

DISCRETE AND FREE TWO-GENERATED SUBGROUPS OF SL_2 OVER NON-ARCHIMEDEAN LOCAL FIELDS

MATTHEW J. CONDER

ABSTRACT. We present a practical algorithm which, given a non-archimedean local field K and any two elements $A, B \in SL_2(K)$, determines after finitely many steps whether or not the subgroup $\langle A, B \rangle \leq SL_2(K)$ is discrete and free of rank two. This makes use of the Ping Pong Lemma applied to the action of $SL_2(K)$ by isometries on its Bruhat-Tits tree. The algorithm itself can also be used for two-generated subgroups of the isometry group of any locally finite simplicial tree, and has applications to the constructive membership problem. In an appendix joint with Frédéric Paulin, we give an erratum to [16, Proposition 1.6], which details some translation length formulae that are fundamental to the algorithm.

1. INTRODUCTION

The problem of deciding whether or not two elements of $SL_2(\mathbb{R})$ generate a free group of rank two has been widely studied in the literature. For instance, the subgroups generated by matrices of the form $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}$ are known to be free of rank two whenever $|\alpha| \geq 2$; this is an easy consequence of the Ping Pong Lemma, applied to the action of $SL_2(\mathbb{R})$ on the hyperbolic plane \mathbb{H}^2 via Möbius transformations. On the other hand, there are many rational values of α in the interval $(-2, 2)$ for which the subgroup generated by the above matrices is not free, and it is an open question to decide whether or not this holds for every such rational α ; see, amongst other papers, [3] and [13].

A key observation in [15] is that arguments involving the Ping Pong Lemma can show that some two-generated subgroups of $SL_2(\mathbb{R})$ which are free are also discrete, with respect to the topology inherited from \mathbb{R}^4 . This helped lead to the discovery of necessary and sufficient conditions, depending on matrix trace, for a two-generated subgroup of $SL_2(\mathbb{R})$ (or, equivalently, of $PSL_2(\mathbb{R})$) to be discrete and free of rank two; see [17] or [18]. Moreover, given any two elements $A, B \in SL_2(\mathbb{R})$, Nielsen transformations can be performed in a ‘trace minimising’ manner to determine whether or not these conditions are satisfied for the subgroup $\langle A, B \rangle \leq SL_2(\mathbb{R})$. This observation (also made in [11] in the context of determining discreteness) forms the basis of a practical algorithm given explicitly in [9], which determines after finitely many steps whether or not a given two-generated subgroup of $SL_2(\mathbb{R})$ (or $PSL_2(\mathbb{R})$) is discrete and free. It is also noted in [9] that this algorithm can be used to solve the constructive membership problem for discrete and free two-generated subgroups of $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$; namely, given such a subgroup G and an element X in the corresponding overgroup, one can determine algorithmically whether or not X lies in G , and if it does, give an explicit representation of X as a word in the generators of G .

Discrete and free two-generated subgroups of SL_2 over other fields, particularly other locally compact fields, are not as well studied. There has been some work done in the case of $\mathrm{SL}_2(\mathbb{C})$ (for instance, see [4]) but the action of this group on hyperbolic space \mathbb{H}^3 is much more complicated to study. Over a non-archimedean local field K , however, the group $\mathrm{SL}_2(K)$ acts by isometries and without inversions on the corresponding Bruhat-Tits tree, and such actions on simplicial trees are very well understood. Given two elements $A, B \in \mathrm{SL}_2(K)$, we will show that Nielsen transformations can be performed in a ‘translation length minimising’ manner until either the subgroup $\langle A, B \rangle \leq \mathrm{SL}_2(K)$ is shown to contain an elliptic element (which is either of finite order or generates an indiscrete infinite cyclic subgroup), or hyperbolic generators of $\langle A, B \rangle$ are found which satisfy the hypotheses of the Ping Pong Lemma. This helped us form the basis of a practical algorithm (Algorithm 4.1) which determines after finitely many steps whether or not a given two-generated subgroup of $\mathrm{SL}_2(K)$ (or, equivalently, of $\mathrm{PSL}_2(K)$) is discrete and free. We will show that this algorithm can also be used more generally (in the context of isometry groups of locally finite simplicial trees) and gives a further algorithm solving the constructive membership problem for such groups that are discrete and free.

In Section 2, we provide some background information on non-archimedean local fields and the group $\mathrm{SL}_2(K)$ defined over such a field K . We describe the Bruhat-Tits tree associated to such groups and some general theory of groups acting on simplicial trees by isometries and without inversions.

Section 3 details the key results leading to Algorithm 4.1; in particular, we show that a discrete and free subgroup of $\mathrm{SL}_2(K)$ cannot contain any elliptic elements, and present a form of the Ping Pong Lemma that gives conditions for a pair of hyperbolic elements to generate a discrete and free subgroup. We also give some important translation length formulae, one of which corrects a formula given in [16, Proposition 1.6]. In the appendix, joint with the author of [16], we give a corrected statement and proof of this proposition.

In Section 4 we present Algorithm 4.1, and prove that it terminates after finitely many steps. We discuss its implementation and give some examples which compare and contrast it with the algorithm from [9].

In Section 5, we show that the same method can be applied to determine whether or not two-generated subgroups of the isometry group of a locally finite simplicial tree are free and discrete, with respect to the topology of pointwise convergence (which, in this setting, is equivalent to the compact-open topology). For any of these subgroups (including those of $\mathrm{SL}_2(K)$) which are discrete and free, we show that there is also a practical algorithm to solve the constructive membership problem.

2. BACKGROUND

A *local field* is a field which is locally compact with respect to the topology induced by some non-trivial absolute value. Such a field K is said to be *non-archimedean* if the corresponding absolute value $|\cdot|$ is non-archimedean, meaning it satisfies the *ultrametric inequality*

$$|a + b| \leq \max\{|a|, |b|\},$$

for all $a, b \in K$. We note that equality holds when $|a| \neq |b|$.

Any local field that does not satisfy the ultrametric inequality is said to be *archimedean*, and is isomorphic to either \mathbb{R} or \mathbb{C} with the same topology as that

induced by the standard absolute values; see [5, Chapter 3, Theorem 1.1]. Non-archimedean local fields are a little different, and have an equivalent characterisation in terms of valuations.

A *valuation* on a field K is a group homomorphism $v: K^\times \rightarrow \mathbb{R}$ such that, when extended by defining $v(0) = \infty$, the ultrametric inequality holds for all $x, y \in K$:

$$v(x + y) \geq \min\{v(x), v(y)\}.$$

We say that v is *discrete* if $v(K^\times) \cong \mathbb{Z}$. Given any valuation v on a field K , the *ring of integers* $\mathcal{O} = \{x \in K : v(x) \geq 0\}$ is a principal ideal domain with unique maximal ideal $\mathcal{P} = \{x \in K : v(x) > 0\}$. The quotient $k = \mathcal{O}/\mathcal{P}$ is called the *residue field* of K . Furthermore, setting $|x|_v = c^{-v(x)}$ for some $c \in (1, \infty)$ defines a non-archimedean absolute value on K . A field K , equipped with discrete valuation v , that is complete with respect to $|\cdot|_v$ and has finite residue field k is a non-archimedean local field. The converse also holds, giving two equivalent definitions of a non-archimedean local field; see [5, Chapter 4] for further details.

