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# Bi-immunity over Different Size Alphabets

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## Abstract

In this paper we study various notions of bi-immunity over alphabets with $b \geq 2$ elements and recursive transformations between sequences on different alphabets which preserve them. Furthermore, we extend the study from sequences bounded by a constant to sequences over the alphabet of all natural numbers, which may or may not be bounded by a recursive function, and relate them to the Turing degrees in which they can occur.

*Keywords:* randomness, immune sequence, bi-immune sequence, immune function, bi-immune function, martingale

## 1. Introduction

Randomness is an important resource in science, statistics, cryptography, gambling, medicine, art and politics. For a long time pseudo-random number generators (PRNGs) – computer algorithms designed to simulate randomness – have been the main, if not the only, sources of randomness. As early as 1951 von Neumann noted [52] that: "Anyone who attempts to generate random numbers by deterministic means is, of course, living in a state of sin." This statement was not meant to stop people from using PRNGs, but to caution against mistakenly believing that PRNGs produce "true" randomness. With the development of algorithmic information theory [21, 39, 23] classes of different quality of random strings/sequences have been studied and von Neumann intuition was rigorously proved: mathematically there is no "true" random string/sequence [15].

In many domains requiring random numbers it is crucial to have high quality randomness. This is obvious in cryptography, where good randomness is vital to the security of data and communication, but is equally true in other areas such as medicine, where decisions of consequence may be made based on scientific and statistical studies relying essentially on randomness. Problems with the poor quality of randomness of various PRNGs are well known and can have serious consequences: a classical example is the discovery in 2012 of a weakness in a worldwide-used encryption system which was traced to a PRNG [38].

These practical requirements have driven a recent surge of interest in developing random number generators "better than PRNGs", in particular, quantum random number generators (QRNGs) [17, 28]. QRNGs are generally considered to be, by their very nature, "better" than classical RNGs and "should excel" precisely on properties of randomness where algorithmic PRNGs obviously fail: incomputability and inherent unpredictability. To date only one class of QRNGs has been proved to satisfy these desiderata by Abbott, Calude, Svozil [4, 5, 37]. This type of QRNGs is based on a located form [1, 3, 6, 7, 8] of the Kochen-Specker Theorem [35], a result true only in Hilbert spaces of dimension at least three. These QRNGs – which locate and repeatedly measure a value-indefinite quantum observable – produce more than

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incomputable sequences (over alphabets with at least three letters), more precisely, bi-immune sequences[1], that is, sequences for which no algorithm can compute more than finitely many exact values. As almost all applications need quantum random binary strings, there is a stringent demand of randomness-preserving algorithms transforming non-binary strings into binary ones. This is the context motivating the following questions studied in this paper: (a) which sequences on non-binary alphabets are immune or bi-immune?, (b) how can one algorithmically transform a bi-immune sequence over a non-binary alphabet into a binary bi-immune sequence?

Historically, the notion of immunity grew out of attempts to solve Post's problem [43]; it has since been studied in other areas such as algorithmic randomness [31, 22, 9], the theory of minimal index sets [50] as well as the theory of numberings and $\Sigma^0_1$-dense sets [12]. Traditionally, algorithmic information theory was presented for binary strings and sequences [23, 41]. In Calude [15] the theory was developed in the general case of an alphabet with at least two elements, so the invariance under the change of the size of the alphabet became important. Early results go back to Borel normality, which is not invariant under the change of the base; in contrast, Martin-Löf randomness is invariant Calude and Jürgensen [16] and Staiger [49]. In [19] the relations between four classes of real numbers, Liouville numbers, computable reals, Borel absolutely-normal numbers and Martin-Löf random reals are studied.

In this context we investigate various generalised notions of (bi-)immunity for sequences over finite and infinite alphabets, in particular sequences that do not grow too quickly in the sense that a single recursive function bounds each term of such a sequence. The following questions will be studied: (c) how does the Turing degree of a (bi-)immune sequence bounded by a recursive function $h$ (or *recursively bounded* (bi-)immune sequence) depend on $h$?, (d) which oracles are powerful enough to compute recursively-bounded (bi-)immune sequences?, (e) what is the computational power of recursively-bounded (bi-)immune sequences compared to that of the Halting Problem?, (f) are the Turing degrees of recursive-bounded bi-immune sequences closed upwards?

## 2. Notation

For background on algorithmic randomness, we refer the reader to books of Schnorr, Calude, Downey and Hirschfeldt, Nies [47, 15, 23, 41]. The set of positive integers will be denoted by $\mathbb{N}$; $\mathbb{N} \cup \{0\}$ will be denoted by $\mathbb{N}_0$. Consider the alphabet $A_b = \{0, 1, \ldots, b-1\}$, where $b \geq 2$ is an integer; the elements of $A_b$ are to be considered the digits used in natural positional representations of numbers in the interval $B$ at base $b$ where $B$ is the unit interval of real numbers. By $A_b^*$ and $A_b^\omega$ we denote the sets of (finite) strings and (infinite) sequences over the alphabet $A_b$. Strings will be denoted by $\sigma, x, y, u, w$; the length of the string $x = x_1 x_2 \ldots x_m$, $x_i \in A_b$, is denoted by $|x|_b = m$ (the subscript $b$ will be omitted if it is clear from the context); $A_b^m$ is the set of all strings of length $m$. Sequences will be denoted by $\mathbf{w} = w_1 w_2 \ldots$; the prefix of length $m$ of $\mathbf{w}$ is $\mathbf{w} \upharpoonright m = w_1 w_2 \ldots w_m$. Sequences can be also viewed as $A_b$-valued functions defined on $\mathbb{N}$. Further, we consider a generalised kind of sequence called an *h-bounded sequence* for some recursive function $h$; for such a sequence $\mathbf{w} = w_1 w_2 \ldots$, one has $w_i < h(i)$ for each $i \in \mathbb{N}$ ($h(0)$ is excluded for notational convenience). An *h-bounded function* is any (possibly partial) function $g$ satisfying $g(i) < h(i)$ for each $i \in \mathrm{dom}(g)$. We denote by $\preceq$ the prefix relation (between two strings or a string and a sequence). The complement of $U \subseteq \mathbb{N}_0$ will be denoted by $\overline{U}$, that is, $\overline{U} = \mathbb{N}_0 \setminus U$.

Any unexplained recursion-theoretic notation can be found in the textbooks of Rogers, Soare and Odifreddi [44, 48, 42]. We assume knowledge of elementary computability theory over different size alphabets [15].

For any string $y \in A_b^*$, the class of $b$-ary infinite sequences extending $y$ is denoted by $y \cdot A_b^\omega = \{\mathbf{w} \in A_b^\omega : y \preceq \mathbf{w}\}$; as before, the subscript $b$ will be omitted if it is clear from the context. Extending this notation, if $W$ is any set of strings belonging to $A_b^*$, then $W \cdot A_b^\omega = \{\mathbf{w} \in A_b^\omega : (\exists y \in W)[y \preceq \mathbf{w}]\}$ where $\cdot$ is the concatenation of strings with other strings or sequences. Given alphabets $A_b$ and $A_{b'}$, a *morphism* (or *homomorphism*) of $A_b$ into $A_{b'}$ is a mapping $\mu : A_b^* \to A_{b'}^*$ such that $\mu(xy) = \mu(x)\mu(y)$ for all $x, y \in A_b^*$. A

[1] The weakest form of algorithmic randomness [23].

morphism $\mu$ of $A_b^*$ into $A_{b'}^*$ is *alphabetic* if, for every $a \in A_b$, $\mu(a)$ is either a letter of $A_{b'}$ or the empty word, and it is *non-erasing* if no $\mu(a), a \in A_b$, is the empty word. We extend a morphism $\mu : A_b^* \to A_b^*$ as follows in a natural way to sequences $\mathbf{w} \in A_b^*$: $\mu(\mathbf{w}) = \mu(w_1) \cdot \mu(w_2) \cdots \mu(w_i) \cdots \in A_b^* \cup A_b^\omega$.

The *value* of a string $w_1 w_2 \dots w_n \in A_b^*$ is the real number $v_b(w_1 w_2 \dots w_n) = \sum_{i=1}^{n} w_i b^{-i} \in \mathbb{R}$. The value of the sequence $\mathbf{w} = w_1 w_2 \dots \in A_b^\omega$ is the real number $v_b(\mathbf{w}) = \sum_{i=1}^{\infty} w_i b^{-i} \in \mathbb{R}$. Clearly, $v_b(\mathbf{w} \restriction n) \to v_b(\mathbf{w})$ as $n \to \infty$.

If $v_b(\mathbf{w})$ is irrational, then $v_b(\mathbf{w}') = v_b(\mathbf{w})$ implies $\mathbf{w}' = \mathbf{w}$. Some rational numbers have two different representations. Since our interest is in incomputable reals and rational numbers are far from being incomputable, this issue will not cause a problem.

Let $\mathcal{P}$ denote the class of all partial-recursive functions of one argument over $\mathbb{N}_0$, let $\mathcal{P}^2$ denote the class of all partial-functions of two arguments over $\mathbb{N}_0$, and let $\mathcal{R}$ denote the class of all recursive functions of one argument over $\mathbb{N}_0$.

Any function $\psi \in \mathcal{P}^2$ is called a *numbering of partial-recursive functions*. Set $\psi_e = \lambda i.\psi(e, i)$ and $\mathcal{P}_\psi := \{\psi_e : e \in \mathbb{N}_0\}$. A numbering $\varphi \in \mathcal{P}^2$ is said to be an *acceptable numbering* or *Gödel numbering* of all partial-recursive functions if $\mathcal{P}_\varphi = \mathcal{P}$ and for every numbering $\psi \in \mathcal{P}^2$, there is a $f \in \mathcal{R}$ such that $\psi_e = \varphi_{f(e)}$ for all $e \in \mathbb{N}_0$ (see [44]). Throughout this paper, $\varphi$ denotes a fixed acceptable numbering and $\varphi_e$ denotes the partial-function computed by the $e$-th program in the numbering $\varphi$. $\Phi$ denotes a fixed Blum complexity measure [13] for the numbering $\varphi$. For every $e$, $W_e$ denotes the domain of $\varphi_e$.

Let $e, i \in \mathbb{N}_0$; if $\varphi_e(i)$ is defined then we write $\varphi_e(i)\downarrow$ and also say that $\varphi_e(i)$ *converges*. Otherwise, $\varphi_e(i)$ is said to *diverge* (abbr. $\varphi_e(i)\uparrow$).

A *martingale* is a function $mg : A_b^* \to \mathbb{R}^+ \cup \{0\}$ that satisfies for every $x \in A_b^*$ the equality $\sum_{a \in A_b} mg(x \cdot a) = b \cdot mg(x)$. For a martingale $mg$ and a sequence $\mathbf{w} \in A_b^\omega$, the martingale $mg$ *succeeds* on $\mathbf{w}$ if $\sup_n mg(\mathbf{w} \restriction n) = \infty$.

Let $D_0, D_1, D_2, \dots$ be a canonical indexing of all finite sets. For any two sets $U$ and $V$, $U$ is *truth-table reducible* or *tt-reducible* to $V$, denoted $U \leq_{tt} V$, if for some recursive functions $f$ and $g$, $U(i) = g(\langle a, i\rangle)$ for all $i$, where $a$ is the canonical index of $D_{f(i)} \cap V$. $U$ is *bounded truth-table reducible* or *btt-reducible* to $V$, denoted $U \leq_{btt} V$, if $U \leq_{tt} V$ and there is some number $m$ such that $|D_{f(i)}| \leq m$ for all $i$ (where $f$ is as in the definition of tt-reducibility). In the latter definitions, the role of $f$ is to select the elements to be queried, while $g$ evaluates the value of the truth-table condition. $U$ is *tt-equivalent* (resp. btt-equivalent) to $V$ if $U \leq_{tt} V$ (resp. $U \leq_{btt} V$) and $V \leq_{tt} U$ (resp. $V \leq_{btt} U$). A set $U$ has *PA degree* (or is *PA-complete*) if $U$ computes a $\{0, 1\}$-valued diagonally non-recursive (d.n.r.) function, that is, a $\{0, 1\}$-valued function $f$ such that $f(e) \neq \varphi_e(e)$ for any $e$ such that $\varphi_e(e)\downarrow$. Equivalently, a set $U$ has *PA degree* if one can compute relative to oracle $U$ a total extension of any partial-recursive $\{0, 1\}$-valued function, that is, for any $\{0, 1\}$-valued function $\psi$, there is a total function $g \leq_T U$ such that $g(i) = \psi(i)$ whenever $\psi(i)\downarrow$; moreover, $g$ may be chosen to be $\{0, 1\}$-valued.

An *r.e. open set* is an open set generated by an r.e. set of binary strings. Regarding $W_e$ as a subset of $A_2^*$, one has an enumeration $W_0 \cdot A_2^\omega, W_1 \cdot A_2^\omega, W_2 \cdot A_2^\omega, \dots$ of all r.e. open sets. A *uniformly r.e. sequence* $(G_m)_{m<\omega}$ *of open sets* is given by a recursive function $f$ such that $G_m = W_{f(m)} \cdot A_2^\omega$ for each $m$. A *Martin-Löf test* is a uniformly r.e. sequence $(G_m)_{m<\omega}$ of open sets such that $(\forall m < \omega)[\lambda(G_m) \leq 2^{-m}]$; here $\lambda$ denotes the uniform measure, that is, for every $\sigma \in A_2^\omega$, $\lambda(\sigma \cdot A_2^\omega) = 2^{-|\sigma|}$. A sequence $\mathbf{w} \in A_2^\omega$ *fails* the test if $\mathbf{w} \in \bigcap_{m<\omega} G_m$; otherwise $\mathbf{w}$ *passes* the test. $\mathbf{w}$ is *Martin-Löf random* if $\mathbf{w}$ passes each Martin-Löf test [40].

Martin-Löf randomness may be defined analogously for non-binary sequences over a finite alphabet; however, this work will consider Martin-Löf randomness only for binary sequences. Thus, throughout this paper, by "Martin-Löf random sequence" will always be meant "Martin-Löf random binary sequence".

