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THE APPLICATION OF WAVE BASED TECHNIQUES TO INTENSITY MEASUREMENT IN BEAMS AND PLATES

by

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A thesis submitted in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Engineering

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ABSTRACT

THE APPLICATION OF WAVE BASED TECHNIQUES TO INTENSITY MEASUREMENT IN BEAMS AND PLATES

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The estimation of flexural wave intensity requires a knowledge of the internal forces present within the material, and these may be related to the spatial derivatives of displacement through the use of simple bending theories. A fundamental problem of intensity measurement lies in the estimation of these spatial derivatives. Commonly a finite difference approximation is used, however alternative techniques based on the decomposition of the motion into wave components can offer improved versatility and accuracy, along with greater insight into the dynamics of the system. These techniques are the subject of this thesis.

Wave decomposition is used to facilitate the estimation of flexural wave intensity in a beam, both in the far field and in regions where one and two near-fields exist. Different arrangements of transducers are compared in terms of the conditioning of the problem of estimating wave amplitudes from measured variables. It is shown that the choice of measured variables has an effect on the conditioning, and this is verified experimentally.

Wave-based techniques for far field plate intensity measurements, both using the commonly used assumption of one-dimensional propagation and assuming two-dimensional propagation, are investigated. A new technique is proposed in which it is assumed that the displacement of the plate can be expressed as the sum of plane waves, and that the wave amplitudes are a function of propagation direction which is approximated to a truncated complex Fourier series. The intensity can then be written in terms of the Fourier series coefficients. The effect that the use of different measurement systems has on the conditioning of the evaluation of the coefficients is discussed. Results of experimental measurements using the proposed techniques are presented.
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A vector of wave amplitudes
A cross-sectional area of beam
$A_p, B_p$ plane propagating wave amplitudes (plate)
$A^+, A^-$ propagating wave amplitudes (beam)
$A_N^+, A_N^-$ evanescent (near-field) wave amplitudes (beam)
$A_N^+, A_N^-$ evanescent (near-field) wave amplitudes at their points of origin (beam)
$A(\theta)$ complex wave amplitude as a function of direction
a linear acceleration
C vector of Fourier series coefficients
$C_p^*$ near-field wave amplitude (plate)
$C_n$ complex Fourier series coefficient
D flexural stiffness per unit width of plate
$D_d$ flexural stiffness per unit width of diaphragm
d strain gauge half-length, blocking mass length
E Young's modulus of beam or plate
$E_d$ Young's modulus of diaphragm
$E_x, E_y$ error in estimate of intensity component
e base of natural logarithm
F,G,H,T transformation matrices
F applied force
G shear modulus
$H_o^{(2)}(\ )$ Hankel function of the second kind of zero'th order
h beam or plate thickness
I second moment of area
\( I_x(t), I_y(t) \) \hspace{1cm} \text{instantaneous energy flow per unit width (intensity) in} \\
\hspace{1.5cm} x \text{ and } y \text{-directions}

\( I_{q,r}(t), I_{M_x}(t), I_{T_y}(t) \) \hspace{1cm} \text{shear, bending and twisting moment components of intensity}

\( i \) \hspace{1cm} \text{complex operator}

\( J \) \hspace{1cm} \text{moment of inertia}

\( J_n \) \hspace{1cm} \text{Bessel function of } n \text{th order}

\( K_d(x,r) \) \hspace{1cm} \text{kinetic energy density}

\( K_n \) \hspace{1cm} \text{modified Bessel function of } n \text{th order}

\( k, k_p \) \hspace{1cm} \text{wavenumbers}

\( k_x, k_y, k_z \) \hspace{1cm} \text{trace wavenumbers}

\( M, M_x, M_y \) \hspace{1cm} \text{bending moments}

\( M_{xy} \) \hspace{1cm} \text{twisting moment}

\( P(t) \) \hspace{1cm} \text{instantaneous energy flow (beam)}

\( P_d(x,t) \) \hspace{1cm} \text{potential energy density}

\( P_Q(t), P_M(t) \) \hspace{1cm} \text{shear and moment components of instantaneous energy flow}

\( P(k_r) \) \hspace{1cm} \text{Hankel transform of applied pressure } p(r)

\( P(k_x, k_y) \) \hspace{1cm} \text{Fourier transform (in } x \text{ and } y \text{) of applied pressure } p(x,y)

\( p \) \hspace{1cm} \text{applied pressure}

\( Q, Q_x, Q_y \) \hspace{1cm} \text{shear forces}

\( \text{Re, Im} \) \hspace{1cm} \text{real and imaginary parts of a complex number}

\( R, \phi \) \hspace{1cm} \text{displacements in polar space}

\( r, \phi \) \hspace{1cm} \text{coordinates in polar space}

\( S \) \hspace{1cm} \text{area of integration}

\( S_s \) \hspace{1cm} \text{sensitivity to variations in parameter } s

\( t \) \hspace{1cm} \text{time}

\( U_n, V_n \) \hspace{1cm} \text{related to Bessel functions (see Section 6.5)}

\( \text{U, V, W} \) \hspace{1cm} \text{vectors of measured variables}

\( W(k_r) \) \hspace{1cm} \text{Hankel transform of displacement } w(r)

\( W(k_x, k_y) \) \hspace{1cm} \text{Fourier transform (in } x \text{ and } y \text{) of displacement } w(x,y)
$u,v,w$  
  displacements in Cartesian space

$x,y,z$  
  coordinates in Cartesian space

$\varepsilon_x, \varepsilon_y$  
  normal strains

$\gamma_{xy}$  
  shear strain

$\nu$  
  Poisson's ratio

$\sigma_x, \sigma_y$  
  normal stresses

$\tau_{xy}$  
  shear stress

$\tau$  
  time delay

$\omega$  
  angular frequency

$\rho$  
  density

$\delta(r)$  
  Dirac delta function

$\theta, \theta_A, \theta_B$  
  directions of wave propagation

$\theta(x)$  
  rotation at point $x$

$\theta_z$  
  angle of rotation

$\phi$  
  phase difference

$\Delta$  
  transducer separation

$\Gamma$  
  factorial function

$\nabla^2$  
  Laplace operator (2-D) $= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\langle \rangle$  
  time-average

$\langle \rangle^T$  
  transpose of a matrix

$\langle \rangle''$  
  conjugate transpose of a matrix

Subscripts

$w$  
  wave-based approximation

$FD$  
  finite difference-based approximation

$FS$  
  Fourier series approximation
Part A

Structural Intensity
- Background and Theory
Chapter 1: Introduction

1.1 Introduction

It could be said that the study of vibration is essentially negative in its nature, as in most cases our interest in vibration or dynamic characteristics stems from a desire to eliminate (or at least minimise) vibration. That vibrations are undesirable in most cases is beyond doubt. They are capable of causing damage to structures both through fatigue and, in extreme cases, excessive stresses, while sensitive equipment may give erroneous readings or be damaged due to the acceleration levels experienced. Additionally, the effects of vibration on humans should not be overlooked, it being capable of causing both discomfort and physical harm. It is therefore of particular concern that today's competitive marketplace demands that products be lighter, faster and more efficient than their forerunners. These characteristics, while desirable, generally make the products more prone to noise and vibration problems. There is thus a considerable commercial interest in noise and vibration control technology.

Ideally, the reduction of vibration should be considered at the design stage. By appropriate use of such techniques as Finite Element Analysis and Statistical Energy Analysis, an attempt can be made to incorporate the desired dynamic characteristics in the structure. In many cases, however, a vibration problem occurs in an existing structure. This problem may manifest itself in various ways, such as excessive displacement, acceleration or sound levels, but is a consequence of excessive vibrational energy. An obvious starting point in tackling a vibration problem is therefore to reduce the input of energy to the structure, either by modification of the source (e.g. by balancing of rotating machinery) or by the use of isolating mounts. Similarly, sensitive equipment may be isolated from the vibrating structure, effectively reducing the vibrational energy supplied to the equipment.
If these approaches are inappropriate or insufficient, further improvements may be achieved through modification of the structure. Vibration treatments which come under this category include stiffening, detuning, decoupling, damping, and the use of active control. Through the use of appropriate vibration treatment it should be possible to reduce the vibrational energy in the critical regions. It is quite possible, however, for a particular treatment to have the reverse effect, or to reduce one problem but introduce or exacerbate another.

It is at this stage that the deficiencies of conventional vibration measurements become apparent. The dynamic behaviour of a complex structure may be the result of a combination of longitudinal, torsional and flexural vibrations, all capable of contributing to the net energy flow within that structure. Different vibration types in dissimilar structural components may therefore contribute to the response at a point. The measurement of energy flow (power, or alternatively power per unit width for structural members that are effectively two dimensional) offers a unifying concept by which different vibration types in different media can be compared. It should be noted that large amplitudes of vibration do not necessarily indicate large energy flow - a standing wave, being composed of two identical waveforms travelling in opposite directions, results in a zero net energy flow though characterised by large displacements at the antinodes. The measurement of energy flow therefore requires alternative (and somewhat more complicated) measurement techniques.

Vibrational energy flow is a product of local stresses within the structural components and local velocities. In general the velocities are relatively easily measured with modern transducers, however the measurement of internal stresses poses more of a problem. As it is not possible to place transducers within the structural member to measure these stresses directly they must be estimated from the external parameters that we can measure, such as acceleration and strain. This limitation means that vibrational energy flow measurements can only be made on structural elements for which the relationships between internal stresses and external displacements or strains are known.
The measurement of energy flow (or structural intensity) in solids is directly analogous to the measurement of acoustic intensity in fluids, yet structural intensity measurement is still in the developmental stages while acoustic intensity measurement systems are commercially available. This is primarily because there are several additional complications associated with structural intensity measurement which make it inherently more difficult than the acoustic equivalent. First, while acoustic waves in inviscid fluid mediums are purely of longitudinal type, a solid medium can support longitudinal, torsional and flexural waves, each obeying a different wave equation. Any rigorous evaluation of structural intensity should include all three wave types, and this requires an elaborate system of transducers to separate the wave components. Secondly, a pressure transducer can (usually) readily be placed to give a direct measurement of 'internal stress' in a fluid, whereas internal stresses in solids must be inferred from external measurements as previously mentioned. Thirdly, the flexural wave, which is commonly most significant in structure-borne sound transmission, being readily excited and an effective means of radiating sound into the surrounding acoustic medium, obeys a fourth order differential equation. This allows the existence of near-fields (non-propagating, evanescent waves), which are not present for the wave types obeying second order differential equations.
1.2 Structural Intensity Measurement - A Survey

The following survey describes the more significant works and techniques as pertaining to the specific research described in this thesis - namely flexural wave intensity measurement in beams and plates. There have, of course, been numerous valuable contributions to intensity measurement in other applications, particularly regarding other wave types and other structural members such as pipes and shells. However, for the sake of brevity and clarity, these have not been included in the survey.

The principles of structural energy flow measurement in homogeneous and isotropic beams and plates undergoing flexural vibration were first given by Noiseux [1], adopting thin beam theory and thin plate theory to estimate internal stresses from external measurements. Noiseux noted that the energy flow due to flexural vibration in beams can be considered the sum of two components - the component due to the shear force and that due to the bending moment. The time-average of these components is equal in regions remote from discontinuities, meaning that under these circumstances it is only necessary to estimate one component of intensity, and this requires only two accelerometers. For flexural wave intensity measurements on a plate a simple approximation, which is equivalent to assuming that far field conditions exist and that wave propagation is one-dimensional, was proposed. Under these circumstances it was shown that the component of the intensity vector in a particular direction on a plate can be estimated using two accelerometers. It was noted that, in the far field, any error in this assumption has a zero spatial average in a uniform wavefield. Thus an estimate of the total energy flow can be obtained from a number of such measurements even though the individual measurements are in error.

It was the work of Pavic [2] that took the principle of structural intensity measurement to a useable form. Through the use of finite difference approximations to estimate the necessary spatial derivatives of displacement, Pavic presented expressions for structural intensity in beams and plates undergoing flexural vibration. These expressions did not include any simplifications to reduce the number of spatial derivatives required and therefore could be used for measurements
close to discontinuities as well as in the far field. Analogue circuitry for the time-domain processing of measurements was presented, along with three simplifications for use in the presence of one-dimensional propagating waves only. While the mathematical reasoning behind each of these three approaches differed, all used a finite difference approximation to give an estimate of the intensity in terms of two measurements of displacement or its temporal derivatives.

Both the aforementioned authors utilised processing in the time domain and referred only to flexural waves. Verheij [3], however, used finite difference approximations in conjunction with processing in the frequency domain. This gave expressions for intensity in a beam under flexural, longitudinal, and torsional vibration in terms of cross-spectral densities. Intensity estimates of longitudinal and torsional vibrations, together with flexural vibrations in the far field, were each given in terms of the cross-spectral density of two acceleration measurements, while near-field estimates used three cross-spectra from four acceleration measurements. Thus, with the exception of measurements in the near-field, use of these formulations means that the intensity calculation may be performed on the commonly available two-channel spectrum analyser, with no special equipment required. Processing in the frequency domain also simplifies the correction of phase mismatch between channels, as this is generally frequency dependent.

Rasmussen and Rasmussen [4] used the principles proposed by Noiseux and Pavic, in combination with processing in the frequency domain, to give an estimate of a component of the intensity vector on a plate (under flexural vibration) using two closely spaced accelerometers. Processing in this case was performed with a Brüel and Kjær Type 3360 sound intensity analyser. Quinlan [5] used a similar approach to estimate active and reactive intensity components, together with the potential energy density, in terms of the cross-spectrum of two acceleration measurements. Processing was performed with a conventional two-channel signal analyser. These approaches, being based on that proposed by Noiseux, are founded on the assumptions of far field conditions and one-dimensional wave propagation.
Redman-White [6] performed a comprehensive investigation into the potential of various proposed measurement systems, considering both systematic and random error. It was concluded that, for practical purposes, simple measurement systems using two or four linear accelerometers were preferable to more rigorous approaches.

A large number of papers (eg. [7-10]) have been published describing theoretical and experimental work following and developing the ideas set down by Pavic, Verheij, and Rasmussen and Rasmussen. Using finite difference approximations to estimate spatial derivatives, these currently form the basis of most practical structural intensity measurements.

The use of wave decomposition to determine the intensity in a beam has also been proposed [11-13]. Within the limitations of thin beam theory, the displacement of a beam between adjacent discontinuities can be described exactly by four complex wave amplitudes and these may be estimated from a number of measurements of acceleration or other convenient parameters. The intensity, and other parameters such as reflection coefficients and energy density, may therefore be written in terms of the wave amplitudes. Wave-based principles have also been applied on plates [14], however as an infinite number of waves may exist the approach is, in general, no longer exact.

Pavic [15] discussed the transducer requirements for structural intensity measurement, concluding that both accelerometers and strain gauges are suitable, with processing being minimised if a combination of the two are used. The field of non-contact sensing for intensity measurement has also been receiving significant attention, using both laser velocimetry [16][17] and acoustic near-field measurements [18][19]. Non-contacting measurement obviously holds considerable potential, as the number of measurements required for accurate intensity estimation in practical applications may lead to excessive mass loading of the structure when conventional transducers are used, and the repeated mounting of a large number of transducers is very time consuming.
Simple bending theory has been used in most analyses of flexural wave intensity, which limits applicability to homogeneous and isotropic structural elements at small deformations and lower frequencies. While this is adequate in many situations a more rigorous theoretical analysis, thus applicable at higher frequencies, has been applied to homogeneous and isotropic plates by Maysenhölter [20].