For a non-archimedean local field K , the maximal ideal \mathcal{P} is generated by a *uniformiser* $\pi \in \mathcal{O}$ such that $v(\pi) = 1$, and hence the residue field k is of the form $\mathcal{O}/\pi\mathcal{O}$. For a fixed finite set S of representatives of k , every $a \in K^\times$ can be uniquely expressed a sum

$$a = \sum_{i=N}^{\infty} a_i \pi^i,$$

with each $a_i \in S$, and for some integer N such that $a_N \neq 0$; see [5, Chapter 4]. It follows that non-archimedean local fields satisfy the *Bolzano-Weierstrass property*, that is, every bounded sequence (in terms of the corresponding absolute value) has a convergent subsequence.

A common example of a non-archimedean local field is the p -adic numbers, defined using the p -adic valuation v_p on \mathbb{Q} . Namely, if p is a prime and $x \in \mathbb{Q}$ is of the form $p^r \frac{a}{b}$ with $p \nmid a, b$, then $v_p(x) = r$. The corresponding absolute value is usually defined by $|x|_p = p^{-r}$, and the p -adic numbers \mathbb{Q}_p are the completion of \mathbb{Q} with respect to $|\cdot|_p$. Every non-archimedean local field is isomorphic to a finite extension of either \mathbb{Q}_p or the field of formal Laurent series $\mathbb{F}_p((t))$ for some prime p ; see [5, Exercise 25 of Chapter 4 and Lemma 1.1 of Chapter 8].

Given a non-archimedean local field K with associated valuation v , there is a locally finite simplicial tree T_v , called the *Bruhat-Tits tree*, upon which the group $SL_2(K)$ acts. The vertices of T_v are equivalence classes of free \mathcal{O} -modules of rank two (called *lattices*), where lattices L and L' are *equivalent* if $L = xL'$ for some $x \in K^\times$. Furthermore, given a lattice L , each equivalence class of lattices has a unique representative $L_0 \subseteq L$ for which L/L_0 is isomorphic (as an \mathcal{O} -module) to $\mathcal{O}/\pi^n\mathcal{O}$, for some $n \in \mathbb{Z}_{\geq 0}$. This gives rise to the edge structure of T_v , by having edges between the vertices represented by L and L_0 if and only if $n = 1$; for further details, see [19, Chapter II].

There is a natural action of $GL_2(K)$ on the set of lattices, and this gives rise to a faithful action of $PGL_2(K)$ on T_v by isometries. Moreover, the subgroups $SL_2(K)$ and $PSL_2(K)$ act on T_v *without inversions*, that is, no element swaps adjacent vertices; see [14, Corollary II.3.14]. Isometries of a simplicial tree T acting without inversions can be classified based on their *translation length*: given such an isometry

g , this is the integer

$$l(g) = \min_{x \in V(T)} d(x, gx),$$

where $V(T)$ denotes the vertex set of T , and d is the standard path metric on T . Note that $l(g) = l(g^{-1})$ and $l(hgh^{-1}) = l(g)$ for all such isometries g, h of T . Moreover, if $l(g) = 0$, then g fixes a vertex of T and g is said to be *elliptic*. If $l(g) > 0$ then g is said to be *hyperbolic*.

Proposition 2.1. *Suppose that g is a hyperbolic isometry of a simplicial tree T . Then $\{p \in V(T) : d(p, gp) = l(g)\}$ is the vertex set of a straight path in T (called the axis of g) on which g acts by translations of length $l(g)$. Moreover, if a vertex $q \in V(T)$ is at distance k from the axis of g , then $d(q, gq) = l(g) + 2k$.*

Proof. See [19, Chapter I, Proposition 24]. □

Corollary 2.2. *An edge $p - q$ in T is contained in the axis of a hyperbolic element g if and only if $d(p, gp) = d(q, gq)$.*

Elements of $\mathrm{SL}_2(K)$ can be classified as either elliptic or hyperbolic via their action on the Bruhat-Tits tree T_v , and this depends only on the trace:

Proposition 2.3. *If $A \in \mathrm{SL}_2(K)$, then $l(A) = -2 \min\{0, v(\mathrm{tr}(A))\}$.*

Proof. See [14, Proposition II.3.15]. □

3. DISCRETE AND FREE SUBGROUPS

In this section we fix a non-archimedean local field K with valuation v , and present key results which underpin our algorithm that determines whether or not a given two-generated subgroup of $\mathrm{SL}_2(K)$ is discrete and free of rank two. As with the algorithm for two-generated subgroups of $\mathrm{SL}_2(\mathbb{R})$ in [9], we use Nielsen transformations on pairs of generating elements, but in this case we aim to minimise translation lengths until either an elliptic element or a suitable pair of hyperbolic elements is encountered (a similar ‘reduction’ process is used in Section 4 of [8] in the context of free groups of rank two acting on \mathbb{R} -trees). We also show that a group containing an elliptic element cannot be both discrete and free, and give some translation length formulae which allow us to check when a pair of hyperbolic elements generate a discrete and free group of rank two.

First recall that a *Nielsen transformation* takes an n -tuple of elements (g_1, \dots, g_n) of a group and performs some finite sequence of the following operations:

- Swap g_i and g_j (for $i \neq j$);
- Replace g_i by g_i^{-1} ;
- Replace g_i by $g_j^{-1}g_i$ (for $i \neq j$).

This preserves generation of the subgroup generated by g_1, \dots, g_n .

Recall also that a *topological group* is a group equipped with a topology such that the inversion and multiplication maps are continuous. A topological group is said to be *discrete* if the corresponding topology is discrete. Since multiplication by any element is a homeomorphism, such a group is discrete if and only if the set $\{1\}$ is open. Hence any metrisable topological group (in particular, $\mathrm{SL}_2(K)$ - via the subspace topology and metric it inherits from K^4) is discrete if and only if any sequence of elements in the group converging to the identity is eventually constant.

Proposition 3.1. *Let $A \in SL_2(K)$. Then the subgroup $\langle A \rangle \leq SL_2(K)$ is discrete if and only if either A has finite order or $v(\text{tr}(A)) < 0$.*

Proof. Set $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $t = \text{tr}(A)$. If A has finite order then it generates a discrete group, so suppose that $v(t) < 0$, that is, $|t|_v > 1$. Using the ultrametric inequality, we may also assume that $|a|_v > 1$. Let a_n denote the top left entry of the matrix A^n for each $n \in \mathbb{N}$. By the Cayley-Hamilton Theorem we have $A^n = tA^{n-1} - A^{n-2}$ so, if $|a_{n-1}t|_v > |a_{n-2}|_v$, then the ultrametric inequality implies

$$|a_n t|_v > |a_{n-1}t - a_{n-2}|_v = |a_{n-1}t|_v > |a_{n-1}|_v.$$

Since $|a_1 t|_v > 1 = |a_0|_v$, this inductively proves that $|a_n t|_v > |a_{n-1}|_v$ and hence that $|a_{n+1}|_v = |a_n t|_v$ for all $n \in \mathbb{N}$. Thus $|a_n|_v$ tends to ∞ as n does, so $\langle A \rangle$ is discrete.

