A strong Schottky lemma on n generators for CAT(0) spaces

Matthew J. Conder and Jeroen Schillewaert

Abstract

We give a criterion for a set of n hyperbolic isometries of a CAT(0) metric space X to generate a free group on n generators. This extends a result by Alperin, Farb and Noskov who proved this for 2 generators under the additional assumption that X is complete and has no fake zero angles. Moreover, when X is locally compact, the group we obtain is also discrete. We then apply these results to Euclidean buildings.

1 Introduction

We generalise the main theorem of [1] as follows:

Theorem A. *Let X be a* CAT(0) *metric space. Let $g_1, \dots, g_n$ be hyperbolic isometries of X with axes $A_1, \dots A_n$, where $n \geq 2$. Suppose that for each distinct pair of distinct axes A_i, A_j either:*

(I) *$S_{ij} = A_i \cap A_j$ is a bounded segment, and the two angles $\theta_{ij}^-, \theta_{ij}^+$ between A_i and A_j measured from the two endpoints of S_{ij} are both equal to π (as in the left-hand diagram of Figure 1); or*

(II) *A_i and A_j are disjoint, and there is a geodesic B_{ij} between A_i and A_j such that all four angles between B_{ij} and A_i, A_j are equal to π (as in the right-hand diagram of Figure 1).*

Additionally, suppose that for each $1 \leq i \leq n$ there is an open segment $D_i \subseteq A_i$ of length equal to the translation length of g_i such that

$$\bigcup_{j \neq i} p_i(A_j) \subseteq D_i,$$

[0]Department of Mathematics, University of Auckland, 38 Princes Street, 1010 Auckland, New Zealand. Both authors are supported by a University of Auckland FRDF grant. The first author is also supported by the Woolf Fisher Trust.

where $p_i\colon X \to A_i$ is the geodesic projection map. Then the subgroup of $\mathrm{Isom}(X)$ *generated by $g_1, \ldots, g_n$ is free of rank n and, when X is locally compact, it is also discrete.*

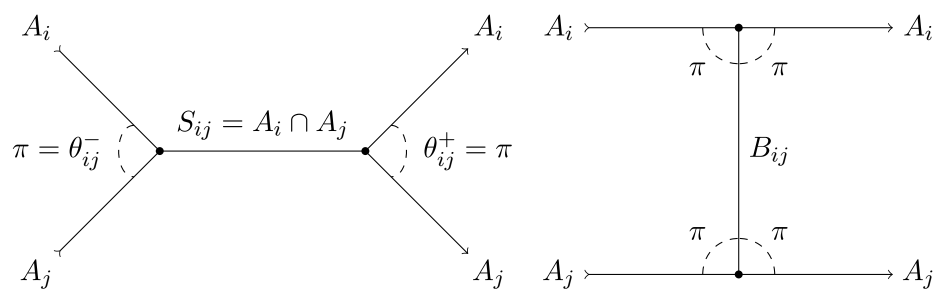

Figure 1: Cases (I) and (II) of Theorem A.

Remark 1.1. *By the angle between two geodesic paths, we mean the upper (or Alexandrov) angle, as defined in* [3, I.1.12].

Remark 1.2. *We only ever consider a topology on* $\mathrm{Isom}(X)$ *when X is locally compact. The topology we use is the compact-open topology, which is equivalent to the topology of pointwise convergence in this setting and this gives* $\mathrm{Isom}(X)$ *the structure of a topological group which acts continuously on X; see* §2.4 Theorem 1 *and* §3.4 Corollary 1 *of* [2, Chapter X].

Theorem A generalises the theorem stated in [1] as we allow for an arbitrary finite number of generators and we no longer require that X is complete and has no fake zero angles. Moreover, we also prove discreteness when X is locally compact, and this generalises a result by Lubotzky for isometries of trees [9, Prop 1.6].

In fact, the main theorem of [1] follows directly from Theorem A when $n = 2$: in case (I), the projection condition implies that S_{ij} has length strictly less than the translation length of both g_1 and g_2 and, in case (II), the projection condition always holds since the projection onto each axis is the unique corresponding endpoint of the geodesic B_{12}.