## 3. Degrees of Bi-immunity Over Different Size Finite Alphabets

We recall that an infinite set $U \subseteq \mathbb{N}_0$ is *immune* (in the sense of recursion theory) if it contains no infinite recursively enumerable (r.e.) subset; $U$ is *bi-immune* set if both $U$ and $\overline{U}$ are immune [44, 42]. Bi-immune sets are highly non-recursive in the sense that no partial-recursive function with an infinite domain can be extended to the characteristic function of such a set. The notion of algorithmic randomness is also closely related to that of immunity: every Martin-Löf random sequence $\mathbf{w}$, for example, is *effectively* bi-immune in the sense that there is a recursive function that computes for every $e$ such that $W_e$ is contained in $\mathbf{w}^{-1}(1)$ (resp. $\mathbf{w}^{-1}(0)$) an upper bound on the size of $W_e$. Even stronger than the notion of immunity is that of *hyperimmunity*: an infinite set $U$ is *hyperimmune* if it is infinite and there is no recursive function $f$ such that $|U \cap \{0, \ldots, f(n)\}| \geq n$ for all $n$. In what follows, we generalise the notions of immunity and bi-immunity to sequences. One may take a cue from how Martin-Löf randomness for binary sequences is adapted to sequences over an arbitrary base $b \geq 2$ by identifying a sequence $\mathbf{w} \in A_b^\omega$ with the real number $\sum_{i=0}^{\infty} w_i b^{-i-1}$; these definitions of Martin-Löf randomness and asymptotic Kolmogorov complexity (constructive dimension) are base-invariant [16, 49]. Unfortunately, as we will show later in Propositions 21 and 23, there are reals that are bi-immune in one base but not in another base; thus the concept of bi-immunity is – like the concepts of Borel normality and disjunctiveness (see [20, 45, 46, 34]) – base-dependent if one directly adapts the definition of bi-immune sets to sequences.

Further, motivated by non-binary quantum random number generators [1, 7] we study which recursive transformations between sequences on different alphabets preserve bi-immunity. A specific case of interest is the ternary and binary sequences: which recursive transformations between ternary and binary sequences preserve bi-immunity?

In this paper we introduce and study a formalisation of bi-immunity for sequence over an alphabet with $b \geq 2$ elements. Broadly speaking, a sequence $\mathbf{w} \in A_b^\omega$ is $b$-graph-immune (resp. $b$-graph-bi-immune) if no algorithm that outputs only elements of $A_b$ can generate infinitely many correct (resp. incorrect) values of its elements (pairs, $(i, w_i)$).[2] This condition can be formalised directly by the following definition (given in [11]):

**Definition 1.** *A sequence* $\mathbf{w} \in A_b^\omega$ *is* $b$-graph-immune *(resp.* $b$-graph-bi-immune*) if there exists no partial-recursive function* $\varphi$ *from* $\mathbb{N}$ *to* $A_b$ *having an infinite domain* $\mathrm{dom}(\varphi)$ *with the property that* $\varphi(i) = w_i$ *(resp.* $\varphi(i) \neq w_i$*) for all* $i \in \mathrm{dom}(\varphi)$.

Note that $b$-graph-bi-immunity does not only imply that the complement is immune, but also that the graph itself is immune, see Proposition 4 below, the reason we have called it graph-bi-immunity. Clearly, graph-bi-immunity is a stronger form of incomputability.

**Remark 2.** If $\mathbf{w} \in A_b^\omega$ does not contain a certain letter $c \in A_b$ then the recursive function $\varphi(i) = c$ witnesses that $\mathbf{w}$ cannot be $b$-graph-bi-immune.

In case of $b$-graph-immunity the situation is different. Therefore, we introduce a more restrictive type of $b$-graph-immunity, known as *strong $b$-graph-immunity*:

**Definition 3.** *A sequence* $\mathbf{w} \in A_b^\omega$ *is strongly $b$-graph-immune if it is $b$-graph-immune and for every $c < b$ there are infinitely many $i$ with $w_i = c$.*

For the next proposition, we define $b$-*graph*$(\mathbf{w}) := \{b \cdot (n-1) + w_n : n \in \mathbb{N}\} \subseteq \mathbb{N}_0$. This proposition provides various characterisations for the notion of $b$-graph-immune and $b$-graph-bi-immune sequences; the reader should note that we will generalise these notions in Section 7 to the case where the bound $b$ is not a constant but where it is either absent (alphabet is $\mathbb{N}_0$) or where the size of the alphabet depends on the index of the item in the sequence. Also there a characterisation similar to the next proposition is possible.

**Proposition 4.** *The following three items characterise $b$-graph-immunity, strong $b$-graph-immunity and $b$-graph-bi-immunity, respectively.*

**(a)** $\mathbf{w}$ *is $b$-graph-immune if one of the following equivalent characterisations holds:*

1. *for all* $a \in A_b$, $\mathbf{w}^{-1}(a)$ *is immune or finite;*

[2] The modifier 'graph' comes from the fact that the immunity of a sequence $\mathbf{w}$ is equivalent to the immunity (in the usual recursion-theoretic sense) of its associated *b-graph*, defined as $\{b \cdot (n-1) + w_n : n \in \mathbb{N}\}$; see Proposition 4.

2. *b-graph*$(\mathbf{w})$ *is immune.*

**(b)** $\mathbf{w}$ *is strongly b-graph-immune if and only if for all* $a \in A_b$, $\mathbf{w}^{-1}(a)$ *is immune.*

**(c)** $\mathbf{w}$ *is b-graph-bi-immune if one of the following equivalent characterisations holds:*

1. *for all* $a \in A_b$, $\mathbf{w}^{-1}(a)$ *is bi-immune;*
2. *for all non-empty* $A \subset A_b$, $\bigcup_{a\in A} \mathbf{w}^{-1}(a)$ *is immune;*
3. *for all non-empty* $A \subset A_b$, $\bigcup_{a\in A} \mathbf{w}^{-1}(a)$ *is bi-immune;*
4. *b-graph*$(\mathbf{w})$ *is bi-immune;*
5. *b-graph*$(\mathbf{w})$ *is co-immune.*

**Proof.** **(a)** Assume that $\mathbf{w}$ is not $b$-graph-immune. Then there is a partial-recursive function $\varphi$ with infinite domain such that $\varphi(i) = w_i$ on the domain of $\varphi$; one can now select a value $a \in A_b$ such that $\varphi$ takes $a$ infinitely often and let $\psi$ be the restriction of $\varphi$ to the set of inputs which are mapped by $\varphi$ to $a$. It follows that the domain of $\psi$ is an infinite r.e. subset of $\mathbf{w}^{-1}(a)$. Thus Item 1 is not satisfied. Now if Item 1 is not satisfied, then some $\mathbf{w}^{-1}(a)$ is neither immune nor finite, hence $\mathbf{w}^{-1}(a)$ has an infinite recursive subset $R$. Now $\{b\cdot(n-1)+a : n \in R\}$ is an infinite recursive subset of $b$-graph$(\mathbf{w})$.

Finally, if $b$-graph$(\mathbf{w})$ is not immune, as it is infinite, it has an infinite recursive subset $R$. Then $\varphi(n) = a$ if and only if $b\cdot(n-1)+a \in R$ defines a partial-recursive function witnessing that $\mathbf{w}$ is not $b$-graph-immune.

**(b)** This statement is only an obvious variant of the definition.

**(c)** Let $\mathbf{w}^{-1}(a)$ be not bi-immune. If there exists an infinite recursive subset $R \subseteq \{n : w_n \neq a\}$, then define the partial-recursive function $\varphi : R \to A_b$ via $\varphi(n) = a, n \in R$. Otherwise, there is an infinite recursive subset $R \subseteq \{n : w_n = a\}$, so define the partial-recursive function $\varphi : R \to A_b$ via $\varphi(n) = a', n \in R, a' \neq a$. In either case, $\varphi$ witnesses that $\mathbf{w}$ is not $b$-graph-bi-immune.

If, for all $a \in A_b$, the set $\mathbf{w}^{-1}(a)$ is bi-immune then its complement $\bigcup_{a'\neq a} \mathbf{w}^{-1}(a')$ and all its infinite subsets $\bigcup_{a'\in A} \mathbf{w}^{-1}(a'), a \notin A$, are immune, so Item 1 implies Item 2.

If all sets $\bigcup_{a\in A} \mathbf{w}^{-1}(a), \emptyset \neq A \neq A_b$, are immune, so are their complements. Hence Item 2 implies Item 3.

Let $b$-graph$(\mathbf{w})$ be not bi-immune. Then there is an infinite recursive subset $R \subseteq \mathbb{N}_0$ such that $R \subseteq b$-graph$(\mathbf{w})$ or $R \cap b$-graph$(\mathbf{w}) = \emptyset$. Without loss of generality, let $R \subseteq \{b\cdot(n-1)+a : n \in \mathbb{N}\}, a \in A_b$. Consider $R' = \{n : n \in \mathbb{N} \wedge b\cdot(n-1)+a \in R\}$. Then, in case $R \subseteq b$-graph$(\mathbf{w})$ the set $R'$ is an infinite recursive subset of $\mathbf{w}^{-1}(a)$, and in case $R \cap b$-graph$(\mathbf{w}) = \emptyset$ the set $R'$ is disjoint to $\mathbf{w}^{-1}(a)$. Thus, Item 3 implies Item 4.

Item 4 trivially implies Item 5.

Finally, let $\mathbf{w}$ be not $b$-graph-bi-immune and $\varphi$ be a partial-recursive function with infinite domain dom$(\varphi)$ such that $\varphi(n) \neq w_n$ for $n \in \text{dom}(\varphi)$. Then $\{b\cdot(n-1)+\varphi(n) : n \in \text{dom}(\varphi)\}$ is an infinite r.e. subset disjoint to $b$-graph$(\mathbf{w})$. ☐

**Remark 5.** In the binary case (that is, $b = 2$) Proposition 4 shows that 2-graph-immunity is equivalent with the property that $\mathbf{w}^{-1}(1)$ and its complement $\mathbf{w}^{-1}(0)$ are immune, and hence bi-immune, in the sense of recursion theory, i.e. they are infinite and do not contain infinite recursively enumerable (equivalently, recursive) sets [44]. Furthermore, we obtain that in the binary case all variants of immunity – 2-graph-bi-immunity, 2-graph-immunity and strong 2-graph-immunity – coincide. This does not hold for larger alphabets.

**Example 6.** An immune sequence $\mathbf{w} \in A_2^\omega$ considered as an element of $A_3^\omega$ is 3-graph-immune but not 3-graph-bi-immune since $\{i \in \mathbb{N} : w_i = 2\} = \emptyset$. In fact, every $b$-graph-bi-immune $\mathbf{w} \in A_b$ as an element of $A_{b+1}$ is $(b+1)$-graph-immune but neither strongly $(b+1)$-graph-immune nor $(b+1)$-graph-bi-immune. ☐

It follows from Proposition 4 that every $b$-graph-bi-immune sequence is strongly $b$-graph-immune. The converse does not hold for $b > 2$ as shown by the following Example 7.

**Example 7.** Let $M_0 \subseteq \mathbb{N}$ be an immune set whose complement (with respect to $\mathbb{N}$) $\mathbb{N} \setminus M_0$ is recursively enumerable, let $g : \mathbb{N} \to \mathbb{N}, g(\mathbb{N}) = \mathbb{N} \setminus M_0$ be an injective recursive mapping, and let $M \subseteq \mathbb{N}$ be a bi-immune set. Set $M_1 = g(M)$ and $M_2 = g(\mathbb{N} \setminus M)$. Then $M_1$ and $M_2$ are immune.

Define a sequence $\mathbf{w} = w_1 w_2 \cdots \in A_3^\omega$ via the preimages $\mathbf{w}^{-1}(a) = M_a, a \in \{0, 1, 2\}$. Then, clearly, every preimage $\mathbf{w}^{-1}(a)$ is immune, but as a recursively enumerable set the union $\mathbf{w}^{-1}(1) \cup \mathbf{w}^{-1}(2) = M_1 \cup M_2$ is not immune.

Observe that the other combinations $M_0 \cup M_1$ and $M_0 \cup M_2$ are immune. Assume e.g. $M' \subseteq M_0 \cup M_1$ to be recursive. Then $M' \cap M_1 = M' \cap g(\mathbb{N}_0)$ as a recursively enumerable subset of $M_1$ is finite. Thus $M' \cap M_0 = M' \setminus (M' \cap M_1)$ is recursive too, hence also finite. □

**Remark 8.** One might also ask how $b$-graph-bi-immunity relates to other notions. Clearly, $b$-graph-bi-immunity is implied by but not equivalent to $b$-randomness. The study of $b$-randomness was motivated by the idea that the sequence should be as near as possible to the typical outcome of a sequence drawn by a $b$-sided coin; such a sequence is formally defined that there are no structures on which an effective martingale can bet successfully [15].

For example, $b$-random sequences contain every finite string infinitely often, thus they contain squares, that is, sequences of the form $uu$ infinitely often. For the binary alphabet, this is shared with all sequences, as even every finite binary word of length 4 or more contains at least one of the following squares as a subword: 00, 11, 0101, 1010. In contrast to this, Morse as well as Thue [51] constructed ternary sequences which do not contain any single square. Subsequent research [26, 29, 33] asked questions like how many squares a prefix of length $n$ of a sequence can contain and Jonoska, Manea and Seki [33] conjectured that if a binary word contains $k$ 1s and $n-k$ 0s with $2 \leq k \leq n/2$, then there are at most $(2k-1)/(2k+2) \cdot n$ distinct squares. Here two squares are distinct if they are different as strings. The value $k = 1$ does not satisfy this conjecture as the string $0^{n-1}1$ has $\lfloor n/2 - 1 \rfloor$ squares while $(2k-1)/(2k+2) = 1/4$.

One might ask how the number of squares in the prefixes of length $n$ of a $b$-graph-bi-immune sequence grows with $n$? The upper bound can be expected to be similar to the case of arbitrary words, as one can take a sequence which has about $n/2$ squares in a prefix and then in a very thin way adjust the bits to make it 2-graph-bi-immune. So one might be more interested in lower bounds which are taken by some sequence instead of all sequences. The following example shows that for $b = 6$, one can make a sequence which is 6-graph-bi-immune.

**Example 9.** There is a 6-graph-bi-immune sequence without any square as a subword. This stands in contrast to random sequences in which the number of squares in prefixes of length $n$ cannot be bounded by any constant.

To construct a square-free 6-graph-bi-immune sequence $\mathbf{w}$, one first constructs, using the undecidable Halting Problem as an oracle, a sequence $i_1, i_2, \ldots$ of natural numbers such that $i_1 = 1$ and $i_{k+1} \geq 3i_k + 9$ and whenever $\varphi_k$ has an infinite domain and is $\{0, 1, 2, 3, 4, 5\}$-valued then $\varphi_k(i_{2k})\downarrow = \varphi_k(i_{2k+1})\downarrow$. Next one defines on each interval $I_k = \{i_k, i_k + 1, \ldots, i_{k+1} - 1\}$ that $\mathbf{w}$ is chosen as follows:

1. $w_{i_k} w_{i_k+1} \ldots w_{i_{k+1}-1}$ is a square-free word;
2. if $k$ is even then the digits $0, 1, 2$ are used else the digits $3, 4, 5$ are used;
3. if $k = 2e$ and $\varphi_e(i_k)\downarrow \in \{0, 1, 2\}$ then $w_{i_k} = \varphi_e(i_k)$;
4. if $k = 2e + 1$ and $\varphi_e(i_k)\downarrow \in \{3, 4, 5\}$ then $w_{i_k} = \varphi_e(i_k)$.

