SMALL VOLUME HYPERBOLIC MANIFOLDS CONSTRUCTED FROM COXETER GROUPS

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ABSTRACT. Using the symmetric group S_{n+1} and related hyperbolic Coxeter groups $[3, 3, \ldots, 3, 6]$, we construct cusped hyperbolic n-manifolds of small volume having small rank fundamental group, for $n \leq 5$. In particular, we find a cusped orientable arithmetic hyperbolic 5-manifold M_*^5 of volume $13\zeta(3)/2$ with $\mathrm{rank}(\pi_1(M_*^5)) = 3$, starting from the ideal hyperbolic birectified 5-simplex described in a recent paper by the second author.

Keywords. Coxeter group, small index torsion-free subgroup, cusped hyperbolic manifold, volume, ideal hyperbolic birectified 6-cell.

1. INTRODUCTION

The aim of this work is to construct and describe cusped hyperbolic manifolds of small volume with small rank fundamental groups. We are especially interested in the case of n-manifolds of dimension $n \leq 5$ that are commensurable with an ideal k-rectified regular simplex tesselating hyperbolic space $\mathbb{H}^n$ (for some $k \geq 0$). This means that both space forms admit a common finite-sheeted cover, or in other words, that the associated discrete groups of hyperbolic isometries are commensurable (in the wide sense).

It is a classical fact that the thrice-punctured sphere Σ_3 is one of minimal volume among all orientable hyperbolic surfaces. One way to construct the surface Σ_3 is to glue together two ideal triangles S_{reg}^∞, each of area π. Its fundamental group $\pi_1(\Sigma_3)$ has rank 2, is arithmetic, and is commensurable with the hyperbolic Coxeter group $[3, \infty]$ and the modular group $\mathrm{PSL}(2, \mathbb{Z})$.

The Gieseking manifold G is a non-orientable hyperbolic 3-manifold with one cusp, constructible from an ideal (0-rectified) regular tetrahedron $S_{\mathrm{reg}}^\infty \subset \mathbb{H}^3$ with dihedral angle $\frac{\pi}{3}$, by glueing facets together in pairs. Its fundamental group $\pi_1(G)$ has rank 2, and admits the presentation $\langle u, v \mid u^2 v^2 = v u \rangle$. Also $\pi_1(G)$ is arithmetic and commensurable with the hyperbolic Coxeter group $[3, 3, 6]$, as well as with the Eisenstein modular group $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ where $\omega = (-1 + \sqrt{-3})/2$ is a primitive cubic root of unity. In [1], Adams

proved that G has minimal volume ν_3 among all cusped hyperbolic 3-manifolds, with $\nu_3 = \mathrm{vol}_3(S_{\mathrm{reg}}^\infty) = \frac{3^{5/2}}{2\pi^2}\,\zeta_{\mathbb{Q}(\sqrt{-3})}(2)$, where $\zeta_K(s)$ denotes the Dedekind zeta function of the number field K.

Due to constructions of Ratcliffe-Tschantz [21, 23] and Kolpakov-Martelli [16], there are many non-isometric orientable and non-orientable cusped hyperbolic 4-manifolds of small volume. These constructions are almost all based on the ideal regular 24-cell $\mathcal{C}$, whose symmetry group is given by the arithmetic Coxeter group $[3,4,3,4]$. The latter is commensurable with the pseudo-modular group $\mathrm{PSL}(2,\mathbb{Z}[i,j])$ whose coefficients are in the ring of Hamilton integers $\mathbb{Z}[i,j]$; see [10, ch. 15.2]. While the 1171 examples of Ratcliffe and Tschantz described in [21] are of minimal volume $\nu_4 = \frac{4\pi^2}{3}$, with 5 or 6 cusps, the manifold of volume $4\nu_4$ constructed by Kolpakov and Martelli has only one cusp. Recently, in [23], Ratcliffe and Tschantz showed that there are exactly four hyperbolic 4-manifolds obtained from $\mathcal{C}$, with a single cusp and of volume ν_4, up to isometry. These four manifolds are all non-orientable.

In [22, Theorem 2], Ratcliffe and Tschantz classified all the orientable hyperbolic 5-manifolds whose fundamental groups are torsion-free subgroups of (minimal possible) index 32 in the congruence two subgroup $\Gamma(2)$ of positive units of the Lorentzian quadratic form $x_1^2 + \cdots + x_5^2 - x_6^2$. The group $\Gamma(2)$ is a hyperbolic Coxeter group associated with a non-compact right-angled fundamental polyhedron of volume $\frac{7\zeta(3)}{8}$ having 16 facets. One of these 5-manifolds has a symmetry group of order 16 that acts freely on the manifold. It turns out that the resulting quotient manifold N^5 has 2 cusps and volume $\frac{7\zeta(3)}{4}$, which is the smallest known volume of a complete hyperbolic 5-manifold; see [22, Section 9].

In this work, we construct cusped hyperbolic n-manifolds for $3 \le n \le 5$, by considering ideal $(n-3)$-rectified regular simplices $r_{n-3}\hat{S}_{\mathrm{reg}}$ that give rise to a tesselation of $\mathbb{H}^n$. The symmetry group of such an object $r_{n-3}\hat{S}_{\mathrm{reg}} \subset \mathbb{H}^n$ is related to the symmetric group S_{n+1} and to the hyperbolic Coxeter groups $\hat{\Gamma}_n = [3,3,\ldots,3,6]$ (or their duals $[6,3,\ldots,3,3]$) of rank $n+1$. Our strategy is to find the smallest index of a torsion-free subgroup in each of these Coxeter groups $\hat{\Gamma}_n$, for small n, and then show that the smallest index of a torsion-free subgroup in the Coxeter group $[6,3,3,3,3,6]$ is equal to 2880.

For $n = 3$, the group $\hat{\Gamma}_3 = [3,3,6]$ is related to the ideal regular tetrahedron $S_{\mathrm{reg}}^\infty = r_0\hat{S}_{\mathrm{reg}} \subset \mathbb{H}^3$, and we rediscover Gieseking's single-cusped manifold G as described above.

For $n = 4$, we find a torsion-free subgroup Γ_*^4 of minimum index 720 in the arithmetic Coxeter pyramid group $\Gamma_4 = [\infty, 3,3,3,6]$ of covolume $\frac{\pi^2}{540}$; see [13]. The related quotient space $M_*^4 = \mathbb{H}^4/\Gamma_*^4$ yields a non-orientable 4-cusped hyperbolic manifold of Euler characteristic $\chi(M_*^4) = 1$. The group Γ_4 is the finite-covolume counterpart of the group $\hat{\Gamma}_4$, and it is incommensurable to the Coxeter simplex group $[3,4,3,4]$; see [8]. Furthermore, Γ_4

contains as a subgroup of index $5! = 120$ the group generated by the reflections in the facets of the ideal 1-rectified Coxeter 5-cell $\mathcal{R} = r_1 \hat{S}_{\text{reg}} \subset \mathbb{H}^4$. The manifold M_*^4 is commensurable to the orientable manifold N^4 with $\chi(N^4) = 1$ constructed by Riolo and Slavich in a different way; see [24, 25].

For $n = 5$, we find there is an orientable 2-cusped hyperbolic manifold M_*^5 of small volume and with small rank fundamental group, closely related to the ideal birectified 6-cell $\mathcal{B} = r_2 \hat{S}_{\text{reg}}$ with dihedral angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$. The polyhedron $\mathcal{B}$ is highly symmetric, and can be barycentrically decomposed into $6! = 720$ Coxeter pyramids with symbol $[6, 3, 3, 3, 3, 6]$, with each one having volume $\frac{13\zeta(3)}{5760}$; see [15].

The fundamental group of the manifold M_*^5 is isomorphic to a torsion-free subgroup of minimal possible index (namely 2880) in the arithmetic group $\Gamma_5 = [6, 3, 3, 3, 3, 6]$. (By the way, Γ_5 also contains a subgroup of infinite index isomorphic to $\hat{\Gamma}_5 = [6, 3, 3, 3, 3]$.) The torsion-free subgroup of index 2880 in Γ_5 was found with the help of the software system MAGMA [2] and methods analogous to those used by the first author in [4]. In particular, the volume of M_*^5 is $\frac{13\zeta(3)}{2}$, and its symmetry group $I(M_*^5)$ has order 24. Furthermore, the group $\pi_1(M_*^5)$ has rank 3, with a presentation in terms of three orientation-preserving isometries, where two are loxodromic elements of equal translation length, and the third is parabolic.

