hat{\mathbb{I}} \cdot \hat{\mathbb{I}} \\ &\quad - \frac{4(d-7)}{(d-5)(d-6)} \hat{\mathbb{I}}^4 + \frac{10d^3 - 110d^2 + 296d - 172}{(d-1)(d-2)^2(d-5)(d-6)} K^2. \end{aligned}$$

Moreover, if $d = 5$ then

$$(4.20) \quad \begin{aligned} \ddot{K} &\overset{\Sigma}{\cong} -4\hat{\mathbb{I}} \cdot \bar{\Delta}\hat{\mathbb{I}} + \frac{20}{3} \hat{\mathbb{I}} \cdot \bar{\nabla}\bar{\nabla} \cdot \hat{\mathbb{I}} + \frac{8}{9} (\bar{\nabla} \cdot \hat{\mathbb{I}})^2 + \bar{\Delta}K \\ &\quad - 4\hat{\mathbb{I}} \cdot C_n^\top + 20\hat{\mathbb{I}}^2 \cdot \bar{P} + 2\bar{J}K - 4H\hat{\mathbb{I}}^3 - 4H\hat{\mathbb{I}} \cdot \hat{\mathbb{I}}\hat{\mathbb{I}} \\ &\quad + 4\hat{\mathbb{I}}\hat{\mathbb{I}} \cdot \hat{\mathbb{I}}\hat{\mathbb{I}} - 2\hat{\mathbb{I}}^2 \cdot \hat{\mathbb{I}}\hat{\mathbb{I}} + \frac{31}{18} K^2 + 8\hat{\mathbb{I}}^{ad} \hat{\mathbb{I}}^{bc} \bar{W}_{abcd}. \end{aligned}$$

Proof. We first prove that $\dot{K} = 2\dot{\Pi} \cdot \dot{\text{III}}$ (note that a proof was already given in [31]). To do so, consider the product $\dot{\Pi} \cdot \dot{\text{III}}$. By inserting these fundamental forms into tractors, we find that $\dot{\Pi} \cdot \dot{\text{III}} = L \cdot \bar{q}(\dot{\text{III}})$. From Equation (4.14) and the fact that $X \cdot L = 0$, we have that $\dot{\Pi} \cdot \dot{\text{III}} = L \cdot \dot{P}^t$. Further, $L \stackrel{\Sigma}{=} P + \mathcal{E}(X) + \mathcal{E}(N)$ and $X \cdot \dot{P}^t = 0 = N \cdot \dot{P}^t$, so $\dot{\Pi} \cdot \dot{\text{III}} \stackrel{\Sigma}{=} P \cdot \dot{P}^t$. By definition, $\dot{P}^t = \bar{r}(I_A^{A'} I_B^{B'} \dot{P}_{A'B'} + I_{AB} U)$ for some $U \in \Gamma(\mathcal{E}\Sigma[-2])$. Thus, because $X \cdot \dot{P} = \mathcal{E}(X)$ (see Equation (4.15)), using Equation (2.11) we have that $P \cdot \dot{P}^t \stackrel{\Sigma}{=} P \cdot \dot{P}$ and in turn $\dot{\Pi} \cdot \dot{\text{III}} \stackrel{\Sigma}{=} P \cdot \dot{P}$. But from Proposition 2.4, we have that $\dot{K} \stackrel{\Sigma}{=} 2P \cdot \dot{P}$, so the first claim of the lemma follows.

To prove the second claim, for which $d > 6$, we first apply Proposition 2.4 twice to $P_{AB} P^{AB}$ and find that

$$\dot{K} \stackrel{\Sigma}{=} 2\dot{P}^2 + 2P \cdot \ddot{P} - \frac{2}{d-6}(\hat{D}P)^2.$$

Applying Lemmas 4.27, 4.28, and 4.29, we obtain the second claim.