On the other hand, suppose A has infinite order and $v(t) \geq 0$, that is, $|t|_v \leq 1$. Let a_n, b_n, c_n and d_n denote the corresponding entries of the matrix A^n . Note that if both $|a_{n-1}|_v$ and $|a_{n-2}|_v$ are bounded above, then so is $|a_n|_v$ by the ultrametric inequality and the Cayley-Hamilton Theorem. It follows by induction that $|a_n|_v$ is bounded above for all $n \in \mathbb{N}$. Similarly, $|b_n|_v, |c_n|_v$ and $|d_n|_v$ are bounded above for all $n \in \mathbb{N}$. The Bolzano-Weierstrass property then implies that $\langle A \rangle$ is not discrete. \square

Corollary 3.2. *If $G \leq SL_2(K)$ is discrete and free then $l(g) > 0$ for all $g \in G$.*

Proof. Suppose that $g \in G$ is elliptic. Then either g has finite order, whereby G is not free, or otherwise Proposition 2.3 implies that $v(\text{tr}(A)) \geq 0$. But then G cannot be discrete by Proposition 3.1. \square

We will frequently make use of the following version of the Ping Pong Lemma. As stated, it applies only to metrisable topological groups acting continuously on a topological space; this makes it more specialised than other statements of the lemma (for instance, see [9] or [13]) but it enables us to determine when such a group is not only free, but discrete as well.

Recall that a topological group G acts *continuously* on a topological space X if the map $G \times X \rightarrow X$ (given by $(g, x) \mapsto gx$) is continuous with respect to the product topology. Note that the action of $SL_2(K)$ on the Bruhat-Tits tree T_v is defined by polynomials and is hence continuous.

Lemma 3.3 (The Ping Pong Lemma). *Let G be a metrisable topological group acting continuously on a topological space X and let $g, h \in G \setminus \{1\}$. Suppose that U_+, U_-, V_+, V_- are non-empty closed pairwise disjoint subsets of X which do not cover X and satisfy*

$$\begin{aligned} g(X \setminus U_-) &\subseteq U_+; & g^{-1}(X \setminus U_+) &\subseteq U_-; \\ h(X \setminus V_-) &\subseteq V_+; & h^{-1}(X \setminus V_+) &\subseteq V_- . \end{aligned}$$

Then the subgroup $H = \langle g, h \rangle \leq G$ is discrete and free of rank two.

Proof. Fix some $x \in D = X \setminus (U_+ \cup U_- \cup V_+ \cup V_-) \neq \emptyset$. If $w \in H$ is a non-trivial word in g, h then note that $w(x) \in X \setminus D$ by hypothesis. In particular, this implies $w \neq 1$ in H and thus H is free of rank two. On the other hand, suppose that H is not discrete. Then one can find a sequence $(h_n)_{n \in \mathbb{N}}$ of non-identity elements of

H which converges to $1 \in H$. Since $h_n(x) \in X \setminus D$ for each $n \in \mathbb{N}$ and G acts continuously on X , this gives a sequence $(h_n(x))_{n \in \mathbb{N}}$ of elements of $X \setminus D$ which converges to $x \in D$. But $X \setminus D$ is closed, so this is impossible. Thus H is discrete and free of rank two. See Figure 1. \square

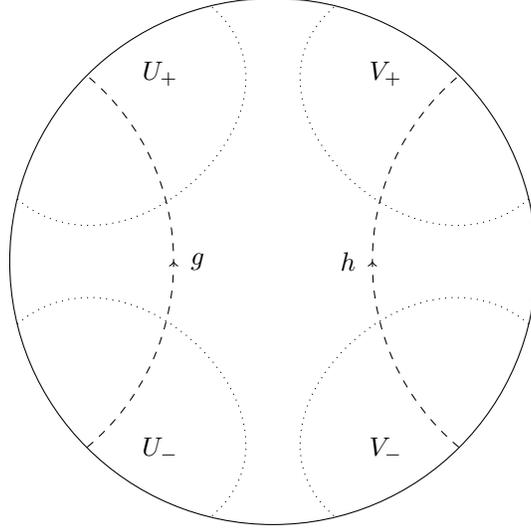


FIGURE 1. The Ping Pong Lemma

Using a version of the Ping Pong Lemma that does not involve discreteness, Lemma 2.6 of [7] shows that two hyperbolic isometries of a \mathbb{R} -tree generate a free group of rank two when their axis overlap is sufficiently small. Lemma 3.2 of [20] generalises this to Λ -trees (where distances take values in some totally ordered abelian group Λ , not necessarily \mathbb{R} or \mathbb{Z}). Here we use our version of the Ping Pong Lemma to prove a similar, but stronger, result for certain hyperbolic isometries of a simplicial tree:

Proposition 3.4. *Let G be a metrisable topological group acting continuously, by isometries and without inversions on a simplicial tree T . Suppose that $A, B \in G$ are hyperbolic, and their axes are either disjoint or intersect along a path of length $0 \leq \Delta(A, B) < \min\{l(A), l(B)\}$. Then the subgroup $\langle A, B \rangle \leq G$ is discrete and free of rank two.*

Proof. First of all, if the axes of A and B are disjoint, then there is a unique path P of minimal distance from a vertex p' on the axis of A to a vertex q' on the axis of B . Choose vertices p and q (on the axes of A and B respectively) so that the interior of the path between p and Ap contains p' , and the interior of the path between q and Bq contains q' (if either A or B has translation length one, then it may be necessary to subdivide each edge of T at its midpoint in order to find such vertices); see the left-hand diagram of Figure 2. On the other hand, if the axes of A and B intersect along a common subpath P of length $\Delta(A, B) < \min\{l(A), l(B)\}$, then choose vertices p and q (on the axes of A and B respectively) such that the interior of the paths between p and Ap , and q and Bq , each contain P (if either

A or B has translation length $\Delta(A, B) + 1$, then it may be necessary to subdivide each edge of T at its midpoint in order to find such vertices); see the right-hand diagram of Figure 2.

In each case, define U_+ (respectively U_-) to be the maximal subtree of T containing all vertices on the axis of A from Ap onwards (respectively up to, and including p) with respect to the direction of translation, but no other vertices on the axis of A . Similarly define V_+ (respectively V_-) as the maximal subtree containing the vertices of the axis of B from Bq onwards (respectively up to, and including, q) but no other vertices on the axis of B . Then, in each case, U_-, U_+, V_- and V_+ are non-empty, pairwise disjoint closed subsets that do not cover T . Moreover, Proposition 2.1 implies that $A(T \setminus U_-) \subseteq U_+$, $A^{-1}(T \setminus U_+) \subseteq U_-$, $B(T \setminus V_-) \subseteq V_+$ and $B^{-1}(T \setminus V_+) \subseteq V_-$. The result then follows from the Ping Pong Lemma. \square

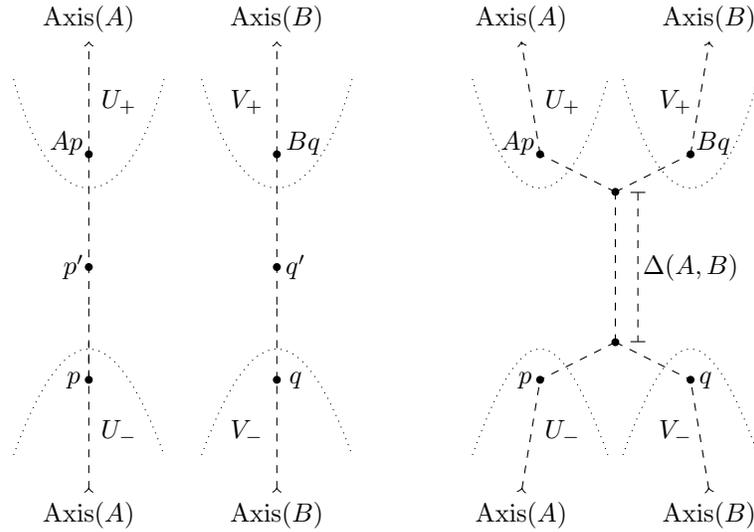


FIGURE 2. Applying the Ping Pong Lemma on trees

Given two hyperbolic isometries A and B of a simplicial tree, determining how their axes interact relies on the following proposition. It is effectively a reformulation of [16, Proposition 1.6] for isometries of simplicial trees, however we provide an extra case (given by case (2)(iii) in our version of the proposition) which was not considered in [16]. In the appendix, we give an erratum to [16, Proposition 1.6] with the author of [16] and prove this extra case in the context of \mathbb{R} -trees.