In comparison with the volume $\frac{7\zeta(3)}{4}$ of the manifold N^5 found by Ratcliffe and Tschantz [22], the manifold M_*^5 is larger than N^5. Observe that the fundamental groups of M_*^5 and N^5, as with the associated Coxeter groups, are incommensurable; see [8]. Now if there were a subgroup H of $I(M_*^5)$ of order $|H| \geq 4$ acting without fixed points on M_*^5, as in [22], then the quotient manifold M_*^5/H would have volume strictly less than that of N^5, but it turns out that there is no such subgroup in $I(M_*^5)$.

The rest of this paper is organised as follows. In Section 2 we provide some necessary background about hyperbolic Coxeter groups, Coxeter polyhedra and their description by Coxeter symbols, and for $3 \leq n \leq 5$, we present the relation between the Coxeter groups $\hat{\Gamma}_n = [3, \ldots, 3, 6]$, their associated groups Γ_n of known finite covolume, and ideal $(n-3)$-rectified regular simplices in $\mathbb{H}^n$. In Section 3, we first prove Theorem 1, concerning the smallest index of a torsion-free subgroup in each of the Coxeter groups $[3, 3, \ldots, 3, 6]$ of rank 3 to 6. In the case of $[3, 3, 3, 3, 6]$, the smallest index of a torsion-free subgroup is 1440, and in contrast to this, we then show that the smallest index of a torsion-free subgroup of the Coxeter group $[6, 3, 3, 3, 3, 6]$ is 2880. Properties of the corresponding hyperbolic 5-manifold M_*^5 of volume $\frac{13\zeta(3)}{2}$ are summarised in Theorem 2. In Section 4 we give additional information about the manifold M_*^5 and present our findings about the symmetry group $I(M_*^5)$ in Theorem 3. We also include a comparison with the 5-manifold Q^5 constructed by Felikson and Tumarkin [7] by means of quiver mutations, and in Section 5, we complete the picture, providing constructions of small volume cusped hyperbolic n-manifolds for $n = 3$ and $n = 4$.

2. The Coxeter groups $[3,3,\ldots,3,6]$ and the hyperbolic reflection groups Γ_n of finite covolume

Let $\mathbb{X}^n$ denote either the sphere $\mathbb{S}^n$, or Euclidean space $\mathbb{E}^n$, or hyperbolic space $\mathbb{H}^n$, of dimension n. We may interpret $\mathbb{X}^n$ via its linear model embedded in a quadratic space Y^{n+1}, so that

$$\mathbb{H}^n = \{\, x \in \mathbb{R}^{n+1} \mid q_n(x) = x_1^2 + x_2^2 + \ldots + x_n^2 - x_{n+1}^2 = -1\,, \, x_{n+1} > 0 \,\}.$$

Furthermore, denote by $\mathrm{Isom}\,\mathbb{X}^n$ the isometry group of $\mathbb{X}^n$. Recall that the group $\mathrm{Isom}\,\mathbb{H}^n$ is isomorphic to the group $\mathrm{O}^+(n,1)$ of positive Lorentz matrices (see [19, Chapter 3]), that is,

$$(2.1) \qquad \mathrm{O}^+(n,1) = \{\, A \in \mathrm{Mat}(n+1,\mathbb{R}) \mid A^t J A = J \,, \, [A]_{n+1,n+1} > 0 \,\}\,,$$

where $J = \mathrm{diag}(1,1,\ldots,1,-1)$ is the diagonal matrix associated with q_n.

For a given subgroup Γ of $\mathrm{Isom}\,\mathbb{H}^n$, the quotient space $\mathbb{H}^n/\Gamma$ is a (complete) hyperbolic n-orbifold if Γ is discrete. In addition, if the group Γ has no torsion elements (that is, no non-trivial elements of finite order), then $\mathbb{H}^n/\Gamma$ is a hyperbolic n-manifold, which we will denote by M.

In this paper, we consider only *finite volume* quotients V of $\mathbb{H}^n$ by discrete groups Γ of hyperbolic isometries (see [3], [19]). Accordingly, the group Γ is finitely-generated, and the volume of V can be identified with the volume of a (closed) fundamental domain for the deck group $\pi_1(V)$ of V, which is isomorphic to Γ. By Selberg's Lemma [19, Chapter 7], the group Γ has a torsion-free (normal) subgroup of finite index, so the space V is finitely covered by a hyperbolic manifold. Finally, recall from [20] that the isometry group $I(V)$ of a finite volume orbifold $V = \mathbb{H}^n/\Gamma$ is a finite group isomorphic to the quotient group $N(\Gamma)/\Gamma$, where $N(\Gamma)$ is the normaliser of Γ in $\mathrm{Isom}\,\mathbb{H}^n$.

Two groups in $\mathrm{Isom}\,\mathbb{H}^n$ are *commensurable* in the wide sense (or commensurable, for short) if the intersection of one group with some conjugate of the other group is of finite index in both groups. Commensurability is an equivalence relation, which preserves properties such as cocompactness, finite covolume, and arithmeticity.

A *geometric Coxeter group* is a discrete subgroup Γ of $\mathrm{Isom}\,\mathbb{X}^n$ generated by a finite set S of reflections s_i in hyperplanes H_i of $\mathbb{X}^n$, for $1 \le i \le |S|$. The cardinality $|S|$ of S is called the *rank* of Γ. The generators in S satisfy the relations $s_i^2 = 1$ and $(s_i s_j)^{m_{ij}} = 1$ where $m_{ij} = m_{ji} \in \{2,3,\ldots,\infty\}$ for all i,j. Here, $m_{ij} = \infty$ means that $s_i s_j$ does not have specified finite order. Associated with Γ is the Coxeter diagram $\Sigma = \Sigma(\Gamma)$, which is a graph with nodes ν_i (corresponding to s_i and H_i), with ν_i and ν_j joined by an edge with label m_{ij} whenever $m_{ij} \ge 3$. (This label is usually omitted when $m_{ij} = 3$.) Observe that in the hyperbolic case, if H_i and H_j are parallel and intersect on $\partial\mathbb{H}^n$ (or lie at hyperbolic distance $l > 0$ from each other), then the nodes ν_i and ν_j are connected by an edge with label ∞ (or by a dotted edge with label l).

Hence the Coxeter diagram Σ of Γ depicts the intersection behaviour of the hyperplanes $H_i = v_i^\perp$ (for $1 \le i \le |S|$) and the facets of the associated

Coxeter polyhedron $P = \cap_{1 \le i \le |S|} H_i^- \subset \mathbb{X}^n$, where the closed half spaces H_i^- are oriented by means of the unit vectors v_i orthogonal to H_i, pointing outwards with respect to P. The polyhedron P is the closure of a fundamental domain of Γ, and the group Γ is said to be *of finite covolume* if P has finite volume.

We use a *Coxeter symbol* to depict a Coxeter diagram of simple combinatorial type. As explained in [11, Appendix], for example, this symbol can take the form $[p_1, p_2, \ldots, p_k]$ or $[q_1, q_2, \ldots, q_l, \infty]$ with integer labels $p_i, q_j \ge 3$, for a linear Coxeter diagram with $k+1$ or $l+2$ nodes and with edges marked by the respective weights. In a similar way, the Coxeter symbol $[\infty, q_1, q_2, \infty]$ with integers $q_1, q_2 \ge 3$ denotes a linear Coxeter diagram with 5 nodes.

The spherical and Euclidean Coxeter groups were classified by Coxeter [5] in 1934. The list of irreducible spherical Coxeter groups is comparatively short. In this paper, the group $\mathcal{A}_n = [3, 3, \ldots, 3]$ of order $(n+1)!$ is of particular interest since it is isomorphic to the symmetric group S_{n+1}, under an isomorphism that takes its generator s_i to the single transposition $(i, i+1)$, for $1 \le i \le n$. Alternatively, it can be viewed as the symmetry group of a regular simplex $S_{\mathrm{reg}} \subset \mathbb{S}^{n-1}$, with each s_i interpreted as a reflection in $\mathbb{R}^n$.