Remark 1.3. *Karlsson remarked without proof* [6, end of Section 6] *that "the condition of no-fake angles in* [1] *can be removed and the translation lengths do not necessarily have to be strictly greater than the length of S". This is part of what we do here, and our proof is similar to the one in* [1].

Without the requirement for completeness, Theorem A may be applied to CAT(0) spaces which are not necessarily complete, such as certain non-discrete Euclidean buildings; see [11] for some background material.

By [7, 4.6.1-4.6.2] isometries of Euclidean buildings map apartments to apartments, and if the building at infinity is thick, they also map Weyl chambers to Weyl chambers [11, 2.25 and 2.27]. As in [1], we call an isometry f *generic* if none of its parallel axes is contained in any wall of any apartment of X. An isometry f is generic if and only if it has a unique invariant apartment $\mathcal{A}_f$ [11, 2.26]. A generic isometry f determines, for any fixed choice of basepoint $x \in \mathcal{A}_f$, a pair of chambers in $\text{link}(x)$. We say that generic isometries f and g are *opposite* if $\mathcal{A}_f \cap \mathcal{A}_g = \{x\}$ and each of the chambers determined by f is opposite in $\text{link}(x)$ to each of the chambers determined by g.

Corollary B. *Let X be a Euclidean building (where X^∞ is thick) and let $f_1, \cdots f_n$ be hyperbolic isometries of X. If $f_1, \cdots f_n$ are pairwise opposite and the pairwise intersection points of their axes are contained in an open ball of radius at most half the minimum of the translation lengths of $f_1, \cdots f_n$, then $f_1, \cdots f_n$ generate a subgroup of* $\text{Isom}(X)$ *which is free of rank n. If X is locally compact, then this subgroup is also discrete.*

Proof. By [11, Prop 1.12], two halfrays of A_i and A_j emanating from x are contained in an apartment. Thus the projection of A_j onto A_i is equal to their intersection point. By our assumption on the intersection points, and the fact that projection does not increase distances [3, II.2.4(4)], the proof is completed using Theorem A. □

Note that the geometric realisation of a simplicial complex (in particular, of a simplicial building) is locally compact if and only if it is locally finite. When G is a linear semi-simple algebraic group defined over a non-archimedean field k, then the Bruhat-Tits building associated to G [14] is locally compact if and only if k is a local field [12, p.464].

Remark 1.4. *Although all simplicial buildings have a metrically complete* CAT(0) *Davis realisation* [5, 11.1]*, a Euclidean building is not necessarily metrically complete, even if it is a Bruhat-Tits building* [10]*. Moreover, the Cauchy completion of a Euclidean building is not necessarily a Euclidean building* [8, 6.9]*. One can instead use the theory of ultralimits to embed a Euclidean building into a metrically complete Euclidean building* [7]*. However, to prove Corollary B we did not need this.*

2 Proof of Theorem A

We will use the following statement of the Ping Pong Lemma. This generalises the version in [4, 3.3] to an arbitrary finite number of elements. For the discreteness part, we also remove the condition that the topological group G is metrisable.

Lemma 2.1 (The Ping Pong Lemma)**.** *Let G be a group acting on a set X, and let $g_1, \ldots, g_n \in G \backslash \{e\}$. Suppose that $X_1^+, X_1^-, \ldots, X_n^+, X_n^-$ are non-empty, pairwise disjoint subsets of X, which do not cover X and for all $1 \le i \le n$ satisfy*

$$g_i(X \backslash X_i^-) \subseteq X_i^+ \quad \textit{and} \quad g_i^{-1}(X \backslash X_i^+) \subseteq X_i^-.$$

Then the subgroup $H = \langle g_1, \ldots, g_n \rangle \le G$ is free of rank n. In the case that X is a topological space and G is a topological group which acts continuously on X, if each of the subsets $X_1^+, X_1^-, \ldots, X_n^+, X_n^-$ is closed in X, then H is also discrete.