Next consider a square $uu$ of length $2h$ and on positions $j, j+1, \ldots, j+2h-1$. Choose $k$ such that the interval $I_{k+1}$ contains the upper end $j + 2h - 1$ of the positions of the square $uu$, that is, the inequalities $i_{k+1} \leq j + 2h - 1 \leq i_{k+2} - 1$ hold. As for all $\ell < h$, $w_{j+\ell} = w_{j+h+\ell}$ and neighbouring intervals use the disjoint sets $\{0, 1, 2\}$ and $\{3, 4, 5\}$ of digits, $j + h - 1$ and $j + 2h - 1$ must either be in the same interval or at least two intervals apart. Note that the chain of inequalities $i_k \leq (i_{k+1} - 9)/3 \leq (j + 2h - 1 - 9)/3 \leq j + h - 1$ follows from the choice of the sequence $i_1, i_2, \ldots$; thus these inequalities postulate that $j + h - 1 \in I_k \cup I_{k+1}$. It follows that $j + h - 1 \in I_{k+1}$, as it cannot be in the neighbouring $I_k$. As both halves of $uu$ use the same digits, $\{j, j+1, \ldots, j+2h-1\} \subseteq I_{k+1}$. By construction, the sequence $\mathbf{w}$ is square-free within the interval $I_{k+1}$ and therefore the square $uu$ cannot be a subword of $\mathbf{w}$. Thus it follows that the full sequence $\mathbf{w}$ is square-free. $\square$

## 4. Base-invariance

In this section, we study the question of whether (bi-)immunity for sequences over a finite alphabet is preserved over different bases. The main insight is that while $b$-graph-immunity is indeed preserved over bases of the form $b^k$, where $k \geq 1$, the same does not hold for $b$-graph-(bi-)immunity and thus for strong $b$-graph-immunity.

The simplest computable transformation of a sequence $\mathbf{w} \in A_3^\omega$ into a binary sequence $\mathbf{x} \in A_2^\omega$ is to delete all occurrences of 2 in $\mathbf{w}$; we call this transformation $\mathrm{delete}_2$. The next lemma shows that $\mathrm{delete}_2$ does not preserve graph-bi-immunity.

**Lemma 10.** *(1) There exists a sequence $\mathbf{w} \in A_3^\omega$ which is not 3-graph-bi-immune such that $\mathrm{delete}_2(\mathbf{w})$ is 2-graph-bi-immune.*

*(2) There exists a 3-graph-bi-immune sequence $\mathbf{w} \in A_3^\omega$ such that $\mathrm{delete}_2(\mathbf{w})$ is not 2-graph-bi-immune.*[3]

**Proof.** For (1) we take a 2-graph-bi-immune sequence $\mathbf{x} \in A_2^\omega$ and define the ternary sequence $\mathbf{w}$ by $w_{2i} = x_i, w_{2i+1} = 2$. For (2) we consider the family of all infinite r.e. subsets $(N_i)_{i\in\mathbb{N}_0}$ of $\mathbb{N}_0$ and choose from $N_i$ the first three elements $n_{3i} < n_{3i+1} < n_{3i+2}$ larger than[4] $n_{3(i-1)+2}$ and let $M_j := \{n_{3i+j} : i \in \mathbb{N}_0\}, j = 0, 1, 2$. Then every $M_j \subseteq \mathbb{N}$ is bi-immune as each of them contains (and does not contain) at least one element from every infinite r.e. subset. Now define $\mathbf{w}$ as follows:

$$w_n = \begin{cases} 0, & \text{if } n \in M_0, \\ 1, & \text{if } n \in M_1, \\ 2, & \text{otherwise.} \end{cases}$$

Then the image under the mapping $\mathrm{delete}_2$ is $\mathrm{delete}_2(\mathbf{w}) = 010101\ldots$. $\square$

**Remark 11.** Lemma 10 (2) was communicated in [10] with a different proof.

Next we start with the preservation of (strongly) $b$-graph-(bi)-immune sequences under morphisms. We also provide sufficient conditions that guarantee a morphism $\mu : A_b \to A_b^*$ preserves (strong) $b$-graph-(bi-)immunity.

We start with a property of morphisms of a special kind. Let $\pi_i : \{w : w \in A_b^* \wedge |w| \geq i\} \to A_b$ be the projection on the $i$th letter, that is, $\pi_i(w_1 \cdots w_\ell) := w_i$ for $i \leq \ell$. We call a morphism $\mu : A_b \to A_b^\ell$ *stable* if for all $i \leq \ell$ and for every $a \in A_b$ there is an $a' \in A_b$ such that $\pi_i(\mu(a')) = a$, that is, the letters at a fixed position $i$ in the words $\mu(a), a \in A_b$, are just a permutation of $A_b$.

**Lemma 12.** *Let $\ell \geq 1$ and let $\mu : A_b \to A_b^\ell$ be a stable morphism. Then $\mu(\mathbf{w})$ is $b$-graph-immune ($b$-graph-bi-immune, respectively) if and only if $\mathbf{w}$ is $b$-graph-immune ($b$-graph-bi-immune, respectively).*

**Proof.** Assume that $\bigcup_{a\in A} \mathbf{w}^{-1}(a), \emptyset \subset A \subset A_b$, contains an infinite recursive subset $M \subseteq \mathbb{N}$ and consider $A^{(1)} = \{\pi_1(\mu(a)) : a \in A\}$. Then $\{\ell \cdot (n-1) + 1 : n \in M\} \subseteq \bigcup_{a'\in A^{(1)}} \mu(\mathbf{w})^{-1}(a')$ and $\{\ell \cdot (n-1) + 1 : n \in M\}$ is also infinite and recursive.

Conversely, let $M \subseteq \mathbb{N}$ be an infinite recursive subset of $\bigcup_{a'\in A'} \mu(\mathbf{w})^{-1}(a')$, $\emptyset \subset A' \subset A_{b^2}$. Then there is a $j \leq \ell$ such that $M' := M \cap \{\ell \cdot (n-1) + j : n \in \mathbb{N}\}$ is also infinite and recursive. Let $A := \{a : \exists a'(a' \in A' \wedge \pi_j(\mu(a)) = a')\}$. Then $\{n : \ell \cdot (n-1) + j \in M'\}$ is an infinite recursive subset of $\bigcup_{a\in A} \mathbf{w}^{-1}(a)$. $\square$

[3] A first proof for this was given in [10].
[4] For completeness, set $n_{-1} = -1$.

**Remark 13.** Lemma 12 does not hold for arbitrary morphisms $\mu$ even if all letters are mapped to words of the same length. Consider e.g. $\mu : A_2 \to A_2^*$ where $\mu(a) := 0a$.

**Lemma 14.** *Let $2 \leq b' \leq b$ and let $\mathbf{w} \in A_b^\omega$ be b-graph-bi-immune. If $\mu$ is a non-erasing alphabetic morphism of $A_b$ onto $A_{b'}$ then $\mu(\mathbf{w}) \in A_{b'}$ is $b'$-graph-bi-immune.*

**Proof.** We have $\mu(A_b) = A_{b'}$ and $\mu(a) \in A_{b'}$ for $a \in A_b$. Consider a nonempty subset $A' \subset A_{b'}$. Then $A = \{a : \mu(a) \in A'\} \neq A_b$ and $\bigcup_{a' \in A'} \mu(\mathbf{w})^{-1}(a') = \bigcup_{\mu(a) \in A'} \mathbf{w}^{-1}(a)$. If $\mathbf{w} \in A_b^\omega$ is $b$-graph-bi-immune, according to Proposition 4, every set $\bigcup_{a' \in A'} \mu(\mathbf{w})^{-1}(a'), \emptyset \neq A' \neq A_{b'}$ is immune, and therefore $\mu(\mathbf{w})$ is $b'$-graph-bi-immune. □

Lemma 14 does not hold for (strongly) $b$-graph-immune sequences.

**Example 15.** We refer to the immune subsets $M_0, M_1, M_2 \subseteq \mathbb{N}$ defined in Example 7 where $M_1 \cup M_2$ is recursively enumerable. Define $\mathbf{w} \in A_3^\omega$ via $\mathbf{w}^{-1}(a) = M_a, a \in \{0, 1, 2\}$, and $\mu(0) = 0, \mu(1) = \mu(2) = 1$. Then $\mathbf{w}$ is strongly 3-graph-immune but $\mu(\mathbf{w})$ is not 2-graph-immune. □

The preimages of alphabetic morphisms preserve $b$-graph-immunity of sequences but not $b$-graph-bi-immunity even if we require that every letter occurs infinitely often in the preimage.

**Lemma 16.** *Let $\mu$ be a non-erasing alphabetic morphism of $A_b$ onto $A_{b'}$. If $\mu(\mathbf{w}) \in A_{b'}$ is $b'$-graph-immune then $\mathbf{w} \in A_b^\omega$ is also b-graph-immune.*

**Proof.** Observe that $\mu(\mathbf{w})^{-1}(a') = \bigcup_{\mu(a)=a'} \mathbf{w}^{-1}(a)$. Consequently, if $\mu(\mathbf{w})^{-1}(a')$ is immune or finite then its subset $\mathbf{w}^{-1}(a)$ is also immune or finite. □

**Example 17.** To show that Lemma 16 cannot be extended to $b$-graph-bi-immunity we refer to Example 7 and the sequence $\mathbf{w}$ defined there, and we use the morphism $\mu : A_3 \to A_2$ defined by $\mu(0) = \mu(1) = 0$ and $\mu(2) = 1$. Since $\mu(\mathbf{w})^{-1}(0) = M_0 \cup M_1$ and $\mu(\mathbf{w})^{-1}(1) = M_2$ are both immune, $\mu(\mathbf{w}) \in A_2^\omega$ is 2-graph-bi-immune, but, as shown in Example 7 the sequence $\mathbf{w} \in A_3^\omega$ is not 3-graph-bi-immune. □

As a special case essential in the design of a quantum random generator (cf. [1, 7, 8]), from Lemma 14 we obtain the following:

**Corollary 18.** *Consider $b \geq 3$ and a non-erasing alphabetic morphism $\mu$ of $A_b$ onto $A_{b-1}$. Then for every b-graph-bi-immune sequence $\mathbf{w} \in A_b^\omega$, the sequence $\mu(\mathbf{w}) \in A_{b-1}$ is $(b-1)$-graph-bi-immune.*

Next we study the preservation of $b$-(bi-)immunity under base change, that is, we consider sequences $\mathbf{w} \in A_b^\omega$ and $\mathbf{v} \in A_{b'}^\omega$ which are expansions of the same real number $r = v_b(\mathbf{w}) = v_{b'}(\mathbf{v})$.

**Proposition 19.** *Let $\mathbf{w} \in A_b^\omega$ be the b-ary expansion of the real $r \in \mathbb{R}$. If $\mathbf{v} \in A_{b^k}, k \geq 1$, is the $b^k$-ary expansion of $r$ and for some $a \in A_{b^k}$ the subset $\mathbf{v}^{-1}(a) \subseteq \mathbb{N}$ is infinite and not immune then there is an $a' \in A_b$ such that $\mathbf{w}^{-1}(a') \subseteq \mathbb{N}$ is infinite and not immune.*

**Proof.** Let $\mathbf{v}^{-1}(a)$ be infinite but not immune, and let $M \subseteq \mathbb{N}$ be an infinite and recursive set such that $M \subseteq \mathbf{v}^{-1}(a)$. Since $\mathbf{w}$ is the $b$-ary expansion of $r$ there is a homomorphism $\mu : A_{b^k} \to A_b^k$ satisfying $\mu(\mathbf{v}) = \mathbf{w}$. Let $\mu(a) = a_1 \cdots a_k, a_i \in A_b$. Then $\mathbf{w}^{-1}(a_1) \supseteq \{k \cdot (n-1) + 1 : n \in M\}$, and consequently $\mathbf{w}^{-1}(a_1)$ is infinite and not immune. □

**Corollary 20.** *Let $\mathbf{w} \in A_b^\omega$ be b-graph-immune and the b-ary expansion of the real $r \in \mathbb{R}$. If $\mathbf{v} \in A_{b^k}^\omega, k \geq 1$, is the $b^k$-ary expansion of $r$ then $\mathbf{v}$ is $b^k$-graph-immune.*

Corollary 20 cannot be extended to $b$-graph-bi-immunity.

**Proposition 21.** *For every base b there is a sequence which is b-graph-bi-immune but only $b^2$-graph-immune in base $b^2$.*

**Proof.** Note that when $\mathbf{w}$ is strongly $b$-graph-bi-immune, so is also $\mathbf{v}$ with $v_{2n-1} = v_{2n} = w_n$. This follows from Lemma 12 since the morphism $\mu : A_b \to A_b^2$ with $\mu(a) = aa$ is stable.

However, if we consider the real $r$ whose $b$-expansion is given by $\mathbf{v}$ then its $b^2$-expansion is given by $n \mapsto w_n \cdot (b+1)$ which has only multiples of $(b+1)$ as digits, thus this sequence is not strongly $b^2$-graph-immune. ☐

One might also have a $b$-graph-bi-immune $\mathbf{x}$ such that the corresponding $\mathbf{w}$ is strongly $b^2$-graph-immune but not $b^2$-graph-bi-immune.

**Example 22.** Let $\mathbf{y} = y_1 y_2 \cdots \in A_2^\omega$ be $b$-graph-bi-immune. Define $\mathbf{x} := y_1 y_2 \cdots \in A_2^\omega$ by

$$x_{2i-1}x_{2i} = \begin{cases} 00, & \text{if } y_i = 0 \wedge i \text{ is odd},\\ 01, & \text{if } y_i = 0 \wedge i \text{ is even},\\ 10, & \text{if } y_i = 1 \wedge i \text{ is even},\\ 11, & \text{if } y_i = 1 \wedge i \text{ is odd}. \end{cases}$$

Then according to Proposition 4, the sequence $\mathbf{x} \in A_2^\omega$ is also 2-graph-bi-immune, e.g. $\{j \in \mathbb{N} : x_j = 0\} = \{2i-1 \in \mathbb{N} : y_i = 0\} \cup \{2i \in \mathbb{N} : y_i = 0 \wedge i \text{ is odd}\} \cup \{2i \in \mathbb{N} : y_i = 1 \wedge i \text{ is even}\}$. Let $\mathbf{w} \in A_4^\omega$ such that $v_2(\mathbf{x}) = v_4(\mathbf{w})$.