For hyperbolic Coxeter groups of finite covolume, Vinberg [26] developed a highly satisfactory theory allowing one to analyse their arithmetic nature and the combinatorial-metrical structure of their Coxeter polyhedra P in terms of the Gram matrix $G = G(P)$. For example, a finite-covolume hyperbolic Coxeter group with *non-compact* polyhedron P is arithmetic (and defined over $\mathbb{Q}$) if and only if all coefficient cycles (related to closed paths on the Coxeter diagram) of $2 \cdot G$ are rational integers. Despite this insight, Coxeter groups of finite covolume in Isom $\mathbb{H}^n$ are classified only for very small rank $|S| \ge n+1$. For an up-to-date list and non-existence bounds, see [6].

Here we focus on finite-covolume hyperbolic Coxeter groups Γ_n given by linear diagrams, with associated non-compact Coxeter polyhedron $P \subset \mathbb{H}^n$, such that Γ_n has minimal rank and contains $\mathcal{A}_n = [3, 3, \ldots, 3]$ as a subgroup.

For example, when $n = 2$ there is the arithmetic Coxeter group $\Gamma_2 = [3, \infty]$ of rank 3 given in Fig. 1, with rotation subgroup Γ_2^+ isomorphic to the rational modular group $\mathrm{PSL}(2, \mathbb{Z})$. The group Γ_2 generates the symmetry group of an ideal triangle $S_{\mathrm{reg}}^\infty \subset \mathbb{H}^2$ of area π.
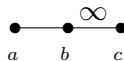

FIGURE 1. The Coxeter group $\Gamma_2 = [3, \infty]$

In geometric terms, by decomposing the triangle S_{reg}^∞ barycentrically into 3! isometric copies of the right-angled triangle $R_2 = \Delta(0, \frac{\pi}{3}, \frac{\pi}{2})$ with angle $\frac{\pi}{3}$ and one vertex on the boundary at infinity of $\mathbb{H}^2$, the generators of the group Γ_2 are the reflections in the geodesic lines bounding R_2.

For $n = 3$, the group Γ_3 of rank 4 has diagram given in Fig. 2 and coincides with the group $\hat{\Gamma}_3 = [3, 3, 6]$. It is of arithmetic nature, and the commutator subgroup of Γ_3 is isomorphic to the Eisenstein modular group $\mathrm{PSL}(2, \mathbb{Z}[\omega])$ where $\omega = (-1 + \sqrt{-3})/2$; see [10, Section 14.3]. The group Γ_3 is generated by the reflections a, b, c, d in the facets of a non-compact Coxeter tetrahedron R_3, with precisely one vertex on the boundary at infinity, given by $A = H_b \cap H_c \cap H_d$. The stabiliser of A in Γ_3 is the Euclidean Coxeter subgroup with symbol $[3, 6]$. The covolume of Γ_3 can be computed in two different ways. Classically, the geometrical approach due to Lobachevsky yields

$$\mathrm{vol}_3(R_3) = \frac{1}{8} \mathrm{JI}(\frac{\pi}{3}) \approx 0.04228 \,,$$

where $\mathrm{JI}(\omega) = -\int_0^\omega \log|2 \sin t| \, dt$ denotes the Lobachevsky function; see [12], for example.
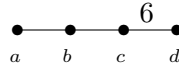

FIGURE 2. The Coxeter group $\Gamma_3 = \hat{\Gamma}_3 = [3, 3, 6]$

The tetrahedron R_3 is the characteristic simplex of an ideal regular tetrahedron S_{reg}^∞ with dihedral angle $\frac{\pi}{3}$ and with centre $D = H_a \cap H_b \cap H_c$. By barycentric decomposition, the Coxeter simplex S_{reg}^∞ can be cut into $4! = 24$ isometric copies of R_3, so that $\mathrm{vol}_3(S_{\mathrm{reg}}^\infty) = 3 \, \mathrm{JI}(\frac{\pi}{3})$. Notice that the group Γ_3 is commensurable with the Coxeter pyramid group $\Gamma_3' = [\infty, 3, 6, \infty]$ in Isom $\mathbb{H}^3$, which implies that their quotient spaces $\mathbb{H}^3/\Gamma_3$ and $\mathbb{H}^3/\Gamma_3'$ are covered by a common manifold; see [8]. The Coxeter pyramid $[\infty, 3, 6, \infty]$ can be decomposed into the tetrahedra $[3, 6, 3]$ and $[6, 3, 6]$ so that its volume equals $\frac{5}{4} \mathrm{JI}(\frac{\pi}{3})$; see [12]. In contrast with Γ_3', the Coxeter pyramid group $\Theta = [\infty, 3, 3, \infty]$ is *not* commensurable with Γ_3 and Γ_3'. (It contains $\mathcal{A}_3$ as well, but as such its rank is not minimal.)

For $n = 4$, there are no non-compact Coxeter simplices of finite volume with linear diagram containing the diagram for $\mathcal{A}_4$ as a subgraph; see [27, Part II, Chapter 5]. There is, however, an infinite volume Coxeter polyhedron $\hat{R}_4 \subset \mathbb{H}^4$ with symbol $[3, 3, 3, 6]$, giving rise to the reflection group $\hat{\Gamma}_4$ of rank 5. By taking into account the exterior of hyperbolic space and by passing to the projective ball model K^4 for $\mathbb{H}^4$ in $\mathbb{R}P^4$, the polyhedron $\hat{R}_4$ can be interpreted as the characteristic simplex of a regular simplex $\hat{S}_{\mathrm{reg}}$ with dihedral angle $\frac{\pi}{3}$, all of whose edges lie outside of $\mathbb{H}^4$ but are tangent at their midpoints to $\partial \mathbb{H}^4$. By taking the (hyperbolic) convex hull of all these edge midpoints of $\hat{S}_{\mathrm{reg}}$, one obtains an ideal hyperbolic 4-polyhedron $\mathcal{R}$ with dihedral angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$ whose symmetry group is isomorphic to a subgroup of S_5. More precisely, this process is given by polar truncation of each of the ultra-ideal vertices of $\hat{S}_{\mathrm{reg}}$, and is called (*simple*) *rectification* of $\hat{S}_{\mathrm{reg}}$, indicated by $r_1 \hat{S}_{\mathrm{reg}} = \mathcal{R}$; see [15, Section 3.2], [25]. This process also gives rise to a (truncated) finite volume Coxeter polyhedron $R_4 \subset \mathbb{H}^4$ with

symbol $[\infty, 3, 3, 3, 6]$, with precisely one ideal vertex, and which barycentrically decomposes $\mathcal{R}$ into $5! = 120$ isometric copies. The Coxeter diagram of R_4 and the associated reflection group Γ_4 is depicted in Fig. 3.
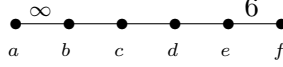

FIGURE 3. The Coxeter group $\Gamma_4 = [\infty, 3, 3, 3, 6]$

Next, by [13, Appendix], since the volume of the polyhedron R_4 is $\frac{\pi^2}{540}$, the volume of the ideal rectified regular simplex $r_1\hat{S}_{\text{reg}} = \mathcal{R} \subset \mathbb{H}^4$ is given by

$$(2.2) \qquad \operatorname{vol}_4(\mathcal{R}) = 5! \operatorname{vol}_4([\infty, 3, 3, 3, 6]) = \frac{2\pi^2}{9} \ .$$

Observe that the Coxeter group Γ_4 is *not* commensurable with the Coxeter group $[3, 4, 3, 4]$. The Coxeter simplex $[3, 4, 3, 4]$ has volume $\frac{\pi^2}{864}$ and takes part (as characteristic simplex) in the barycentric decomposition of the ideal right-angled 24-cell $\mathcal{C} \subset \mathbb{H}^4$ into 1152 isometric copies (see [8]). In particular, the volume of $\mathcal{C}$ is $\frac{4\pi^2}{3}$. Both groups Γ_4 and $[3, 4, 3, 4]$ are arithmetic, and are closely related to certain pseudo-modular groups of quaternionic Clifford matrices as described in [10, Section 15.2].