Proof. Set $Y = X_1^+ \cup X_1^- \cup \cdots \cup X_n^+ \cup X_n^-$ and choose $x \in X \backslash Y$. If w is a non-trivial word in $g_1, \ldots, g_n$, then $w(x) \in Y$, therefore $w \neq e$ in G. Hence H is free of rank n.

For the second part, note that H acts continuously on X, that is, the map $H \times X \to X$ is continuous with respect to the product topology. It follows that the inverse image of the open set $X \setminus Y$ is open in $H \times X$. But the intersection of this inverse image with the open set $H \times X \setminus Y$ is $\{e\} \times X \setminus Y$, thus $\{e\}$ is open in H and hence H is discrete. □

Lemma 2.2. *Let $[x, y]$ and $[y, z]$ be geodesics in a* CAT(0) *space. If $\angle_y(x, z) = \pi$, then the concatenation $[x, z] = [x, y] \cup [y, z]$ is a geodesic.*

Proof. By [3, II.1.7(4)] the corresponding angle in the relevant comparison triangle is also π and thus $d(x, z) = d(x, y) + d(y, z)$. □

Proof of Theorem A. Since geodesics are complete convex subsets in CAT(0) spaces, the projection maps p_i we use are well-defined [3, II.2.4].

Note that for each $1 \le i \le n$, the open segment D_i is a fundamental domain for the action of g_i on A_i. Let A_i^+ denote the union of all translates of $\overline{D_i}$ under positive powers of g_i. Similarly, let A_i^- denote the union of all translates of $\overline{D_i}$ under negative powers of g_i. Then A_i^+ and A_i^- are disjoint geodesic rays with $A_i \backslash D_i = A_i^+ \sqcup A_i^-$. Set $X_i^+ = p_i^{-1}(A_i^+)$ and $X_i^- = p_i^{-1}(A_i^-)$ for each i. We will show that these subsets satisfy the hypotheses of the first part of Lemma 2.1.

It is straightforward to check that the subsets $X_1^{\pm}, \ldots, X_n^{\pm}$ are non-empty, closed and that they do not cover X. Each X_i^+ is also disjoint from X_i^-, so to apply Lemma 2.1 we must show that the sets $X_i^{\pm}$ are disjoint from $X_j^{\pm}$ for $i \neq j$. Since we can replace g_i and g_j by their inverses, if necessary, it suffices to show that X_i^+ and X_j^+ are disjoint.

To this end, suppose that $x \in X_i^+ \cap X_j^+$ for some $i \neq j$. Then $p_i(x) \in A_i^+$ and $p_j(x) \in A_j^+$. Note that $p_i(x) \neq p_j(x)$, as otherwise $p_i(x) \in D_i \subseteq A_i \backslash A_i^+$. A similar argument shows that $x \notin A_i \cup A_j$.

In case (I), let y_i and y_j be the (not necessarily distinct) endpoints of S_{ij} which are closest to $p_i(x)$ and $p_j(x)$ respectively. In case (II), let $A_i \cap B_{ij} = \{y_i\}$ and $A_j \cap B_{ij} = \{y_j\}$. By Lemma 2.2, the geodesic $[p_i(x), p_j(x)]$ is the concatenation of geodesics $[p_i(x), y_i] \cup [y_i, y_j] \cup [y_j, p_j(x)]$. In particular, $\angle_{p_i(x)}(x, p_j(x)) = \angle_{p_i(x)}(x, y_i) \geq \frac{\pi}{2}$ and $\angle_{p_j(x)}(x, p_i(x)) = \angle_{p_j(x)}(x, y_j) \geq \frac{\pi}{2}$ by [3, II.2.4(3)]. But the triangle with distinct vertices $x, p_i(x), p_j(x)$ has a Euclidean comparison triangle with corresponding angles which are also at least $\frac{\pi}{2}$ by [3, II.1.7(4)], and this is a contradiction.