By construction $\mathbf{w}$ contains at even positions only the letters 1 and 2 and at odd positions only the letters 0 and 3. Thus Proposition 19 and Proposition 4 show that $\mathbf{w}$ is strongly 4-graph-immune but not 4-graph-bi-immune. ☐

The following proposition shows that another natural algorithmic transformation fails to preserve graph-bi-immunity.

**Proposition 23.** *There exists a real whose base* 8*-expansion is strongly* 8*-graph-bi-immune while its base* 4 *expansion is not* 4*-graph-bi-immune.*

**Proof.** Let $c$ denote the mirror image of the binary complement of $b$, so possible pairs $bc$ in the system of base 8 are $07, 13, 25, 31, 46, 52, 64, 70$ and from now on, $bc$ is always one pair of these octal digits. Next we define the stable morphism $\mu : A_8 \to A_{8^2}$ via $\mu(b) = bc$ and choose an 8-bi-immune sequence $\mathbf{v}$. According to Lemma 12 the image $\mathbf{w} = \mu(\mathbf{v})$ is also 8-bi-immune.

However, the base 4 counterpart $\mathbf{y} \in A_4^\omega$ of $\mathbf{w}$ translates every block $w_{2n}w_{2n+1}$ into three quaternary digits where the middle digit is either 1 or 2 as this is binary 01, 10 and the pairs $bc$ are such selected that the end digit of $b$ in binary differs from the first digit of $c$ in binary. Thus $\mathbf{y}^{-1}(1) \cup \mathbf{y}^{-1}(2)$ contains the infinite recursive subset $\{3(n-1)+2 : n \in \mathbb{N}\}$, and according to Proposition 4 the sequence $\mathbf{y}$ is not 4-bi-immune. ☐

## 5. Blind Martingales

In this section we use blind martingales to study recursive transformations preserving bi-immunity.

A martingale is called blind if its bet on $u \in A_b^*$ only depends on the length $|u|$ and not on the actual history coded in $u$; furthermore, the share of the capital betted on a digit $a \in A_b$ is also blindly computed, but the scaling in dependence of the available capital can, of course, be done.

We start with the definition of the *blind martingale*:

**Definition 24.** *A martingale over $A_b$ is referred to as* blind *if there is a family $(\Gamma_\ell)_{\ell \in \mathbb{N}_0}, \emptyset \neq \Gamma_\ell \subseteq A_b$, such that, for $u \in A_b^*$ and $a \in A_b$ it holds*

$$mg(u \cdot a) = \begin{cases} \frac{b}{|\Gamma_{|u|}|} \cdot mg(u), & \text{if } a \in \Gamma_{|u|},\\ 0, & \text{otherwise.} \end{cases}$$

*A blind martingale is* recursive *if the mapping $f : \mathbb{N}_0 \to 2^{A_b}$ with $f(\ell) = \Gamma_\ell$ is recursive.*

We note that $\Gamma_\ell = A_b$ is equivalent to abstaining from betting.

**Proposition 25.** *(a) A sequence $\mathbf{w} \in A_b^\omega$ is b-graph-bi-immune if and only if there is no blind recursive martingale succeeding on $\mathbf{w}$.*

*(b) A sequence $\mathbf{w} \in A_b^\omega$ is b-graph-immune if and only if there is no blind recursive martingale succeeding on $\mathbf{w}$ with $|\Gamma_\ell| = 1$ for infinitely many $\ell \in \mathbb{N}_0$.*

**Proof.** (a) If $\mathbf{w}$ is not $b$-graph-bi-immune then there is a nonempty subset $\Gamma \subset A_b$ for which $\bigcup_{a\in\Gamma} \mathbf{w}^{-1}(a)$ is infinite and not immune. Let $M \subseteq \bigcup_{a\in\Gamma} \mathbf{w}^{-1}(a)$ be infinite and recursive. Then the martingale

$$mg(u \cdot a) = \begin{cases} mg(u), & \text{if } |u| + 1 \notin M, \\ \frac{b}{|\Gamma|} \cdot mg(u), & \text{if } a \in \Gamma \text{ and } |u| + 1 \in M, \\ 0, & \text{otherwise.} \end{cases}$$

succeeds on $\mathbf{w}$.

Conversely, let a blind recursive martingale succeed on $\mathbf{w}$. Since $A_b$ is finite, there is an infinite recursive set $M \subseteq \mathbb{N}_0$ such that for some subset $A \subset A_b$, for all $\ell \in M$, $\Gamma_\ell = A$. Consequently, $M \subseteq \bigcup_{a\in A} \mathbf{w}^{-1}(a)$, and according to Proposition 4, $\mathbf{w}$ is not strongly $b$-graph-bi-immune.

(b) Assume $\mathbf{w}$ to be not $b$-graph-immune. Then the subset $\Gamma \subset A_b$ can be chosen to be a singleton, and the construction is the same as in part (a).

Let a blind recursive martingale succeed on $\mathbf{w}$ with $|\Gamma_\ell| = 1$ for infinitely many $\ell \in \mathbb{N}_0$. As in case (a) there is an infinite recursive set $M \subseteq \mathbb{N}_0$ such that for some $a \in A_b$ and all $\ell \in M$, $\Gamma_\ell = \{a\}$, that is, $M \subseteq \mathbf{w}^{-1}(a)$. Again Proposition 4 shows that $\mathbf{w}$ is not $b$-graph-immune. □

## 6. Mappings Preserving Strong *b*-graph Immunity

For any function $f : \mathbb{N} \to \mathbb{N}$, say that $f$ *preserves strong b-graph-immunity* if for any strongly $b$-graph-immune sequence $\mathbf{w} \in A_b^\omega$, the sequence $\mathbf{v}$ defined by $v_i = w_{f(i)}$ for all $i \in \mathbb{N}$ is strongly $b'$-graph-immune for some $b' \in \{2, \ldots, b\}$.

**Theorem 26.** *1. Suppose $b \geq 3$. Then for all recursive functions $f : \mathbb{N} \to \mathbb{N}$, $f$ preserves strong b-graph-immunity if and only if range$(f)$ is co-finite and $f^{-1}(j) := \{i \in \mathbb{N} : f(i) = j\}$ is finite for all $j \in \mathbb{N}$.*

*2. Suppose $b = 2$. Then for all recursive functions $f : \mathbb{N} \mapsto \mathbb{N}$, $f$ preserves strong b-graph-immunity if and only if range$(f)$ is infinite and $f^{-1}(j) := \{i \in \mathbb{N} : f(i) = j\}$ is finite for all $j \in \mathbb{N}$.*

**Proof.** *Assertion 1.* Let $f$ be any recursive function. Suppose *range*$(f)$ is co-finite and $f^{-1}(j) := \{i \in \mathbb{N} : f(i) = j\}$ is finite for all $j \in \mathbb{N}$. Take any strongly $b$-graph-immune sequence $\mathbf{w} \in A_b^\omega$. By the definition of strong $b$-graph-immunity, *range*$(\mathbf{w}) = A_b$ and every $a \in A_b$ occurs infinitely often in $\mathbf{w}$. As *range*$(f)$ is co-finite, it follows that every $a \in A_b$ occurs infinitely often in the sequence $\mathbf{v} \in A_b^\omega$ given by $v_i = w_{f(i)}$ for all $i \in \mathbb{N}$. Thus for each $a \in A_b$, $\mathbf{v}^{-1}(a)$ is infinite. Since $f^{-1}(j) := \{i \in \mathbb{N} : f(i) = j\}$ is finite for all $j \in \mathbb{N}$, it follows that if $M$ were an infinite recursively enumerable subset of $\mathbf{v}^{-1}(a)$, then $\{f(i) : i \in M\}$ would be an infinite recursively enumerable subset of $\mathbf{w}^{-1}(a)$, contradicting the immunity of $\mathbf{w}^{-1}(a)$. Therefore $\mathbf{v}$ is strongly $b$-graph-immune.

Next, suppose that *range*$(f)$ is co-infinite. We first prove the statement "*range*$(f)$ is co-infinite $\Rightarrow$ $f$ does not preserve strong $b$-graph-immunity" for the case $b = 3$, and then explain at the end how to extend the proof to the case $b > 3$. Consider two cases.

**Case 1:** *range*$(f)$ is finite. Take any strongly $b$-graph-immune sequence $\mathbf{w} \in A_b^\omega$. Without loss of generality, assume that $\{i : f(i) = f(1)\}$ is infinite (otherwise, one may replace 1 by any $i_0 \in \mathbb{N}$ for which $\{i : f(i) = f(i_0)\}$ is infinite in the subsequent argument; such an $i$ exists because *range*$(f)$ is finite). Then $\{i : f(i) = f(1)\}$ is an infinite recursively enumerable subset of $\mathbf{v}^{-1}(v_1) = \mathbf{v}^{-1}(w_{f(1)})$, and so $\mathbf{v}$ is not strongly $b'$-graph-immune for any $b' \in \{2, \ldots, b\}$.

**Case 2:** $\mathit{range}(f)$ is infinite.

Consider any bi-immune set $U$ such that $\mathbb{N} \setminus (\mathit{range}(f) \cup U)$ is infinite. We will show later that such a set $U$ exists. Let $s = \min(\mathit{range}(f) \cap U)$; such an $s$ exists due to the bi-immunity of $U$. Now define a sequence $\mathbf{w} \in A_3^\omega$ as follows. For all $i \in \mathbb{N}$,

$$w_i = \begin{cases} 0, & \text{if } i \in \{s\} \cup (\mathbb{N} \setminus (\mathit{range}(f) \cup U)), \\ 1, & \text{if } i \in U \setminus \{s\}, \\ 2, & \text{if } i \in \mathit{range}(f) \setminus U\,. \end{cases}$$

Let $\mathbf{v}$ be the sequence defined by $v_i = w_{f(i)}$ for all $i \in \mathbb{N}$. Then by construction, $\mathbf{v}^{-1}(0) = \{j \in \mathbb{N} : f(j) = s\}$; the latter set being recursively enumerable (possibly even finite), it follows that $\mathbf{v}$ cannot be a strongly $b'$-graph-immune sequence for any $b' \in \{2, \ldots, b\}$. On the other hand, $\mathbf{w}$ is a strongly 3-graph-immune sequence because:

- $\mathbf{w}^{-1}(0) = \{s\} \cup (\mathbb{N} \setminus (\mathit{range}(f) \cup U))$, which is infinite due to $\mathbb{N} \setminus (\mathit{range}(f) \cup U)$ being infinite by assumption, and $\{s\} \cup (\mathbb{N} \setminus (\mathit{range}(f) \cup U)) \subseteq^* \mathbb{N} \setminus U$. Since $\mathbb{N} \setminus U$ is immune, $\mathbf{w}^{-1}(0)$ must also be immune.
- $\mathbf{w}^{-1}(1) = U \setminus \{s\}$ is an infinite subset of $U$ and so it is immune.
- $\mathbf{w}^{-1}(2) = \mathit{range}(f) \setminus U$ is an infinite subset of $\mathbb{N} \setminus U$; otherwise, $\mathit{range}(f) \subseteq^* U$, which would contradict the immunity of $U$. Therefore, since $\mathbb{N} \setminus U$ is immune, $\mathbf{w}^{-1}(2)$ is also immune.

It remains to show that a set $U$ as chosen above exists. Let $I_0, I_1, I_2, \ldots$ be a one-one enumeration of all infinite recursively enumerable sets. For all $i \in \mathbb{N}$, define $U$ and pairs $(s_{2i-1}, t_{2i-1}), (s_{2i}, t_{2i})$ in stages as follows.

- $(s_{2i-1}, t_{2i-1})$ is any pair of distinct elements belonging to $I_j$ for the least $j$ such that $s_{2i-1}$ and $t_{2i-1}$ are different from any $s_{i'}$ or $t_{i'}$ with $i' < 2i-1$, and $\bigcup_{i'<2i-1}\{s_{i'}\} \subset I_j$ or $\bigcup_{i'<2i-1}\{s_{i'}\} \subset \mathbb{N} \setminus I_j$. Put $s_{2i-1}$ into $U$.
- $(s_{2i}, t_{2i})$ is any pair of distinct elements belonging to $I_j$ for the least $j$ such that $s_{2i}$ and $t_{2i}$ are different from any $s_{i'}$ or $t_{i'}$ with $i' < 2i$, and $s_{2i} \in \mathit{range}(f)$ and $t_{2i} \notin \mathit{range}(f)$. Such $j, s_{2i}$ and $t_{2i}$ exist because the infinitude and coinfinitude of $\mathit{range}(f)$ together imply that there are infinitely many infinite recursively enumerable sets that infinitely intersect both $\mathit{range}(f)$ and $\mathbb{N} \setminus \mathit{range}(f)$. Put $s_{2i}$ into $U$.

By construction, every infinite recursively enumerable set $I_j$ intersects both $U$ and $\mathbb{N} \setminus U$. Thus $U$ is bi-immune. Furthermore, $\mathbb{N} \setminus U$ intersects $\mathbb{N} \setminus \mathit{range}(f)$ infinitely often. Consequently, $\mathbb{N} \setminus (\mathit{range}(f) \cup U)$ is infinite, as required.

To finish this part of the proof, we explain how to convert the strongly 3-graph-immune sequence $\mathbf{w}$ into a strongly $b$-graph-immune one $\mathbf{w}'$ for any $b > 3$. In the definition of $\mathbf{w}$, replace the last condition "$w_i = 2$ if $i \in \mathit{range}(f) \setminus U$" by "$w'_i = k + 2$ if $i \in (\mathit{range}(f) \setminus U) \cap V_k$", where $\{V_0, \ldots, V_{b-3}\}$ is a partition of $\mathit{range}(f) \setminus U$ into $b-2$ infinite sets. For all other values of $i$, $w'_i$ is defined to be $w_i$. Each $V_i$ is an infinite subset of the immune set $\mathbb{N} \setminus U$, and is thus immune too. Therefore $\mathbf{w}' \in A_b^\omega$ and $\mathbf{w}'^{-1}(i)$ is immune for all $i \in \{0, \ldots, b\}$. The same argument as before shows that the sequence $\mathbf{v}'$ with $v'_i = w'_{f(i)}$ for all $i \in \mathbb{N}$ cannot be strongly $b'$-graph-immune for any $b' \in \{2, \ldots, b\}$.