For $n = 5$, as in the case $n = 4$, there are no finite-covolume hyperbolic Coxeter groups of rank 6 with linear diagram containing the diagram for $\mathcal{A}_5$ as a subgraph, but there is an infinite volume Coxeter polyhedron $\hat{R}_5 \subset \mathbb{H}^5$ with symbol $[3, 3, 3, 3, 6]$, which belongs to the ideal *birectified regular 5-simplex* $r_2\hat{S}_{\text{reg}} =: \mathcal{B}$ with dihedral angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$. More concretely, by passing to the projective ball model K^5 for $\mathbb{H}^5$, the polyhedron $\hat{R}_5$ can be associated with the characteristic simplex of a regular simplex $\hat{S}_{\text{reg}}$ in the ambient space of K^5, with all of its 2-dimensional faces lying outside but tangent at their centres to the boundary sphere at infinity ∂K^n. The truncation of the ultra-ideal vertices v_i of $\hat{S}_{\text{reg}}$ by means of their polar hyperplanes π_i (for $1 \leq i \leq 6$) has the property that $\measuredangle(\pi_i, \pi_k) = \frac{\pi}{3}$ whenever $i \neq k$. This process yields a finite volume hyperbolic Coxeter polyhedron with dihedral angles $\frac{\pi}{3}$ and $\frac{\pi}{2}$, coinciding with the (hyperbolic) convex hull of all ideal centres of the 2-faces of $\hat{S}_{\text{reg}}$, the ideal birectified regular simplex $\mathcal{B}$, whose symmetry group is isomorphic to a subgroup of S_6. For more details, see [15, Section 3.2].

The truncation of $\hat{S}_{\text{reg}}$ induces a truncation of $\hat{R}_5$ and provides a finite volume Coxeter polyhedron $R_5 \subset \mathbb{H}^5$, with precisely one vertex on $\partial \mathbb{H}^5$. The associated reflection group Γ_5 has the generators a, b, c, d, e, f, g according to Fig. 4, where the mirror H_a associated with a coincides with a truncation hyperplane of $\hat{S}_{\text{reg}}$.
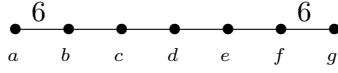
FIGURE 4. The group $\Gamma_5 = [6, 3, 3, 3, 3, 6]$

The combinatorial type of R_5 is that of a pyramid over a product of two triangles whose apex D is the ideal point given by the intersection of the hyperplanes H_a, H_b, H_c and H_e, H_f, H_g of R_5. More precisely, the Euclidean vertex figure of D is of type $[6, 3] \times [3, 6]$. In [8], all such Coxeter pyramid groups were classified up to commensurability, and further, it was shown that the group Γ_5 does *not* belong to the commensurability class of the Coxeter simplex group $[3, 4, 3, 3, 3]$. Also it is known by [9] that the quotient space of $\mathbb{H}^5$ by the (arithmetic) discrete reflection group $[3, 4, 3, 3, 3]$ has minimal volume among *all* cusped hyperbolic 5-orbifolds. This minimal value has previously been computed in [14], and is given by

$$(2.3) \qquad \operatorname{covol}_5([3, 4, 3, 3, 3]) = \frac{7\zeta(3)}{46080} \ .$$

The group Γ_5 contains as a (normal) subgroup of index 4 the Coxeter pyramid group Λ_5 given by Fig. 5. This follows easily by reflecting the Coxeter pyramid given by Fig. 4 in the mirror H_a and then in H_g. In other words, the group Λ_5 is generated by the elements aba, b, c, d, e, f, gfg of Γ_5, and it will play a role in Section 4.
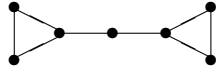
FIGURE 5. The subgroup Λ_5 of index 4 in Γ_5

Furthermore, Γ_5 is arithmetic and closely related to the so-called hybrid quaternionic modular group $\operatorname{PS}_\Delta \operatorname{L}(2, \mathbb{Z}[\omega, j])$ where Δ is the Dieudonné determinant of a quaternionic matrix and $\omega = (-1 + \sqrt{-3})/2$ as usual; see [10, Section 15.5]. The covolume of the group Γ_5 is much harder to compute than the covolume of $[3, 4, 3, 3, 3]$ and its commensurable groups. In [15], using the proof of [15, Theorem 2] and relations in the crystallographic Napier cycles defined by the group Γ_5, the second author was able to compute the covolume of $\Gamma_5 = [6, 3, 3, 3, 3, 6]$ and that of $\operatorname{PS}_\Delta \operatorname{L}(2, \mathbb{Z}[\omega, j])$. Indeed

$$(2.4) \qquad \operatorname{covol}_5([6, 3, 3, 3, 3, 6]) = \frac{13}{5760} \zeta(3) \ .$$

As a consequence, the volume of the ideal birectified regular 5-simplex $\mathcal{B}$ is $6! \cdot \operatorname{covol}_5([6, 3, 3, 3, 3, 6]) = \frac{13\zeta(3)}{8}$.

In Fig. 6 and Fig. 7, we list the different hyperbolic Coxeter groups as described above.

Coxeter group	Related group in $\mathrm{Isom}\,\mathbb{H}^n$	covol_n
$[3]$ (spherical)	$\Gamma_2 = [3, \infty]$	$\frac{\pi}{6}$
$[3,6]$ (Euclidean)	$\Gamma_3' = [\infty, 3, 6, \infty]$	$\frac{5}{4}\,\text{JI}(\frac{\pi}{3})$

FIGURE 6. The hyperbolic Coxeter groups Γ_2 and Γ_3'

n	$\hat{\Gamma}_n$	Γ_n	covol_n
3	$[3,3,6]$	$[3,3,6]$	$\frac{1}{8}\,\text{JI}(\frac{\pi}{3})$
4	$[3,3,3,6]$	$[\infty,3,3,3,6]$	$\frac{\pi^2}{540}$
5	$[3,3,3,3,6]$	$[6,3,3,3,3,6]$	$\frac{13}{5760}\zeta(3)$

FIGURE 7. The hyperbolic Coxeter groups $\hat{\Gamma}_n$ and Γ_n

3. TORSION-FREE SUBGROUPS OF SMALL INDEX IN CERTAIN COXETER GROUPS

In this section we find the smallest index of a torsion-free subgroup in each of the Coxeter groups $[6,3,3,\ldots,3]$ (and equivalently, in their duals $[3,3,\ldots,3,6]$) of ranks 3 to 6, and explain why the smallest index of a torsion-free subgroup in the $[6,3,3,3,3,6]$ Coxeter group Γ_5 is 2880.

To do this, we start by noting that Γ_5 is canonically generated by seven reflections a, b, c, d, e, f, g, which satisfy the defining relations that can be read off the Coxeter diagram given in Fig. 4. We need to know the representatives of conjugacy classes of torsion elements in this group, and we can find those in a similar way to the approach taken in [4] for finding a small index torsion-free subgroup in the $[5,3,3,3]$ Coxeter group.

Specifically, we observe that the group Γ_5 has a natural action as a reflection group in 5-dimensional hyperbolic space $\mathbb{H}^5$, and hence that every torsion element is conjugate to a torsion element of one or more of the subgroups generated after removing an element from the canonical generating set for Γ_5. In the case of $[5,3,3,3]$, every such subgroup was finite, but in this case, that does not happen. Actually it never happens, because each of the Coxeter groups $[6,3,3,3,3]$, $[6,3,3,3]$, $[6,3,3]$ and $[6,3]$ and their duals (obtained by reversing the Coxeter diagram) is infinite. Nevertheless, we can make the analogous observation about torsion elements in each of these Coxeter groups, using their actions on 5-, 4- and 3-dimensional hyperbolic space and 2-dimensional Euclidean space, respectively.

For example, the elements a, b and c generate an infinite subgroup isomorphic to the $[6,3]$ Coxeter group, otherwise known as the $(2,3,6)$ triangle group, and each of its torsion elements is conjugate to a torsion element of one or more of the finite subgroups $\langle a,b \rangle \cong D_6$ (of order 12), $\langle a,c \rangle \cong C_2 \times C_2$ (of order 4) and $\langle b,c \rangle \cong D_3$ (of order 6), and hence to one or more of the elements a, b, ab, $(ab)^2$, $(ab)^3$, c, ac and bc. (In fact we can drop b or c from this list, as each of them is conjugate to the other in $\langle b,c \rangle \cong D_3$.)

Next, a, b, c and d generate a subgroup isomorphic to the $[6,3,3]$ Coxeter group, and each of its torsion elements is conjugate to a torsion element of one or more of either $\langle a,b,c \rangle \cong [6,3]$ or $\langle a,b,d \rangle \cong D_6 \times C_2$ (of order 24) or $\langle a,c,d \rangle \cong C_2 \times D_3$ (of order 12) or $\langle b,c,d \rangle \cong [3,3] \cong S_4$ (of order 24).