There still remains the problem that intensity measurements require that the relationship between internal stresses and external deformation be known. While it may be known for simple structural elements such as beams, plates, and shells it is not, in general, known for more complex ones. Addressing this issue, Pavic [21] has proposed the measurement of 'structural surface intensity' - essentially the energy flow on the surface of an arbitrary solid - which does not require the internal stress distribution to be known. However as the relationship between surface and sub-surface energy flow is not generally known the range of application of the technique is unclear.
1.3 Motivation and Philosophy of Research

While there exist expressions for intensity in beams and plates in terms of variables that can be measured (e.g. [2]), there can be many problems involved in practical measurements. This is largely because the estimation of intensity in this way can be very sensitive to errors in the measured data, and current transducers and signal processing are sometimes inadequate for the task.

The sensitivity of a calculation to errors in the input parameters is termed the *conditioning* of the problem, a well conditioned problem being relatively insensitive to errors. If the conditioning of the intensity calculation can be improved then measurement errors become less critical, allowing better intensity estimates, or permitting measurements to be performed in more demanding conditions than would otherwise be possible.

The estimation of structural intensity is usually achieved using finite difference approximations to estimate the required spatial derivatives. Implicit in this approach is the fitting of polynomials to the measurement points, the spatial derivatives of these polynomials then being found. It is thus assumed that the deformation can be described by (in this case low order) polynomials. There is potentially a significant difference between the true and assumed deformations, however. If the measurement points are close together then this difference is small, and the errors inherent in the intensity calculation due to the finite difference approximation are small also. If, however, the measurement points are widely spaced then the difference between the true and assumed deformations is large, and the errors due to the finite difference approximation are also large. It would therefore appear that a small distance between measurement locations is sufficient to ensure an adequate estimate of intensity. While this is true in terms of systematic errors, the reverse is true with regard to random errors. The finite difference approximation entails the addition and subtraction of individual measurements, and when these measurements are taken at closely spaced locations they are almost identical. Under these circumstances the difference between measurements can be dominated by measurement errors and noise, and the intensity
calculation is poorly conditioned. Since a poorly conditioned problem will be prone to large random errors, the choice of transducer spacing must be a compromise between low systematic error and low random error [22–24].

As an alternative to the finite difference approach, the work described in this thesis exploits the fact that most structures are continuous and wave-bearing, and that the waves may be considered the mechanism by which energy propagates. A description of the response within a region in terms of wave component amplitudes therefore allows the local intensity, and any other vibrational parameters such as the energy density, to be calculated at all points within the region where the wave description remains valid.

In applying wave-based principles to structural intensity measurement it is assumed that the deformation can be described using trigonometric functions (and exponential functions if near-fields are accommodated). These functions are fitted to the measurements, giving a better approximation of the deformation than that given by the low order polynomials used in the finite difference approach. This allows a better estimate of the required spatial derivatives, and hence of the intensity, for a given transducer spacing. Alternatively, a larger transducer separation may be used, giving improved conditioning and therefore reduced sensitivity to random errors, without raising systematic errors to an unacceptable level.

The common theme linking the various parts of this research is the application of wave-based techniques to the estimation of intensity in beams and plates, with particular interest in the resultant conditioning of the problem and how this is affected by the choice of measured variables and measurement locations.
1.4 Contents of Thesis

This thesis is divided into three parts. Part A introduces the concept of structural intensity and describes the theoretical basis for intensity measurement, giving the derivation of the necessary equations. In Part B, methods of implementing the equations derived in Part A in practical measurement systems are discussed, while in Part C experimental work is described and discussed.

In Part A, the following chapter gives the derivation of expressions for displacement in beams and plates in terms of waves. The expressions for intensity in beams and plates are given in Chapter 3, along with observations regarding the behaviour of the intensity vector under a variety of field conditions.

Chapters 4, 5 and 6, constituting Part B, describe wave-based approaches to intensity measurement on beams, on plates when assuming one-dimensional propagation, and on plates assuming general far field conditions respectively. Various transducer configurations are considered in terms of systematic errors and conditioning.

In Part C, Chapters 7 and 8 describe experimental intensity measurements using the proposed techniques on a beam and a plate respectively. The main emphasis and findings of the thesis are summarised in Chapter 9, and recommendations for further study are made in Chapter 10.
Chapter 2: Wave Theory

2.1 Introduction

The equations of motion for flexural vibration in beams and plates are derived in this chapter, along with the solutions to these equations in certain situations. The solutions justify the assumption that, under particular conditions, the motion of a beam or plate may be expressed in terms of plane waves. This assumption is fundamental to the proposed wave-based approach to structural intensity measurement.

The derivation of the equations of motion requires descriptions of the behaviour of beams and plates under bending. These are provided by thin beam theory and thin plate theory [25], and are based on the following fundamental assumptions:

i. There is no deformation of the middle plane (neutral plane) of the beam or plate.

ii. Points lying initially on a normal to the neutral plane remain so after deformation. This implies no shear deformation.

iii. Normal stresses perpendicular to the neutral plane are negligible.

While there exist more rigorous and general models of beam and plate bending, most approaches to structural intensity measurement use these elementary bending theories. The assumptions involved restrict intensity measurements to small deformations in thin homogeneous beams and plates, and effectively impose an upper frequency limit on the measurement. In many practical situations, however, these simple models are adequate.
2.2 Flexural Waves in Beams

Let the mid-plane of a beam lie in the \( x-y \) plane, as shown in Figure 2.1. Deflection of the beam will induce a slope, \( \partial w / \partial x \), in that mid-plane. It is assumed that, since the deflection is small, points on the mid-plane are not displaced in the \( x \)-direction. As a consequence of this and the assumption that the shear deformation is negligible, the displacement in the \( x \)-direction of a point a distance, \( z \), from the mid-plane is given by

\[
u = -z \frac{\partial w}{\partial x}.
\]

The strain in the \( x \)-direction at this point is therefore, by definition,

\[
\epsilon_x = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}.
\]

The stress-strain relationship under uniaxial stress is given by Hooke's law as

\[
\sigma_x = E\epsilon_x,
\]

and substituting equation 2.2.2 into equation 2.2.3 shows the normal stress at a distance, \( z \), from the mid-plane of the beam to be
\[ \sigma_x = -Ez \frac{\partial^2 w}{\partial x^2}. \]  

2.2.4

Integrating the moment due to this stress about the mid-plane over the cross-section of the beam to give the total moment, \( M \), as defined in Figure 2.2, yields

\[ M = \int \int \sigma_x z dS = -EI \frac{\partial^2 w}{\partial x^2}, \]  

2.2.5

where \( E \) is Young's modulus and \( I = \int \int z^2 dS \) is the second moment of area of the section.

Figure 2.2: Forces and moments acting on an element of beam.

The relationships between the forces shown in Figure 2.2 are determined by the requirements of equilibrium. Referring to Figure 2.2, the requirement of translational equilibrium in the vertical plane implies that

\[ -\frac{\partial Q}{\partial x} + \rho A \frac{\partial^2 w}{\partial t^2} = F(x,t), \]  

2.2.6

where \( \rho \) is the beam density, \( A \) is the cross-sectional area of the beam, \( Q \) is the shear force, and \( F(x,t) \) the excitation force per unit length. Similarly, by requiring rotational equilibrium and ignoring both rotational inertia and second order terms in \( dx \), it can be shown that
\[ Q = \frac{\partial M}{\partial x}. \] \hspace{1cm} \text{(2.2.7)}

Combining equations 2.2.5 - 2.2.7 gives the equation of motion for the beam,

\[ EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = F(x,t). \] \hspace{1cm} \text{(2.2.8)}

The solution to equation 2.2.8 is, of course, dependent on the excitation, \( F(x,t) \), to which the beam is subjected. In the absence of excitation, however, equation 2.2.8 becomes

\[ \frac{\partial^4 w}{\partial x^4} \frac{\omega^2 \rho A}{EI} w = 0, \] \hspace{1cm} \text{(2.2.9)}

where \( \omega \) is the angular frequency, if time-harmonic motion is assumed. The general solution is then

\[ w(x,t) = (A^+ e^{-ikx} + A^- e^{ikx} + A^+_N e^{-kx} + A^-_N e^{kx}) e^{i\omega t}, \] \hspace{1cm} \text{(2.2.10)}

where

\[ k^* = \frac{\omega^2 \rho A}{EI}. \] \hspace{1cm} \text{(2.2.11)}

\( k \) being the wavenumber. In equation 2.2.10, \( A^+ \) and \( A^- \) may be interpreted as the complex amplitudes of waves propagating in the positive and negative \( x \)-direction respectively. They therefore exhibit harmonic variation in both time and space. Amplitudes \( A^+_N \) and \( A^-_N \) describe the evanescent or near-field waves, which vary harmonically with time but decay exponentially in space, the superscript indicating the direction of decay. These near-fields are generally the result of a discontinuity (such as a change in section) of the beam, or some perturbation such as an applied force. Their presence is necessary to ensure that the required boundary conditions are met,
and their exponential decay from the point of origin means that, remote from discontinuities, only propagating waves are of significant amplitude. The term 'far field' is used to describe these conditions, while near-field conditions are said to exist at a point if the amplitude of an evanescent wave is significant at that point.

Such an interpretation is strictly only valid if the wavenumber, \( k \), is purely real. If the modulus, \( E \), is complex, as it is in the presence of structural damping, then the propagating wave components also exhibit an amplitude that decays spatially, while the evanescent components exhibit a spatial modulation. However as the loss factor in most structural materials is very small [26] the imaginary part of the wavenumber is typically very small relative to the real part, and the wavenumber may be considered purely real.
2.3 Flexural Waves in Plates

Let the mid-plane (which is also the neutral plane of bending) of a plate lie in the \( x-y \) plane as shown in Figure 2.3. If the plate is deflected in the \( z \)-direction the slopes of the mid-plane in the \( x \)- and \( y \)-directions are given by \( \partial w / \partial x \) and \( \partial w / \partial y \) respectively. The assumption of small deflection means that the displacement in the \( x-y \) plane of points on the mid-plane of the plate is negligible. This, combined with the assumption that there is no shear deformation, means that the displacements in the \( x-y \) plane of points a distance, \( z \), from the mid-plane are given by

\[
\begin{align*}
  u &= -z \frac{\partial w}{\partial x}, \\
  v &= -z \frac{\partial w}{\partial y}.
\end{align*}
\]

The strains at a distance, \( z \), from the mid-plane are therefore

\[
\begin{align*}
  \varepsilon_x &= \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}, \\
  \varepsilon_y &= \frac{\partial v}{\partial y} = -z \frac{\partial^2 w}{\partial y^2}, \\
  \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z \frac{\partial^2 w}{\partial x \partial y}.
\end{align*}
\]

Figure 2.3: Coordinate system for plate.
As it is assumed that normal stresses perpendicular to the neutral plane are negligible a state of plane stress exists, and the stress-strain relationships are given by the generalised Hooke's law as

\[
\varepsilon_x = \frac{1}{E}(\sigma_x - v\sigma_y), \quad \varepsilon_y = \frac{1}{E}(\sigma_y - v\sigma_x), \quad \gamma_{xy} = \frac{1}{G} \tau_{xy} = \frac{2(1+v)}{E} \tau_{xy},
\]

where \(G\) is the shear modulus, or, by rearranging,

\[
\sigma_x = \frac{E}{(1-v^2)}(\varepsilon_x + v\varepsilon_y), \quad \sigma_y = \frac{E}{(1-v^2)}(v\varepsilon_x + \varepsilon_y), \quad \tau_{xy} = \frac{E}{2(1+v)} \gamma_{xy}.
\]

Substituting equations 2.3.2 into equations 2.3.4 shows the stresses at a distance, \(z\), from the mid-plane of the plate to be

\[
\sigma_x = \frac{-Ez}{(1-v^2)} \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right),
\]

\[
\sigma_y = \frac{-Ez}{(1-v^2)} \left( v \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right),
\]

\[
\tau_{xy} = \frac{-Ez}{(1+v)} \frac{\partial^2 w}{\partial x \partial y}.
\]

The stresses given by equations 2.3.5 vary linearly through the thickness of the plate, generating moments per unit width. Following the sign conventions and nomenclature shown in Figure 2.4, these moments may be found by integrating the moment applied by the stresses about the neutral plane over all values of \(z\), giving

\[
M_x = \frac{1}{3} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_x z dz = -D \left[ \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right],
\]
\[ M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_y zdz = -D \left[ v \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] \]

and \[ M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} zdz = -D(1 - v) \frac{\partial^2 w}{\partial x \partial y} \]

where

\[ D = \frac{1}{(1-v^2) \int_{-\frac{h}{2}}^{\frac{h}{2}} E z^2 dz} = \frac{E h^3}{12(1-v^2)} \]

is the flexural stiffness per unit width of the plate.

Figure 2.4: Forces and moments acting on an element of plate.

The relationships between the various forces and moments shown in Figure 2.4 may be found by determining the necessary conditions for equilibrium. The requirement of translational equilibrium implies that
\[ p(x, y, t)dx \, dy + (Q_x + dQ_x)dy + (Q_y + dQ_y)dx - Q_x dy - Q_y dx = \rho h dx dy \frac{\partial^2 w}{\partial t^2}. \]  

2.3.8

Noting that

\[ dQ_x = \frac{\partial Q_x}{\partial x} dx, \quad dQ_y = \frac{\partial Q_y}{\partial y} dy, \]  

2.3.9

equation 2.3.8 may be simplified to give

\[ p(x, y, t) + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = \rho h \frac{\partial^2 w}{\partial t^2}. \]  

2.3.10

Similarly, requiring rotational equilibrium about the x- and y-axes while ignoring the rotational inertia of the element yields

\[ (M_x + dM_x)dy - M_x dy - Q_x dx dy + \left( M_{yx} + dM_{yx} \right) dx - M_{yx} dx + \]  

\[ (Q_y + dQ_y)dx \frac{dx}{2} - Q_y dx \frac{dx}{2} + p(x, y, t) dx \, dy \frac{dx}{2} = 0 \]  

2.3.11a

and

\[ (M_y + dM_y)dx - M_y dx - Q_y dx dy + \left( M_{xy} + dM_{xy} \right) dy - M_{xy} dy + \]  

\[ (Q_x + dQ_x)dy \frac{dy}{2} - Q_x dy \frac{dy}{2} + p(x, y, t) dx \, dy \frac{dy}{2} = 0. \]  

2.3.11b

Noting that

\[ dM_x = \frac{\partial M_x}{\partial x} dx, \quad dM_y = \frac{\partial M_y}{\partial y} dy, \quad dM_{xy} = \frac{\partial M_{xy}}{\partial x} dx, \quad dM_{yx} = \frac{\partial M_{yx}}{\partial y} dy \]  

2.3.12

and ignoring second order terms in \( dx \) and \( dy \), the relationships between shear forces and bending moments are therefore given by
\[ Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} \]  \hspace{2cm} 2.3.13a

and \[ Q_y = \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x}. \]  \hspace{2cm} 2.3.13b

Substituting equations 2.3.13 in equation 2.3.10 and rearranging gives

\[ -\left( \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{yx}}{\partial x \partial y} \right) - \left( \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} \right) + \rho h \frac{\partial^2 w}{\partial t^2} = p(x, y, t), \]  \hspace{2cm} 2.3.14

which simplifies to become

\[ -\left( \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} \right) + \rho h \frac{\partial^2 w}{\partial t^2} = p(x, y, t) \]  \hspace{2cm} 2.3.15

since \( M_{xy} = M_{yx} \). Substituting equations 2.3.6 in equation 2.3.15, then gives

\[ D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \rho h \frac{\partial^2 w}{\partial t^2} \approx p(x, y, t), \]  \hspace{2cm} 2.3.16

or \[ D \nabla^2 \nabla^2 w + \rho h \frac{\partial^2 w}{\partial t^2} = p(x, y, t). \]  \hspace{2cm} 2.3.17

where \( \nabla^2 \) is the Laplace operator. Equation 2.3.17 is the equation of motion of an infinite flat plate excited by an arbitrary pressure distribution. This is not as useful as it first might appear, as in practice we deal with finite rather than infinite elements, however by considering suitable cases certain useful results may be obtained.
2.3.1 Plate Response Using the Hankel Transform

Using polar coordinates, let the pressure distribution on the plate be

\[ p(r, \phi, t) = p(r, t)e^{-in\phi} \tag{2.3.18} \]

where \( n \) is an integer, and assume that the response is given by

\[ w(r, \phi, t) = w(r, t)e^{-in\phi}. \tag{2.3.19} \]

Since it is known that

\[ \nabla^2 w(r, \phi, t) = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) [w(r, t)e^{-in\phi}] \]

\[ = e^{-in\phi} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right) [w(r, t)], \tag{2.3.20} \]

and similarly

\[ \nabla^2 \nabla^2 w(r, \phi, t) = e^{-in\phi} \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right)^2 \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right) [w(r, t)], \tag{2.3.21} \]

if we denote the operator

\[ \Delta^2 = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right), \tag{2.3.22} \]

the equation of motion, 2.3.17 can be rewritten as
\[
\frac{D}{\rho h} \Delta^2 \Delta^2 w(r,t) + \frac{\partial^2 w}{\partial t^2} = \frac{p(r,t)}{\rho h},
\]

2.3.23

eliminating the variable, \(\phi\).