Finally, suppose there is some $j \in \mathit{range}(f)$ such that $f^{-1}(j)$ is infinite. Fix any such $j$. Take any bi-immune set $U'$. Without loss of generality, assume that $j \in U'$ (otherwise, one may replace $U'$ by $\mathbb{N} \setminus U'$ in the subsequent argument). Let $\{U'_0, \ldots, U'_{b-2}\}$ be any partition of $\mathbb{N} \setminus U'$ into $b-1$ infinite sets. Let $\mathbf{w} \in A_b^\omega$ be the sequence for which $w_i = 0$ if $i \in U'$ and $w_i = k+1$ if $i \in U'_k$. The bi-immunity of $U'$ implies that $\mathbf{w}^{-1}(a)$ is immune for every $a \in A_b$, and so $\mathbf{w}$ is strongly $b$-graph-immune. If $\mathbf{v}$ is the sequence given by $v_i = w_{f(i)}$ for all $i \in \mathbb{N}$, then $f^{-1}(j) = \{i \in \mathbb{N} : f(i) = j\}$ is an infinite recursively enumerable subset of $\mathbf{v}^{-1}(0)$. Therefore $\mathbf{v}$ cannot be a strongly $b'$-graph-immune sequence for any $b' \in \{2, \ldots, b\}$.

*Assertion 2.* Suppose $b = 2$, and $f$ is any recursive function such that $range(f)$ is infinite and $f^{-1}(j)$ is finite for all $j \in \mathbb{N}$. As mentioned earlier, all variants of immunity coincide over binary alphabets; thus it suffices to consider 2-graph-immune sequences in the following proof. Let $\mathbf{w} \in A_2^\omega$ be any 2-graph-immune sequence. By the 2-graph-immunity of $\mathbf{w}$, $range(f) \cap \mathbf{w}^{-1}(0)$ and $range(f) \cap \mathbf{w}^{-1}(1)$ are both infinite. Thus the sequence $\mathbf{v}$ defined by $v_i = w_{f(i)}$ for all $i \in \mathbb{N}$ belongs to $A_2^\omega$, and $\mathbf{v}^{-1}(0)$ and $\mathbf{v}^{-1}(1)$ are both infinite. If $M$ were an infinite recursively enumerable subset of $\mathbf{v}^{-1}(0)$, then $\{f(i) : i \in M\}$ would be contained in $\mathbf{w}^{-1}(0)$; moreover, since $f^{-1}(j)$ is finite for all $j \in \mathbb{N}$, $\{f(i) : i \in M\}$ would be an infinite recursively enumerable subset of $\mathbf{w}^{-1}(0)$, contradicting the 2-graph-immunity of $\mathbf{w}$. A similar argument shows that $\mathbf{v}^{-1}(1)$ cannot contain any infinite recursively enumerable subset. Thus $\mathbf{v}$ is 2-graph-immune, as required.

If $range(f)$ is finite, then the argument in Case 1 of the proof of Assertion 1 shows that $f$ cannot be 2-graph-immune-preserving. Finally, if $range(f)$ is infinite and there is some $j \in range(f)$ such that $f^{-1}(j)$ is infinite, then an argument similar to that in the proof of Assertion 1 shows that $f$ is not 2-graph-immune-preserving.

□

**Remark 27.** Suppose a function $f : \mathbb{N} \to \mathbb{N}$ is said to be *strongly b-graph-weakly-immune-preserving* if for any strongly $b$-graph-immune sequence $\mathbf{w} \in A_b^\omega$, the sequence $\mathbf{v}$ defined by $v_i = w_{f(i)}$ for all $i \in \mathbb{N}$ is *b-graph-immune* (in contrast to being strongly $b'$-graph-immune for some $b' \in \{2, \ldots, b\}$). Then any one-one increasing recursive function $f : \mathbb{N} \to \mathbb{N}$ is strongly $b$-weakly-immune-preserving: for each $a \in A_b$, either $\mathbf{v}^{-1}(a) = \{i : w_{f(i)} = a\}$ is finite, or $\{i : w_{f(i)} = a\}$ is infinite; in the latter case, if there were an infinite recursively enumerable subset $M$ of $\{i : w_{f(i)} = a\}$, then, since $f$ is one-one and increasing, the set $\{f(i) : i \in M\}$ would be an infinite recursively enumerable subset of $\mathbf{w}^{-1}(a)$, which would contradict the immunity of $\mathbf{w}^{-1}(a)$.

## 7. Immunity and Bi-immunity for Sequences Over Infinite Alphabets

In this section we introduce and study various notions of (bi-)immunity for sequences over an infinite alphabet. Immunity and bi-immunity for sequences over infinite alphabets are defined almost exactly as they are for sequences over finite alphabets: a graph-immune (resp. graph-bi-immune) sequence $\mathbf{w}$ is one such that no algorithm (with no restriction on the output range) can generate infinitely many, and only correct (resp. incorrect) values of its elements – pairs of the form $(i, w_i)$. Graph-immunity of $\mathbf{w}$ is equivalent to immunity, in the usual recursion-theoretic sense, of the graph of $\mathbf{w}$ as a subset of $\mathbb{N} \times \mathbb{N}_0$; this is analogous to the earlier observation (Proposition 4) that $\mathbf{w}$ is $b$-graph-immune if and only if $b$-*graph*$(\mathbf{w})$ is immune as a set. We also consider sequences that are strictly bounded above by a single recursive function $h$ with $h(i) \geq 2$ for all $i$, or *h-bounded* sequences. Unless otherwise specified, when we refer to a $h$-graph-(bi-)immune sequence, $h$ is always taken to be a generic recursive function such that $h(i) \geq 2$ for all $i$. The terms of such a recursively-bounded sequence may range over an infinite alphabet, though they do not grow too quickly in that they are bounded by a single recursive function. Since no $h$-bounded sequence is graph-bi-immune, as witnessed by $h$ itself, it is fairly natural to define immunity and bi-immunity for $h$-bounded sequences with respect to $h$-bounded partial-recursive functions with an infinite domain. An interesting question, which is partially addressed in this section, is whether, and if so how, the choice of the bound function $h$ influences the computational power of the class of $h$-graph-(bi-)immune sequences. We proceed with the formal definitions of graph-(bi-)immunity.

**Definition 28.** *Let $h$ be a recursive function such that $h(i) \geq 2$ for all $i$. An $h$-*bounded sequence *is any sequence $\mathbf{w} = w_1 w_2 \ldots$ satisfying $w_i < h(i)$ for each $i \in \mathbb{N}$. Let $\mathbf{w} = w_1 w_2 \ldots$ be a sequence.*

*(i)* $\mathbf{w}$ *is* graph-immune *if for every partial-recursive function $g$ with an infinite domain, there is an $i \in \mathrm{dom}(g)$ with $w_i \neq g(i)$.*

*(ii)* $\mathbf{w}$ *is* graph-bi-immune *if for every partial-recursive function $g$ with an infinite domain, there are $i, j \in \mathrm{dom}(g)$ with $w_i = g(i)$ and $w_j \neq g(j)$.*

*(iii)* $\mathbf{w}$ *is $h$-*graph-immune *if* $\mathbf{w}$ *is $h$-bounded and for every partial-recursive function $g$ such that the domain of $g$ is infinite and $g$ is $h$-bounded, there is an $i \in \mathrm{dom}(g)$ with $w_i \neq g(i)$.*

*(iv)* **w** *is* $h$-graph-bi-immune *if* **w** *is $h$-bounded and for every partial-recursive function $g$ such that the domain of $g$ is infinite and $g$ is $h$-bounded, there are $i, j \in \mathrm{dom}(g)$ with $w_i = g(i)$ and $w_j \neq g(j)$.*

**Remark 29.** (I) Definition 28(i) is just a reformulation of the fact that $\{(i, w_i) : i \in \mathbb{N}\}$ is immune as a subset of $\mathbb{N} \times \mathbb{N}_0$. However, Definition 28(ii) does not imply that $\{(i, w_i) : i \in \mathbb{N}\}$ is bi-immune as a subset of $\mathbb{N} \times \mathbb{N}$ since, for example, $\{(1, c) : c \neq w_1\}$ is already an infinite recursive subset of $(\mathbb{N} \times \mathbb{N}) \setminus \{(i, w_i) : i \in \mathbb{N}\}$.

(II) Flajolet and Steyaert introduced the concept of immunity into computational complexity theory by defining an infinite set $U$ to be *immune* for a complexity class $\mathcal{C}$ if $U$ contains no infinite subset belonging to $\mathcal{C}$; an infinite, coinfinite set $U$ is *bi-immune* for $\mathcal{C}$ if $U$ and $\overline{U}$ are both immune for $\mathcal{C}$ [24, 25]. The notion of $h$-graph-immunity may be formulated in a similar fashion: **w** is $h$-graph-immune if $\{\langle i, w_i \rangle : i \in \mathbb{N}\}$ is immune for $\Big\{\{\langle i, \varphi_e(i)\rangle : i \in \mathbb{N}_0\} : e \in \mathbb{N}_0 \wedge \big|\mathrm{dom}(\varphi_e)\big| = \infty \ \wedge (\forall i \in \mathrm{dom}(\varphi_e))[\varphi_e(i) < \ h(i)]\Big\}$. The notions of graph-(bi-)immunity, $h$-graph-bi-immunity and strong $b$-graph-(bi-)immunity may be defined analogously.

Here are some examples of graph-(bi-)immune sequences, as well as $h$-graph-(bi-)immune sequences.

**Example 30.** (I) If $U$ is limit-recursive and non-recursive, then its convergence-module sequence $w^U$ given by $w_i^U := \min\{s' \geq i : \forall s \geq s' \, \forall j \leq i \, [U_s(j) = U(j)]\}$ is a graph-immune sequence, where for each $j$, the uniformly recursive approximation $U_s(j)$ converges to $U(j)$.

(II) Let $\varphi_{e_1}, \varphi_{e_2}, \ldots$ be an enumeration of all partial-recursive functions with infinite domain. For every $i$, let $(a_i, b_i)$ be a pair of elements in the domain of $\varphi_{e_i}$ such that $\{a_i, b_i\} \cap \{a_j, b_j\} = \emptyset$ whenever $i \neq j$. Then for every sequence **w** such that for each $i$, **w** and $\varphi_{e_i}$ agree on exactly one of $\{a_i, b_i\}$ (for example, $w_{a_i} = \varphi_{e_i}(a_i)$ and $w_{b_i} = \varphi_{e_i}(b_i) + 1$), **w** is graph-bi-immune. Thus there are $2^{\aleph_0}$ graph-bi-immune sequences.

(III) Let $h$ be a recursive function with $h(i) \geq 2$ for all $i$. Let $\varphi_{d_1}, \varphi_{d_2}, \ldots$ be an enumeration of all partial-recursive functions with infinite domain such that $\varphi_{d_i}(j)\downarrow < h(j)$ for each $j \in \mathrm{dom}(\varphi_{d_i})$. Let $a_1, a_2, \ldots$ be a strictly increasing sequence such that $\varphi_{d_i}(a_i)\downarrow$ for each $i$. Then the sequence **w** defined by $w_{a_i} = \varphi_{d_i}(a_i)$ for each $i \in \mathbb{N}$ and $w_j = 0$ for each $j \notin \{a_1, a_2, \ldots\}$ is $h$-graph-bi-immune. ☐

We begin by providing equivalent characterisations of ($h$-)graph-(bi-)immunity; these characterisations will be useful later in some proofs.

**Proposition 31.** *Let* $\mathbf{w} = w_1 w_2 \ldots$ *be a sequence.*

(I) **w** *is graph-immune if and only if every partial-recursive $g$ with infinite domain satisfies that $g(i) \neq w_i$ for infinitely many $i \in \mathrm{dom}(g)$.*

(II) **w** *is graph-bi-immune if and only if every partial-recursive $g$ with infinite domain satisfies that $g(i) = w_i$ for infinitely many $i \in \mathrm{dom}(g)$.*

(III) **w** *is graph-bi-immune if and only if for every partial-recursive function $g$ with infinite domain, there is an $i \in \mathrm{dom}(g)$ such that $w_i = g(i)$.*

(IV) *Assertions (I), (II) and (III) hold also for $h$-graph-(bi-)immunity, where* **w** *and $g$ are $h$-bounded for any recursive function $h$ satisfying $h(i) \geq 2$ for all $i$.*

**Proof.** *Assertion (I).* Let $g$ be a partial-recursive function with infinite domain. Suppose on the contrary that $g(i) \neq w_i$ for only finitely many $i \in \mathrm{dom}(g)$. Let $U = \{i \in \mathrm{dom}(g) : g(i) \neq w_i\}$. Define $f$ as follows

$$f(i) = \begin{cases} w_i, & \text{if } i \in U, \\ g(i), & \text{otherwise.} \end{cases} \tag{1}$$

Since $U$ is finite, $f$ is partial-recursive. Moreover, $f(i) = w_i$ for all $i \in \mathrm{dom}(f)$, where $\mathrm{dom}(f) = \mathrm{dom}(g)$ is infinite. This contradicts that $\mathbf{w}$ is graph-immune. Hence, every partial-recursive $g$ with infinite domain satisfies that $g(i) \neq w_i$ for infinitely many $i \in \mathrm{dom}(g)$.

The proof of the converse is trivial.

*Assertion (II).* We prove the contrapositive. Let $g$ be a partial-recursive function with infinite domain such that $g(i) = w_i$ for only finitely many $i \in \mathrm{dom}(g)$. Define $f$ as follows

$$f(i) = \begin{cases} \mathrm{abs}(g(i) - 1), & \text{if } g(i) = w_i, \\ g(i), & \text{otherwise.} \end{cases} \tag{2}$$

Since there are finitely many $i$ such that $g(i) = w_i$, $f$ is partial-recursive. Moreover, $\mathrm{dom}(f) = \mathrm{dom}(g)$ is infinite and $f(i) \neq w_i$ for all $i \in \mathrm{dom}(f)$. Thus $\mathbf{w}$ is not graph-bi-immune. Now, suppose that $\mathbf{w}$ is not graph-bi-immune. Then, there is a partial-recursive function $g'$ with infinite domain such that $g'(i) = w_i$ for all $i \in \mathrm{dom}(g')$ or there is a partial-recursive function $g''$ with infinite domain such that $g''(i) \neq w_i$ for all $i \in \mathrm{dom}(g')$. In the first case define $\hat{g}$ as $\hat{g}(i) = \mathrm{abs}(g'(i) - 1)$. Then, $\hat{g}$ is partial-recursive and $\mathrm{dom}(\hat{g}) = \mathrm{dom}(g')$ is infinite but $\hat{g}(i) \neq w_i$ for all $i \in \mathrm{dom}(\hat{g})$.