In this way, we can build a set of representatives of conjugacy classes of torsion elements of each of the seven subgroups generated after removing an element from the given generating set $S = \{a,b,c,d,e,f,g\}$ for Γ_5. This, however, produces a very long list (which interested readers could take some time to find for themselves).

A much better approach is to take the maximally finite subgroups of Γ_5 generated by subsets of S, as given in Fig. 8.

$$\langle a,b,d,e,f \rangle \cong \langle a,b \rangle \times \langle d,e,f \rangle \cong D_6 \times S_4 \text{ (of order 288)},$$
$$\langle a,b,d,e,g \rangle \cong \langle a,b \rangle \times \langle d,e \rangle \times \langle g \rangle \cong D_6 \times D_3 \times C_2 \text{ (of order 144)},$$
$$\langle a,b,d,f,g \rangle \cong \langle a,b \rangle \times \langle d \rangle \times \langle f,g \rangle \cong D_6 \times C_2 \times D_6 \text{ (of order 288)},$$
$$\langle a,c,d,e,f \rangle \cong \langle a \rangle \times \langle c,d,e,f \rangle \cong C_2 \times S_5 \text{ (of order 240)},$$
$$\langle a,c,d,e,g \rangle \cong \langle a \rangle \times \langle c,d,e \rangle \times \langle g \rangle \cong C_2 \times S_4 \times C_2 \text{ (of order 96)},$$
$$\langle a,c,d,f,g \rangle \cong \langle a \rangle \times \langle c,d \rangle \times \langle f,g \rangle \cong C_2 \times D_3 \times D_6 \text{ (of order 144)},$$
$$\langle b,c,d,e,f \rangle \cong S_6 \text{ (of order 720)},$$
$$\langle b,c,d,e,g \rangle \cong \langle b,c,d,e \rangle \times \langle g \rangle \cong S_5 \times C_2 \text{ (of order 240)},$$
$$\langle b,c,d,f,g \rangle \cong \langle b,c,d \rangle \times \langle f,g \rangle \cong S_4 \times D_6 \text{ (of order 288)},$$

FIGURE 8. Maximally finite subgroups of the $[6,3,3,3,3,6]$ Coxeter group Γ_5 obtained after deleting generators

Now every torsion-free subgroup H of Γ_5 intersects each of these nine subgroups trivially, and so in the natural permutation representation of Γ_5 on (right) cosets of H (by right multiplication), each of these nine subgroups acts fixed-point-freely (that is, with no non-trivial element fixing a point), and hence with orbits all having the same length as the order of the subgroup. It follows that the index of H in Γ_5 is divisible by the order of each one, and so must be divisible by $\mathrm{LCM}(288,144,240,96,720) = 32 \cdot 9 \cdot 5 = 1440$.

The same arguments as above show that the index of every torsion-free subgroup of one of the Coxeter groups $[6,3]$, $[6,3,3]$, $[6,3,3,3]$ and $[6,3,3,3,3]$ is divisible by 12, 24, 720 and 1440, respectively. Moreover, we can easily use this information to prove the following.

Theorem 1. *The smallest index of a torsion-free subgroup in one of the Coxeter groups* $[6,3]$*,* $[6,3,3]$*,* $[6,3,3,3]$ *and* $[6,3,3,3,3]$ *is 12, 24, 720 and 1440, respectively.*

Proof. Let $\{a,b,c\}$, $\{a,b,c,d\}$, $\{a,b,c,d,e\}$ or $\{a,b,c,d,e,f\}$ be the canonical generating set for the relevant Coxeter group, of rank 3, 4, 5 or 6, respectively. Then computations using MAGMA reveal the following:

(1) The subgroup generated by $\{abacb, bababc\}$ has index 12 in the $[6,3]$ Coxeter group, and intersects each of the subgroups $\langle a,b\rangle$, $\langle a,c\rangle$ and $\langle b,c\rangle$ trivially (because each of those maximally finite subgroups acts fixed-point-freely on the corresponding coset space), and hence is torsion-free.

(2) The subgroup generated by $\{abc, babcd\}$ has index 24 in the $[6,3,3]$ Coxeter group, and intersects each of the subgroups $\langle a,b,d\rangle$, $\langle a,c,d\rangle$ and $\langle b,c,d\rangle$ trivially (because each of those maximally finite subgroups acts fixed-point-freely on the coset space), and hence is torsion-free.

(3) The subgroup generated by $\{abc, babcd, bedcbababcbababcdeca\}$ has index 720 in the $[6,3,3,3]$ Coxeter group, and intersects each of the subgroups $\langle a,b,d,e\rangle$, $\langle a,c,d,e\rangle$ and $\langle b,c,d,e\rangle$ trivially (because each of those maximally finite subgroups acts fixed-point-freely on the coset space), and hence is torsion-free.

(4) The subgroup generated by $\{abcdcbfedcbababa, adcbaedcfedbcaba, abcbd\ edcbfabcdecdaba\}$ has index 1440 in the $[6,3,3,3,3]$ Coxeter group, and intersects each of the subgroups $\langle a,b,d,e,f\rangle$, $\langle a,c,d,e,f\rangle$ and $\langle b,c,d,e,f\rangle$ trivially (because each of those maximally finite subgroups acts fixed-point-freely on the coset space), and hence is torsion-free.

In each of these four cases, the index is equal to the lower bound on it obtained earlier, and hence is the minimum index. $\square$

In contrast, however, we were unable to find a torsion-free subgroup of the anticipated minimum possible index 1440 in the $[6,3,3,3,3,6]$ Coxeter group Γ_5. We did find one of index 2880, again with the help of MAGMA, namely the subgroup generated by $bacbdefdea$, $cbabedgfgeda$ and $cbabfegfgefdea$. In the natural permutation representation of Γ_5 on (right) cosets of this subgroup, each of the nine maximally finite subgroups of Γ_5 in Fig. 8 acts fixed-point-freely on the coset space, so the given subgroup contains no torsion element. The abelianisation of this subgroup is $\mathbb{Z}_7^3$, obtainable using the `AbelianQuotientInvariants` function in MAGMA.

(In fact, up to conjugacy this subgroup was the only torsion-free subgroup of index 2880 in Γ_5 that we could find, using a range of approaches (including searching for subgroups of index $\frac{2880}{k}$ in subgroups of index k for mid-range divisors k of 2880, and intersections of subgroups of small index, and pre-images of subgroups of appropriate orders in small quotients). It is plausible that the subgroup we found is the only torsion-free subgroup of index 2880 in Γ_5 up to conjugacy, but we could not prove this.)

The quotient M_*^5 of $\mathbb{H}^5$ by this subgroup is a manifold with two cusps. Indeed, the Coxeter polyhedron associated with the group Γ_5 has precisely one ideal vertex v_0, with stabiliser subgroup $\Gamma_5(v_0) = \langle a, b, c \rangle \times \langle e, f, g \rangle$. The cusps of this manifold are in one-to-one correspondence with the orbits of the group $\Gamma_5(v_0)$ in the action of Γ_5 on the 2880 cosets of the torsion-free subgroup, and an easy MAGMA computation shows that there are two such orbits, of lengths 576 and 2304.

This left us with the somewhat challenging task of showing that there is no torsion-free subgroup of index 1440. We explain below how we did that.

Suppose there is one, say H, and again consider the natural permutation representation of Γ_5 on the (right) coset space $\Omega = (\Gamma_5 : H)$. In this representation, every finite subgroup of Γ_5 acts fixed-point-freely on Ω, and in particular, the subgroup $J = \langle b, c, d, e, f \rangle$ induces two (regular) orbits of length $|S_6| = 720$. These two sub-orbits are linked together by the permutation induced by a to give a transitive permutation representation of $\langle a, b, c, d, e, f \rangle$ satisfying the 'fixed-point-free orbits' test from item (4) in the proof of Theorem 1, and are also linked together by the permutation induced by g to give a transitive permutation representation of $\langle b, c, d, e, f, g \rangle$ satisfying its 'dual' (under the automorphism of Γ_5 that reverses its Coxeter diagram).

We used this information to proceed as follows:

Step 1: We found all transitive permutation representations of the $[6, 3, 3, 3, 3]$ Coxeter group of degree 1440 in which the maximal finite subgroups act fixed-point freely, as in the proof of Theorem 1, by determining all ways in which two regular representations of the $[3, 3, 3, 3]$ Coxeter group S_6 can be linked together in the appropriate way. This took several days of computing time using MAGMA, even after breaking it up into 945 independent sub-cases, depending on how the orbits of the $[3, 3, 3, 3]$ Coxeter subgroup of order 120 generated by $\{c, d, e, f\}$ are linked together by the permutation induced by the generator a (which commutes with each of c, d, e, f). It produced 19704 transitive representations of degree 1440.