The Hankel transform of the function \(\Delta^2 w(r,t)\) is given by

\[
\int_0^\infty r(\Delta^2 w(r,t)) J_n(k,r) dr = \int_0^\infty \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right] \left[ w(r,t) \right] J_n(k,r) dr
\]

\[= -k_n^2 W(k_r,t),
\]

2.3.24

where \(J_n\) is the \(n\)th order Bessel function of the first kind and \(W(k_r,t)\) is the Hankel transform of displacement \(w(r,t)\), \(k\), being the radial wavenumber. If we denote

\[
q(r,t) = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right) w(r,t),
\]

2.3.25

and let \(Q(k_r,t)\) be the Hankel transform of \(q(r,t)\), then the Hankel transform of the function \(\Delta^2 \Delta^2 w(r,t)\) is

\[
\int_0^\infty r(\Delta^2 \Delta^2 w(r,t)) J_n(k,r) dr = \int_0^\infty \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right] \left[ q(r,t) \right] J_n(k,r) dr,
\]

\[= -k_n^2 Q(k_r,t)
\]

\[= -k_n^2 \int_0^\infty \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{n^2}{r^2} \right] \left[ w(r,t) \right] J_n(k,r) dr
\]

\[= k_n^4 W(k_r,t).
\]

2.3.26
Applying the Hankel transform to the equation of motion 2.3.23 therefore gives

$$\frac{D}{\rho h}k_r^4 W(k_r, t) + \frac{\partial^2 W(k_r, t)}{\partial t^2} = \frac{1}{\rho h} P(k_r, t), \quad 2.3.27$$

where

$$P(k_r, t) = \int_0^\infty r p(r, t) J_0(k_r r) dr. \quad 2.3.28$$

In the case of excitation by a single time-harmonic point force at the coordinate origin

$$p(r, \phi, t) = p(r, t) = F \delta(r) e^{im_1} \quad 2.3.29$$

as there is no angular dependence, and hence

$$P(k_r, t) = \int_0^\infty r F \delta(r) e^{im_1} J_0(k_r r) dr$$

$$= \frac{1}{2\pi} F e^{im_1} \quad 2.3.30$$

since \(2\pi \int_0^\infty r \delta(r) dr = 1\) and \(J_0(0) = 1\), \(\delta(r)\) being the Dirac delta function. The equation of motion 2.3.27 therefore becomes

$$\frac{D}{\rho h}k_r^4 W(k_r, t) + \frac{\partial^2 W(k_r, t)}{\partial t^2} = \frac{F e^{im_1}}{2\pi \rho h}. \quad 2.3.31$$

Furthermore, if the motion is time-harmonic then

$$W(k_r, t) = W(k_r) e^{im_1}. \quad 2.3.32$$
and equation 2.3.31 can be rewritten as

$$\frac{D}{\rho h} k_r^4 W(k_r) - \omega_0^2 W(k_r) = \frac{F}{2\pi \rho h}.$$  \hspace{1cm} 2.3.33

or

$$W(k_r) = \frac{F}{2\pi \rho h} \left( \frac{D}{\rho h} k_r^4 - \omega_0^2 \right).$$  \hspace{1cm} 2.3.34

Letting $b^2 = \frac{D}{\rho h}$ and rearranging equation 2.3.34 then gives

$$W(k_r) = \frac{F b}{4\pi D \omega_0} \left( \frac{1}{k_r^2 - \frac{\omega_0}{b}} - \frac{1}{k_r^2 + \frac{\omega_0}{b}} \right).$$  \hspace{1cm} 2.3.35

The response at radius, $r$, is found by applying the inverse Hankel transform to equation 2.3.35, giving

$$w(r) = \frac{F b}{4\pi D \omega_0} \int_0^\infty k_r J_0(k_r r) \left( \frac{1}{k_r^2 - \frac{\omega_0}{b}} - \frac{1}{k_r^2 + \frac{\omega_0}{b}} \right) dk_r,$$

$$= \frac{F b}{4\pi D \omega_0} \left[ \int_0^\infty k_r J_0(k_r r) dk_r - \int_0^\infty k_r J_0(k_r r) dk_r \right].$$  \hspace{1cm} 2.3.36

The integrals in equation 2.3.36 can be evaluated using the relation [27]

$$\int_0^\infty \frac{k_r^{\nu+1} J_0(ak_r)}{(k_r^2 + z^2)^{\mu+1}} dk_r = \frac{a^{\mu} \Gamma(\nu+\mu + \mu) K_{\nu-\mu}(az)}{2^{\mu} \Gamma(\mu+1)},$$  \hspace{1cm} 2.3.37

where $K_{\nu-\mu}$ is a modified Bessel function and $\Gamma(\mu+1) = \mu!$ is the factorial function. This gives
\[ w(r) = \frac{-iF}{8k_p^2 D} \left[ H_0^{(2)}(k_p r) - H_0^{(2)}(-ik_p r) \right] \]  \hspace{1cm} 2.3.38

or \[ w(r,t) = \frac{-iFe^{i\omega t}}{8k_p^2 D} \left[ H_0^{(2)}(k_p r) - H_0^{(2)}(-ik_p r) \right], \]  \hspace{1cm} 2.3.39

where

\[ k_p^4 = \frac{\omega_0^2 \rho h}{D} \]  \hspace{1cm} 2.3.40

and \( H_0^{(2)} \) denotes the zero'th order Hankel function of the second kind,

\[ H_0^{(2)}(k_p r) = J_0(k_p r) - iY_0(k_p r). \]  \hspace{1cm} 2.3.41

The displacement field of an infinite plate excited by a time-harmonic point force can thus be expressed in terms of the sum of two Hankel functions. Expressions of this form can be considered the Green's functions for time-harmonic bending waves in a plate, an arbitrary force distribution being described as the sum of a number of point sources. In addition to true excitation, plate behaviour at boundaries and local discontinuities may be described by duplicating with point sources a force distribution that results in the required boundary or local conditions.

Further simplification is possible if only regions remote from excitation and discontinuities are considered. In these cases the arguments of the Hankel functions are large, and it is known that for large \( z \) \[ H_0^{(2)}(z) = \frac{2}{\pi z} e^{-i\left(\frac{\pi}{4}\right)} . \]  \hspace{1cm} 2.3.42

It is immediately apparent that if the argument has a negative imaginary component then exponential decay with distance occurs. In many materials the structural damping is very small,
making the wavenumber almost purely real and the argument \((-ik_p r)\) almost purely imaginary. The Hankel function, \(H_0^{(2)}(-ik_p r)\), therefore exhibits an approximately exponential decay with \(r\) at large arguments, while \(H_0^{(2)}(k_p r)\) decays as \(1/\sqrt{r}\). Thus only the Hankel function with the real argument, \(H_0^{(2)}(k_p r)\), is significant at large \(r\), and equation 2.3.39 becomes

\[
w(r, t) \approx \frac{-iFe^{i\omega t}}{8k_p^2 D} \sqrt{\frac{2}{\pi k_p r}} e^{-\left(\frac{s-r}{4}\right)}.
\]

2.3.43

Consider now a section, \(S\), of the plate surface, upon which the excitation may be represented by a distribution of point sources \(F(x_0, y_0)\). If \((x, y)\) represents the point of interest on the plate, then the radial distance of this point from the sources is

\[r = \sqrt{(x-x_0)^2 + (y-y_0)^2},\]

2.3.44

and provided that \(k_p r\) is large the displacement at point \((x, y)\) is given by

\[
w(x, y, t) = \iint_S \frac{-iF(x_0, y_0)e^{i\pi}e^{i\omega t}}{8k_p^2 D} \sqrt{\frac{2}{\pi k_p r(x_0, y_0)}} e^{-i\omega r(x_0, y_0)} dx_0 dy_0.
\]

2.3.45

Furthermore, if the dimensions of the area of excitation, \(S\), are small relative to \(r\), then \(\sqrt{r(x_0, y_0)}\) may be considered constant over \(S\), and equation 2.3.45 may be rewritten as

\[
w(x, y, t) = \frac{-ie^{i\pi}e^{i\omega t}}{8k_p^2 D} \sqrt{\frac{2}{\pi k_p r}} \iint_S F(x_0, y_0)e^{-i\omega r(x_0, y_0)} dx_0 dy_0.
\]

2.3.46

At the point \((x, y)\) this may be interpreted as a set of plane waves emanating from region \(S\) in the direction of \((x, y)\). If boundaries and sources of excitation are sufficiently remote from the point of interest then each may be represented by a number of such regions. Furthermore, within a small vicinity of point \((x, y)\) there will be little variation in \(r\), and thus a very similar wave
description applies throughout that vicinity. The response at, and within a vicinity of, point \((x, y)\) can then be expressed as the sum of the contributions of the individual regions and hence as the sum of plane waves from various directions.

### 2.3.2 Plate Response Using the Fourier Transform

Definitions of the Fourier transform vary with regard to where a factor of \(1/2\pi\) is included. In the following discussion the Fourier transform of a function \(f(x)\) is defined as

\[
F(k_x) = \int_{-\infty}^{\infty} f(x)e^{ik_x x} dx, \tag{2.3.47}
\]

and the inverse Fourier transform as

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k_x)e^{-ik_x x} dk_x. \tag{2.3.48}
\]

Taking the Fourier transform (in \(x\) and \(y\)) of the equation of motion, 2.3.17, gives

\[
\left[ D(k_x^4 + 2k_x^2k_y^2 + k_y^4) - \rho \omega_0^2 \right] W(k_x, k_y) = P(k_x, k_y). \tag{2.3.49}
\]

We can therefore write

\[
W(k_x, k_y) = \frac{P(k_x, k_y)}{D(k_x^4 + 2k_x^2k_y^2 + k_y^4) - \rho \omega_0^2}, \tag{2.3.50}
\]

or

\[
W(k_x, k_y) = \frac{1}{D} \frac{P(k_x, k_y)}{\left( k_x^2 + k_y^2 \right)^2 - k_p^2}, \tag{2.3.51}
\]

where
\[
k'_p = \frac{\omega_0^2 p h}{D}.
\]

Further insight can be gained by simplifying equation 2.3.51 to the form

\[
W(k_x, k_y) = \frac{P(k_x, k_y)}{2k'_p D} \left[ \frac{1}{(k_x^2 + k_y^2) - k'_p} - \frac{1}{(k_x^2 + k_y^2) + k'_p} \right]
\]

and by applying the inverse Fourier transform in \(k_x\) and \(k_y\) to give

\[
w(x, y) = \frac{1}{8\pi^2 k'_p D} \int_{-\infty}^{\infty} P(k_x, k_y) \left[ \frac{1}{(k_x^2 + k_y^2) - k'_p} - \frac{1}{(k_x^2 + k_y^2) + k'_p} \right] e^{-ik_x x} e^{-ik_y y} dk_x dk_y.
\]

Whether it is possible or practical to evaluate equation 2.3.54 analytically using contour integration is dependent on the ability to evaluate the integral along the required branch cuts. It is clear, however, that the double integral will be dominated by combinations of trace wavenumbers, \(k_x\) and \(k_y\), that result in one of the denominators being approximately zero, and these are given by

\[
k'_p = \pm (k_x^2 + k_y^2).
\]

Considering initially only the far field solution, we are interested in those trace wavenumbers that, in the absence of damping, are purely real. This means that the second denominator is never zero. Expression 2.3.54 can therefore be simplified to

\[
w'_p (x, y) = \frac{1}{8\pi^2 k'_p D} \int_{-\infty}^{\infty} P(k_x, k_y) \left[ \frac{1}{(k_x^2 + k_y^2) - k'_p} \right] e^{-ik_x x} e^{-ik_y y} dk_x dk_y
\]

and the preferred relationship between trace wavenumbers is given by

\[
k'_p = (k_x^2 + k_y^2).
\]
Since only real trace wavenumbers are being considered, the trace wavenumbers given by equation 2.3.57 will be less than or equal to the plate wavenumber. They may therefore be written

\[ k_x = k_p \cos \theta, \quad k_y = k_p \sin \theta. \quad 2.3.58 \]

Changing to polar coordinates, with \( k_x = k_r \cos \phi \), \( k_y = k_r \sin \phi \), equation 2.3.56 becomes

\[ w_{ff}(x, y) = \frac{1}{8\pi^2 k_p^2 D} \int_0^{2\pi} \int_0^{\infty} P(k_r, \phi) \left[ \frac{1}{k_r^2 - k_p^2} \right] e^{-ik_r (x \cos \phi + y \sin \phi)} k_r dk_r d\phi, \quad 2.3.59 \]

and since the integral is dominated by its value when \( k_r = k_p \), equation 2.3.59 may be rewritten as

\[ w_{ff}(x, y) = \frac{1}{8\pi^2 k_p D} \int_0^{2\pi} P(k_p, \phi) \left[ \frac{1}{k_p^2 - k_p^2} \right] dk_p e^{-ik_p (x \cos \phi + y \sin \phi)} d\phi. \quad 2.3.60 \]

Equation 2.3.60 is of the form

\[ w_{ff}(x, y) = \int_0^{2\pi} A(\theta) e^{-ik_p (x \cos \theta + y \sin \theta)} d\theta, \quad 2.3.61 \]

where

\[ A(\theta) = \frac{P(k_p, \theta)}{8\pi^2 k_p D} \int_0^{2\pi} \int_0^{\infty} \frac{1}{k_r^2 - k_p^2} dk_r, \quad 2.3.62 \]

and may be interpreted as approximating the far field displacement in a plate as the sum of plane waves, with the wave amplitude being given as a function of direction by \( A(\theta) \).