Finally, we turn to the third claim with $d = 5$. Because Lemmas 4.28 and 4.29 do not hold when $d = 5$, we need to use a different method. Also if $\Sigma \hookrightarrow (M^5, \mathbf{c})$ is a generic conformally embedded hypersurface, the tensor $\dot{\text{IV}}$ is only a canonical conditional fundamental form, so in particular it cannot appear in an otherwise conformally-invariant expression for \dot{K} . Thus, to compute \dot{K} in this case, we resort to a Riemannian computation, and use that when $d = 5$ (see [31]),

$$I \cdot \hat{D}^2 K^e \stackrel{\Sigma}{=} \left[\Delta^\top - 2\bar{J} + \frac{1}{3}K + 2\nabla_n^2 + 4(2H\nabla_n - P_{nn} - \frac{1}{3}K + \frac{5}{2}H^2) \right] (\nabla n + sP + g\rho)^2.$$

The expression for $\nabla_n^2 \rho$ along Σ may also be found in [31]. The remaining terms were handled by using the computer algebra system FORM [51]; this computation is documented in [10]. \square

It only remains to prove Theorem 1.5 from the introduction.

Proof of Theorem 1.5. The quantity \dot{K} is a density of weight -4 and hence can be invariantly integrated over the hypersurface Σ when $d = 5$. Moreover the Codazzi–Mainardi Equation (3.4) can be used to verify the identity

$$\bar{\nabla}^c W_{c(ab)\hat{n}}^\top = \bar{\Delta} \dot{\Pi}_{ab} - \frac{d-1}{d-2} \bar{\nabla}_{(a} \bar{\nabla} \cdot \dot{\Pi}_{b)\circ} - (d-1) \dot{\Pi}_{(a} \bar{P}_{b)\circ} - \bar{J} \dot{\Pi}_{ab} - \dot{\Pi}^{cd} \bar{W}_{cabd},$$

which is valid in any bulk dimension $d \geq 4$. It is straightforward to check that the left hand side of the above display is conformally invariant precisely when $d = 5$. It can then be used to verify that the integrand in Display (1.5) differs from a non-zero multiple of \dot{K} only by total divergences and manifestly invariant terms. (Note that the choice of $g \in \mathbf{c}$ is used to trivialize density bundles in the integrand of Display (1.5).) \square

ACKNOWLEDGEMENTS

A.W. was also supported by a Simons Foundation Collaboration Grants for Mathematicians ID 317562 and 686131, and thanks the University of Auckland for warm hospitality. A.W. and A.R.G. gratefully acknowledge support from the Royal Society of New Zealand via Marsden Grants 16-UOA-051 and 19-UOA-008.

REFERENCES

- [1] Pierre Albin. Renormalizing curvature integrals on Poincaré–Einstein manifolds. *Adv. Math.*, 221(1):140–169, 2009.
- [2] Michael T. Anderson. L^2 curvature and volume renormalization of AHE metrics on 4-manifolds. *Math. Res. Lett.*, 8(1-2):171–188, 2001.
- [3] Lars Andersson, Piotr T. Chruściel, and Helmut Friedrich. On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein’s field equations. *Comm. Math. Phys.*, 149(3):587–612, 1992.
- [4] M. Atiyah, R. Bott, and V. K. Patodi. On the heat equation and the index theorem. *Invent. Math.*, 19:279–330, 1973.
- [5] Patricio Aviles and Robert C. McOwen. Complete conformal metrics with negative scalar curvature in compact Riemannian manifolds. *Duke Math. J.*, 56(2):395–398, 1988.