Step 2: We recognised that the representations in Step 1 give all transitive permutation representations of the $[3, 3, 3, 3, 6]$ Coxeter group with the analogous property, simply by applying the restriction to $\langle a, b, c, d, e, f \rangle$ of the automorphism θ of Γ_5 that reverses the order of its generating set, taking (a, b, c, d, e, f) to (g, f, e, d, c, b).

Step 3: We considered all pairs of the representations found in steps 1 and 2, to first test if the permutations induced respectively by a and g commute, in which case they give a transitive permutation representation of the $[6, 3, 3, 3, 3, 6]$ Coxeter group Γ_5, and then (if that test succeeded) to check the resulting permutation representation gives a fixed-point-free representation of each of the maximally finite subgroups of Γ_5 in the list in Fig. 8. This was straightforward, and required looking only at the orbits of the groups induced by the 2nd, 3rd, 5th and 6th of the finite subgroups

listed in Fig. 8 (namely the ones containing both a and g), but also took some considerable time, because of the sheer number of representations coming out of Steps 1 and 2.

Sadly, in Step 3 we found no way of combining pairs from Steps 1 and 2 to give a representation of Γ_5 of the anticipated kind. As it happened, 3120 of the pairs produced permutation representations in which the permutations induced by a and g commute, but all of them failed the fixed-point-free orbits test.

In summary, the above arguments and computations prove the following.

Theorem 2. *The smallest index of a torsion-free subgroup in the Coxeter group* $[6,3,3,3,3,6]$ *is* 2880, *achieved for example by the torsion-free subgroup generated by* $bacbdefdea$, $cbabedgfgeda$ *and* $cbabfegfgefdea$. *This subgroup gives rise to a hyperbolic* 5-*manifold of volume* $2880 \cdot \frac{13\zeta(3)}{5760} = \frac{13\zeta(3)}{2}$, *with two cusps and with first homology group* $\mathbb{Z}_7^3$.

4. The manifold M_*^5 and its fundamental group

Here we take a closer look at the manifold described in Theorem 2. The elements

$$u := bacbdefdea, \quad v := cbabedgfgeda \quad \text{and} \quad w := cbabfegfgefdea$$

in the Coxeter group $\Gamma_5 = [6,3,3,3,3,6]$ generate a torsion-free subgroup Γ_5^* of index 2880 and are of even lengths $10, 12$ and 14, respectively. Hence the quotient space $M_*^5 = \mathbb{H}^5/\Gamma_5^*$ is an orientable cusped hyperbolic 5-manifold of volume $\frac{13\zeta(3)}{2}$.

Now let us analyse the fundamental group $\pi_1(M_*^5) \cong \Gamma_*^5$ in more detail. Because the group $\Gamma_5 = [6,3,3,3,3,6]$ is an arithmetic reflection group (defined over $\mathbb{Q}$), the group $\pi_1(M_*^5)$ is an arithmetic lattice. One may check using the Reidemeister-Schreier `Rewrite` function in Magma that the elements of the generating-set $S_* := \{u, v, w\}$ satisfy the set R_* of defining relations given in Fig. 9, so that $\Gamma_*^5 = \langle S_* \,|\, R_* \rangle$.

In order to understand the geometry of the generators u, v, w of $\pi_1(M_*^5)$, we represent the generators $a, b, \ldots, g$ of $[6,3,3,3,3,6]$ by means of positive Lorentz matrices $A, B, \ldots, G \in \mathrm{O}^+(5,1)$, as given in Fig. 10; see [8, p. 96]. Here, a matrix $M \in \mathrm{O}^+(5,1)$ is written as a 6-tuple $[m_1, m_2, \ldots, m_6]$ of vectors $m_1, m_2, \ldots, m_6$ with respect to the canonical basis $\{e_1, e_2, \ldots, e_6\}$ of $\mathbb{R}^6$.

$$
\begin{aligned}
1 =\ & v^{-1}uw^{-1}u^{-2}vw^{-1}u^2wu^{-1}v^2wv^{-2} \\
=\ & w^2u^{-1}v^2u^{-1}vuw^{-1}uv^{-1}uv^{-2}uw^{-1}u^{-2} \\
=\ & uw^{-1}uv^{-1}uwv^{-2}uw^{-1}uv^{-2}wuwv^{-2}uw^{-1} \\
=\ & v^3w^{-1}u^{-1}vwu^{-1}v^2u^{-1}v^2uw^{-1}uv^{-2}u^2wu^{-1}v \\
=\ & u^{-1}vwu^{-1}v^2uwu^{-1}vuw^{-1}uw^{-1}u^{-1}vw^{-1}u^2wu^{-1}v^2 \\
=\ & w^{-1}uw^{-1}u^{-1}vu^{-1}wu^{-1}v^{-1}uw^{-1}u^{-1}w^{-1}u^{-1}w^{-1}vuw^{-1}uv^{-1}w^{-1}uw^{-1} \\
=\ & w^{-1}uw^{-1}u^{-1}w^{-1}u^{-1}vu^{-1}wu^{-1}v^{-1}uw^{-1}u^{-1}w^{-1}uw^{-2}vuw^{-1}uv^{-1} \\
=\ & vuw^{-1}uv^{-1}uwv^{-2}uw^{-1}v^{-1}uv^{-2}uw^{-1}v^{-1}uwv^{-1}uwu^{-1} \\
=\ & uwu^{-1}wuwv^{-2}uw^{-1}v^{-1}w^2u^{-1}v^2w^{-1}u^{-1}w^{-1}uw^{-1}u^{-1}vw^{-1} \\
=\ & v^2w^{-1}uw^{-2}vuw^{-1}uv^{-1}w^{-1}v^2u^{-1}w^{-1}vwu^{-1}v^2w^{-1}u^{-1}w^{-1} \\
=\ & wvwu^{-1}v^2w^{-1}u^{-1}w^{-1}vuw^{-1}uv^{-3}uw^{-1}u^{-1}wv^{-1}uwu^{-1} \\
=\ & w^{-1}u^{-1}w^{-1}vuwu^{-1}wuwv^{-2}uw^{-1}uvw^{-1}u^{-1}vu^{-1}wu^{-1}v^{-1}u \\
=\ & v^{-1}uw^{-1}u^{-1}w^{-1}vuw^{-1}uv^{-1}w^{-1}uw^{-1}u^{-1}wu^{-1}v^2w^{-1}u^{-1}w^{-1}uw^{-1}u^{-1}v^{-1} \\
=\ & wu^{-1}wvu^{-1}wu^{-1}v^{-1}wuwv^{-2}uw^{-3}vuw^{-1}uv^{-1}w^{-1}v^2u^{-1} \\
=\ & wu^{-1}vu^{-1}wu^{-1}v^{-1}uw^{-1}u^{-2}vwu^{-1}wvu^{-1}wu^{-1}v^{-1}w^2v^{-1}uwu^{-1} \\
=\ & w^{-1}uw^{-1}u^{-1}vw^{-1}uwu^{-1}v^{-1}uw^{-1}u^{-2}v^2u^{-1}w^2u^{-1}wuwv^{-2}uw^{-1}u \\
=\ & uwu^{-1}vuw^{-1}uv^{-1}wu^{-1}v^2w^{-1}u^{-1}w^{-1}uw^{-2}uv^{-2}wvu^{-1}wu^{-1}v^{-1}w \\
=\ & u^{-1}wu^{-1}v^2w^{-1}u^{-1}w^{-1}uw^{-1}u^{-1}vw^{-1}u^{-1}w^{-1}uw^{-1}u^{-1}vw^{-1}u^{-1}vwu^{-1}v^2w^{-1}u^{-1}w^{-1} \\
=\ & vwu^{-1}v^2w^{-1}u^{-1}w^{-1}uw^{-1}u^{-2}vwu^{-1}v^2uwu^{-1}vu^2wu^{-1}vuw^{-1}uv^{-1}u \\
=\ & v^{-2}u^2wu^{-1}v^3uwu^{-1}wuv^{-2}uw^{-1}u^{-2}wvuwu^{-1}wvu^{-1}wu^{-1}v^{-1}wu \\
=\ & u^{-1}wu^{-1}v^2w^{-1}u^{-1}w^{-1}uw^{-1}u^{-1}vw^{-1}u^{-1}wv^{-1}uwu^{-1}wu^{-1}vu^{-1}wu^{-1}v^2w^{-1}u^{-1}w^{-1}u^2wu^{-1}v^2 \\
=\ & v^{-1}uw^{-1}u^{-1}wv^{-1}uwu^{-1}w^2uwu^{-1}v^2wv^{-2}uw^{-1}u^{-1}wv^{-1}uwu^{-1}wv^{-2}uw^{-1}u^{-1}wv^{-2}.
\end{aligned}
$$