Proposition 3.5. *Let A and B be hyperbolic isometries of a simplicial tree, such that AB and $A^{-1}B$ act without inversions. Then precisely one of the following holds:*

- (1) *The axes of A and B do not intersect. If k is the minimum distance between the two axes, then*

$$l(AB) = l(A^{-1}B) = l(A) + l(B) + 2k.$$

- (2) The axes of A and B intersect along a (possibly infinite) path of length $\Delta = \Delta(A, B) \geq 0$,

$$\max\{l(AB), l(A^{-1}B)\} = l(A) + l(B),$$

and either:

- (i) $\Delta < \min\{l(A), l(B)\}$ and $\min\{l(AB), l(A^{-1}B)\} = l(A) + l(B) - 2\Delta$;
or
(ii) $\Delta > \min\{l(A), l(B)\}$ and $\min\{l(AB), l(A^{-1}B)\} = |l(A) - l(B)|$;
or
(iii) $\Delta = \min\{l(A), l(B)\}$, either the axes of B and $A^{-1}BA$ (if $l(A) \leq l(B)$) or the axes of A and $B^{-1}AB$ (if $l(A) > l(B)$) intersect along a (possibly infinite) path of length $\Delta' \geq 0$ and

$$\min\{l(AB), l(A^{-1}B)\} = \begin{cases} |l(A) - l(B)| - 2\Delta' & \text{if } \Delta' < \frac{|l(A) - l(B)|}{2} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Proposition 3.5 follows from Proposition A.1, with essentially the same proof. The only difference is that the proof of the third subcase of Proposition A.1 (2)(ii) (corresponding to the second case of Proposition 3.5 (2)(iii)) uses the fact that an isometry of an \mathbb{R} -tree which fixes the midpoint m of some path is elliptic. In the context of simplicial trees, however, this midpoint m could be a vertex or the midpoint of an edge. One can check that the assumption that both AB and $A^{-1}B$ act without inversions is sufficient to ensure that this midpoint m is indeed a vertex and thus $\min\{l(AB), l(A^{-1}B)\} = 0$, as desired. \square

We note that the missing case from [16, Proposition 1.6] was discovered when considering various examples in $\mathrm{SL}_2(\mathbb{Q}_7)$. Namely, given the matrices

$$X = \begin{bmatrix} 7^3 & 0 \\ 0 & \frac{1}{7^3} \end{bmatrix}, Y = \begin{bmatrix} \frac{2}{7^7} & 7^3 \\ \frac{1}{7^3} & 7^7 \end{bmatrix},$$

setting $A = XY$ and $B = X^3Y^3$ yields hyperbolic elements with respective translation lengths of 8 and 32. Moreover, the axes of A^{-1} and B overlap with opposite directions of translation. However $l(A^{-1}B) = 16$, and this is inconsistent with the formula given in case (2)(ii) of [16, Proposition 1.6]; this value is neither $l(B) - l(A)$ nor of the form $l(A) + l(B) - 2\Delta$ for some $\Delta < 8$.

Corollary 3.6. *Let G be a metrisable topological group acting continuously, by isometries and without inversions on a simplicial tree. If $A, B \in G$ are hyperbolic and $|l(A) - l(B)| < \min\{l(AB), l(A^{-1}B)\}$, then $\langle A, B \rangle \leq G$ is discrete and free of rank two.*

Proof. We consider the cases given in Proposition 3.5. If the axes of A and B do not intersect then

$$l(AB) = l(A^{-1}B) \geq l(A) + l(B) > |l(A) - l(B)|.$$

If the axes of A and B do intersect, and $\Delta(A, B) < \min\{l(A), l(B)\}$, then

$$\min\{l(AB), l(A^{-1}B)\} = l(A) + l(B) - 2\Delta(A, B) > |l(A) - l(B)|.$$

Otherwise, we have $\Delta(A, B) \geq \min\{l(A), l(B)\}$ and

$$\min\{l(AB), l(A^{-1}B)\} \leq |l(A) - l(B)|.$$

Hence $|l(A) - l(B)| < \min\{l(AB), l(A^{-1}B)\}$ if and only if the axes of A and B either do not intersect, or intersect along a path of length $0 \leq \Delta(A, B) < \min\{l(A), l(B)\}$. By Proposition 3.4, this implies $\langle A, B \rangle \leq G$ is discrete and free of rank two. \square

We conclude this section by noting that determining whether or not a finitely generated subgroup of $SL_2(K)$ is discrete and free is equivalent to the same problem for the corresponding subgroup in $PSL_2(K)$ (which inherits the quotient topology from $SL_2(K)$).

Proposition 3.7. *Let K be a local field and suppose $G \leq SL_2(K)$ is n -generated. Then G is discrete and free of rank n if and only if the corresponding subgroup $\overline{G} \leq PSL_2(K)$ (its image under the quotient map) is discrete and free of rank n .*

Proof. It is easy to check that G is discrete if and only if \overline{G} is. So consider the restriction of the quotient map $\pi: SL_2(K) \rightarrow PSL_2(K)$ to the epimorphism $\pi_G: G \rightarrow \overline{G}$. Note that $\pi(g) = 1$ if and only if $g = \pm I_2$. So if G is free of rank n then π_G is 1-to-1. Thus $G \cong \overline{G}$ and so \overline{G} must also be free of rank n .

Similarly, if \overline{G} is free of rank n then, by the universal property of free groups, there exists a unique homomorphism $\overline{G} \rightarrow G$ sending the generators of \overline{G} back to their corresponding elements in G . This is an inverse to π_G , showing that $G \cong \overline{G}$ and so G must also be free of rank n . \square

4. THE ALGORITHM

In this section we present our algorithm, which determines after finitely many steps whether or not a two-generated subgroup of $SL_2(K)$ is discrete and free of rank two. The key idea is to use Proposition 2.3 to compute translation lengths on the Bruhat-Tits tree, and perform Nielsen transformations on the generators until these produce either an elliptic element, or two hyperbolic elements satisfying the hypotheses of Corollary 3.6. By Proposition 3.7, the algorithm can also be applied to two-generated subgroups of $PSL_2(K)$ by taking representatives in $SL_2(K)$.