- [6] Toby N. Bailey, Michael G. Eastwood, and A. Rod Gover. Thomas's structure bundle for conformal, projective and related structures. *Rocky Mountain J. Math.*, 24(4):1191–1217, 1994.
- [7] Toby N. Bailey, Michael G. Eastwood, and C. Robin Graham. Invariant theory for conformal and CR geometry. *Ann. of Math. (2)*, 139(3):491–552, 1994.
- [8] Olivier Biquard. Métriques d'Einstein asymptotiquement symétriques. *Astérisque*, (265):vi+109, 2000.
- [9] Samuel Blitz, A. Rod Gover, and Andrew Waldron. Extrinsic Paneitz Operators and Q -curvatures. *in preparation*.
- [10] Samuel Blitz, A. Rod Gover, and Andrew Waldron. FORM documentation for *Conformal fundamental forms and the asymptotically Einstein condition*. See ancillary files included with this submission., 2021.
- [11] Thomas Branson and A. Rod Gover. Conformally invariant non-local operators. *Pacific J. Math.*, 201(1):19–60, 2001.
- [12] Jeffrey S. Case. Boundary operators associated with the Paneitz operator. *Indiana Univ. Math. J.*, 67(1):293–327, 2018.
- [13] Jeffrey S. Case and Yi Wang. Boundary operators associated to the σ_k -curvature. *Adv. Math.*, 337:83–106, 2018.
- [14] Pascal Cherrier. Problèmes de Neumann non linéaires sur les variétés riemanniennes. *J. Funct. Anal.*, 57(2):154–206, 1984.
- [15] Sean N. Curry and A. Rod Gover. An introduction to conformal geometry and tractor calculus, with a view to applications in general relativity. In *Asymptotic analysis in general relativity*, volume 443 of *London Math. Soc. Lecture Note Ser.*, pages 86–170. Cambridge Univ. Press, Cambridge, 2018.
- [16] Charles Fefferman and C. Robin Graham. Conformal invariants. Number Numéro Hors Série, pages 95–116. 1985. The mathematical heritage of Élie Cartan (Lyon, 1984).
- [17] Charles Fefferman and C. Robin Graham. *The ambient metric*, volume 178 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012.
- [18] Aaron Fialkow. Conformal differential geometry of a subspace. *Trans. Amer. Math. Soc.*, 56:309–433, 1944.
- [19] Michael Glaros, A. Rod Gover, Matthew Halbasch, and Andrew Waldron. Variational calculus for hypersurface functionals: singular Yamabe problem Willmore energies. *J. Geom. Phys.*, 138:168–193, 2019.
- [20] A. Rod Gover. Aspects of parabolic invariant theory. Number 59, pages 25–47. 1999. The 18th Winter School “Geometry and Physics” (Srń, 1998).
- [21] A. Rod Gover. Invariant theory and calculus for conformal geometries. *Adv. Math.*, 163(2):206–257, 2001.
- [22] A. Rod Gover. Almost Einstein and Poincaré-Einstein manifolds in Riemannian signature. *J. Geom. Phys.*, 60(2):182–204, 2010.
- [23] A. Rod Gover, Emanuele Latini, and Andrew Waldron. Poincaré-Einstein holography for forms via conformal geometry in the bulk. *Mem. Amer. Math. Soc.*, 235(1106):vi+95, 2015.
- [24] A. Rod Gover and Lawrence J. Peterson. Conformally invariant powers of the Laplacian, Q -curvature, and tractor calculus. *Comm. Math. Phys.*, 235(2):339–378, 2003.
- [25] A. Rod Gover and Lawrence J. Peterson. Conformal boundary operators, t -curvatures, and conformal fractional Laplacians of odd order. *Pac. J. Math. in press*, 2021.
- [26] A. Rod Gover and Andrew Waldron. The $\mathfrak{so}(d+2, 2)$ minimal representation and ambient tractors: the conformal geometry of momentum space. *Adv. Theor. Math. Phys.*, 13(6):1875–1894, 2009.
- [27] A. Rod Gover and Andrew Waldron. Boundary calculus for conformally compact manifolds. *Indiana Univ. Math. J.*, 63(1):119–163, 2014.
- [28] A. Rod Gover and Andrew Waldron. Conformal hypersurface geometry via a boundary Loewner–Nirenberg–Yamabe problem. *Commun. Anal. Geom. to appear*, 2015.
- [29] A. Rod Gover and Andrew Waldron. Submanifold conformal invariants and a boundary Yamabe problem. In *Extended abstracts Fall 2013—geometrical analysis, type theory, homotopy theory and univalent foundations*, volume 3 of *Trends Math. Res. Perspect. CRM Barc.*, pages 21–26. Birkhäuser/Springer, Cham, 2015.
- [30] A. Rod Gover and Andrew Waldron. Renormalized volume. *Comm. Math. Phys.*, 354(3):1205–1244, 2017.
- [31] A. Rod Gover and Andrew Waldron. A calculus for conformal hypersurfaces and new higher Willmore energy functionals. *Adv. Geom.*, 20(1):29–60, 2020.
- [32] C. Robin Graham. Volume renormalization for singular Yamabe metrics. *Proc. Amer. Math. Soc.*, 145(4):1781–1792, 2017.
- [33] C. Robin Graham and John M. Lee. Einstein metrics with prescribed conformal infinity on the ball. *Adv. Math.*, 87(2):186–225, 1991.
- [34] C. Robin Graham and Maciej Zworski. Scattering matrix in conformal geometry. *Invent. Math.*, 152(1):89–118, 2003.
- [35] D. Grant. A conformally invariant third order neumann-type operator for hypersurfaces. Master's thesis, University of Auckland, 2003.
- [36] M. Henningson and K. Skenderis. The holographic Weyl anomaly. *J. High Energy Phys.*, (7):Paper 23, 12, 1998.
- [37] Euihun Joung, Massimo Taronna, and Andrew Waldron. A calculus for higher spin interactions. *J. High Energy Phys.*, (7):186–211, 2013.