Thus in both cases there is a partial-recursive function $f \in \{g'', \hat{g}\}$ with infinite domain such that $f(i) \neq w_i$ for all $i \in \mathrm{dom}(f)$.

*Assertion (III).* Suppose that for every partial recursive function $g$ with infinite domain, there is an $i \in \mathrm{dom}(g)$ such that $w_i = g(i)$. Let $g$ be a partial recursive function. Define $g' : i \to \mathrm{abs}(g(i) - 1)$. Then, for every partial recursive function $g$ with infinite domain, there is a $j \in \mathrm{dom}(g) = \mathrm{dom}(g')$ such that $w_j = g'(j) = \mathrm{abs}(g(j) - 1) \neq g(j)$. So $\mathbf{w}$ is graph-bi-immune.

The proof of the converse is trivial.

*Assertion (IV).* The above proofs also apply for the $h$-bounded version, since if $\mathbf{w}$ and $g$ are both bounded by $h$, then so are the functions constructed in the proofs. ☐

The following series of propositions will establish methods for constructing new $h$-graph-(bi-)immune sequences from given ones. In the subsequent proposition, it is shown that any recursive finite-one function preserves graph-bi-immunity of each $h$-graph-bi-immune sequence, albeit with respect to a recursive bound function that may be different from $h$ in general.

**Proposition 32.** *Assume that $\mathbf{w}$ is $h$-graph-bi-immune and $f$ a recursive finite-one function. Then the function $i \mapsto w_{f(i)}$ is $\tilde{h}$-graph-bi-immune, where $\tilde{h}(i) = h(f(i))$ for all $i$.*

**Proof.** First, note that since $w_i < h(i)$ for all $i$, $w_{f(i)} < \tilde{h}(i)$ for all $i$. Suppose that $\tilde{g}$ is a partial-recursive function with infinite domain such that $\tilde{g}(i) < \tilde{h}(i)$ for all $i \in \mathrm{dom}(\tilde{g})$. Let $f'$ be a partial-recursive function defined such that $f'(i)$ is the first $j \in \mathrm{dom}(\tilde{g})$ found that satisfies $f(j) = i$. Define $g(i) = \tilde{g}(f'(i))$. Then, $g$ is a partial-recursive function with domain $f(\mathrm{dom}(\tilde{g}))$ and $g(i) = \tilde{g}(f'(i)) < \tilde{h}(f'(i)) = h(i)$ for all $i \in \mathrm{dom}(g)$. Since $f$ is finite-one and $\tilde{g}$ has infinite domain, $\mathrm{dom}(g)$ is also infinite. Then there are $i, j \in \mathrm{dom}(g)$ with $w_i = g(i)$ and $w_j \neq g(j)$. Then, $f'(i), f'(j) \in \mathrm{dom}(\tilde{g})$ and $w_{f(f'(i))} = w_i = g(i) = \tilde{g}(f'(i))$ and $w_{f(f'(j))} = w_j \neq g(j) = \tilde{g}(f'(j))$. So, by Proposition 31, the function is $\tilde{h}$-graph-bi-immune. ☐

**Proposition 33.** *Assume that $h, \tilde{h}$ are recursive functions, $\mathbf{w}$ is $h$-graph-bi-immune and $\forall i\,[2 \leq \tilde{h}(i) \leq h(i)]$. Let $\tilde{w}_i = w_i \mod \tilde{h}(i)$ for all $i$. Now $\tilde{\mathbf{w}}$ is $\tilde{h}$-graph-bi-immune.*

**Proof.** Let $g$ be a partial-recursive function with infinite domain such that $g(i) < \tilde{h}(i)$ for all $i \in \mathrm{dom}(g)$. Since $\mathbf{w}$ is $h$-graph-bi-immune and $\tilde{h}(i) \leq h(i)$, by Proposition 31, $g(i) = w_i$ for infinitely many $i$. Since $g$ is strictly bounded by $\tilde{h}$, for all $i$ such that $g(i) = w_i$, we also have that $\tilde{w}_i = w_i$. Hence, $g(i) = \tilde{w}_i$ for infinitely many $i$. So, by Proposition 31, $\tilde{\mathbf{w}}$ is $\tilde{h}$-graph-bi-immune. ☐

**Proposition 34.** *If $\mathbf{w}$ is graph-bi-immune and $h$ is a recursive function such that $h(i) \geq 2$ for all $i$, then $\tilde{\mathbf{w}}$ with $\tilde{w}_i = w_i \mod h(i)$ is $h$-graph-bi-immune.*

**Proof.** Let $g$ be a partial-recursive function with infinite domain such that $g(i) < h(i)$ for all $i \in \mathrm{dom}(g)$. Since $\mathbf{w}$ is graph-bi-immune, by Proposition 31, $g(i) = w_i$ for infinitely many $i \in \mathrm{dom}(g)$. Since $g$ is strictly bounded by $h$, for all $i \in \mathrm{dom}(g)$, if $g(i) = w_i$, then $w_i = \tilde{w}_i$. Hence, $g(i) = \tilde{w}_i$ for infinitely many $i$. So, by Proposition 31, $\tilde{\mathbf{w}}$ is $h$-graph-bi-immune. ☐

**Proposition 35.** *If there is a $U$-recursive sequence $\mathbf{w}$ and an unbounded recursive function $h$ such that $h(i) \geq 2$ for all $i$, and $\mathbf{w}$ is $h$-graph-bi-immune then for any recursive function $\tilde{h}$ with $\forall i\,[\tilde{h}(i) \geq 2]$ it holds that there is a $\tilde{\mathbf{w}} \leq_T U$ such that $\tilde{\mathbf{w}}$ is $\tilde{h}$-graph-bi-immune.*

**Proof.** Let $f(i)$ be the first number $j$ found such that $h(j) \geq \tilde{h}(i)$ and if $i > 0$, $j > f(i-1)$. Since $h$ is unbounded, $f$ is recursive and one-one. Then by Proposition 32, the sequence $i \mapsto w_{f(i)}$ is $h'$-graph-bi-immune where $h'(i) = h(f(i))$ for all $i$. By the definition of $f$, $h'(i) \geq \tilde{h}(i)$ for all $i$. So, by Proposition 33, the sequence $\tilde{\mathbf{w}} : i \mapsto w_{f(i)} \mod \tilde{h}(i)$ is $\tilde{h}$-graph-bi-immune. Moreover, since $\tilde{\mathbf{w}}$ is recursive in $\mathbf{w}$, $\tilde{\mathbf{w}} \leq_T U$. This completes the proof. ☐

The next theorem shows that for every many-one recursive function $h$, the class of $h$-graph-immune sequences is fairly rich; in fact, every non-recursive Turing degree contains such a sequence. The proof is effective in that it shows how to construct such a sequence from any given set in the non-recursive degree.

**Theorem 36.** *Let $h$ be a recursive function such that $h(i) \geq 2$ for all $i$. If $h$ is finite-one then every non-recursive Turing degree contains an $h$-graph-immune sequence.*

**Proof.** Let $\boldsymbol{a}$ be a non-recursive Turing degree. Let $U$ be a set in $\boldsymbol{a}$. Define $w_i = \sum_{m:2^{m+1}<h(i)} 2^m \cdot U(m)$ where $U(m)$ takes the value 1 if $m \in U$ and 0 otherwise.

Let $g$ be a partial-recursive function with infinite domain, bounded by $h$. Suppose that $g(i) = w_i$ for all $i \in \mathrm{dom}(g)$. Since $h$ is finite-one, for any $i$ there must be a $j \in \mathrm{dom}(g)$ such that $h(j) > 2^{i+1}$. Then, $U(i)$ is the $(i+1)$-st digit counted from the right of the binary representation of $g(j)$. So, $U$ is Turing reducible to every recursive enumeration of the graph of $g$. Such recursive enumerations exist and therefore then $U$ would be recursive, a contradiction. Hence, $\mathbf{w}$ must be $h$-graph-immune.

Clearly, $\mathbf{w} \leq_T U$. Moreover, we can determine whether or not $i \in U$ from $\mathbf{w}$ where $h(j) > 2^{i+1}$ as shown earlier. Hence, $\mathbf{w}$ is in $\boldsymbol{a}$. ☐

The next result characterises the Turing degrees containing at least one $h$-graph-immune sequence for any recursive function $h$ such that $h$ takes at least one value infinitely often.

**Theorem 37.** *Let $h$ be a recursive function such that $h(i) \geq 2$ for all $i$. If $h$ takes some value infinitely often then a Turing degree contains an $h$-graph-immune function if and only if it contains a bi-immune set.*

**Proof.** We will use the following lemma to prove the backward direction.

**Lemma 38.** *Let $h, \tilde{h}$ be recursive functions such that $\forall i[\tilde{h}(i) \geq h(i) \geq 2]$. If sequence $\mathbf{w}$ is $h$-graph-immune, then $\mathbf{w}$ is $\tilde{h}$-graph-immune.*

**Proof.** Let $g$ be a partial-recursive function strictly bounded by $\tilde{h}$ with infinite domain. Suppose that $g$ is strictly bounded by $h$. Then, there is an $i \in \mathrm{dom}(g)$ with $w_i \neq g(i)$. Otherwise, there is an $i \in \mathrm{dom}(g)$ such that $g(i) \geq h(i) > w_i$. So, $w_i \neq g(i)$. ☐

Let $\boldsymbol{a}$ be a bi-immune Turing degree. Then, there is a bi-immune set $V$ in $a$. By Proposition 4, the characteristic function of $V$ is 2-graph-immune. Thus, by the above lemma, the characteristic function of $V$ is $h$-graph-immune.

Conversely, suppose that $\boldsymbol{a}$ contains an $h$-graph-immune sequence $\mathbf{w}$. By definition, there is a $c$ such that $h$ takes the value $c$ infinitely often. Then, there is a one-one recursive function $f$ such that $h(f(i)) = c$ for all $i$. Suppose that there is a partial-recursive function $g$ with infinite domain, bounded by $c$ such that $g(i) = w_{f(i)}$ for all $i \in \mathrm{dom}(g)$. Then, there is a partial-recursive function $g' : i \mapsto g(f^{-1}(i))$ where $g(f^{-1}(j)) = w_j$ for all $j \in \mathrm{dom}(g') = f(\mathrm{dom}(g))$. Since $f$ is one-one, $\mathrm{dom}(g')$ is also infinite. This contradicts that $\mathbf{w}$ is $h$-graph-immune. So, $\mathbf{w}(f)$ is $c$-graph-immune. Note that $\mathbf{w}(f)$ is Turing reducible to $\mathbf{w}$.

To show that the degree of $\mathbf{w}$ is bi-immune, we use the following lemma.

**Lemma 39.** *Let $\mathbf{w}^c$ be a c-graph-immune sequence. Then, there is a sequence reducible to $\mathbf{w}^c$ which is 2-graph-immune.*

**Proof.** Suppose that $\mathbf{w}^c$ is $c$-graph-bi-immune. Then, by Proposition 33, the sequence $i \mapsto w_i^c \mod 2$ is 2-graph-bi-immune and so 2-graph-immune. This sequence is Turing reducible to $\mathbf{w}^c$.

Otherwise, suppose that there exists a partial-recursive function $g$ with infinite domain and bounded by $c$ such that $g(i) \neq w_i^c$ for all $i \in \mathrm{dom}(g)$. There exists an $a$ such that $g^{-1}(a)$ is infinite. Without loss of generality, assume that $a = c - 1$. Now we can find a one-one recursive function $f$ such that $g'(i) = g(f(i)) = c - 1$ for all $i$. Then, $w_i^{c-1} = w_{f(i)}^c \neq g(f(i)) = c - 1$ for all $i$. By the $c$-graph-immunity of $\mathbf{w}^c$, $\mathbf{w}^{c-1}$ is thus $(c-1)$-graph-immune. Moreover, $\mathbf{w}^{c-1} \leq_T \mathbf{w}^c$.

By iterating this process repeatedly, we can find a sequence $\mathbf{w}^2$ which is 2-graph-immune and Turing reducible to $\mathbf{w}^c$. ☐

Hence, by the lemma, there is a sequence reducible to $\mathbf{w}$ which is 2-graph-immune and thus is a characteristic sequence of a bi-immune set. By the upward closure of bi-immune degrees (as shown in [30, 32]), the degree $\boldsymbol{a}$ containing $\mathbf{w}$ is also bi-immune. ☐

The following theorem shows that for any unbounded recursive function $h$ with $h(i) \geq 2$ for all $i$, Martin-Löf random sequences of hyperimmune-free degree cannot compute any $h$-graph-bi-immune sequence.

**Theorem 40.** *Let $h$ be a recursive unbounded function which is always at least 2. Then no Martin-Löf random sequence $\mathbf{v}$ which has a hyperimmune-free degree can compute an $h$-graph-bi-immune sequence $\mathbf{w}$.*

**Proof.** Recall from [39] that $\mathbf{v}$ is Martin-Löf random if and only if the prefix-free Kolmogorov complexity $H$ satisfies the inequality $H(v_1 v_2 \ldots v_n) \geq n$ for all sufficiently large $n$.

Now assume that $\mathbf{v}$ has hyperimmune-free Turing degree and $\mathbf{w} \leq_T \mathbf{v}$. Then $\mathbf{w}$ is truth-table reducible to $\mathbf{v}$ (see, for example, [42, Proposition VI.6.18]). Furthermore, there is a recursive function $f$ such that $f$ is strictly ascending and $h(f(n)) > n^3$, as $h$ is unbounded. Furthermore one can for the truth-table reduction choose a use-function which is recursive and one-one; here a use-function is a function which bounds all the queries of the truth-table reduction.

Now let $g$ be a partial-recursive function with the recursive domain $\{f(0), f(1), \ldots\}$ such that $g(f(n))$ is that value $m$ below $h(f(n))$ for which the number of tuples of length $use(f(n))$ mapped by the truth-table reduction to $m$ is the smallest among all possible values. So there are at most $2^{use(f(n))}/n^3$ many strings mapped to $g(f(n))$ by the truth-table reduction and the prefix of $\mathbf{v}$ up to $use(f(n))$ must be among these strings for those $n$ where $w_{f(n)} = g(f(n))$ and there exist infinitely many of those in the case that $\mathbf{w}$ is $h$-graph-bi-immune. So one can describe the string $v_1 v_2 \ldots v_{use(f(n))}$ in a prefix-free way by $H(n)$ bits giving $n$ in a prefix-free way and then compute from $n$ the value $use(f(n))$ and the right choice among the $2^{use(f(n))}/n^3$ possibilities can be selected with a binary number of length $use(f(n)) - 3\log(n)$ plus constant bits.