Algorithm 4.1. *Let K be a non-archimedean local field. Given two elements $A, B \in SL_2(K)$, we proceed as follows. If $G = \langle A, B \rangle \leq SL_2(K)$ is discrete and free of rank two then the algorithm will return true and output a generating pair for G which satisfy the hypotheses of the Ping Pong Lemma; otherwise it will return false.*

- (1) Set $X = A, Y = B$. If $l(X) = 0$ or $l(Y) = 0$ then return false.
- (2) If $l(X) > l(Y)$ then swap X and Y .
- (3) Compute $m = \min\{l(XY), l(X^{-1}Y)\}$.
- (4) If $m = 0$ then return false.
- (5) If $m \leq l(Y) - l(X)$ then replace Y by an element from $\{XY, X^{-1}Y\}$ which has translation length m and return to (2).
- (6) Otherwise return true and the generating pair (X, Y) .

Theorem 4.2. *Algorithm 4.1 terminates after finitely many steps and produces the correct output.*

Proof. If at any point the algorithm encounters an elliptic element then G is not discrete and free by Corollary 3.2. So suppose that the algorithm only ever encounters hyperbolic elements. Then it must reach step (5). If $m > l(Y) - l(X)$ then, by Corollary 3.6, G is discrete and free and the elements X and Y satisfy the hypotheses of the Ping Pong Lemma. Otherwise the algorithm performs a Nielsen

transformation, and outputs a new pair of generators for G on which to run the algorithm.

If this sequence of Nielsen transformations never terminates, then there is an infinite sequence $(x_n, y_n) = (l(X_n), l(Y_n))$ of integral translation length pairs which satisfies $0 < x_n \leq y_n$ for all $n \in \mathbb{N}$ and is decreasing in each component; such a sequence must converge. Moreover, for each pair (X_n, Y_n) of generators, we are in either case (2)(ii) or the first subcase of (2)(iii) of Proposition 3.5. Hence, at each stage (x_n, y_n) is replaced by either $(y_n - x_n - k_n, x_n)$ or $(x_n, y_n - x_n - k_n)$ for some $0 \leq k_n < y_n - x_n$. In particular, this implies that $x_{n+1} + y_{n+1} = y_n - k_n$ for all $n \in \mathbb{N}$. Rearranging and taking limits, it follows that $\lim_{n \rightarrow \infty} x_n = -\lim_{n \rightarrow \infty} k_n \leq 0$, a contradiction since each x_n is a positive integer. Hence this algorithm must eventually terminate, proving the theorem. \square

In terms of implementing this algorithm in a computational package such as MAGMA, the software needs to be able to perform matrix multiplications over K , and compute traces and valuations. Since each non-zero element of K can be expressed uniquely in the form $\sum_{i=N}^{\infty} a_i \pi^i$ for some integer N with $a_N \neq 0$ and some uniformiser π , computing valuations and performing both addition and multiplication over K is straightforward. But there is a clear obstacle in the computational storage space needed for elements of K with an infinite expression of the above form. This can theoretically be overcome by storing elements of K in terms of the data $\{\pi; a_N, a_{N+1}, \dots, a_M\}$ up to some appropriate finite M .

Indeed, given matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix},$$

one iteration of Algorithm 4.1 requires firstly computing $l(A) = -2 \min\{0, v(a+d)\}$ and $l(B) = -2 \min\{0, v(e+h)\}$. Since any non-negative valuation gives a translation length of 0, calculating these accurately requires storing the entries of A and B only up to the coefficient of π^0 (that is, $M = 0$ will suffice). On the other hand, assuming that $0 < l(A) \leq l(B)$, the first iteration of Algorithm 4.1 will also require computing $l(AB) = -2 \min\{0, v(ae + bg + cf + dh)\}$ and $l(A^{-1}B) = -2 \min\{0, v(de - bg - cf + ah)\}$. Storing the entries of A and B up to the coefficient of $\pi^{-\min\{0, v(a), v(b), \dots, v(h)\}}$ is sufficient to compute these valuations accurately. It follows inductively that storing the π^i -coefficients of entries of A and B up to $M = -r \min\{0, v(a), v(b), \dots, v(h)\}$ is enough to correctly apply r iterations of Algorithm 4.1. Thus, given any two elements of $\mathrm{SL}_2(K)$, choosing large enough M (compared with $-\min\{0, v(a), v(b), \dots, v(h)\}$) allows the algorithm to run correctly; if, however, at any point the number of iterations exceeds $\frac{M}{-\min\{0, v(a), v(b), \dots, v(h)\}}$, then a higher bound M will need to be chosen and the algorithm restarted.

The examples we discuss below avoid this issue entirely for the case where $K = \mathbb{Q}_p$ for some prime p . By restricting our interest to pairs of matrices in $\mathrm{SL}_2(\mathbb{Q})$, we can perform matrix multiplication and compute traces in the usual sense, and then consider p -adic valuations separately. In this particular case, it is interesting to view the subgroups generated as subgroups of both $\mathrm{SL}_2(\mathbb{Q}_p)$ and $\mathrm{SL}_2(\mathbb{R})$, and compare the properties of each. For instance, it is a well known consequence of the

Ping Pong Lemma that the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

generate a discrete and free subgroup of $SL_2(\mathbb{R})$, whereas the matrices

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

do not. However, neither of these pairs of matrices generate a discrete and free subgroup of $SL_2(\mathbb{Q}_p)$ for any prime p since a matrix of the form $\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix}$ over a non-archimedean local field is elliptic.

One iteration of Algorithm 4.1 also shows that, for any prime p , the matrices

$$A = \begin{bmatrix} p & p-1 \\ \frac{-1}{p} & \frac{1}{p^2} \end{bmatrix}, B = \begin{bmatrix} \frac{2}{p^4} & p^3 \\ \frac{1}{p^3} & p^4 \end{bmatrix}$$

generate a subgroup of $SL_2(\mathbb{Q}_p)$ which is discrete and free of rank two. Using the same matrices as input for the algorithm in [9] shows that they do not generate a free and discrete subgroup of $SL_2(\mathbb{R})$ (this follows since $\text{tr}(AB) = \frac{p+1}{p^3} < 2$, so AB is conjugate to a rotation matrix). On the other hand, for any prime $p \neq 2$, the matrices

$$A = \begin{bmatrix} p & p-1 \\ \frac{-1}{p} & \frac{1}{p^2} \end{bmatrix}, B = \begin{bmatrix} \frac{2}{p^3} & p^4 \\ \frac{1}{p^4} & p^3 \end{bmatrix}$$

generate subgroups of both $SL_2(\mathbb{Q}_p)$ and $SL_2(\mathbb{R})$ which are discrete and free of rank two; this follows from Corollary 3.6 and [9, Theorem 4.4 (b)(iv)] respectively.

Each of these examples requires only one iteration of Algorithm 4.1, but this is certainly not always the case. Indeed, given a prime $p \neq 2$ and positive integer r , the matrices

$$A = \begin{bmatrix} p^3 & 0 \\ 0 & \frac{1}{p^3} \end{bmatrix}, B = \begin{bmatrix} \frac{2}{p^{3r+1}} & p^3 \\ \frac{1}{p^3} & p^{3r+1} \end{bmatrix}$$

generate a discrete and free subgroup of $SL_2(\mathbb{Q}_p)$, and this requires $r+2$ iterations of Algorithm 4.1.

5. GENERALISATIONS AND APPLICATIONS

In this final section we discuss a generalisation of Algorithm 4.1 to two-generator subgroups of the isometry group of any locally finite simplicial tree, and some applications to the constructive membership problem. Recall that, given a finitely generated subgroup $G = \langle g_1, \dots, g_n \rangle$ of some group H , and an element $h \in H$, the constructive membership problem involves determining whether or not h is an element of G , and if it is, finding a word in g_1, \dots, g_n that represents h .