If near-fields are present then undamped trace wavenumbers may be complex. The second denominator in the integral in equation 2.3.54 then also has zeroes in the range of integration and
therefore also makes a significant contribution to the response. If it assumed that the near-field wave possesses purely propagating characteristics in the $x$-direction, the trace wavenumber being given by

$$k_x = k_p \cos \theta,$$  \hspace{1cm} 2.3.63

then the component of the response that can be attributed to the presence of near-fields is due almost entirely to the second term in the integral, namely

$$w_{nf} (x, y) = -\frac{1}{8\pi^2 k_p^2 D} \int \int P(k_x, k_y) \left[ \frac{1}{(k_x^2 + k_y^2 + k_p^2)} \right] e^{i k_x x} e^{-i k_y y} dk_x dk_y.$$  \hspace{1cm} 2.3.64

Equation 2.3.64 will be dominated by the combination of trace wavenumbers given by

$$k_x^2 + k_y^2 = -k_p^2,$$  \hspace{1cm} 2.3.65

and it follows that the preferred trace wavenumber in the $y$-direction will be

$$k_y = \pm i k_p \left( 1 + \cos^2 \theta \right)^{0.5}.$$  \hspace{1cm} 2.3.66

It is therefore apparent that the trace wavenumber in the $y$-direction is, in magnitude, greater than or equal to the plate wavenumber. A wave travelling purely in the $y$-direction ($\theta = \pi/2$) thus results in the magnitude of the trace wavenumber equalling that of the plate wavenumber, this being analogous to the case of the beam. Other angles of propagation, however, result in an imaginary trace wavenumber in the $y$-direction that exceeds the magnitude of the plate wavenumber. This gives a more rapid decay of the near-field than was found on the beam.
Chapter 3: Vibration and Energy Flow

3.1 Introduction

Vibrational energy flow is a consequence of motion and internal forces. In this chapter the relationships between surface deformation and internal forces, which were derived in the previous chapter, are combined with the appropriate velocities to give expressions for vibrational energy flow in beams and plates in terms of local displacement and its spatial and temporal derivatives.

In the case of beams, where there exists a closed form solution for the displacement in regions in which no excitation is applied, a general expression for intensity is given in terms of wave amplitudes and the effects of different field conditions are discussed.

Since there does not exist a simple closed form solution for an arbitrarily vibrating plate it is not possible to investigate the effects of different field conditions in the same manner as on the beam. However, two simple cases have been studied: plane wave propagation in one dimension, and two plane waves of differing direction. These give an insight into the behaviour of the intensity vector.
3.2 Energy Flow Due to Flexural Vibrations in Beams

In the following discussion, and the remainder of the thesis, it is assumed that the beam is vibrating in one plane only. In reality, displacement due to flexural vibrations could take place in two dimensions, particularly if the beam did not have a direction of relatively low flexural stiffness. An obvious example of this is a length of circular pipe, which has no preferred plane of vibration. Under these circumstances the energy flows due to vibration in two orthogonal planes would need to be calculated individually.

In the case of flexural vibrations in beams, there are two internal forces which contribute to the energy flow - the shear force and the bending moment. Using equations 2.2.5 and 2.2.7 relating shear force, bending moment and deformation, and the conventions shown in Figure 2.2, we can express the individual contributions to the energy flow as the product of a force and a corresponding velocity, giving

\[ P_q(t) = -Q \frac{dw}{dt} \]

\[ = EI \frac{\partial^3 w}{\partial x^3} \frac{dw}{dt} \]

and

\[ P_M(t) = M \frac{\partial^2 w}{\partial x \partial t} \]

\[ = -EI \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial t}. \]

In practice the time-averages of these components are generally of more interest than their instantaneous values, and these will be written

\[ \langle P_q(t) \rangle = EI \left( \frac{\partial^3 w}{\partial x^3} \frac{dw}{dt} \right), \quad \langle P_M(t) \rangle = -EI \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial t} \right), \]

3.2.3
where \( \langle \cdot \rangle \) denotes a time-average. Further insight into equations 3.2.3 can be gained by recalling that, in a section of uniform beam where there is no external excitation, the displacement under flexural vibration is given by equation 2.2.10 as

\[
w(x,t) = (A^+ e^{-ikx} + A^- e^{ikx} + A_N^+ e^{-kx} + A_N^- e^{kx})e^{i\omega t},
\]

and that if \( F(t) \) and \( G(t) \) are complex time-harmonic quantities such that \( F(t) = Fe^{i\omega t} \) and \( G(t) = Ge^{i\omega t} \), then the time-average

\[
\langle F(t)G(t) \rangle = 0.5 \text{Re}(F^*G).
\]

By using the relation given in equation 3.2.5 and substituting equation 3.2.4 in equations 3.2.3, the time-averaged energy flow associated with the shear force and the bending moment can be shown to be

\[
\langle P_Q(t) \rangle = 0.5EI\omega k^3 \text{Re}
\left[
A^+ - \left|A^+\right|^2 + A^+ \frac{A_N^+ e^{-i(1)kx} + A_N^- e^{i(1)kx}}{-A_N^+ e^{i(1)kx} + A_N^- e^{-i(1)kx} - iA_N^+ e^{-i(1)kx} + A_N^- e^{i(1)kx} + iA_N^- e^{-i(1)kx} - A_N^+ e^{i(1)kx} - A_N^- e^{-i(1)kx} + iA_N^+} - iA^+ \frac{A_N^+ e^{-i(1)kx} - A_N^- e^{i(1)kx}}{-A_N^+ e^{i(1)kx} - A_N^- e^{-i(1)kx} - iA_N^+ e^{-i(1)kx} + A_N^- e^{i(1)kx} + iA_N^- e^{-i(1)kx} - A_N^+ e^{i(1)kx} - A_N^- e^{-i(1)kx} + iA_N^+}\right]
\]

and

\[
\langle P_M(t) \rangle = 0.5EI\omega k^3 \text{Re}
\left[
A^+ - \left|A^+\right|^2 - iA^+ \frac{A_N^+ e^{-i(1)kx} - A_N^- e^{i(1)kx}}{-A_N^+ e^{i(1)kx} - A_N^- e^{-i(1)kx} + iA_N^+ e^{-i(1)kx} - A_N^- e^{i(1)kx} + iA_N^- e^{-i(1)kx} - A_N^+ e^{i(1)kx} - A_N^- e^{-i(1)kx} + iA_N^+} - iA^+ \frac{A_N^+ e^{-i(1)kx} - A_N^- e^{i(1)kx}}{-A_N^+ e^{i(1)kx} + A_N^- e^{-i(1)kx} - iA_N^+ e^{-i(1)kx} + A_N^- e^{i(1)kx} + iA_N^- e^{-i(1)kx} - A_N^+ e^{i(1)kx} + A_N^- e^{-i(1)kx} + iA_N^+}\right]
\]

respectively if a real wavenumber, and hence negligible damping, is assumed.

It is apparent from equations 3.2.6 and 3.2.7 that in the far field, when \( A_N^+ \) and \( A_N^- \) are negligible, the two time-averaged components of intensity are equal. Having gained an insight into the individual behaviour of the components, further simplification is possible by their combination, giving
\[
\langle P(t) \rangle = \langle P_o(t) \rangle + \langle P_M(t) \rangle = E\omega k^3 \left[ |A^+|^2 - |A^-|^2 + 2 \text{Im}(A^+_N A^-_N) \right]. \tag{3.2.8}
\]

In practice near-fields decay exponentially from their point of origin, and generally originate at a discontinuity or a point of application of force. The near-field components can therefore be expressed in terms of their amplitudes, \(A^+_N\) and \(A^-_N\), at their respective points of origin, giving

\[
\langle P(t) \rangle = E\omega k^3 \left[ |A^+|^2 - |A^-|^2 + 2e^{-il} \text{Im}(A^+_N A^-_N) \right], \tag{3.2.9}
\]

where \(l\) is the separation of the discontinuities. Clearly as this separation increases, the energy flow due to the presence of near-fields decreases. It should be noted, though, that in the highly reverberant case when \(|A^+|^2 \approx |A^-|^2\), small near-field terms may still make a significant contribution to the total energy flow. If, however, only one near-field exists then it makes no contribution to this flow, and equation 3.2.9 can be further simplified to give

\[
\langle P(t) \rangle = E\omega k^3 \left[ |A^+|^2 - |A^-|^2 \right]. \tag{3.2.10}
\]

It must be noted that, as stated previously, the preceding comments are only valid in the absence of damping, when the wavenumber, \(k\), is purely real. If the wavenumber is complex then each wave component contributes to the intensity, both individually and in conjunction with the other wave components that are present. In general, however, the loss factor in most structural materials is small enough for the imaginary component of the wavenumber to be considered negligible.

Since the wave amplitudes and wavenumber completely describe the displacement of the beam they may be used to estimate not only energy flow but also parameters such as reflection
coefficients and energy densities. For example, the time-averaged kinetic and potential energy density may be written in terms of wave components as

\[ \langle K_d(x,t) \rangle = \frac{1}{2} \rho A \left( \frac{\partial w}{\partial t} \right)^2 = \frac{1}{4} \rho A \omega^2 \left| A^* e^{-ikx} + A^- e^{ikx} + A_n^* e^{-kx} + A_n^- e^{kx} \right|^2 \] 3.2.11

and

\[ \langle P_d(x,t) \rangle = \frac{1}{2} EI \left( \frac{\partial^2 w}{\partial x^2} \right)^2 = \frac{1}{4} ELk^4 \left| A^* e^{-ikx} + A^- e^{ikx} - A_n^* e^{-kx} - A_n^- e^{kx} \right|^2, \] 3.2.12

respectively, and in the far field they reduce to

\[ \langle K_d(x,t) \rangle = \frac{1}{4} \rho A \omega^2 \left( |A^*|^2 + |A^-|^2 + 2 |A^*||A^-| \cos(2kx - \phi) \right) \] 3.2.13

and

\[ \langle P_d(x,t) \rangle = \frac{1}{4} ELk^4 \left( |A^*|^2 + |A^-|^2 + 2 |A^*||A^-| \cos(2kx - \phi) \right), \] 3.2.14

where \( \phi \) is the phase lead of \( A^* \) over \( A^- \). The equality of these two components in the far field is readily apparent, since \( ELk^4 = \rho A \omega^2 \), as is their spatial variation about a non-zero mean.
3.3 Energy Flow Due to Flexural Vibrations in Plates

Whereas the flexural motion of a beam generates two internal forces, the shear force and the bending moment, that are mechanisms for energy flow, there exist three such mechanisms within a plate undergoing flexural vibration - the shear force, the bending moment and the twisting moment. There are thus three contributing terms to the expression for energy flow in a flexurally vibrating plate, each being the product of an internal force and a corresponding velocity. Furthermore, for plate energy flow measurements, the flexural stiffness, $EI$, of the beam is replaced by the flexural stiffness per unit width, $D$, of the plate. The calculated quantity is thus energy flow per unit width (or power per unit width), commonly referred to as structural intensity.

Using the coordinate system and sign convention shown in Figure 2.3 and the relationships between shear forces, moments and deformation given in equations 2.3.6 and 2.3.13, the individual components of time-averaged intensity in the $x$-direction can be shown to be

$$
\langle I_{q_x}(t) \rangle = \left\langle -Q_x \frac{\partial w}{\partial t} \right\rangle
$$

$$
= \left\langle D \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial x \partial y^2} \right) \frac{\partial w}{\partial t} \right\rangle,
$$

$$
3.3.1
$$

$$
\langle I_{m_x}(t) \rangle = \left\langle M_x \frac{\partial^3 w}{\partial x \partial t} \right\rangle
$$

$$
= -D \left( \frac{\partial^2 w}{\partial x^2} + v \frac{\partial^2 w}{\partial y^2} \right) \frac{\partial^2 w}{\partial x \partial t},
$$

$$
3.3.2
$$

and

$$
\langle I_{m_y}(t) \rangle = \left\langle M_y \frac{\partial^2 w}{\partial y \partial t} \right\rangle
$$

$$
= -D (1 - v) \left( \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial y \partial t} \right).
$$

$$
3.3.3
$$
Combining the intensity components due to the shear force, bending moment, and twisting moment therefore shows the total time-averaged intensity in the \(x\)-direction to be given by

\[
\langle I_x(t) \rangle = D \left[ \left( \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \frac{\partial w}{\partial t} \right] - \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \frac{\partial^2 w}{\partial x \partial t} \right] - (1 - \nu) \left[ \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial y \partial t} \right]. \tag{3.3.4a}
\]

Similarly, by interchanging \(x\) and \(y\)-indices, the time-averaged intensity in the \(y\)-direction may be shown to be

\[
\langle I_y(t) \rangle = D \left[ \left( \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} \right) \frac{\partial w}{\partial t} \right] - \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \frac{\partial^2 w}{\partial y \partial t} \right] - (1 - \nu) \left[ \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial t} \right]. \tag{3.3.4b}
\]

The 2-dimensional nature of the plate problem means that the displacement cannot be described by a small number of variables, as in the beam case. In fact an infinite number of variables would be required for a general description, as even for the case of plane wave excitation an infinite number of propagation directions, each with a unique amplitude and phase, are possible. This increased complexity means that it is no longer possible to obtain an expression which offers an intuitive understanding of intensity, such as equation 3.2.9 for the beam. It is both possible and informative, however, to evaluate the intensity for two elementary cases - one-dimensional plane wave excitation and simple two-dimensional plane wave excitation.

### 3.3.1 Plane Waves in One Dimension

If we consider a positive- and negative-going wave pair whose path of propagation lies at an angle, \(\theta\), to the \(x\)-axis, as shown in Figure 3.1, then the displacement of the plate may be expressed as
\[ w(x,y,t) = \left( A_p e^{-ik_p (x \cos \theta + y \sin \theta)} + B_p e^{ik_p (x \cos \theta + y \sin \theta)} \right) e^{i\omega t}. \]  \hspace{1cm} 3.3.5

![Figure 3.1: One-dimensional plane wave propagation.](image)

Again assuming that the structural damping within the material is negligible, and hence that the plate wavenumber is purely real, the time-averaged shear force, bending moment, and twisting moment components of intensity are therefore

\[ \langle I_{Q_x}(t) \rangle = 0.5D\omega k_p^2 \left| A_p \right|^2 - \left| B_p \right|^2 \cos \theta, \]  \hspace{1cm} 3.3.6

\[ \langle I_{M_z}(t) \rangle = 0.5D\omega k_p^2 \left| A_p \right|^2 - \left| B_p \right|^2 \left( \cos^2 \theta + \nu \sin^2 \theta \right) \cos \theta, \]  \hspace{1cm} 3.3.7

and \[ \langle I_{M_x}(t) \rangle = 0.5D\omega k_p^2 \left| A_p \right|^2 - \left| B_p \right|^2 \left( 1 - \nu \right) \sin^2 \theta \cos \theta, \]  \hspace{1cm} 3.3.8

respectively. The total time-averaged intensity in the \( x \)-direction is therefore the sum of these individual components,

\[ \langle I_x(t) \rangle = \langle I_{Q_x}(t) \rangle + \langle I_{M_z}(t) \rangle + \langle I_{M_x}(t) \rangle \]

\[ = D\omega k_p^2 \left| A_p \right|^2 - \left| B_p \right|^2 \cos \theta, \]  \hspace{1cm} 3.3.9
and similarly the time-averaged intensity in the y-direction can be shown to be

$$\langle I_y(t) \rangle = D \omega k_p^3 \left( |A_p|^2 - |B_p|^2 \right) \sin \theta. \quad 3.3.10$$

It can be seen that, for one-dimensional wave propagation in the absence of damping, there is no spatial variation of intensity. Furthermore the shear force component of time-averaged intensity is equal to the sum of the bending and twisting moment components. There is thus, naturally, a close analogy with flexural vibrations in a beam.

### 3.3.2 Plane Waves in Two Dimensions

Consider two plane waves with complex amplitudes $A_p$ and $B_p$ and directions of propagation $\theta_A$ and $\theta_B$ respectively, as shown in Figure 3.2.

![Two-dimensional plane wave propagation](image)

Figure 3.2: Two-dimensional plane wave propagation.