FIGURE 9. Defining relations for the subgroup generated by S_*

$$A =[-e_1,e_2,e_3,e_4,e_5,e_6]$$

$$B =[-\frac{1}{2}e_1+\frac{\sqrt{3}}{2}e_2,\frac{\sqrt{3}}{2}e_1+\frac{1}{2}e_2,e_3,e_4,e_5,e_6]$$

$$C =[e_1,-e_2+2e_5+2e_6,e_3,e_4,2e_2-e_5-2e_6,-2e_2+2e_5+3e_6]$$

$$D =[e_1,e_2,\frac{1}{2}e_3-\frac{1}{\sqrt{2}}e_4-\frac{1}{2}e_5,-\frac{1}{\sqrt{2}}e_3-\frac{1}{\sqrt{2}}e_5,-\frac{1}{2}e_3-\frac{1}{\sqrt{2}}e_4+\frac{1}{2}e_5,e_6]$$

$$
\begin{aligned}
E =\ & [e_1,e_2,\frac{1}{2}e_3+\frac{\sqrt{3}}{2}e_4+\frac{3-\sqrt{6}}{2}e_5+\frac{3-\sqrt{6}}{2}e_6,\frac{\sqrt{3}}{2}e_3-\frac{1}{2}e_4+\frac{3(\sqrt{2}-\sqrt{3})}{2}e_5+\frac{3(\sqrt{2}-\sqrt{3})}{2}e_6, \\
& \frac{3-\sqrt{6}}{2}e_3+\frac{\sqrt{3}(-3+\sqrt{6})}{2}e_4-(\frac{13}{2}-3\sqrt{6})e_5-(\frac{15}{2}-3\sqrt{6})e_6, \\
& -\frac{3-\sqrt{6}}{2}e_3-\frac{\sqrt{3}(-3+\sqrt{6})}{2}e_4+(\frac{15}{2}-3\sqrt{6})e_5+(\frac{17}{2}-3\sqrt{6})e_6]
\end{aligned}
$$

$$F =[e_1,e_2,-e_3+2e_5+2e_6,e_4,2e_3-e_5-2e_6,-2e_3+2e_5+3e_6]$$

$$
\begin{aligned}
G =\ & [e_1,e_2,-\frac{1}{2}e_3-\frac{\sqrt{3}}{2}e_4+\frac{3+\sqrt{6}}{2}e_5+\frac{3+\sqrt{6}}{2}e_6,-\frac{\sqrt{3}}{2}e_3+\frac{1}{2}e_4+\frac{\sqrt{2}+\sqrt{3}}{2}e_5+\frac{\sqrt{2}+\sqrt{3}}{2}e_6, \\
& \frac{3+\sqrt{6}}{2}e_3+\frac{\sqrt{2}+\sqrt{3}}{2}e_4-(\frac{3}{2}+\sqrt{6})e_5-(\frac{5}{2}+\sqrt{6})e_6, \\
& -\frac{3+\sqrt{6}}{2}e_3-\frac{\sqrt{2}+\sqrt{3}}{2}e_4+(\frac{5}{2}+\sqrt{6})e_5+(\frac{7}{2}+\sqrt{6})e_6]
\end{aligned}
$$

FIGURE 10. The generators of $[6,3,3,3,3,6]$ as elements in $O^+(5,1)$

In this way, the three elements u, v, w can be represented conveniently by matrices U, V, W in $\mathrm{O}^+(5,1)$. It turns out that the characteristic polynomial of each of these three matrices has precisely one pair of zeros $\tau, \tau^{-1} \in \mathbb{R}$ with absolute values $\neq 1$, while all other zeros lie on the unit circle. The value $\tau > 1$ is the exponential of the translation length along the axis of the loxodromic element in question (in the direction of the attractive fixed point). Hence the elements u, v, w are all loxodromic. Furthermore, one may check that the element z given by

$$z := vw^{-1} = (ecdgfgdce)(fegfgef)$$

is parabolic. Since c and d commute with both f and g (see Fig. 4), the element z is a product of the form $z = [r, f] = rfrf$ where $r := egfge$ is the reflection with respect to the hyperbolic hyperplane $e_3^\perp \cap \mathbb{H}^5$ given by the Lorentz matrix $\mathrm{diag}(1, 1, -1, 1, 1, 1)$.

Next, as mentioned in the Introduction, the manifold M_*^5 is closely related to the highly symmetric ideal birectified regular 5-simplex $\mathcal{B} = r_2 \hat{S}_{\mathrm{reg}}$. The presentation $\langle S_* \,|\, R_* \rangle$ for $\pi_1(M_*^5)$ given above does not help to determine the order of the symmetry group $I(M_*^5)$ of M_*^5, or to detect non-trivial subgroups acting fixed-point freely, but nevertheless we can still prove the following theorem, which resulted from following up a helpful suggestion by the referee.

Theorem 3. *The symmetry group $I(M_*^5)$ of M_*^5 is isomorphic to the symmetric group S_4, of order 24, and contains no non-trivial subgroup that acts without fixed points on M_*^5.*

Proof. First, we note that by a theorem of Mühlherr [17, Theorem 3.3], the automorphism group of the Coxeter group $\Gamma_5 = [6, 3, 3, 3, 6]$ is the semi-direct product of Γ_5 by the cyclic group C_2 induced by the automorphism taking (a, b, c, d, e, f, g) to (g, f, e, d, c, b, a), corresponding to the non-trivial symmetry of its Coxeter diagram.

It follows that the normaliser of Γ_5 in $\mathrm{Isom}\,\mathbb{H}^5$ is isomorphic to this semi-direct product, and that makes it easy for us to find $I(M_*^5) \cong N(\Gamma_*^5)/\Gamma_*^5$. In particular, our subgroup Γ_*^5 has index 5760 in $N(\Gamma_5)$, and a MAGMA computation shows that the normaliser $N(\Gamma_*^5)$ of Γ_*^5 in $N(\Gamma_5)$ has index 240 in $N(\Gamma_5)$, and that $N(\Gamma_*^5)/\Gamma_*^5$ is isomorphic to S_4, of order $24 = 5760/240$.

To complete the proof, we show that the only torsion-free subgroup of $N(\Gamma_*^5)$ containing Γ_*^5 is Γ_*^5 itself.

For suppose there is a larger one, say H. Then since Γ_*^5 has index 24 in $N(\Gamma_*^5)$ and index 5760 in $N(\Gamma_5)$, we see that H has index at most 2880 in $N(\Gamma_5)$. But also $K := H \cap \Gamma_5$ is a torsion-free subgroup of Γ_5, so K has index at least 2880 in Γ_5 and hence index at least 5760 in $N(\Gamma_5)$. It follows that K has index 2 in H and index 2880 in $N(\Gamma_5)$, and so $K = H \cap \Gamma_5 = \Gamma_*^5$.