Given any proper metric space X (for instance, a locally finite simplicial tree) the isometry group $\text{Isom}(X)$ (viewed as a subspace of X^X , the space of all continuous maps $X \rightarrow X$ equipped with the product topology) is a metrisable topological group; see [6, Lemmas 5.B.3 and 5.B.5]. This topology is often known as the *topology of pointwise convergence*, in the sense that a sequence (f_i) in $\text{Isom}(X)$ converges to $f \in \text{Isom}(X)$ if and only if the sequence $(f_i(x))$ converges to $f(x)$ for

each $x \in X$. Note that, for a non-archimedean local field K , the group $\mathrm{PSL}_2(K)$ (as a subgroup of the isometry group of the corresponding Bruhat-Tits tree) inherits the topology of pointwise convergence, and this coincides with the standard topology on $\mathrm{PSL}_2(K)$ used in this paper.

In the setting of isometry groups, the topology of pointwise convergence is equivalent to the well-known *compact-open topology*; see [6, Lemmas 5.B.1 and 5.B.2]. The pointwise convergence property of these equivalent topologies leads to an analogue of Corollary 3.2 for isometries of a locally finite simplicial tree. Note that, by subdividing each edge of the tree at its midpoint, if necessary, every element of such an isometry group can be assumed to act without inversions.

Proposition 5.1. *Let T be a locally finite simplicial tree and suppose that $G \leq \mathrm{Isom}(T)$ is discrete (with respect to the topology of pointwise convergence) and free. Then G contains no elliptic elements.*

Proof. Suppose G contains some elliptic element g , which fixes some vertex p of T . There are only finitely many vertices adjacent to p and g acts to permute these. This implies there is some integer n_1 for which g^{n_1} fixes p and all adjacent vertices. One continues inductively to obtain a sequence (g^{n_i}) of elements of $\mathrm{Isom}(T)$, where g^{n_i} fixes all vertices at distance at most i from p . But then $(g^{n_i}(x))$ converges to x for each vertex x of T , so (g^{n_i}) converges to the identity. Thus either g has finite order or G is not discrete. \square

For any proper metric space X , the natural map $\mathrm{Isom}(X) \times X \rightarrow X$ is continuous; see [6, Lemma 5.B.4 (2)]. This implies that Corollary 3.6 can also be applied to the isometry group of a locally finite simplicial tree, when equipped with the topology of pointwise convergence. Thus we have the following generalisation of Algorithm 4.1:

Algorithm 5.2. *Given two elements A and B in the isometry group of a locally finite simplicial tree T , and a method of computing translation lengths, we proceed through steps (1) – (6) of Algorithm 4.1. If $G = \langle A, B \rangle \leq \mathrm{Isom}(T)$ is discrete (with respect to the topology of pointwise convergence) and free of rank two, then the algorithm will return true and output a generating pair for G which satisfies the hypotheses of the Ping Pong Lemma; otherwise it will return false.*

Theorem 5.3. *Algorithm 5.2 terminates after finitely many steps and produces the correct output.*

Proof. The only difference from the proof of Theorem 4.2 is that if the algorithm encounters an elliptic element then G cannot be both discrete and free by Proposition 5.1, instead of Corollary 3.2. \square

Algorithm 5.2 can be applied, for instance, to certain amalgamated free products. Suppose that $\Gamma = H *_C K$ is the amalgamated free product of groups H and K over some subgroup C which is finite index in both H and K . It is well-known that, given fixed transversals T_H and T_K of right coset representatives of C in H and K respectively, each element $g \in \Gamma$ has a unique normal form

$$g = cx_1 \dots x_n,$$

for some integer $n \geq 0$, where $c \in C$ and for each $i \geq 1$, either $x_i \in T_H$ and $x_{i+1} \in T_K$ or vice versa. Moreover, Γ acts faithfully, by isometries, and without inversions, on a locally finite tree T with vertices given by cosets of the form gH or gK and edges given by cosets gC , for $g \in \Gamma$; see [19, Chapter I, Section 4].

Consider the shortest normal form $cx_1 \dots x_{n_0}$ of all conjugates of g in Γ ; such a form is *cyclically reduced* in the sense that either $n_0 = 0, 1$ or x_1 and x_{n_0} lie in different transversals. If n_0 is 0 or 1, then g is conjugate into either A or B and hence $l(g) = 0$. On the other hand, if $n_0 > 1$ then $l(g) = n_0$, which is an even integer; this follows from [2, Lemma 2.25] and [16, Proposition 1.7]. Thus, given such a group Γ and a method of computing a cyclically reduced normal form of each element (such algorithms exist since the transversals T_H and T_K are finite), Algorithm 5.2 can be applied to determine whether or not any two-generated subgroup of Γ is both discrete and free.

We conclude this paper by showing that, as is the case in [9], these algorithms to determine whether or not a subgroup of a certain group is both discrete and free of rank two have applications to the constructive membership problem. This requires the notion of a *fundamental domain*: given a group G acting on a topological space X , this is an open set $D \subseteq X$ such that, if \overline{D} denotes the closure of D in X , then

- (i) $\bigcup_{g \in G} g\overline{D} = X$;
- (ii) $gD \cap hD = \emptyset$ for all distinct $g, h \in G$.

In the proof of Proposition 3.4, given a metrisable topological group G (acting continuously, by isometries, and without inversions on a simplicial tree T) and two hyperbolic elements $A, B \in G$ whose axes are either disjoint or intersect along a sufficiently short path, we found vertices p and q (on the axes of A and B respectively) and considered their images Ap and Bq in order to construct subtrees $U_+, U_-, V_+, V_- \subseteq T$ satisfying the conditions of the Ping Pong Lemma; see Figure 2. Note that in each case D_A , which we define to be the interior of the path between p and Ap (this is isometric to an open interval in \mathbb{R} with integral endpoints, and is hence open in T), is a fundamental domain for the action of $\langle A \rangle$ on $\text{Axis}(A)$. Similarly the open set D_B , defined to be the interior of the path between q and Bq , is a fundamental domain for the action of $\langle B \rangle$ on $\text{Axis}(B)$.

If the axes of A and B do not intersect, then set D to be the union of D_A and D_B with the path between p' and q' ; otherwise, set $D = D_A \cup D_B$. Then the union of images of \overline{D} under the action of $\langle A, B \rangle$ forms a subtree $S \subseteq T$ for which D is a fundamental domain for the action of $\langle A, B \rangle$ on S ; see the proof of [7, Lemma 2.6] for further details. If one replaces the role of T by this subtree S , then $D = T \setminus (U_+ \cup U_- \cup V_+ \cup V_-)$, where U_+, U_-, V_+, V_- are as in Figure 2. Moreover, it follows from the proof of Proposition 3.4 that there is at least one vertex in D . These observations yield the following algorithm:

Algorithm 5.4. *Given a discrete and free two-generated subgroup $G = \langle A, B \rangle$ of $SL_2(K)$ (respectively the isometry group of a locally finite simplicial tree T , along with a method of computing translation lengths) and an element C of the corresponding overgroup, we proceed as follows. If $C \in G$ then the algorithm will return true and output a word $w = w(a, b)$ (where a, b are abstract elements generating a free group F of rank two) such that $w(A, B) = C$; otherwise it will return false.*

- (1) Run Algorithm 4.1 (respectively Algorithm 5.2) on G to obtain generators $X = X(A, B), Y = Y(A, B)$ which satisfy the hypotheses of Proposition 3.4.
- (2) Replacing T by an appropriate subtree, if necessary, find a fundamental domain $D = T \setminus (U_+ \cup U_- \cup V_+ \cup V_-)$ for the action of G on T , and choose a vertex $z' \in D$.
- (3) Set $w = 1 \in F$ and $z = Cz'$.