It remains to prove that $g_i(X \backslash X_i^-) \subseteq X_i^+$ and $g_i^{-1}(X \backslash X_i^+) \subseteq X_i^-$ for each $1 \leq i \leq n$. As in [1], we first note that p_i commutes with g_i. Indeed, for $x \in X$, $p_i(g_i(x))$ is the unique point on A_i which realises the distance $d(g_i(x), A_i)$. It follows that $p_i(g_i(x)) = g_i(p_i(x))$, since

$$d(g_i(x), A_i) = d(g_i(x), g_i(A_i)) = d(x, A_i) = d(x, p_i(x)) = d(g_i(x), g_i(p_i(x))).$$

Hence if $x \in X \backslash X_i^-$, then $p_i(g_i(x)) = g_i(p_i(x)) \in A_i^+$ i.e. $g_i(x) \in X_i^+$. Similarly, p_i commutes with g_i^{-1} and, if $x \in X \backslash X_i^+$, then $p_i(g_i^{-1}(x)) = g_i^{-1}(p_i(x)) \in A_i^-$ i.e. $g_i^{-1}(x) \in X_i^-$. Thus $g_1, \ldots, g_n$ generate a free group of rank n by the first part of Lemma 2.1.

Finally, we prove discreteness when X is locally compact. The action of $\mathrm{Isom}(X)$ on X is continuous by Remark 1.2 and each of the subsets $X_i^{\pm}$ is closed in X by [3, II.2.4(4)]. Hence the second part of Lemma 2.1 completes the proof of Theorem A. □

3 Two examples

There are isometries of locally compact CAT(0) metric spaces which generate groups which are free but not discrete.

Example 3.1. The matrices $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ generate a free group of rank two (the *Sanov subgroup* [13]). However, viewing them as matrices over the p-adic numbers $\mathbb{Q}_p$, both A and B are infinite order elliptic isometries of the Bruhat-Tits tree T_p corresponding to $\mathbb{Q}_p$. Hence this subgroup of $\mathrm{PSL}_2(\mathbb{Q}_p) \leq \mathrm{Isom}(T_p)$ is not discrete.

We conclude with an example satisfying the conditions of Corollary B.

Example 3.2. Let x be a vertex in a locally finite Euclidean building X of type $\tilde{A}_2$. The link of x is a (not necessarily classical) projective plane π of order n. Chambers in π correspond to incident point-line pairs. Two chambers (p_1, L_1) and (p_2, L_2) are opposite in π if and only if $p_1 \notin L_2$ and $p_2 \notin L_1$.

Consider the set of $n+1$ lines $L_1, \ldots, L_{n+1}$, which each pass through a fixed point p of π and points $p \neq p_i \in L_i$. The corresponding chambers $C_i = (p_i, L_i)$ are pairwise opposite in the link of x. For any $1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$, we can choose k pairs of chambers $P_1, \ldots, P_k$. For each pair $P_i = (C, C')$, select interior points $q \in C$ and $q' \in C'$ which are opposite in the geometric realisation. By [11, 1.12], the Weyl chambers corresponding to the members of P_i are contained in an unique apartment and the halfrays corresponding to q and q' form a geodesic A_i. If these A_i are the axes of hyperbolic isometries f_i then, by construction, these isometries are generic and pairwise opposite and hence, by Corollary B, they generate a subgroup of $\mathrm{Isom}(X)$ which is both discrete and free of rank k.

For example, consider the Bruhat-Tits building Δ associated to $G = \mathrm{SL}_3(\mathbb{Q}_p)$. If f is a generic isometry with unique invariant apartment $\mathcal{A}_f$ then sfs^{-1} has $s\mathcal{A}_f$ as its unique invariant apartment. Choose Weyl chambers and apartments as above. Since G acts strongly transitively on Δ [14] it suffices to find one generic isometry f and then the appropriate conjugates of f will generate a free group, which moreover is discrete since Δ is locally finite. We can take f to be, for instance, the diagonal matrix with entries 1, p and p^{-1}. A more explicit description of the appropriate conjugates of f could then be obtained using similar techniques to [1, 15].

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