In particular, H contains Γ_*^5 as a subgroup of index 2, and so H/Γ_*^5 ies in one of the three conjugacy classes of subgroups of order 2 in $N(\Gamma_*^5)/\Gamma_*^5 \cong S_4$. A further MAGMA computation shows that up to conjugacy, H is generated

by Γ_*^5 and one of the three elements

$$z_1 := abgfgfababefgcdbcb, \quad z_2 := abgfgfgfba \quad \text{and} \quad z_3 := abgfgfababgft.$$

On the other hand, H is not a subgroup of Γ_5, so it must contain an element lying outside Γ_5, and therefore H cannot be generated by Γ_*^5 and one of z_1 or z_2. But also the element z_3 has order 2, since

$$(abgfgfababgft)^2 = abgfgfababgfgfababgfgfgab = (ab)^6(fg)^6 = 1,$$

and this is impossible because H is torsion-free. Thus no such subgroup H exists. $\qquad\square$

The ideal birectified regular 5-simplex $\mathcal{B}$ can be dissected into 720 copies of the Coxeter pyramid with Coxeter symbol $[6,3,3,3,3,6]$. Its associated reflection group Γ_5 contains the Coxeter pyramid group Λ_5 as a subgroup of index 4; see Section 2. The Coxeter diagram of Λ_5 depicted in Fig. 11 is distinguished by having edges only of weight 3, and arising as a mutation of the quiver indexed by the Weyl group A_7. More concretely, in [7], Felikson and Tumarkin used techniques of quiver mutations from the theory of cluster algebras in order to construct manifolds whose symmetry groups contain a given finite Weyl group. In this way, they are able to determine all finite volume hyperbolic manifolds arising from quivers of finite type whose nodes are connected by at most one arrow (or quivers of finite type given by simply-laced graphs). In fact, the fundamental groups of their manifolds are torsion-free subgroups of certain finite-covolume hyperbolic Coxeter groups whose Coxeter diagrams have weights only 2 and 3. In particular, the corresponding Coxeter polyhedra have mutually intersecting facets forming dihedral angles equal to either $\frac{\pi}{2}$ or $\frac{\pi}{3}$; this family of polyhedra has been classified by Prokhorov [18]. The single relevant example distinct from simplices in $\mathbb{H}^5$ is the Coxeter polyhedron associated with Λ_5.
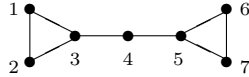

FIGURE 11. The generators of the group Λ_5

The corresponding manifold Q^5 found by Felikson and Tumarkin arises as follows. Consider the natural set $S = \{s_1, \ldots, s_7\}$ of reflections generating the group Λ_5 as indexed in Fig. 11. These satisfy a set of relations that can be read off from the Coxeter diagram of Λ_5. Following [7, Section 3, (R3)], construct the two *cycle relators*

$$r_1 := (s_3 s_1 s_2 s_1)^2 = (cbabab)^2 \quad \text{and} \quad r_2 := (s_5 s_6 s_7 s_6)^2 = (efgfgf)^2,$$

and consider the normal closure N_C of $\{r_1, r_2\}$ in Λ_5. By the *Manifold Property* [7, Theorem 6.2], the group N_C is a torsion-free subgroup of index 40320 in Λ_5 with quotient manifold $Q^5 = \mathbb{H}^5/N_C$.

Moreover, it can be shown that Q^5 has 70 cusps and symmetry group $I(Q^5)$ containing the Weyl group A_7 of order 8!; see [7, Section 5.3, Table 5.1]. As for its volume, the observation (2.4) allows us to deduce that

$$\mathrm{vol}_5(Q^5) = 4 \cdot 8! \cdot \mathrm{vol}_5([6,3,3,3,3,6]) = 364\,\zeta(3) \approx 437.54871\,.$$

In fact the Felikson-Tumarkin manifold Q^5 is a cover of our manifold M_*^5, because the torsion-free subgroup N_C of index 40320 in Λ_5 is contained in the torsion-free subgroup of index 2880 we found in proving Theorem 2. This is easily verifiable using MAGMA.

5. SOME FURTHER OBSERVATIONS

We finish this paper by exploiting the conclusions of Theorem 1 about small index torsion-free subgroups for the related cases of the hyperbolic Coxeter groups $[\infty,3,3,3,6]$, $[3,3,6]$ and $[\infty,3,6,\infty]$.

Theorem 4. *The smallest index of a torsion-free subgroup in the Coxeter group $[\infty,3,3,3,6]$ is 720, achieved for example by the torsion-free subgroup generated by def, $acdcbfeab$, $abcdefedfefdefefdcb$, $bcdedfedfefdefefcdcb$ and $bcdefedcbabcdefedcab$, where a,b,c,d,e and f are the canonical generators for the Coxeter group. This subgroup gives rise to a non-orientable hyperbolic 4-manifold of smallest volume $\frac{4\pi^2}{3}$, with four cusps and with first homology group $\mathbb{Z} \oplus \mathbb{Z}_2^4$.*

Proof. First, the Euler characteristic of the Coxeter group $\Gamma_4 = [\infty,3,3,3,6]$ is $1/720$, so the minimum index of a torsion-free subgroup is at least 720. On the other hand, a MAGMA computation shows that the subgroup given in the statement of the theorem has index 720, and contains no conjugate of any of the representative torsion elements a, b, c, d, e, f, ac, ad, ae, af, bc, bd, be, bf, cd, ce, cf, de, df, ace, acf, adf, bdf, $(ef)^2$, $(ef)^3$, $a(ef)^3$, $ac(ef)^3$, $bcde$, $bc(ef)^2$, $b(ef)^3$ and $c(ef)^3$, and has abelianisation $\mathbb{Z} \oplus \mathbb{Z}_2^4$.

The associated hyperbolic 4-manifold is non-orientable, and it has four cusps. Indeed the Coxeter polyhedron related to Γ_4 has precisely one ideal vertex v_0, with stabiliser subgroup $\Gamma_4(v_0)$ isomorphic to $\langle a,b \rangle \times \langle d,e,f \rangle$. The cusps of the manifold are in one-to-one correspondence with the orbits of the group $\Gamma_4(v_0)$ in the action of Γ_4 on the cosets of the torsion-free subgroup, and a MAGMA computation shows that there are exactly four such orbits, of lengths 48, 96, 144 and 432. □

In the case of the Coxeter group $[3,3,6]$, and by means of a simple MAGMA computation, we rediscover the following well known fact mentioned in the Introduction.

Theorem 5. *The smallest index of a torsion-free subgroup in the Coxeter group $[3,3,6]$ is 24, achieved by the torsion-free subgroup generated by $u = abcdc$ and $v = bcd$, which satisfy $u^2v^2 = vu$, where a,b,c and d are the canonical generators for the Coxeter group. This subgroup is unique up to*

conjugation, and yields the fundamental group of Gieseking's 1-cusped non-orientable hyperbolic 3-manifold G of volume ν_3, with infinite cyclic first homology group $\mathbb{Z}$.

Associated with the Euclidean Coxeter group $[6,3]$ is the hyperbolic Coxeter pyramid group $[\infty, 6, 3, \infty] \subset \mathrm{Isom}\mathbb{H}^3$ of covolume $\frac{5}{4}\,\mathrm{JI}(\frac{\pi}{3})$. Part of our Theorem 1 can be extended as follows.

Theorem 6. *The smallest index of a torsion-free subgroup in the Coxeter group $[\infty, 6, 3, \infty]$ is 24, achieved for example by the orientation-preserving torsion-free subgroup generated by ab, acdcbc and acebcd, and also by the torsion-free subgroup generated by ab, acbcd and acdcebc (which does not preserve orientation), where a, b, c, d and e are the canonical generators for the Coxeter group. These subgroups gives rise to one orientable and one non-orientable hyperbolic 3-manifold, each with three cusps and volume $30\,\mathrm{JI}(\frac{\pi}{3}) \approx 10.14941$, and each with first homology group $\mathbb{Z}^3$.*

Proof. First, note that the subgroup generated by b, c and e is finite and isomorphic to $D_6 \times C_2$, of order 24, and so the minimum index of a torsion-free subgroup is at least 24. On the other hand, MAGMA computations show that each of the two subgroups given in the statement of the theorem has index 24, and contains no conjugate of any of the representative torsion elements a, b, c, d, e, ac, ad, ae, bc, bd, be, cd, ce, acd and bce, and has abelianisation $\mathbb{Z}^3$.

Each of the two associated hyperbolic 3-manifolds has three cusps. Indeed the Coxeter polyhedron related to the group $[\infty, 6, 3, \infty]$ has two ideal vertices v_1 and v_2, with stabiliser subgroups isomorphic to $H_1 = \langle b, c, d \rangle \cong [6,3]$ and $H_2 = \langle a, b \rangle \times \langle d, e \rangle$, respectively. Finally, a MAGMA computation determines the number of orbits of the stabilisers in the action of $[\infty, 6, 3, \infty]$ on the cosets of each of two torsion-free subgroups of index 24 in $[\infty, 6, 3, \infty]$. In the case of H_1, there is a single orbit of length 24, while in the case of H_2, there are two orbits of lengths 8 and 16. Hence there are three cusps in total for each of the two torsion-free subgroups. $\qquad\square$

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