- (4) While $z \notin D$:
- (i) If $z \in U_{\pm}$, then replace z by $X^{\mp 1}z$ and w by $wa^{\pm 1}$;
 - (ii) If $z \in V_{\pm}$ then replace z by $Y^{\mp 1}z$ and w by $wb^{\pm 1}$.
- (5) If $w(X(A, B), Y(A, B)) = C$ and $z = z'$ then return true and the word $w = w(X(a, b), Y(a, b))$; otherwise return false.

Theorem 5.5. *Algorithm 5.4 terminates after finitely many steps and produces the correct output.*

Proof. We already know that step (1) is correct and terminates after finitely many steps, and step (2) is discussed in the paragraphs preceding the statement of the algorithm. The proof that the rest of the algorithm terminates after finitely many steps, and is correct, is as in [9, Algorithm 1]. \square

Algorithm 5.4 is another practical algorithm which can be implemented, so long as there is a method to determine whether or not a vertex lies in the fundamental domain D and, if it doesn't, which of the subtrees U_-, U_+, V_-, V_+ it belongs to. Note that the proof of Proposition 2.1 implies that, for any hyperbolic isometry A of a simplicial tree T and any vertex x of T (for instance, for the Bruhat-Tits tree T_v , one could take x to be the vertex representing the standard lattice \mathcal{O}^2), the midpoint of the path between x and Ax lies on the axis of A . Similarly, one can obtain a vertex on the axis of a second hyperbolic element B . Thus, after translating these vertices along each axis by appropriate powers of A and B , and comparing distances between them, one should be able to obtain a rough idea of the vertices lying on each axis and hence a method of distinguishing between vertices in U_-, U_+, V_-, V_+ and D .

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APPENDIX A. THE TRANSLATION LENGTH OF THE PRODUCT OF HYPERBOLIC
ISOMETRIES OF \mathbb{R} -TREES

MATTHEW J. CONDER AND FRÉDÉRIC PAULIN

As noticed by the first author of this appendix in the first version of this paper, Assertion (ii) of Proposition 1.6 (2) in [16] is incorrect. Explicit counter-examples are given after the proof of Proposition 3.5. This appendix serves as an erratum of the paper [16] where Proposition 1.6 (2)(ii) therein should be replaced by Assertion (2)(ii) of the following Proposition A.1. Except this replacement, the remainder of the paper [16] is unchanged.

The second author of this appendix is extremely grateful to the first one for finding the mistake and for fixing it.

We keep the notation of [16] in this appendix, in order to facilitate the checking process. In particular, if γ is an hyperbolic isometry of T , then $l(\gamma)$ is its translation length and A_γ is its translation axis. Most of the statements in the following result also follow from [1, Propositions 8.1, 8.3].

Proposition A.1. *Let γ, δ be two hyperbolic isometries of an \mathbb{R} -tree T .*

(1) *Assume that $A_\gamma \cap A_\delta = \emptyset$. Let D be the length of the connecting arc S between A_γ and A_δ . Then S is contained in the translation axis of $\gamma\delta$, and the isometry $\gamma\delta$ translates $S \cap A_\delta$ towards $S \cap A_\gamma$. We have*

$$l(\gamma\delta) = l(\gamma) + l(\delta) + 2D.$$

(2) *Assume that $A_\gamma \cap A_\delta \neq \emptyset$. Let $D \in [0, +\infty]$ be the length of the intersection $A_\gamma \cap A_\delta$, with $D = 0$ if this intersection is reduced to a point, and $D = \infty$ if this intersection is noncompact.*

(i) *Either if $D > 0$ and the translation directions of γ and δ on $A_\gamma \cap A_\delta$ coincide, or if $D = 0$, then*

$$l(\gamma\delta) = l(\gamma) + l(\delta).$$

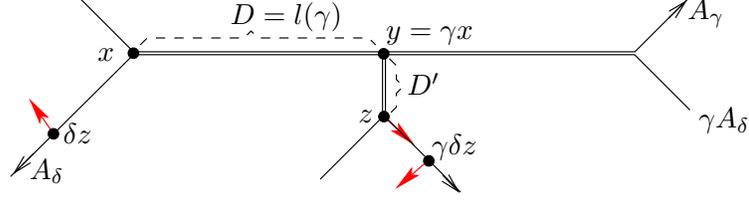
(ii) *Assume that $D > 0$ and the translation directions of γ and δ are opposite on $A_\gamma \cap A_\delta$. Let $D' \in [0, +\infty]$ be the length of the (possibly empty or infinite) segment $A_\delta \cap \gamma A_\delta$ (resp. $A_\gamma \cap \delta A_\gamma$) if $l(\delta) > l(\gamma)$ (resp. $l(\delta) < l(\gamma)$), then*

- $l(\gamma\delta) = l(\gamma) + l(\delta) - 2D$ if $\min\{l(\gamma), l(\delta)\} > D$,
- $l(\gamma\delta) = |l(\gamma) - l(\delta)|$ if $\min\{l(\gamma), l(\delta)\} < D < \max\{l(\gamma), l(\delta)\}$ or $\max\{l(\gamma), l(\delta)\} \leq D$,
- $l(\gamma\delta) = 0$ if $\min\{l(\gamma), l(\delta)\} = D < \max\{l(\gamma), l(\delta)\} \leq D + 2D'$,
- $l(\gamma\delta) = \max\{l(\gamma), l(\delta)\} - D - 2D'$ if $\min\{l(\gamma), l(\delta)\} = D$ and $\max\{l(\gamma), l(\delta)\} > D + 2D'$.

In all four cases, we have $l(\gamma\delta) < l(\gamma) + l(\delta)$.

Proof. We may assume that $l(\gamma) \leq l(\delta)$. The proofs of Assertions (1) and (2)(i), as well as the first two cases of Assertion (2)(ii), are the same ones as in [16], see also [1, Propositions 8.1, 8.3].

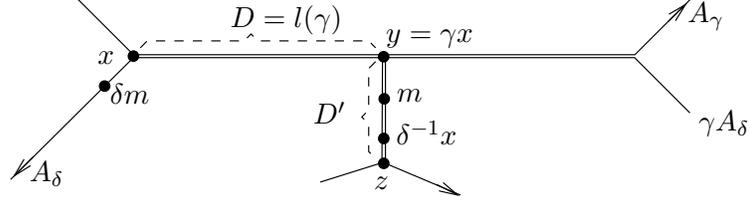
Hence we assume that $l(\gamma) = D < l(\delta)$. In particular D is finite and nonzero, and $A_\gamma \cap A_\delta$ is a compact segment which may be written $[x, y]$ with $y = \gamma x$. We denote by z the point in T such that $[y, z] = \gamma A_\delta \cap A_\delta$, if this segment is compact, or the point at infinity of T such that $[y, z[= \gamma A_\delta \cap A_\delta$ otherwise.