The length of this binary number can also be computed from $n$. Thus there is a prefix-free code using $H(n) + use(f(n)) - 3\log(n) + d$ bits where $d$ is a constant to describe $v_1 v_2 \ldots v_{use(f(n))}$ infinitely often; as $H(n) \leq 2\log(n) + d'$ where $d'$ is some constant for almost all $n$, there are infinitely many $n$ where $H(v_1 v_2 \ldots v_{use(f(n))}) \leq use(f(n)) + d'' - \log(n)$ for some constant $d''$ and so, for binary sequences $\mathbf{v}$ of hyperimmune-free degree, either $\mathbf{v}$ is not Martin-Löf random or there is no $h$-graph-bi-immune sequence Turing reducible to $\mathbf{v}$. ☐

**Remark 41.** There are Martin-Löf random sequences that have hyperimmune-free degree, so Theorem 40 is not vacuously true. By the characterisation of Martin-Löf randomness via prefix-free Kolmogorov complexity, for any fixed $b$, if $\mathbf{v}^b := \{\mathbf{v} : (\forall n)[H(\mathbf{v} \upharpoonright n) > n - b]\}$, then every member of $\mathbf{v}^b$ is Martin-Löf random. Furthermore, $\mathbf{v}^b$ is a $\Pi_1^0$-class since it is closed and the corresponding tree $T_{\mathbf{v}^b} = \{x : (x \cdot A_2^\omega) \cap \mathbf{v}^b \neq \emptyset\}$ is co-r.e. It is known (see, for example, [41, Theorem 1.8.42]) that every non-empty $\Pi_1^0$ class has a member that is recursively dominated.

The fact that there exist Martin-Löf random sequences with hyperimmune-free degree also implies that the condition in Theorem 40 that the function $h$ be unbounded cannot be lifted: otherwise, taking

$h(i) = 2$ for all $i$, any Martin-Löf random sequence with hyperimmune-free degree would automatically be $h$-graph-bi-immune.

**Remark 42.** Kučera [36] and Gács [27] independently showed that *any* sequence is weak truth-table reducible to some Martin-Löf random sequence. In particular, an $h$-graph-bi-immune sequence is always weak truth-table reducible to a Martin-Löf random sequence. Thus the condition in Theorem 40 that $\mathbf{v}$ be of hyperimmune-free degree is essential.

In contrast to Theorem 40, the next result shows that for any PA-complete set $U$, there is a sequence $\mathbf{w} \leq_T U$ for which $\mathbf{w}$ is $h$-graph-bi-immune.

**Theorem 43.** *Let $h$ be a recursive function with $h(i) \geq 2$ for all $i$. Let $U$ be a PA-complete set. Then there is a sequence $\mathbf{w} \equiv_T U$ such that $\mathbf{w}$ is $h$-graph-bi-immune.*

**Proof.** The proof is based on the fact that PA-complete sets can compute an infinite branch in a finitely branching infinite co-r.e. tree [42, Theorem V.5.35]. The tree will at input $i$ branch with all functions which on input $i$ take one of the values $?, 0, 1, \ldots, h(i) - 1$. Furthermore, let the interval $I_\ell = \{3\ell, 3\ell + 1, 3\ell + 2\}$ and fix a recursive enumeration $\psi_0, \psi_1, \ldots$ of all partial-recursive functions with recursive domains; here $\psi_e$ can either code an undefined place with ? or remain undefined from some point $i$ onwards. The specific domain of $\psi_e$ are those $i$ where $\psi_e(i)$ outputs a natural number (and not ?).

Now a string $\sigma$ satisfies the requirement $E(e)$ if and only if there is an $i \in \mathrm{dom}(\sigma)$ such that $\psi_e(i) \mod h(i) = \sigma(i)$ and $\psi_e(i) \neq ?$. A string $\sigma$ gets cancelled if either there is a requirement $E(e)$ for which there are at least $e + 1$ intervals $I_\ell$ completely covered by the domain of $\sigma$ and which intersect the specific domain of $\psi_e$ but $E(e)$ is not satisfied or if there is an interval $I_\ell$ completely inside the domain of $\sigma$ on which $\sigma$ does not take at least twice the value ?. The cut-off branches of the tree $T$ are all those which extend some cancelled string $\sigma$.

Note that one can, using the oracle for the Halting Problem $K$, construct an infinite branch of this tree such that no prefix $\sigma$ gets cancelled: The algorithm is to find in each $I_\ell$ the smallest $e$ such that on one $i \in I_\ell$, $\psi_e(i)$ is defined and the prefix $\sigma$ up to the beginning of $I_\ell$ does not satisfy the requirement $E(e)$. Let $s_k$ be the smallest such $i \in I_\ell$. Then one lets $\sigma(s_k) = \psi_e(s_k) \mod h(i)$ and $\sigma(j) = ?$ for the two other members $j$ of $I_\ell$.

Note that this priority algorithm blocks the requirement $E(e)$ on at most $e$ many intervals where $\psi_e$ is defined on some member of $I_\ell$; on the first such interval where the requirement is not blocked, a coincidence with $\psi_e$ is put and therefore the requirement is satisfied before the requirement can cancel the branch constructed. Furthermore, it is made sure that always at least two values in $I_\ell$ are assigned a ?.

Note that the tree $T$ of all $\sigma$ which never get cancelled and never have a prefix which gets cancelled is a co-r.e. tree which has an infinite branch and which is finitely branching, due to the bound function $h$. As argued two paragraphs ago, this tree $T$ has infinite branches and since $T$ is co-r.e., the class of all infinite branches of $T$ is a $\Pi^0_1$ class and consequently $U$ allows to compute one such branch $\tilde{\mathbf{w}}$. Now on any interval $I_\ell$ and $i \in I_\ell$, if $\tilde{w}_i = ?$ then $w_i = U(\ell)$ else $w_i = \tilde{w}_i$. The so constructed $\mathbf{w}$ is Turing equivalent to $U$, as $U(\ell)$ is the majority-value of $\mathbf{w}$ on $I_\ell$.

Now consider a partial-recursive function $g$ with infinite domain which is bounded by $h$. This $g$ extends some $\psi_e$ which has an infinite recursive domain; that $\psi_e$ coincides with $\mathbf{w}$ on some $i \in \mathrm{dom}(\psi_e)$. Thus $g$ agrees with $\mathbf{w}$ at least once. Thus $\mathbf{w}$ is $h$-graph-bi-immune. ☐

The notion of a *diagonally non-recursive (d.n.r.) function*, that is, a function $f$ such that $f(e) \neq \varphi_e(e)$ whenever $\varphi_e(e)\downarrow$, arises quite naturally in the study of Martin-Löf randomness. For example, every Martin-Löf random set weak truth-table computes a d.n.r. function [36]. The following observation follows from the definition of $h$-graph-bi-immunity together with the fact that there are infinitely many recursive functions $f$ such that $f(i) < h(i)$ for all $i$.

**Proposition 44.** *Let $h$ be a recursive function with $h(i) \geq 2$ for all $i$. Then no $h$-graph-bi-immune sequence is d.n.r.*

We recall that the Boolean algebra of r.e. sets does not contain any bi-immune set: this follows from an argument by induction, using the fact that the difference between two r.e. sets cannot be bi-immune. A similar observation extends to $h$-graph-bi-immune sequences, as the next proposition shows.

**Proposition 45.** *If $h$ is a recursive function satisfying $h(i) \geq 2$ for all $i$, then the Boolean algebra of r.e. sets does not contain the graph of any $h$-graph-bi-immune sequence.*

**Proof.** Consider any Boolean combination $C_{\mathbf{w}}$ of r.e. sets equal to the graph of some sequence $\mathbf{w}$ such that $w_i < h(i)$ for all $i$; without loss of generality, assume $C_{\mathbf{w}} := \bigcup_{1 \leq i \leq \ell} U_i \setminus V_i$, where, for all $i$, $U_i$ and $V_i$ are r.e. sets for which $U_i \setminus V_i \subseteq \{\langle i', j \rangle : i' \in \mathbb{N}, j < h(i')\}$. Assume further that for each $i$, there are infinitely many $i'$ such that for some $j$, $\langle i', j \rangle \in U_i \setminus V_i$; this assumption will be lifted at the end of the proof. It will be shown by induction that for each $k \leq \ell$, there is a partial-recursive function $g$ with infinite domain and $g(i) < h(i)$ for each $i \in \mathrm{dom}(g)$ such that (i) $\mathit{graph}(g) \subseteq \bigcup_{i \leq k} U_i \setminus V_i$ or (ii) $\mathit{graph}(g) \subseteq \{\langle i, j \rangle : j < h(i)\} \setminus \bigcup_{i \leq k} U_i \setminus V_i$. The induction statement holds for $k = 0$ (the empty union); now assume it holds for some $k$, and let $g$ be a partial-recursive function with infinite domain such that (i) or (ii) holds. If (i) holds, then $\mathit{graph}(g) \subseteq \bigcup_{i \leq k} U_i \setminus V_i \cup (U_{k+1} \setminus V_{k+1}) = \bigcup_{i \leq k+1} U_i \setminus V_i$, so the induction statement for $k+1$ automatically follows. Suppose (ii) holds. Consider two cases.

**Case 1:** $\mathit{graph}(g) \subseteq^* \{\langle i, j \rangle : j < h(i)\} \setminus (U_{k+1} \cup V_{k+1})$. Then there is a partial-recursive function $g'$ and a finite set $F$ with $\mathit{graph}(g') = \mathit{graph}(g) \setminus F$ and $\mathit{graph}(g') \subseteq \{\langle i, j \rangle : j < h(i)\} \setminus \bigcup_{i \leq k+1} U_i \setminus V_i$, so the induction statement (for some partial-recursive $g'$ satisfying (ii)) holds for $k+1$.

**Case 2:** Not Case 1. Then $\mathit{graph}(g) \cap (U_{k+1} \cup V_{k+1})$ is infinite. If $\mathit{graph}(g) \cap V_{k+1}$ is also infinite, then one could enumerate an infinite subgraph $\mathit{graph}(g')$ of $\mathit{graph}(g) \cap V_{k+1}$ for some partial-recursive function $g'$; therefore $\mathit{graph}(g') \subseteq \{\langle i, j \rangle : j < h(i)\} \setminus \bigcup_{i \leq k+1} U_i \setminus V_i$, and again the induction statement (for some partial-recursive $g'$ satisfying condition (ii)) holds for $k+1$. Suppose $\mathit{graph}(g) \cap V_{k+1}$ is finite. Then $\mathit{graph}(g) \cap (U_{k+1} \cup V_{k+1}) = (\mathit{graph}(g) \cap (U_{k+1} \setminus V_{k+1})) \cup (\mathit{graph}(g) \cap V_{k+1}) =^* \mathit{graph}(g) \cap (U_{k+1} \setminus V_{k+1})$.[5] It follows that $\mathit{graph}(g) \cap (U_{k+1} \setminus V_{k+1})$ is an infinite r.e. set equal to the graph of some partial-recursive function $g'$ with $g'(i) < h(i)$ for all $i$, so the induction statement (for some partial-recursive $g'$ satisfying condition (i)) holds for $k+1$.

This completes the proof by induction. To conclude the proof of the original statement, take the union of $C_{\mathbf{w}}$ and the graph of any function $f$ with finite domain such that $f(i) < h(i)$ for all $i$, and consider the case that $\{\langle i, j \rangle : j < h(i)\} \setminus C_{\mathbf{w}}$ contains the graph of some partial-recursive function $g$ with infinite domain and $g(i) < h(i)$ for all $i$ (if, instead, $C_{\mathbf{w}}$ contains such a function $g$, then there is nothing more to prove). Then $\{\langle i, j \rangle : j < h(i)\} \setminus (C_{\mathbf{w}} \cup \mathit{graph}(f)) =^* \{\langle i, j \rangle : j < h(i)\} \setminus C_{\mathbf{w}}$, so $\{\langle i, j \rangle : j < h(i)\} \setminus (C_{\mathbf{w}} \cup \mathit{graph}(f))$ contains the graph of some partial-recursive function $g'$ with infinite domain and $g'(i) < h(i)$ for all $i$, as required. ☐

In the next series of results, we compare the computational power of $h$-graph-bi-immune sequences to that of the Halting Problem $K$ by studying various types of reducibilities between them. The following proposition shows that $K$ is truth-table equivalent to some $h$- graph-bi-immune sequence. Since, as mentioned earlier, every set is weak truth-table reducible to some Martin-Löf random set, and, as shown by Calude and Nies [18], no Martin-Löf random set truth-table computes $K$, it follows that an $h$-bi-immune sequence may not be truth-table reducible to any Martin-Löf random set.

**Proposition 46.** *Suppose $h$ is a recursive function such that $h(i) \geq 2$ for all $i$. Then there is an $h$-graph-bi-immune sequence $\mathbf{w}$ such that $\mathbf{w} \equiv_{tt} K$. In particular, no Martin-Löf random sequence $\mathbf{v}$ satisfies $\mathbf{w} \leq_{tt} \mathbf{v}$.*

**Proof.** We construct a sequence $\mathbf{w}$ satisfying two requirements for each $s$: (1) $\varphi_s(s)\downarrow$ if and only if exactly one of $\{w_{2s+1}, w_{2s+2}\}$ equals 0; (2) if $\mathrm{dom}(\varphi_s)$ is infinite and $\varphi_s(i) < h(i)$ for all $i$, then there is some $j$ satisfying $w_j = \varphi_s(j)$. Requirement (1) codes $K$ into the values of $\mathbf{w}$, while Requirement (2) ensures that no

[5] For any sets $U$ and $V$, we write $U =^* V$ to mean that $U$ is a finite variant of $V$, that is, $(U \setminus V) \cup (V \setminus U)$ is finite.

$h$-bounded partial-recursive function $g$ with infinite domain satisfies $g(i) \neq w_i$ for all $i \in \mathrm{dom}(g)$ (this would in turn ensure that $\mathbf{w}$ is $h$-graph-bi-immune).