The displacement of the plate under these circumstances is given by

$$w(x, y, t) = \left( A_p e^{-ik_p(x \cos \theta_A + y \sin \theta_A)} + B_p e^{-ik_p(x \cos \theta_B + y \sin \theta_B)} \right) e^{i\omega t}, \quad 3.3.11$$

and the individual components of time-averaged intensity in the $x$-direction are
\[
\langle I_{Qx}(t) \rangle = 0.5 D \omega k_p^2 \left[ |A_p|^2 \cos \theta_A + |B_p|^2 \cos \theta_B + |A_p||B_p| (\cos \theta_A + \cos \theta_B) \times 
\cos(\phi + k_p x \cos \theta_B - \cos \theta_A) + k_p y (\sin \theta_B - \sin \theta_A) \right].
\]

\[
\langle I_{Mx}(t) \rangle = 0.5 D \omega k_p^2 \left[ |A_p|^2 (\cos^2 \theta_A + \nu \sin^2 \theta_A) \cos \theta_A + 
|B_p|^2 \left( \cos^2 \theta_B + \nu \sin^2 \theta_B \right) \cos \theta_B + |A_p||B_p| \times 
\left( \cos^2 \theta_A + \nu \sin^2 \theta_A \right) \cos \theta_B + \left( \cos^2 \theta_B + \nu \sin^2 \theta_B \right) \cos \theta_A \right) \times 
\cos(\phi + k_p x \cos \theta_B - \cos \theta_A) + k_p y (\sin \theta_B - \sin \theta_A) \right],
\]

and
\[
\langle I_{Tx}(t) \rangle = 0.5 D \omega k_p^2 (1 - \nu) \left[ |A_p|^2 \sin^2 \theta_A \cos \theta_A + |B_p|^2 \sin^2 \theta_B \cos \theta_B + 
|A_p||B_p| \sin \theta_A \sin \theta_B (\cos \theta_A + \cos \theta_B) \right. \times 
\left. \cos(\phi + k_p x \cos \theta_B - \cos \theta_A) + k_p y (\sin \theta_B - \sin \theta_A) \right].
\]

where \( \phi \) is the phase lead of \( A_p \) over \( B_p \). The total time-averaged intensity in the \( x \)-direction, being the sum of these three components, is therefore
\[
\langle I_x(t) \rangle = D \omega k_p^2 \left[ |A_p|^2 \cos \theta_A + |B_p|^2 \cos \theta_B + 0.5 |A_p||B_p| (\cos \theta_A + \cos \theta_B + 
\cos^2 \theta_A \cos \theta_B + \cos \theta_A \cos^2 \theta_B + \nu (\sin^2 \theta_A \cos \theta_B + \sin^2 \theta_B \cos \theta_A) + 
(1 - \nu) \sin \theta_A \sin \theta_B (\cos \theta_A + \cos \theta_B) \right] \times 
\cos(\phi + k_p x \cos \theta_B - \cos \theta_A) + k_p y (\sin \theta_B - \sin \theta_A) \right].
\]

Similarly it can be shown that the time-averaged intensity in the \( y \)-direction is
\[
\langle I_y(t) \rangle = D \omega k_p^2 \left[ |A_p|^2 \sin \theta_A + |B_p|^2 \sin \theta_B + 0.5 |A_p||B_p| (\sin \theta_A + \sin \theta_B + 
\sin^2 \theta_A \sin \theta_B + \sin \theta_A \sin^2 \theta_B + \nu (\cos^2 \theta_A \sin \theta_B + \cos^2 \theta_B \sin \theta_A) + 
(1 - \nu) \cos \theta_A \cos \theta_B (\sin \theta_A + \sin \theta_B) \right] \times 
\cos(\phi + k_p x \cos \theta_B - \cos \theta_A) + k_p y (\sin \theta_B - \sin \theta_A) \right].
\]
Ignoring the possibility of two waves propagating in the same direction, equations 3.3.15a and b can be viewed as having two groups of components - those that are spatially invariant and those that are dependent upon location. The spatially invariant terms (those involving $\left|A_p\right|^2$ and $\left|B_p\right|^2$) are precisely those which would be found if each wave were considered separately. In addition to these, however, is the term involving the product of the wave magnitudes, $\left|A_p\right|\left|B_p\right|$. This term is spatially modulated, the magnitude being determined principally by the wave magnitudes and directions, and the modulation being determined by the variation of the relative phase of the individual waveforms with location. The spatially invariant term therefore describes the spatial average of the intensity in a uniform wavefield, while the spatial variation is a local interference effect.

If, however, there are two waves propagating in the same direction then the term involving the product of the wave amplitudes, $\left|A_p\right|\left|B_p\right|$, becomes spatially invariant. Under these circumstances $\theta_A = \theta_B$, and therefore the modulating term,

$$\cos(\phi + k_p x (\cos \theta_A - \cos \theta_B) + k_p y (\sin \theta_B - \sin \theta_A)) = \cos(\phi),$$  \hspace{1cm} 3.3.16

is constant. The other situation in which there is no spatial variation of intensity is when the modulated term has a zero magnitude, and it is apparent from equation 3.3.15a that this occurs for the $x$-component of intensity when

$$\left(\cos \theta_A + \cos \theta_B + \cos^2 \theta_A \cos \theta_B + \cos \theta_A \cos^2 \theta_B + v \left(\sin^2 \theta_A \cos \theta_B + \sin^2 \theta_B \cos \theta_A\right) + (1 - v) \sin \theta_A \sin \theta_B (\cos \theta_A + \cos \theta_B)\right) = 0.$$  \hspace{1cm} 3.3.17

Using trigonometric identities equation 3.3.17 can be shown to have solutions

$$\theta_A = \theta_B \pm \pi, \hspace{1cm} \theta_A + \theta_B = \pm \pi.$$  \hspace{1cm} 3.3.18

Similarly, the modulated term in the $y$-component of intensity has zero magnitude when
\[(\sin \theta_A + \sin \theta_B + \sin^2 \theta_A \sin \theta_B + \sin^2 \theta_A \sin^2 \theta_B + \sqrt{\cos^2 \theta_A \sin \theta_B + \cos^2 \theta_B \sin \theta_A}) + (1 - \nu) \cos \theta_A \cos \theta_B (\sin \theta_A + \sin \theta_B)) = 0, \]  
3.3.19

the solutions being given by

\[\theta_A = \theta_B \pm \pi, \quad \theta_A = -\theta_B. \]  
3.3.20

Equations 3.1.17 and 3.3.19 are true concurrently, and thus there is no spatial variation of the intensity vector, if \(\theta_A = \theta_B \pm \pi\). A uniform intensity field therefore only results from one-dimensional propagation. Under any other circumstances there will be spatial variation of the intensity vector, even if the wavefield is uniform.

A further characteristic of intensity in a two-dimensional wavefield is the absence of a general relationship between the components of intensity due to the shear force, the bending moment, and the twisting moment. While in a uniform wavefield the spatial average of the shear force component is equal to the sum of the spatial averages of the bending and twisting moment components, this is not generally true for the intensity at a point. It is apparent from equations 3.3.12 - 3.3.14 that this relationship does apply to the terms involving \(|A_p|^2\) and \(|B_p|^2\) for point measurements, but not generally to the term involving \(|A_p||B_p|\). Thus in order for the component of intensity due to the shear force to be equal to the sum of the bending and twisting moment components the coefficient of the \(|A_p||B_p|\) term must be zero or the contributions of the shear force and the moments to this coefficient must be equal.

It has been shown that the term involving \(|A_p||B_p|\) is zero if the waves are propagating in opposing directions, and this is therefore one situation in which the stated relationship between the individual components exists. Alternatively, if the contributions of the shear force and moments to the \(|A_p||B_p|\) term are to be equal then it can be seen from equations 3.3.12 - 3.3.14 that
\[ \cos \theta_A + \cos \theta_B = \cos \theta_A \left( \cos^2 \theta_B + \nu \sin^2 \theta_B + (1 - \nu) \sin \theta_A \sin \theta_B \right) + \]
\[ \cos \theta_B \left( \cos^2 \theta_A + \nu \sin^2 \theta_A + (1 - \nu) \sin \theta_A \sin \theta_B \right), \]
\[ 3.3.21 \]

the solutions to equation 3.3.21 being given by

\[ \theta_A = \theta_B, \quad \theta_A = \theta_B \pm \pi, \quad \theta_A + \theta_B = \pm \pi. \]
\[ 3.3.22 \]

A similar analysis can be performed for the component of intensity in the y-direction, the equivalent expression to equation 3.3.21 being

\[ \sin \theta_A + \sin \theta_B = \sin \theta_A \left( \sin^2 \theta_B + \nu \cos^2 \theta_B + (1 - \nu) \cos \theta_A \cos \theta_B \right) + \]
\[ \sin \theta_B \left( \sin^2 \theta_A + \nu \cos^2 \theta_A + (1 - \nu) \cos \theta_A \cos \theta_B \right), \]
\[ 3.3.23 \]

and the solutions given by

\[ \theta_A = \theta_B, \quad \theta_A = \theta_B \pm \pi, \quad \theta_A = -\theta_B. \]
\[ 3.3.24 \]

Comparison of equations 3.3.22 and 3.3.24 reveals that the solutions that are common to equations 3.3.21 and 3.2.23 correspond to wave propagation in one dimension. This is thus a prerequisite for the shear force and moments to make equal contributions to the intensity vector.

It is apparent that the assumptions of uniform intensity and equal division of intensity between shear force and moment components, which greatly simplified far field intensity measurements on a beam, are specific to the case of far field conditions and one-dimensional propagation. The use of such assumptions on plates, while allowing simplifications that reduce the number of measurements required, can introduce significant errors, and these are discussed in Chapter 5.
Part B

Structural Intensity Measurement
In the previous chapters expressions were derived for vibrational energy flow in beams and plates in terms of surface deformations. It still remains to measure or estimate those surface deformations, however, before the energy flow can be measured.

A wave-based approach to the measurement of intensity in beams is described in Chapter 4. Different arrangements of transducers are discussed in terms of the conditioning of the calculations required to estimate the wave amplitudes. The overall sensitivity of the intensity calculation to individual parameters is investigated.

The measurement of intensity in plates is discussed in Chapters 5 and 6. Chapter 5 describes possible measurement systems if one-dimensional propagation is assumed and the errors inherent in that assumption. In Chapter 6 a new approach to structural intensity measurement, in which wave amplitude as a function of direction is approximated to a complex Fourier series, is described.
Chapter 4: Intensity Measurement in Beams

4.1 Introduction

The measurement of intensity in beams is conventionally carried out using a finite difference approximation to estimate the necessary variables, as described in Appendix A. Alternatively, the intensity can be calculated from the wave amplitudes using equation 3.2.8. A wave-based approach on beams has several advantages over the finite difference approach, since it avoids the requirement of a small transducer spacing, as discussed in Section 1.3, and therefore can improve the conditioning of the intensity calculation.

To implement this approach it is necessary to estimate the wave amplitudes in some way, and the principles of this are described in Section 4.2. The evaluation of wave amplitudes is expressed in terms of matrices, and this also provides an additional insight into the conditioning of the problem. Use of an array condition number as an indicator of the conditioning that results from the use of a particular measurement system is discussed in Section 4.3. This is used to compare possible measurement systems in Section 4.4, and in Section 4.5 the sensitivity of the intensity calculation to variations in individual parameters is discussed.
4.2 Principles of Wave Decomposition

It is apparent from equation 2.2.10 that the use of a measurement system comprising \( n \) displacement (or velocity or acceleration) transducers at \( n \) locations on a beam gives rise to \( n \) measurements of the form

\[
w(x_j) = A^+ e^{-ikx_j} + A^- e^{ikx_j} + A_N^+ e^{-ikx_j} + A_N^- e^{ikx_j}; \quad j = 1, 2, \ldots, n, \tag{4.2.1}
\]

where \( x_j \) is the location of the \( j \)th transducer, and the time dependence, \( e^{iwx} \), is suppressed. In matrix form

\[
W = FA, \tag{4.2.2}
\]

where

\[
W^T = [w(x_1), w(x_2), \ldots, w(x_n)], \quad A^T = [A^+, A^-, A_N^+, A_N^-],
\]

\[
F = \begin{bmatrix}
    e^{-ikx_1} & e^{ikx_1} & e^{-ikx_1} & e^{ikx_1} \\
    e^{-ikx_2} & e^{ikx_2} & e^{-ikx_2} & e^{ikx_2} \\
    \vdots & \vdots & \vdots & \vdots \\
    e^{-ikx_n} & e^{ikx_n} & e^{-ikx_n} & e^{ikx_n}
\end{bmatrix}, \tag{4.2.3}
\]

and \((\cdot)^T\) denotes the transpose of a matrix. Obviously some measurement situations, for example in the far field, will preclude the existence of one or more wave types, and the complexity of equation 4.2.1 may be reduced accordingly. If the number of wave components present exceeds the number of measurement locations the system is indeterminate, and additional conditions are required to yield a solution. More typically the number of measurements will be chosen to match the number of wave components, making the matrix \( F \) square, and therefore the wave component amplitudes are given by
\[ A = F^{-1}W. \]  \hspace{1cm} 4.2.4

In the case of an overdetermined system, the wave components, \( A \), may be found in a least squares sense by using the Moore-Penrose inverse [28], giving

\[ A = (F''F)^{-1}F''W, \]  \hspace{1cm} 4.2.5

where \( (\cdot)^\dagger \) denotes the conjugate transpose of a matrix.
4.3 Conditioning

The concept of conditioning, as described in Section 1.3, is of particular importance in the estimation of structural intensity. While there exist expressions for the intensity in a variety of situations in terms of measured variables, demanding applications such as reverberant fields can reveal current transducers and signal processing to be inadequate for the task. Quite simply, the accuracy of the measurements in terms of amplitude, phase, and noise level is insufficient to permit a reliable estimate of the intensity. Obviously if the conditioning is improved then the errors in the data become less critical, giving an improved estimate of the intensity or, alternatively, permitting intensity measurements in a wider range of situations.

It is equally feasible to perform a sensitivity analysis of, for example, a finite difference based intensity calculation as it is of a wave-based calculation. The sensitivity of the intensity calculation to certain parameters, however, is affected by the field conditions, and as such it will only be valid for a specific vibrational field. Use of the wave approach permits the influence of the measurement system on the sensitivity to be considered separately from the effects of field conditions, as may be seen in the case of intensity measurement on a reverberant beam.

In a reverberant beam there are waves of virtually the same amplitude propagating in opposing directions. Equation 3.2.10 reveals that the intensity in this case is proportional to the difference of two near-identical quantities, and therefore will be sensitive to any error in the estimation of those quantities. This sensitivity is purely a result of the characteristics of the vibrational field, and is not affected by the measurement system.

In contrast, the evaluation of the wave amplitudes from the measured variables requires a set of simultaneous equations to be solved, and the conditioning of this problem is independent of the specific vibrational field. The conditioning of this part of the problem is determined by the choice of measured variables and measurement locations, and therefore may be considered a property of a particular measurement system.
The identification of wave components is reliant on the existence of the inverse of matrix \( F \) (or \( F^T F \) in the case of an overdetermined system). In the normal case, in which the transducer spacing is uniform and the same physical variable is measured at each location, the matrix is singular when the spacing is zero or an integral number of half-wavelengths. At transducer spacings close to those which result in singularity the matrix is ill-conditioned, making calculations prone to large errors. Since the conditioning of a problem is a measure of the sensitivity of the final result to changes in the input data, the term 'ill-conditioned' in practice refers to instances when typical levels of error (e.g. in transducer spacing or measurement noise) cause unacceptable errors in the calculated wave amplitudes, and hence in the intensity estimate. This imposes limits on the spacing of the transducers, and therefore on the overall dimension of the measurement system.