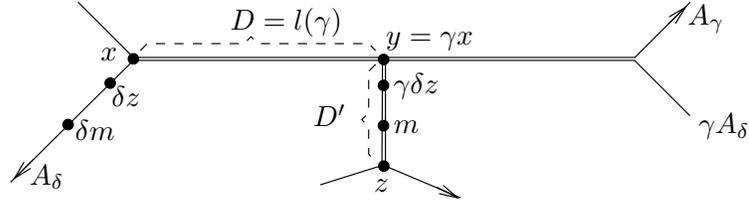
Assume first that $l(\delta) > D + 2D'$, so that in particular D' is finite, $z \in T$ and $D' = d(y, z)$. See the above picture. Since $l(\delta) > D + D'$, the point x belongs to $[z, \delta z]$ and besides $d(x, \delta z) = \ell(\delta) - D - D' > D'$. Therefore $\gamma \delta z$ does not belong to A_δ . The germ at z of the segment from z to $\gamma \delta z$ is hence not sent to the germ at $\gamma \delta z$ of the segment from $\gamma \delta z$ to z . Thus

$$l(\gamma \delta) = d(z, \gamma \delta z) = d(\gamma \delta z, y) - d(y, z) = d(\delta z, x) - d(y, z) = \ell(\delta) - D - 2D',$$

as wanted.



Assume now that $l(\delta) \leq D + D'$. See the above picture. Note that $\delta^{-1}x$ does not belong to A_γ since $l(\delta) > D$, and that $d(\delta^{-1}x, y) = l(\delta) - D \leq D'$. Let m be the midpoint of the segment $[y, \delta^{-1}x]$, so that $d(\delta m, x) = d(m, \delta^{-1}x) = d(m, y)$. Hence $\gamma \delta m$, which is the point of $[y, z]$ (or $[y, z[$ if $D' = +\infty$) at distance $d(\delta m, x)$ from y , is equal to m and $l(\gamma \delta) = 0$, as wanted.



Assume finally that $D + D' < l(\delta) \leq D + 2D'$. See the above picture. In particular D' is finite, $z \in T$ and $D' = d(y, z)$. Note that δz does not belong to A_γ since $l(\delta) > D + D'$, and that

$$d(\delta z, x) = d(\delta z, z) - d(z, y) - d(y, x) = l(\delta) - D - D' \leq D'.$$

Hence $\gamma \delta z \in [y, z]$ and $d(\gamma \delta z, y) = d(\delta z, x) = l(\delta) - D - D'$, so that

$$d(\gamma \delta z, z) = d(z, y) - d(\gamma \delta z, y) = D' - (l(\delta) - D - D') = D + 2D' - l(\delta).$$

Let m be the midpoint of the segment $[\gamma \delta z, z]$, so that $d(m, z) = \frac{1}{2}(D + 2D' - l(\delta))$. Hence

$$d(y, m) = d(y, z) - d(z, m) = \frac{1}{2}(l(\delta) - D).$$

But since m belongs to A_δ and comes after z on A_δ oriented by the translation direction of δ , we have

$$\begin{aligned} d(\delta m, x) &= d(\delta m, \delta z) + d(\delta z, x) = \frac{1}{2}(D + 2D' - l(\delta)) + (l(\delta) - D - D') \\ &= \frac{1}{2}(l(\delta) - D) = d(y, m) \leq D'. \end{aligned}$$

Hence $\gamma\delta m$, which is the point of $[y, z]$ at distance $d(\delta m, x)$ from y , is equal to m and $l(\gamma\delta) = 0$, as wanted. \square

REFERENCES

- [1] R. Alperin and H. Bass. Length functions of group actions on Λ -trees. In *Combinatorial Group Theory and Topology*, volume 111 of *Ann. of Math. Stud.*, pages 265–378. Princeton Univ. Press, 1987.
- [2] S. Alvarez, D. Filimonov, V. Kleptsyn, D. Malicet, C. Meniño Cotón, A. Navas, and M. Triestino. Groups with infinitely many ends acting analytically on the circle. *J. Topol.*, 12(4):1315–1367, 2019.
- [3] A. F. Beardon. Pell’s equation and two generator free Möbius groups. *Bull. London Math. Soc.*, 25(6):527–532, 1993.
- [4] B. H. Bowditch. Markoff triples and quasi-Fuchsian groups. *Proc. London Math. Soc.*, 77(3):697–736, 1998.
- [5] J. W. S. Cassels. *Local Fields*. Cambridge University Press, 1986.
- [6] Y. Cornuier and P. de la Harpe. *Metric Geometry of Locally Compact Groups*. European Mathematical Society, 2016.
- [7] M. Culler and J. W. Morgan. Group actions on \mathbb{R} -trees. *Proc. London Math. Soc.*, 55(3):571–604, 1987.
- [8] M. Culler and K. Vogtmann. The boundary of outer space in rank two. In *Arboreal Group Theory*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 189–230. Springer, 1991.
- [9] B. Eick, M. Kirschmer, and C. Leedham-Green. The constructive membership problem for discrete free subgroups of rank 2 of $SL_2(\mathbb{R})$. *LMS J. Comput. Math.*, 17(1):345–359, 2014.
- [10] D. Gaboriau and G. Levitt. The rank of actions on \mathbb{R} -trees. *Ann. Sci. École Norm. Sup. (4)*, 28(5):549–570, 1995.
- [11] J. Gilman. Two-generator discrete subgroups of $PSL_2(\mathbb{R})$. *Mem. Amer. Math. Soc.*, 117(561), 1995.
- [12] V. Guirardel and G. Levitt. Deformation spaces of trees. *Groups Geom. Dyn.*, 1(2):135–181, 2007.
- [13] R. C. Lyndon and J. L. Ullman. Groups generated by two parabolic linear fractional transformations. *Canadian J. Math.*, 21:1388–1403, 1969.
- [14] J. W. Morgan and P. B. Shalen. Valuations, trees, and degenerations of hyperbolic structures. I. *Ann. of Math. (2)*, 120(3):401–476, 1984.
- [15] M. Newman. Pairs of matrices generating discrete free groups and free products. *Michigan Math. J.*, 15:155–160, 1968.
- [16] F. Paulin. The Gromov topology on \mathbb{R} -trees. *Topology Appl.*, 32(3):197–221, 1989.
- [17] N. Purzitsky. Two-generator discrete free products. *Math. Z.*, 126:209–223, 1972.
- [18] G. Rosenberger. Fuchssche Gruppen, die freies Produkt zweier zyklischer Gruppen sind, und die Gleichung $x^2 + y^2 + z^2 = xyz$. *Math. Ann.*, 199:213–227, 1972.
- [19] J-P. Serre. *Trees*. Springer, 1980. Translated by John Stillwell.
- [20] M. Urbański and L. Zamboni. On free actions on Λ -trees. *Math. Proc. Cambridge Philos. Soc.*, 113(3):535–542, 1993.

M. J. CONDER, DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, CENTRE FOR MATHEMATICAL SCIENCES, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE, CB3 0WB, UNITED KINGDOM

E-mail address: `mjc271@cam.ac.uk`

F. PAULIN, LABORATOIRE DE MATHÉMATIQUE D'ORSAY, UMR 8628 UNIV. PARIS-SUD ET CNRS,
UNIVERSITÉ PARIS-SACLAY, 91405 ORSAY CEDEX, FRANCE
E-mail address: `frederic.paulin@math.u-psud.fr`