- [38] C. R. LeBrun. \mathcal{H} -space with a cosmological constant. *Proc. Roy. Soc. London Ser. A*, 380(1778):171–185, 1982.
- [39] John M. Lee. Fredholm operators and Einstein metrics on conformally compact manifolds. *Mem. Amer. Math. Soc.*, 183(864):vi+83, 2006.
- [40] Charles Loewner and Louis Nirenberg. Partial differential equations invariant under conformal or projective transformations. In *Contributions to analysis (a collection of papers dedicated to Lipman Bers)*, pages 245–272. 1974.
- [41] Juan Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2(2):231–252, 1998.
- [42] Rafe Mazzeo. Regularity for the singular Yamabe problem. *Indiana Univ. Math. J.*, 40(4):1277–1299, 1991.
- [43] Rafe Mazzeo and Frank Pacard. Maskit combinations of Poincaré-Einstein metrics. *Adv. Math.*, 204(2):379–412, 2006.
- [44] Richard B. Melrose. *Geometric scattering theory*. Stanford Lectures. Cambridge University Press, Cambridge, 1995.
- [45] Jan Möllers, Bent Ørsted, and Genkai Zhang. On boundary value problems for some conformally invariant differential operators. *Comm. Partial Differential Equations*, 41(4):609–643, 2016.
- [46] University of Auckland. Tractor calculus and invariants for conformal sub-manifolds. Master’s thesis, University of Auckland, 2005.
- [47] A. Polyakov. Fine structure of strings. *Nuclear Phys. B*, 268(2):406–412, 1986.
- [48] C. Robin Graham and Nicholas Reichert. Higher-dimensional Willmore energies via minimal submanifold asymptotics. *Asian J. Math.*, 24(4):571–610, 2020.
- [49] A. Shaikat. *Unit Invariance as a Unifying Principle of Physics*. PhD thesis, University of California, Davis, 2010.
- [50] Peter Stredder. Natural differential operators on Riemannian manifolds and representations of the orthogonal and special orthogonal groups. *J. Differential Geometry*, 10(4):647–660, 1975.
- [51] Jos Vermaseren. New features of FORM. arXiv math-ph/0010025, 2000.
- [52] Y. Vyatkin. *Manufacturing conformal invariants of hypersurfaces*. PhD thesis, University of Auckland, 2013.
- [53] P. Yang, Dzh. King, and S.-Yu. A. Chang. Renormalized volumes for conformally compact Einstein manifolds. *Sovrem. Mat. Fundam. Napravl.*, 17:129–142, 2006.

^b CENTER FOR QUANTUM MATHEMATICS AND PHYSICS (QMAP), DEPARTMENT OF PHYSICS, UNIVERSITY OF CALIFORNIA, DAVIS, CA95616, USA
Email address: shblitz@ucdavis.edu

[#] DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF AUCKLAND, PRIVATE BAG 92019, AUCKLAND 1142, NEW ZEALAND, AND, MATHEMATICAL SCIENCES INSTITUTE, AUSTRALIAN NATIONAL UNIVERSITY, ACT 0200, AUSTRALIA
Email address: gover@math.auckland.ac.nz

[‡] CENTER FOR QUANTUM MATHEMATICS AND PHYSICS (QMAP), DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CA95616, USA
Email address: wally@math.ucdavis.edu