In detail: at stage $s$, the following steps are carried out in sequence using oracle $K$:

1. Search for the least $e \leq s$ such that $\varphi_e$ has not yet been diagonalised against and $\varphi_e(2s+1)\downarrow < h(2s+1)$ or $\varphi_e(2s+2)\downarrow < h(2s+2)$. If such an $e$ exists, go to Step 2. If no such $e$ exists, go to Step 3.
2. Let $s'$ be the minimum of $\{2s+1, 2s+2\}$ such that $\varphi_e(s')\downarrow$ and set $w_{s'} = \varphi_e(s')$. Let $s''$ be the unique element of $\{2s+1, 2s+2\} \setminus \{s'\}$, and define
$$w_{s''} = \begin{cases} 1, & \text{if } (w_{s'} = 0 \wedge \varphi_s(s)\downarrow) \vee (w_{s'} \neq 0 \wedge \varphi_s(s)\uparrow), \\ 0, & \text{otherwise.} \end{cases}$$
3. If $\varphi_s(s)\downarrow$, set $w_{2s+1} = 0$ and $w_{2s+2} = 1$. If $\varphi_s(s)\uparrow$, set $w_{2s+1} = w_{2s+2} = 0$.

By construction, $\varphi_s(s)\downarrow$ if and only if exactly one of $\{w_{2s+1}, w_{2s+2}\}$ equals 0. Thus $K$ is btt-reducible to $\mathbf{w}$. To see that $\mathbf{w} \leq_{tt} K$, let $g$ and $f$ be recursive functions such that for all $e$, $s$ and $j$,

$$\varphi_e(s)\downarrow < h(s) \Leftrightarrow g(e,s) \in K,$$
$$\varphi_e(s)\downarrow = j \Leftrightarrow f(e,s,j) \in K.$$

Given any number $2s+1$, the tt-reduction from $\mathbf{w}$ to $K$ makes queries to the given oracle for elements in $\{g(e,t) : e \leq s \wedge t \leq 2s+2\} \cup \{f(e,t,z) : e \leq s \wedge t \in \{2s+1, 2s+2\} \wedge z < \max\{h(j) : j \leq 2s+2\}\} \cup \{s\}$. The reduction then determines $w_{2s+1}$ based on the answers to these queries. First, based on the answers to queries for elements in $\{g(e,t) : e \leq s \wedge t \leq 2s+2\}$, one may determine whether there is a least $e \leq s$ such that $\varphi_e$ has not yet been diagonalised against up to stage $s$ and $\varphi_e(2s+1)\downarrow < h(2s+1)$ or $\varphi_e(2s+2)\downarrow < h(2s+2)$; moreover, if such a least $e$ exists, then its value may be determined. If no such $e$ exists, then $w_{2s+1} = 0$. If such an $e$ exists, then the answers to queries for elements in $\{g(e, 2s+1), g(e, 2s+2), s\} \cup \{f(e,t,z) : t \in \{2s+1, 2s+2\} \wedge z < \max\{h(j) : j \leq 2s+2\}\}$ allow one to determine the least $s' \in \{2s+1, 2s+2\}$ such that $\varphi_e(s')\downarrow$, as well as the value of $\varphi_e(s')$ and whether $\varphi_s(s)\downarrow$; it follows from Step 2 of the earlier algorithm that this information may be used to determine $w_{2s+1}$. We note that this procedure for determining $w_{2s+1}$ is recursive for any oracle (not just $K$). A similar tt-reduction applies to any even number. □

**Remark 47.** Although, as shown in the proof of Proposition 46 $K$ is btt-reducible to some $h$-graph-bi-immune sequence, in general no $h$-graph-bi-immune sequence is btt-reducible to $K$. This follows from Proposition 45 and the fact that a set is btt-reducible to $K$ if and only if it is in the Boolean algebra generated by the r.e. sets [42, Proposition III.8.7]. More generally, we observe in the next proposition that no $h$-graph-bi-immune sequence is *bounded Turing reducible* to any r.e. set.

Any tt-reduction from an $h$-graph-(bi-)immune sequence $\mathbf{w}$ to an r.e. set cannot be *positive*; in other words, the tt-condition in any such reduction must contain negation. For otherwise, one could recursively enumerate infinitely many pairs $(i,j)$ for which the tt-condition is true (which implies that $j = w_i$), thereby contradicting the $h$-graph-(bi-)immunity of $\mathbf{w}$.

If $U$ is a non-recursive r.e. set, then any tt-reduction from $U$ to an $h$-graph-(bi-)immune sequence $\mathbf{w}$ cannot be *conjunctive*, that is, the tt-condition is not a conjunction of positive formulas. For otherwise, given a one-one recursive enumeration $x_0, x_1, x_2, \ldots$ of $U$, one obtains a corresponding enumeration $D_{g(x_0)}, D_{g(x_1)}, D_{g(x_2)}, \ldots$ (for some recursive function $g$) of queried sets such that $D_{g(x_i)} \subseteq \mathit{graph}(\mathbf{w})$ for all $i$. Furthermore, $\bigcup_{i \in \mathbb{N}_0} D_{g(x_i)}$ is infinite; otherwise, $\{g(x_i) : i \in \mathbb{N}_0\}$ would be finite and one could then determine recursively whether $x_i \in U$ for each $i$ via the relation $x_i \in U \Leftrightarrow D_{g(i)} \subseteq \mathit{graph}(\mathbf{w})$. Thus there would be an infinite one-one recursive enumeration of a subset of $\mathit{graph}(\mathbf{w})$, contradicting the $h$-(bi-)immunity of $\mathbf{w}$. Similarly, if $\overline{U}$ is a non-recursive r.e. set, then any tt-reduction from $U$ to an $h$-graph-(bi-)immune sequence cannot be *disjunctive*, that is, the tt-condition is not a disjunction of positive formulas.

We recall that a function $f$ is *bounded Turing reducible* to a set $U$ ($f \leq_{bT} U$) if there is a Turing functional $\Phi_e$ and a constant $c$ such that $f = \Phi_e^U$ and for all $i$, $\Phi_e$ on input $i$ makes at most $c$ queries to the oracle $U$.

**Proposition 48.** *No graph-immune sequence and no $h$-graph-immune sequence is bounded Turing reducible to an r.e. set.*

**Proof.** Assume that $\mathbf{w} \leq_{bT} U$ for an r.e. set $U$ with constant $c$. Now one can for each $i$ define the computation-track of $i$ as the oracle answers given by $U$ while computing $w_i$ followed by a 2. These finite strings have at most length $c+1$. Furthermore, one can define similar strings for approximations $U_s$ to $U$ and observe that those computation-tracks which converge in $s$ states converge from below lexicographically to the computation track for $U$ at $i$. Let $\sigma$ be the lexicographically maximal computation track taken by infinitely many $i$, let $X$ be the set of these $i$. There are only finitely many $i$ in a further set $Y$ where some approximation has a computation track which takes the value $\sigma$ as at those $i \in Y$ the computation track is larger. For that reason, the set $X$ is recursively enumerable as the set of all $i \notin Y$ where at some $s$ the computation track $\sigma$ is taken. For the $i \in X$ one can compute $w_i$ by supplying the oracle answers of $U$ according to the bits in $\sigma$ and will eventually obtain the correct value of $\mathbf{w}$. Thus there is a partial-recursive function with the infinite domain $X$ which coincide with $\mathbf{w}$ on its domain. Thus $\mathbf{w}$ is not graph-immune and also not $h$-graph-immune for any $h$. □

In the next proposition, we observe that the bi-immune-free Turing degrees exclude not only traditional bi-immune sets, but also $h$-graph-bi-immune sequences and graph-bi-immune sequences. This contrasts with Theorem 36, where it was shown that every non-recursive Turing degree contains an $h$-graph-immune set whenever $h$ is a many-one recursive function.

**Proposition 49.** *Let $h$ be a recursive function such that $h(i) \geq 2$ for all $i$. The bi-immune-free Turing degrees do not contain any $h$-graph-bi-immune sequence and also no graph-bi-immune sequence.*

**Proof.** Let $U$ be a set of bi-immune-free Turing degree. Assume that $\mathbf{w} \leq_T U$ is graph-bi-immune or $h$-graph-bi-immune for a suitable $h$; now $\tilde{\mathbf{w}}$ given by $\forall i\,[\tilde{w}_i = w_i \mod 2]$ is 2-graph-bi-immune and thus the characteristic function of a bi-immune set. However, $U$ does not Turing compute any bi-immune set. Therefore such an $\mathbf{w}$ cannot exist. □

It is known (see, for example, [41, Proposition 4.3.11]) that the Martin-Löf random Turing degrees are not closed upwards; the following proposition shows, in contrast, that the degrees of $h$-graph-bi-immune sequences are closed upwards.

**Proposition 50.** *Let $h$ be recursive such that $h(i) \geq 2$ for all $i$. If $\mathbf{w}$ is an $h$-graph-bi-immune sequence and $\mathbf{v}$ is a binary sequence in a hyperimmune-free Turing degree which can compute $\mathbf{w}$ then there is a further $h$-graph-bi-immune sequence within the same Turing degree as $\mathbf{v}$.*

**Proof.** Let $B$ be the set of all binary strings $x$ which are a prefix of the sequence $v_1 v_2 v_3 \ldots$ (written $x \preceq \mathbf{v}$) and assume that there is a recursive set $R$ of strings which contains infinitely many members of $B$ and also infinitely many non-members of $B$. In the case that for each $x \notin B$, the set $R$ contains only finitely many strings extending $x$, then one can compute $B$ in the limit, as for each string of length $n$, one guesses always that the string of length $n$ with the most extensions found so far in $R$ is the member of $B$; this algorithm converges for all $n$ to $v_1 v_2 \ldots v_n$. However, the only binary sequences of hyperimmune-free Turing degree which are limit recursive are the recursive sequences (see, for example, [41, Proposition 1.5.12]) and those do not compute an $h$-graph-bi-immune sequences; hence this case does not occur. Thus there is an $x \notin B$ such that infinitely many extensions of $x$ are in $R$; all these are not in $B$ and $R$ has the infinite recursive subset $\{y \in R : x \preceq y\}$ not containing a member of $B$. This fact will be used in the construction of $\tilde{\mathbf{w}}$ – the sequence with the same Turing degree as $\mathbf{v}$ and is $h$-graph-bi-immune.

One makes a recursive bijection from binary strings to the natural numbers following the length-lexicographic ordering, so the empty string gives 0, the string 0 gives 1, the string 1 gives 2 and the string 00 gives 3. Let $num(x)$ be the natural number assigned to $x$. Now one defines

$$\tilde{w_i} = \begin{cases} v_n, & \text{if } i = num(v_1 v_2 \ldots v_{n-1}), \\ w_i, & \text{if } i = num(y) \text{ for some } y \not\preceq \mathbf{v}, \text{ that is, if } i \notin num(B). \end{cases}$$

One can reconstruct $\mathbf{v}$ recursively from $\tilde{\mathbf{w}}$ as $v_n = \tilde{w}_{num(v_1 v_2 \ldots v_{n-1})}$, so $\mathbf{v} \leq_T \tilde{\mathbf{w}}$. Now consider any partial-recursive function $\tilde{g}$ such that the domain of $\tilde{g}$ is infinite and, for all $i \in \mathrm{dom}(\tilde{g})$, $\tilde{g}(i) < h(i)$ and $\tilde{g}(i) \neq \tilde{w}_i$. The domain of $\tilde{g}$ has an infinite recursive subset $R$ which, as explained above, can be chosen to be disjoint from $num(B)$. Now one defines, for all $i \in R$, $g(i) = \tilde{g}(i)$; for all other $x$, $g(i)$ is undefined. It follows that $g(i) < h(i)$ and $g(i) \neq w_i$ for all $i \in R$. Thus if $\tilde{g}$ witnesses that $\tilde{\mathbf{w}}$ is not $h$-graph-bi-immune then $g$ witnesses that $\mathbf{w}$ is not $h$-graph-bi-immune, in contradiction to the choice. Hence $\tilde{\mathbf{w}}$ is $h$-graph-bi-immune. It was already mentioned that $\mathbf{v} \leq_T \tilde{\mathbf{w}}$. It can also be seen that $\tilde{\mathbf{w}} \leq_T \mathbf{v} \oplus \mathbf{w}$ and, as $\mathbf{w} \leq_T \mathbf{v}$, $\tilde{\mathbf{w}} \equiv_T \mathbf{v}$. Here $\mathbf{w} \oplus \mathbf{v}$ denotes the *join* of two binary sequences $\mathbf{w}$ and $\mathbf{v}$, defined to be the sequence $\mathbf{w}_1\mathbf{v}_1\mathbf{w}_2\mathbf{v}_2\mathbf{w}_3\mathbf{v}_3\ldots$ as usually done in recursion theory. □

## 8. Conclusions

The motivation of this study came from the necessity to find an algorithm to transform an infinite ternary graph-bi-immune sequence into a binary graph-bi-immune sequence. This problem has arisen in the design of a QRNG based on measuring a value-indefinite quantum observable [1, 3, 6, 7]. Each ternary sequence generated by such a QRNG is graph-bi-immnune, which shows that the quality of randomness generated is provable higher than the quality of randomness generated by software. Preserving graph-bi-immunity in algorithmic transformations of infinite ternary graph-bi-immune sequences into a binary sequence turned to be a non-trivial problem: to solve it we had to better understand the notion of graph-bi-immunity on non-binary alphabets, the scope of this paper. Corollary 18 has been used in the design of the QRNG by Agüero and Calude in [8] whose protocol is based on measuring a located form [1, 3, 6, 7, 8] of the Kochen-Specker Theorem, a result true only in Hilbert spaces of dimension at least three. Such a QRNG – which locates and repeatedly measures a value-indefinite quantum observable – *always, not only with probability Lebesgue one* – produces graph-bi-immune sequences, that is, sequences for which no algorithm can compute more than finitely many exact values. In fact, no algorithm can compute *any exact value of any* sequence generated by the QRNG [8]. As almost all applications need quantum random binary strings, there is a stringent demand of randomness-preserving algorithms transforming non-binary strings into binary ones.

In this paper we have studied various notions of $b$-graph-bi-immunity over alphabets with $b \geq 2$ elements and recursive transformations between sequences on different alphabets which preserve them. Furthermore, we have extended the study from sequence bounded by a constant to sequences over the infinite alphabet $\mathbb{N}_0$ which may or may not be bounded by a recursive function, and related them to the Turing degrees in which they can occur.

Finally we mention a few open questions. What is the computational power of algorithms using various bi-immune sequences as oracles [2]? In particular, can the Halting Problem be solved with such an algorithm? A weaker question is to replace the Halting Problem with the lesser principle of omniscience [14]: given a recursive binary sequence $(x_n)$ containing at most one 1, decide whether $x_{2n} = 0$ for each $n \geq 1$ or else $x_{2n+1} = 0$ for each $n \geq 1$.

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