In strictly mathematical terms the condition of a matrix may be judged using the 2-norm condition number, which is the ratio of largest to smallest elements in the singular value decomposition of the matrix. This approach can also be used to assess the relative virtues of different transducer configurations after noting two important points. First, that the condition number of the matrix associated with a particular configuration is dependent on the position of the coordinate origin unless near-fields are ignored. It can, however, be shown using Gaussian elimination that the errors resultant from displacing the origin are due only to the error in that displacement, and are not affected by the transducer configuration. Furthermore, this error only affects the phase of the propagating waves, and the magnitudes of the near-field waves in such a way that their cross-spectrum remains the same. It is thus apparent from equation 3.2.8 that this particular error has no effect on the intensity calculation. Secondly, when more than one physical variable is measured, the coefficients multiplying the exponential terms of matrix \( F \) differ in each row, and this also has an effect on the condition number. That this should have no effect on the calculation of wave amplitudes is apparent from the following example. The simultaneous equations obtained from acceleration measurements differ only by a factor of \( -\omega^2 \) from those derived from displacement measurements, and the two may be combined to give results identical to those given by acceleration or displacement alone.
Thus it is proposed that, for the purpose of comparing the potential of different transducer configurations in intensity measurements, the condition number of a modified matrix be used. This modified matrix would be the matrix, \( F \), for an optimally chosen origin (which is the array centre when far field conditions are assumed or the presence of two near-fields is accommodated and a symmetrical transducer configuration is employed), with the rows normalised so that the coefficients multiplying the complex exponential terms are unity. The 2-norm condition number of this matrix provides a relative measure of the sensitivity of the intensity measurements from a particular transducer array to error, both in the measurements of the physical variables and in other quantities such as transducer location and wavenumber. This measure will be referred to as the \textit{array condition number} of the measurement system.
4.4 Measurement Systems

In this section the characteristics, and in particular the array condition number, of various measurement systems are discussed. First traditional systems, comprising arrays of equally spaced accelerometers, are considered. Then hybrid systems, in which more than one vibrational quantity is measured, are described. These can allow improved conditioning when the overall size of the transducer array is restricted.

4.4.1 Linear Accelerometer Arrays

Suppose that it is desired to measure intensity using an array of equally spaced linear accelerometers. In a region remote from discontinuities, when near-fields can be neglected, only two transducers are required, and the transducer spacing may be chosen to minimise the errors in processing. Using the array condition number the optimum transducer spacing is found to be a quarter-wavelength, with poor conditioning occurring at small spacings and those approaching a half-wavelength. It should be noted that the conditioning problem is inherent in the geometry of the array, and not the subsequent processing. As such it will be equally evident when estimating intensity from measured physical variables irrespective of the technique used, be it wave decomposition, finite difference approximation, or cross-spectrum.

In the vicinity of discontinuities allowance must be made for the presence of near-fields, necessitating the use of additional transducers. If a single near-field exists then three (or more) transducers are required, and for the general case where two near-fields exist at least four transducers are necessary in the array to evaluate the wave amplitudes. The condition of the matrix deteriorates with an increasing number of near-field components, however, owing to the inclusion of the associated real exponential terms. This is readily seen from the leading term in the power series expansion of the matrix determinant, as shown in Table 4.1. Since a zero determinant is the defining property of a singular matrix and a poorly conditioned matrix is one that is close to being singular, a determinant tending to zero is indicative of deteriorating matrix
condition. Increasing the possible number of wave components present increases the power to which the leading term in the power series expansion (which will dominate for small transducer separations) is raised. It is therefore apparent that the determinant will reduce with an increasing number of wave components and a reducing transducer spacing.

<table>
<thead>
<tr>
<th>Transducer array</th>
<th>2 Wave Components</th>
<th>3 Wave Components</th>
<th>4 Wave Components</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2 linear accel.</td>
<td>3 linear accel.</td>
<td>4 linear accel.</td>
</tr>
<tr>
<td>Leading term in power series expansion of determinant</td>
<td>$k\Delta$</td>
<td>$(k\Delta)^3$</td>
<td>$(k\Delta)^6$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2 linear + 2 angular accel.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2 linear accel. + 2 strain gauges</td>
</tr>
</tbody>
</table>

Table 4.1: Leading term in power series expansion of determinant for different transducer arrays used to estimate wave amplitudes, in terms of wavenumber, $k$, and transducer spacing, $\Delta$.

The relative values of array condition number for 2, 3 and 4 transducer arrays (assuming $F$ is square) are shown as a function of transducer spacing in Figure 4.1, with a lower array condition number indicating improved conditioning. In addition to the poorer optimum condition for near-field measurements, the presence of the discontinuities in the physical system on which measurements are being taken may restrict the maximum possible array span and thus prevent the array condition from being optimised.

The poorer conditioning if near-fields are to be incorporated is readily understood by considering the finite difference approximation that could be used to estimate the intensity, as described in Appendix A. Conditioning problems result from calculating the difference between two near-
identical quantities, and the more often that this must be done in a particular calculation the poorer the conditioning is likely to be. If far field conditions are assumed then the intensity can be estimated from the displacement and slope at a point. As the slope may be estimated from the difference of two closely spaced measurements there is only one such problematical calculation, and thus the conditioning problem is relatively small. If near-fields are present then it is necessary to estimate higher order spatial derivatives, and these require a greater number of differencing operations. As a result the estimation of these higher order spatial derivatives is inherently more prone to conditioning problems than the estimation of the lower order derivatives, and this is reflected in the poorer conditioning of intensity calculations when near-fields are accommodated.

![Graph showing array condition for uniformly spaced accelerometer systems](image)

Figure 4.1: Array condition for uniformly spaced accelerometer systems: —— 2 accelerometer array (far field); - - - 3 accelerometer array (single near-field); ...... 4 accelerometer array (two near-fields).

The conditioning problem is obviously of particular concern in any application in which a small transducer spacing is required, and thus conditioning cannot be optimised. This includes those approaches based on a finite difference approximation, when increasing the spacing of the measurement points, while improving the conditioning, increases the systematic errors as noted in
Appendix A. If a single wave type is present these systematic errors may be corrected (see Appendix A), and therefore a small spacing is not required. However in the more general cases, when both propagating and near-field waves exist, there is no simple correction and the stipulation of a small transducer spacing is necessary. These situations can result in poor conditioning, and as a consequence the wave-based approach, in which the transducer separation is governed solely by the physical constraints of the component under measurement, can offer significant benefits.

4.4.2 Hybrid Measurement Systems

The condition of a matrix, and hence the array condition number, depends on the numerical values of the matrix elements. These are determined by both the geometry of the transducer array and the physical variables being measured. As a consequence, improved array condition may be achieved if the measurement system includes a variety of transducer types, allowing more than one vibrational quantity (eg. acceleration and strain) to be measured, and giving benefits in terms of wave component identification and intensity measurement. Such a system will be referred to as a hybrid measurement system.

The idea of combining the measurement of different variables to calculate the intensity is not new, having been described by Noiseux [1] and Pavic [15]. However the purpose of this was to simplify the processing by choosing variables that are easily related to the intensity. In the case of wave decomposition it is not necessary to measure parameters that can be related directly to the structural energy flow. The quantities being measured are simply used to evaluate the wave amplitudes, and can be chosen on the basis of the conditioning of the system and ease of measurement.

The assumption of time-harmonic motion means that the measurement of time-derivatives of displacement provides no additional information above displacement itself. There is not necessarily a direct relationship between local displacement and its spatial derivatives, however, and so these provide potential measurement parameters. Of particular interest are rotation and
surface strain, being related to the first and second spatial derivatives respectively. In practice, typical measurements might involve strain and linear and angular acceleration, however for clarity the matrices have been expressed in terms of strain, displacement and rotation.

The second spatial derivative of displacement is related directly to measured surface strain by

\[
\varepsilon(x) = -z \frac{\partial^2 \nu(x)}{\partial x^2} = z k^2 \left( A^+ e^{-ikx} + A^- e^{ikx} - A^+_\nu e^{-k\nu} - A^-_\nu e^{k\nu} \right),
\]

where \( \varepsilon \) is the surface strain and \( z \) is the separation of the strain gauge from the neutral plane. In practice, however, the measurement of strain takes place over a finite length rather than at a point. The measurement therefore yields average strain, \( \bar{\varepsilon} \), over the gauge length, \( 2d \), which may be expressed in terms of wave components as

\[
\bar{\varepsilon}(x) = \frac{z k^2}{2d} \int_{x-d}^{x+d} \left( (A^+ e^{-iky} + A^- e^{iky} - A^+_\nu e^{-k\nu y} - A^-_\nu e^{k\nu y}) dy \right)
\]

\[
= z k^2 \left[ (A^+ e^{-ikx} + A^- e^{ikx}) p(d) - (A^+_\nu e^{-k\nu x} + A^-_\nu e^{k\nu x}) q(d) \right].
\]

where \( p(d) = \sin(kd)/kd \) and \( q(d) = \sinh(kd)/kd \). The four wave component amplitudes can therefore be calculated from two measurements of displacement and two of surface strain by

\[
A = G^{-1} V,
\]

where

\[
G = \begin{bmatrix}
e^{-ikx_1} & e^{ikx_1} & e^{-ikx_2} & e^{ikx_2} \\
e^{-ikx_2} & e^{ikx_2} & e^{-ikx_3} & e^{ikx_3} \\
z k^2 p(d) e^{-ikx_3} & z k^2 p(d) e^{ikx_3} & -z k^2 q(d) e^{-k\nu x_3} & -z k^2 q(d) e^{k\nu x_3} \\
z k^2 p(d) e^{-ikx_4} & z k^2 p(d) e^{ikx_4} & -z k^2 q(d) e^{-k\nu x_4} & -z k^2 q(d) e^{k\nu x_4}
\end{bmatrix}
\]

56
and \( \mathbf{V}^T = [ w(x_1) \quad w(x_2) \quad \bar{\varepsilon}(x_3) \quad \bar{\varepsilon}(x_4) ] \).

The approach described for strain measurements can also be used to incorporate rotational measurements in the evaluation of wave amplitudes. The rotation of the beam at any point is

\[
\theta(x) = \frac{\partial w(x)}{\partial x} = k \left( -iA^+ e^{-ikx} + iA^- e^{ikx} - A^+_N e^{-ikx} + A^-_N e^{ikx} \right).
\]

Four wave component amplitudes can therefore be calculated from two measurements of displacement and two of rotation by

\[
\mathbf{A} = \mathbf{H}^{-1} \mathbf{U},
\]

where

\[
\mathbf{H} = \begin{bmatrix}
    e^{-ik_1} & e^{ik_1} & e^{-ik_2} & e^{ik_2} \\
    e^{-ik_3} & e^{ik_3} & e^{-ik_4} & e^{ik_4} \\
    -ike^{-ik_3} & ike^{ik_3} & -ke^{-ik_4} & ke^{ik_4} \\
    -ike^{-ik_4} & ike^{ik_4} & -ke^{-ik_4} & ke^{ik_4}
\end{bmatrix}
\]

and \( \mathbf{U}^T = [ w(x_1) \quad w(x_2) \quad \theta(x_3) \quad \theta(x_4) ] \).

One advantage of these hybrid systems is that, when the second variable is not directly related to the first, measurements of the different parameters can be taken at the same location (e.g. \( x_1 = x_2, x_3 = x_4 \)). The total array span can therefore be smaller than that necessary for a conventional array with any given transducer spacing or, alternatively, the spacing can be larger for a given array span. This can offer substantial benefits in terms of array condition over a conventional array, in which a single variable is measured, when array space is limited as it may be by the presence of closely spaced discontinuities. For example, Figure 4.2 shows typical array
condition numbers of three, four-transducer measurement systems for use in the presence of two near-fields, where the overall dimension of the array is fixed. Uniform transducer spacing is assumed for the four accelerometer system, while the hybrid linear accelerometer-strain gauge and linear-angular accelerometer systems use maximum spacing for each pair of transducers. The strain gauge length is assumed to be one third of the total array span. It is apparent that the hybrid systems, particularly the linear accelerometer-strain gauge array, can offer much improved array condition over the conventional four accelerometer array.

![Graph](image.png)

**Figure 4.2:** Array condition number for measurements within two near-fields:  
--- 4 accelerometer array; - - - accelerometer-strain gauge array; ...... linear-angular accelerometer array.

This is also indicated by the leading term in the power series expansion of the determinant of the normalised matrix used to calculate the array condition number for the various transducer arrays, as shown in Table 4.1. When the transducer separation in terms of radians, \( k\Delta \), is small this leading term will dominate the determinant. It is apparent under these circumstances that the leading term for the accelerometer-strain gauge array, being \( (k\Delta)^2 \), will be larger than the \( (k\Delta)^4 \) of the linear-angular accelerometer array, which is in turn larger than the \( (k\Delta)^6 \) of the four
accelerometer array. Thus for a given small transducer spacing the hybrid arrays will give larger determinants, and therefore better conditioning, than the four accelerometer array. Furthermore, the use of a hybrid array will normally permit a wider transducer spacing than that allowed by the four accelerometer array, since usually any limitation on transducer spacing will be via a restriction on the overall array span. This will have the effect of increasing the determinant, and hence improving the conditioning, relative to that of the conventional array.

Once again, it is informative to consider the reasons for the improved conditioning offered by the hybrid arrays in terms of finite difference approximations that may be used to estimate the intensity. If two measurements of strain and two of displacement are used, as described in Appendix B, then the required parameters are estimated from the sum or difference of two of the measured quantities. Since each estimation requires at most one differencing operation, as compared to the two required if displacement is measured at all locations, conditioning is less of a problem.
4.5 Sensitivity Analysis

The expression for energy flow in a beam given in equation 3.2.8 can be rewritten in terms of matrices as

\[ \langle P(t) \rangle = EI\omega k^3 A^H QA \]

\[ = EI\omega k^3 W^H (F^{-1})^H QF^{-1} W, \]

where

\[ Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix} \]

The sensitivity of this calculation to any particular parameter, s, is therefore

\[ S_s = \frac{\partial \langle P(t) \rangle}{\partial s} \]

\[ = \frac{\partial}{\partial s} \left( EI\omega k^3 W^H (F^{-1})^H QF^{-1} W \right), \]

or alternatively, in dimensionless form

\[ \frac{s}{\langle P(t) \rangle} S_s = \frac{s}{\langle P(t) \rangle} \frac{\partial}{\partial s} \left( EI\omega k^3 W^H (F^{-1})^H QF^{-1} W \right). \]

As stated previously, the sensitivity is therefore dependent not only on the transducer type and placement (in radians), which completely determine the array condition number, but also on beam properties (manifest in the flexural stiffness and wavenumber) and the relative amplitude and
phase of the wave components that describe the displacement. If the relevant quantities are known, however, the sensitivity may be calculated either by formal differentiation or numerically by using small perturbations about the operating point.

It is apparent from equation 4.5.5 that the inherent errors can be considered in three categories.

i. Errors in the estimate of $EI\omega_k^3$. The effects of these errors are readily apparent and the sensitivity is independent of the measurement system used.

ii. Errors in the evaluation of $(F^{-1})^T QF^{-1}$. These result from imperfect knowledge of the wavenumber and the transducer placement.

iii. Errors in the measured variables, $W$. These include both measurement noise and miscalibration, and are specific to a particular application.

The requirements for minimising the sensitivity to each of these categories of error are not identical, and furthermore in some cases the specific requirements are dependent on the nature of the vibrational field, as noted in Section 4.3.

4.5.1 Sensitivity in Far Field Intensity Calculations

A two accelerometer system and a three accelerometer (least squares) system are considered. The dimensionless sensitivity of the intensity calculation to error in the placing of an outer transducer in the array is shown as a function of the total transducer array span in Figure 4.3. This is unaffected by the degree of reverberance, and it can be seen that the sensitivity to placement error is minimised for the two-accelerometer array when the separation is a quarter-wavelength, as indicated by the array condition number shown in Figure 4.4. If an extra accelerometer is used and a least squares solution found it is apparent that the sensitivity is minimised at a greater overall array span, though lesser individual separation, than in the two accelerometer case. Once
again this occurs when the array condition number is minimised (see Figure 4.4), but this time the matrix is determined using the Moore-Penrose inverse, as described in Section 4.2. It is apparent that both approaches offer very low sensitivity at optimum spacings, however the least squares approach offers low sensitivity over a wider range of transducer spacings.

In reality the transducer placement will often be limited by a maximum attainable absolute accuracy, leading to increasing relative errors with decreasing transducer spacing. This results in increasing absolute sensitivity at small spacings, as shown in Figure 4.5. Once again the minima in the sensitivities are clearly evident, occurring for the transducer array spans that minimise the array condition number shown in Figure 4.4.

The dimensionless sensitivity of the calculation to error in the wavenumber estimate is shown for the two accelerometer array and the three accelerometer least squares approach as a function of total array span in Figure 4.6. This is also unaffected by the degree of reverberance, and the minimum sensitivity occurs with minimum transducer spacing. Increasing the transducer separation increases the sensitivity (since it results in a greater absolute error in the expected phase change between the measurement points), however the three accelerometer least squares approach gives a lower sensitivity than the two accelerometer array.

Since an expression for the intensity in the far field of a beam can be written in terms of the cross-spectrum of two displacements, there is a linear relationship between the amplitude of one such measurement and the calculated intensity, irrespective of transducer spacing and degree of reverberance. If, however, a least squares approach is used the influence of each individual measurement is lessened, resulting in a reduced sensitivity to errors in the amplitudes of individual measurements as shown in Figure 4.7. Assuming that all three transducers have the same accuracy and ignoring all other influences on the chosen transducer spacing it would seem appropriate to choose a spacing which results in equal sensitivity to amplitude errors from all transducers. Once again this occurs at the spacing that results in a minimum array condition number (see Figure 4.4).
Figure 4.3: Dimensionless sensitivity of far field intensity calculation to error in transducer placement: —— 2 accelerometer array; - - - 3 accelerometer (least squares solution) array.

Figure 4.4: Array condition number for far field intensity measurement systems: —— 2 accelerometer array; - - - 3 accelerometer (least squares solution) array.
Figure 4.5: Absolute sensitivity (normalised with respect to energy flow and wavenumber) of far field intensity calculation to error in transducer placement:
--- 2 accelerometer array; - - - 3 accelerometer (least squares solution) array.

Figure 4.6: Dimensionless sensitivity of far field intensity calculation to error in wavenumber estimate: --- 2 accelerometer array; - - - 3 accelerometer (least squares solution) array.
Figure 4.7: Dimensionless sensitivity of far field intensity calculation to amplitude error in three accelerometer (least squares solution) array: —— outer accelerometer; ···· centre accelerometer.

The accurate measurement of relative phase is particularly critical for successful intensity estimation, and the sensitivity to phase errors increases with the degree of reverberance. Furthermore the magnitude and phase of the wave components will influence which transducer spacing will result in minimum sensitivity, this occurring for the two accelerometer array when the measurements differ in phase by $\pi/2$.

4.5.2 Sensitivity in the Presence of Near-Fields

The existence of near-fields introduces further variables which are specific to a particular application and that influence sensitivity. In addition to the influences that are present for far field measurements, the transducer spacing that gives minimum sensitivity also changes with the amplitude and phase of the near-fields. However, as in many practical measurement situations the near-field amplitudes will be relatively small, typical sensitivities can be found through using a numerical model that allows for the presence of near-fields but assuming only propagating waves are present.
Chapter 5: Intensity Measurement in Plates - Propagation in One Dimension Assumed

5.1 Introduction

The measurement of intensity in plates is significantly more complicated than for beams. While it is possible to obtain an exact expression for flexural wave intensity in a beam in terms of a small number of measurements, this is not the case in plates. Any attempt to determine the intensity in a plate thus requires approximations to reduce the number of measurements required. The most common assumption is that far field conditions exist and that wave propagation is one-dimensional. Under these circumstances the component of the intensity vector in a particular direction can be estimated from the cross-spectrum of two individual measurements.

Section 5.2 describes the theoretical basis for the simplifications that are possible if one-dimensional wave propagation is assumed. Possible measurement systems are discussed in Section 5.3, and systematic errors, resulting both from violating the assumption of one-dimensional propagation and from the approximations necessary to implement a particular measurement system, are described in Section 5.4. The conditioning associated with the measurement systems is discussed in Sections 5.5. In Section 5.6 a method, based on one of the measurement systems proposed in Section 5.3, of estimating the intensity vector (as distinct from a component of that vector) from three measurements is described. Finally, in Section 5.7, the possibility of incorporating the effects of near-fields is discussed.
5.2 Measurement Principles

It was shown in Section 3.3.2 that if far field conditions exist and propagation is one-dimensional then the components of intensity due to the shear force and the moments are equal. The component of intensity in a particular direction, \( x \), is therefore twice that due to the shear force, and, referring to equation 3.3.1, may be written as

\[
\langle I_x(t) \rangle_{ID} = 2 \langle I_{Q_x}(t) \rangle = 2D \left( \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) \frac{\partial w}{\partial t} \tag{5.2.1}
\]

\[
= 2D \left( -k_p^2 \frac{\partial w}{\partial x} (i\omega) \right) \tag{5.2.2}
\]

\[
= -D\omega k_p^2 \text{Im} \left( w^* \frac{\partial w}{\partial x} \right). \tag{5.2.3}
\]

The intensity in a particular direction can thus be expressed in terms of the cross-spectrum of the displacement and slope of the plate [2]. Equation 5.2.3 is in the same form as the expression for far field energy flow in a beam given by equation A.11 in Appendix A, the flexural stiffness of the beam being replaced by flexural stiffness per unit width of the plate. This is because, if propagation is one-dimensional, the plate behaves in exactly the same way as a beam apart from the additional stiffness endowed by Poisson's effect.
5.3 Measurement Systems

In order to implement a measurement system using equation 5.2.3 it is necessary to measure or estimate the displacement and slope of the plate at a point. This may be achieved in various ways, a number of which are described in this section.

5.3.1 Combined Linear and Angular Acceleration Measurement

For time-harmonic motion the linear acceleration, $a$, at a point is related to displacement, $w$, by

$$a = -\omega^2 w,$$  \hspace{1cm} 5.3.1

and the angular acceleration is

$$\ddot{\theta}_x = \frac{\partial^3 w}{\partial x \partial t^2} = -\omega^2 \frac{\partial w}{\partial x}. $$  \hspace{1cm} 5.3.2

The displacement and slope may therefore be written as

$$w = -\frac{a}{\omega^2}, \hspace{1cm} \frac{\partial w}{\partial x} = -\frac{\ddot{\theta}_x}{\omega^2}, $$  \hspace{1cm} 5.3.3

and these may be substituted in equation 5.2.3 to give

$$\langle I_x(t) \rangle_{LA} = \frac{-Dk_p^2}{\omega^4} \text{Im}(a^* \dot{\theta}_x). $$  \hspace{1cm} 5.3.4

Equation 5.2.3 can therefore be adapted with no loss of theoretical accuracy to utilise measurements of linear and angular acceleration at a point.
5.3.2 Finite Difference Approximation

The displacement and slope at a point can be estimated from two closely spaced measurements of displacement using a finite difference approximation. If \( w_1 \) and \( w_2 \) are measurements of displacement at points on the x-axis separated by a distance, \( \Delta \), then

\[
\frac{\partial w}{\partial x} \approx \frac{w_2 - w_1}{\Delta}.
\]

These approximations may then be used in equation 5.2.3 to estimate the component of intensity in the x-direction at a point midway between the two measurement points as

\[
\langle I_x(t) \rangle_{FD} = -\frac{D\omega k^3}{\Delta} \text{Im}(w_1^* w_2),
\]

or, since measurements are commonly performed using accelerometers rather than displacement transducers,

\[
\langle I_x(t) \rangle_{FD} = -\frac{Dk^2}{\omega^3 \Delta} \text{Im}(a_1^* a_2).
\]

5.3.3 Wave-Based Approximation

As an alternative to the finite difference approach, the motion of the plate can be assumed to be the sum of two propagating plane waves travelling in opposing directions along the x-axis, giving

\[
\hat{w}(x) = A_p e^{-ik_dx} + B_p e^{ik_dx}.
\]

The hypothetical wave amplitudes, \( A_p \) and \( B_p \), can be found by setting

\[
w_1 = A_p e^{-ik_d \Delta/2} + B_p e^{ik_d \Delta/2}, \quad w_2 = A_p e^{ik_d \Delta/2} + B_p e^{-ik_d \Delta/2}.
\]
and solving these as simultaneous equations. This gives

\[ A'_p = \frac{1}{2i \sin(k_p \Delta)} \left( w_1 e^{ik_p \Delta/2} - w_2 e^{-ik_p \Delta/2} \right) \]

\[ B'_p = \frac{1}{2i \sin(k_p \Delta)} \left( w_2 e^{ik_p \Delta/2} - w_1 e^{-ik_p \Delta/2} \right) \]

and the displacement and slope may be estimated as

\[ w = \hat{w}(0) = A'_p + B'_p, \quad \frac{\partial w}{\partial x} = \frac{\partial \hat{w}}{\partial x} = ik_p (A'_p + B'_p). \]

Substituting equations 5.3.10 - 5.3.12 into equation 5.2.3 to give an estimate of the component of intensity in the \( x \)-direction at the point midway between the two measurement points yields

\[ \langle I_x(t) \rangle_w = -\frac{D \omega k_p^3}{\sin(k_p \Delta)} \text{Im}(w_1^* w_2) \]

or, if acceleration is measured,

\[ \langle I_x(t) \rangle_w = -\frac{D \omega k_p^3}{\omega^3 \sin(k_p \Delta)} \text{Im}(a_1^* a_2). \]
5.4 Systematic Errors

Two types of error are considered in this section: those that result from violating the assumption of one-dimensional wave propagation and those that are a consequence of errors in estimating the displacement and slope necessary when implementing equation 5.2.3.

5.4.1 Errors Due to Assuming One-Dimensional Wave Propagation

While the assumption of one-dimensional propagation greatly simplifies the estimation of intensity, it is unlikely to be valid in many realistic situations. Real-life structures are typically subject to excitation from multiple sources and exhibit low damping, making two-dimensional propagation, where possible, almost inevitable due to reflections and multiple transmission paths. The assumption of one-dimensional propagation under these circumstances will introduce an error in the intensity estimate which will depend on the amplitude, phase and direction of the waves that are present.

If there exist two plane waves, with complex amplitudes $A_p$ and $B_p$ propagating at angles $\theta_A$ and $\theta_B$ to the x-axis, as shown in Figure 3.2, then the true component of intensity in x-direction is given in equation 3.3.15 as

$$\langle I_x(t) \rangle = D \omega k_p^3 \left[ |A_p|^2 \cos \theta_A + |B_p|^2 \cos \theta_B + 0.5 |A_p| |B_p| \left( \cos \theta_A + \cos \theta_B + \cos^2 \theta_A \cos \theta_B + \cos \theta_A \cos \theta_B + v \left( \sin^2 \theta_A \cos \theta_B + \sin^2 \theta_B \cos \theta_A \right) + (1 - v) \sin \theta_A \sin \theta_B (\cos \theta_A + \cos \theta_B) \right) \times \cos(\phi + k_p x (\cos \theta_B - \cos \theta_A) + k_p y (\sin \theta_B - \sin \theta_A)) \right].$$

where $\phi$ is the phase lead of $A_p$ over $B_p$. In contrast, the estimate of this component given by equation 5.2.3 is
\[
\langle I_s(t) \rangle_{1D} = D \omega k^3_p \left[ |A_p|^2 \cos \theta_A + |B_p|^2 \cos \theta_B + |A_p| |B_p| (\cos \theta_A + \cos \theta_B) \right] \cos (\phi + k_p x (\cos \theta_B - \cos \theta_A) + k_p y (\sin \theta_B - \sin \theta_A)).
\]

5.4.2

The error due to assuming one-dimensional propagation is therefore

\[
E_s = \langle I_s(t) \rangle_{1D} - \langle I_s(t) \rangle
\]

5.4.3

\[
= \frac{D \omega k^3_p}{2} |A_p| |B_p| (1 - v) \sin (\theta_B - \theta_A) (\sin \theta_B - \sin \theta_A) \times \cos (\phi + k_p x (\cos \theta_B - \cos \theta_A) + k_p y (\sin \theta_B - \sin \theta_A)).
\]

5.4.4

It can be seen that both the true intensity (equation 5.4.1) and the estimate that results from assuming one-dimensional propagation (equation 5.4.2) exhibit a spatial variation about the same mean, differing only in the magnitude of that variation. Each individual intensity estimate using the assumption of one-dimensional propagation will therefore exhibit an error, but in a uniform wavefield the average of a number of such estimates can give a good estimate of the mean intensity or total energy flow. Since in many cases it is knowledge of this energy flow, rather than of the intensity at individual points, that is required, it is necessary to estimate the intensity at a number of points in the region, irrespective of the accuracy of those estimates. In a uniform wavefield the assumption of one-dimensional propagation will therefore allow a greatly simplified measurement system, with little loss of accuracy in terms of the useful data gained and without increasing the number of intensity measurements required.

If, however, the wavefield is not uniform over a sufficiently large area then each intensity estimate will involve not only a different value of the spatially varying component of intensity, but also a different local mean intensity. Under these circumstances the averaging of a number of intensity estimates will tend to reduce the effects of the errors caused by assuming one-dimensional propagation, but will not eliminate them.
5.4.2 Errors in Estimating Displacement and Slope

The measurement system described in Section 5.3.1, combining linear and angular acceleration measurements at a point, provides, within measurement accuracy, the necessary variables to evaluate equation 5.2.3 exactly. The finite difference and wave-based approaches, described in Sections 5.3.2 and 5.3.3 respectively, only give approximations of the required displacement and slope, and the accuracy of these approximations depends on the vibrational field that is present. The errors that result from these approximations cannot be eliminated through averaging, and it is therefore desirable that suitable measures be taken to minimise their effect.

Since equation 5.2.3 is based upon the assumption of far field conditions and one-dimensional wave propagation, it is worthwhile to initially consider how the finite difference and wave-based approximations perform under these circumstances. If far field conditions and one-dimensional wave propagation are present then the displacement of the plate is given by

\[ w(x, y) = A_p e^{-ik_p(x \cos \theta + y \sin \theta)} + B_p e^{ik_p(x \cos \theta + y \sin \theta)} \tag{5.4.5} \]

where \( \theta \) is the angle between the \( x \)-axis and the propagation direction of wave \( A_p \), and the component of intensity in the \( x \)-direction is given by substituting equation 5.4.5 in equation 5.2.3 to yield

\[ \langle I_x(t) \rangle_D = D\omega k_p^2 \cos \theta \left( |A_p|^2 - |B_p|^2 \right). \tag{5.4.6} \]

Since the measurement points are assumed to lie on the \( x \)-axis and the coordinate origin can be chosen to be at the point midway between those measurement points, the measurements may be written as

\[ w_1 = A_p e^{ik_p \Delta x \cos(\theta)/2} + B_p e^{-ik_p \Delta x \cos(\theta)/2}, \quad w_2 = A_p e^{-ik_p \Delta x \cos(\theta)/2} + B_p e^{ik_p \Delta x \cos(\theta)/2}. \tag{5.4.7} \]
Substituting equations 5.4.7 in equation 5.3.6 shows the intensity estimate given by the finite difference approximation to be

$$\langle I_x(t) \rangle_{FD} = D\omega k_p^2 \frac{\sin(k_x \Delta \cos \theta)}{k_p \Delta} \left(|A_p|^2 - |B_p|^2\right).$$  \hspace{1cm} 5.4.8

The component of the intensity vector as estimated using the finite difference approximation is therefore related to the intended estimate (which is the true intensity since the assumption of one-dimensional propagation is valid) by

$$\langle I_x(t) \rangle_{FD} = \langle I_x(t) \rangle_{ID} \frac{\sin(k_x \Delta \cos \theta)}{k_p \Delta \cos \theta}.$$  \hspace{1cm} 5.4.9

This relationship is similar to the relationship between the true energy flow and that estimated using a finite difference approximation for far field beam measurements, as given in equation A.15 (Appendix A). The difference lies in the inclusion of the 'cos \theta' term, this arising from the fact that the x-axis is now not necessarily aligned with the line of propagation so the apparent transducer separation from the wave's viewpoint becomes $k_p \Delta \cos \theta$, rather than $k_p \Delta$.

It is apparent from equation 5.4.9 that if propagation is one-dimensional the error introduced by the use of the finite difference approximation is dependent on the angle between the direction of propagation and x-axis (ie. the line upon which the transducers lie). Furthermore, only in the worst possible case (when $\cos \theta = \pm 1$), does the relative error reach the magnitude experienced in the equivalent beam measurements. However, under most circumstances $\theta$ will not be known so, unlike for the beam measurements, it is not possible to apply an exact correction for this error.

If propagation is two-dimensional then the error due to the finite difference approximation is more complicated. In the simplest possible case of two propagating plane waves, as shown in Figure 3.2, the use of the finite difference approximation gives
\[
\langle I_s(t) \rangle_{FD} = D \omega k_p^3 \left[ |A_p|^2 \frac{\sin(k_p \Delta \cos \theta_A)}{k_p \Delta} + |B_p|^2 \frac{\sin(k_p \Delta \cos \theta_B)}{k_p \Delta} + 2 \text{Re}(A_p^* B_p) \frac{1}{k_p \Delta} \sin \left( \frac{k_p \Delta (\cos \theta_A + \cos \theta_B)}{2} \right) \right].
\]

5.4.10

When this is compared to the true intensity given in equation 3.3.15 it can be seen that the terms involving single wave components (and the mixed term if \( \theta_A = \theta_B \)), which are those that form the spatial average of the intensity, are each modified by a factor of \( \sin(k_p \Delta \cos \theta) / k_p \Delta \cos \theta \), where \( \theta \) is the specific wave incident angle. Any observations regarding relative errors under one-dimensional propagation can therefore also be applied to the individual wave components of a two-dimensional field if only the spatial average of intensity is considered.

A similar analysis can be performed for the wave approach. Under far field conditions and one-dimensional propagation the relationship between the intended estimate of intensity in the \( x \)-direction and the wave-based estimate is given by

\[
\langle I_s(t) \rangle_w = \langle I_s(t) \rangle_{1D} \frac{\sin(k_p \Delta \cos \theta)}{\sin(k_p \Delta) \cos \theta},
\]

5.4.11

and if propagation is two-dimensional

\[
\langle I_s(t) \rangle_w = D \omega k_p^3 \left[ |A_p|^2 \frac{\sin(k_p \Delta \cos \theta_A)}{\sin(k_p \Delta)} + |B_p|^2 \frac{\sin(k_p \Delta \cos \theta_B)}{\sin(k_p \Delta)} + 2 \text{Re}(A_p^* B_p) \frac{1}{\sin(k_p \Delta)} \sin \left( \frac{k_p \Delta (\cos \theta_A + \cos \theta_B)}{2} \right) \right].
\]

5.4.12

Comparison of equations 5.4.9 - 5.4.12 reveals that the wave approach and the finite difference approach differ by a factor of \( k_p \Delta / \sin(k_p \Delta) \), a result directly analogous to that seen for beam measurements. However, in the case of the plate, neither approach gives an exact result, the relative errors in the presence of one-dimensional propagation being
\[ E_{x(FD)} = \frac{\langle I_x(t) \rangle_{FD} - \langle I_x(t) \rangle_{1D}}{\langle I_x(t) \rangle_{1D}} = \frac{\sin(k_p \Delta \cos \theta)}{k_p \Delta \cos \theta} - 1, \quad \text{5.4.13} \]

\[ E_{x(W)} = \frac{\langle I_x(t) \rangle_{W} - \langle I_x(t) \rangle_{1D}}{\langle I_x(t) \rangle_{1D}} = \frac{\sin(k_p \Delta \cos \theta)}{\sin(k_p \Delta \cos \theta)} - 1, \quad \text{5.4.14} \]

for the finite difference and wave approaches respectively. Equations 5.4.13 and 5.4.14 are plotted as functions of wave incident angle for a range of transducer spacings in Figure 5.1. It can be seen that the finite difference approximation results in the intensity being underestimated, while the wave approach overestimates the intensity. Since each approach has a consistent bias a compensating factor, in the form of a representative value of \( \theta \), could be applied to either approach to reduce the maximum relative error, as proposed by Redman-White [6] for the finite difference approach. It is also apparent that increasing the transducer spacing increases the error irrespective of the approach used, and that the maximum errors are similar for the two approaches if small spacings are used.

![Figure 5.1](image.png)

**Figure 5.1:** Effect of transducer spacing, \( k_p \Delta \), on relative systematic errors for wave based (above x-axis) and finite difference (below x-axis) approximations as a function of propagation direction: —— \( k_p \Delta = 0.05\lambda \), - - - \( k_p \Delta = 0.1\lambda \), ----- \( k_p \Delta = 0.15\lambda \), ---- \( k_p \Delta = 0.2\lambda \).
It should be remembered that, in general, the purpose of determining the component of intensity in a particular direction is to estimate the intensity vector from two such measurements in orthogonal directions. Typically one of these directions will be more closely aligned with the intensity vector than the other. The component of intensity in this particular direction will be larger, and will have more influence on the estimation of the vector, than the component in the orthogonal direction. It is therefore more important to estimate this dominant intensity component accurately than its orthogonal counterpart. This is obvious if we consider the case in which one of the two orthogonal directions is perfectly aligned with the vector. The component of intensity in this direction therefore completely defines the vector, and relative errors in this measurement are critical while relative errors in the orthogonal component are of no consequence.

Figure 5.2: Effect of transducer spacing, $k_p \Delta$, on absolute systematic errors for wave based (above x-axis) and finite difference (below x-axis) approximations as a function of propagation direction: —— $k_p \Delta = 0.05 \lambda$, --- $k_p \Delta = 0.1 \lambda$, ----- $k_p \Delta = 0.15 \lambda$, ---- $k_p \Delta = 0.2 \lambda$.

Figure 5.1 reveals that the finite difference approach gives minimum relative error when the x-axis and the propagation direction are orthogonal. Conversely, the maximum relative error occurs
when the two directions coincide. The relative error is therefore large when estimating the dominant intensity component and small when estimating the less influential component.

The opposite is found in the wave-based approach. Maximum relative errors occur when the transducers lie on a line orthogonal to the direction of propagation, and minimum errors when parallel to the propagation direction. Relative errors are thus small when estimating the dominant component of the intensity vector, and large when estimating the less influential component.

These effects are more apparent if absolute, rather than relative, errors are considered. Figure 5.2 shows the absolute error (normalised with respect to the magnitude of the intensity vector) associated with the finite transducer spacing for the finite difference and wave-based approaches. It is apparent that the wave-based approach gives a lower maximum absolute error than the finite difference approach, and a lower absolute error when estimating the dominant component of the intensity vector (i.e. when the incident angle is less than $\pi/4$).

The reason for the different characteristics of the finite difference and wave-based approaches can be found in the assumptions inherent in the estimation of displacement and slope. The finite difference approximations described by equations 5.3.5 are exact for a straight line between the two measurement points. There will thus be no error in the finite difference approximation only if there is no curvature in the deformed shape of the plate between the measurement points. If propagation is one-dimensional then this condition is met only if the transducers lie on a line parallel to the wavefronts, and thus orthogonal to the line of propagation and the intensity vector. If the transducers are placed in any other way there will be curvature of the plate between the measurement points, and therefore an error in the finite difference approximation and a consequent error in the intensity estimate. When the transducers are placed on a line parallel to the direction of wave propagation (corresponding to $\theta = 0$) there is maximum curvature between the two measurement points and thus maximum error in the finite difference approximation and the resulting intensity estimate.
The assumption of one-dimensional propagation means that the component of intensity in the x-direction is determined solely by the behaviour of the plate along the x-axis. It is therefore not necessary for the whole plate to conform to the wave description given in equation 5.3.8 for the wave approach to be exact. It is only required that equation 5.3.8 describe the behaviour of the plate along the line between the measurement points.

If propagation is one-dimensional then a wave description of the form given in equation 5.3.8 is applicable regardless of the direction of propagation, different propagation directions simply resulting in different trace wavenumbers. Knowing the trace wavenumber, which may be estimated using three transducers as described by Meyer et al [29] and Wagstaff et al [30], it is therefore possible to estimate the slope and displacement correctly at a point. When applying the wave-based approach with two transducers the trace wavenumber in the direction of interest (ie. the x-axis) is not known, however, and is assumed equal to the plate wavenumber. This assumption is true if the x-axis coincides with the line of propagation, so under these circumstances the required deformation is known exactly and there is thus no error in the wave approximation. If the x-axis does not coincide with the line of propagation then the trace wavenumber is not equal to the plate wavenumber. Since an incorrect trace wavenumber is used there is an error in the assumed deformation of the plate, and hence an error in the resulting intensity estimate. This error reaches a maximum when the x-axis is orthogonal to the direction of propagation (ie. parallel to the wavefronts). Under these circumstances there is no curvature between the two measurement points, however the use of the plate wavenumber assumes maximum possible curvature between the points, and therefore results in maximum error.
5.5 Conditioning

The concepts of conditioning and the array condition number that were introduced for beam intensity measurements in Chapter 4 can also be applied to plate intensity measurements.

The relative conditioning of the proposed systems can be compared by considering the calculation necessary to estimate the amplitudes of the waves that are assumed to be present. If far field conditions and one-dimensional propagation exist then the displacement of the plate is given by

\[ w(x, y) = A_p e^{-i(k_x x + k_y y)} + B_p e^{i(k_x x + k_y y)}. \]  

5.5.1

Considering first the combined measurement of linear and rotational acceleration, the analytical expressions for these variables at the origin are

\[ a(0,0) = -\omega^2 (A_p + B_p), \quad \dot{\theta}_x (0,0) = i k_x \omega^2 (A_p - B_p). \]  

5.5.2

These simultaneous equations may be written in matrix form as

\[
\begin{pmatrix}
  a(0,0) \\
  \dot{\theta}_x (0,0)
\end{pmatrix} = \begin{pmatrix}
  A_p \\
  B_p
\end{pmatrix},
\]

5.5.3

where

\[
T = \begin{bmatrix}
  -\omega^2 & -\omega^2 \\
  ik_x \omega^2 & -ik_x \omega^2
\end{bmatrix}.
\]  

5.5.4

The 2-norm condition number of the transformation matrix, \( T \), is equal to the trace wavenumber, \( k_x \). However if the matrix, \( T \), is normalised as proposed in Section 4.3 it becomes
\[
\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
\]

The 2-norm condition number of the normalised matrix, \(\mathbf{T}\), which is the array condition number introduced in Section 4.3, is unity. This indicates that, if far field conditions and one-dimensional wave propagation are assumed, the estimation of the wave amplitudes, and the hence intensity, from the combined measurement of linear and rotational acceleration at a point provides optimal conditioning. It should be remembered, however, that this does not make allowance for the conditioning of the calculation of rotational acceleration. If this is derived from the difference of two linear acceleration measurements then it, too, may be subject to ill-conditioning and, if so, would result in a 'noisy' rotational acceleration measurement.

The estimation of intensity from two measurements of linear acceleration, be it by finite difference approximation or wave-based approach, offer the same conditioning since they differ simply by a constant factor. In the case of the wave-based approach it is assumed that the displacement of the plate between the measurement points is given by

\[
w(x, y) = A_p e^{-ik_p x} + B_p e^{ik_p x},
\]

so the assumed analytical expressions for displacement at the measurement points are

\[
w_1 = A_p e^{ik_p \Delta / 2} + B_p e^{-ik_p \Delta / 2}, \quad w_2 = A_p e^{-ik_p \Delta / 2} + B_p e^{ik_p \Delta / 2}.
\]

These simultaneous equations can be written in matrix form as

\[
\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} e^{ik_p \Delta / 2} & e^{-ik_p \Delta / 2} \\ e^{-ik_p \Delta / 2} & e^{ik_p \Delta / 2} \end{pmatrix} \begin{pmatrix} A_p \\ B_p \end{pmatrix},
\]
the transformation matrix being identical to that used to evaluate wave amplitudes on a beam if only propagating waves are present. The variation of the condition number of this matrix with transducer spacing was shown in Figure 4.1, with conditioning being optimised when the spacing is a quarter-wavelength and deteriorating to singularity when the transducers coincide or are a half-wavelength apart.
5.6 Estimation of the Intensity Vector

The previous sections of this chapter have concerned the estimation of a particular directional component of the intensity vector from the cross-spectrum of two measurements. It is, of course, possible to obtain an estimate of the intensity vector by determining another directional component from another such pair of measurements. Alternatively, the use of three displacement, velocity, or acceleration transducers means that there are three possible cross-spectra that can be calculated. These may be used to give three estimates of directional components of the intensity vector, via either the finite difference approximation or the wave approach. If the transducers are arranged in a triangle, as shown in Figure 5.3, then it is possible to obtain, at each of three different locations, \( a, b, \) and \( c, \) an estimate of a different directional component of the intensity vector at that location. If it is assumed that the intensity is uniform over the region where the transducers are placed then there is surplus information for the determination of the intensity vector, since it is only necessary to know the component of intensity in two directions. The system is therefore overdetermined, and an optimum solution may be found.

The approach described obviously has much in common with that proposed by Zhou et al [31]. Here however, a wave-based, rather than a finite difference, approximation is used.

![Figure 5.3: Estimation of the intensity vector using three accelerometers.](image)
Using the wave-based approach, the estimates of the individual intensity components are given by

\[
I_a = -\frac{D\omega k_p^3}{\sin(k_p \Delta)} \text{Im}(w_1^* w_2), \quad I_b = -\frac{D\omega k_p^3}{\sin(k_p \Delta)} \text{Im}(w_2^* w_3),
\]

\[
I_c = -\frac{D\omega k_p^3}{\sin(k_p \Delta)} \text{Im}(w_3^* w_1). \tag{5.6.1}
\]

If the triangle is equilateral and the intensity is uniform in the region then (ignoring any systematic errors) the magnitudes of the intensity components will be related to the true intensity by

\[
I_a = I_0 \cos \theta,
\]

\[
I_b = I_0 \cos \left( \theta - \frac{2\pi}{3} \right) = I_0 \left( \cos \theta \cos \frac{2\pi}{3} + \sin \theta \sin \frac{2\pi}{3} \right),
\]

\[
I_c = I_0 \cos \left( \theta - \frac{4\pi}{3} \right) = I_0 \left( \cos \theta \cos \frac{4\pi}{3} + \sin \theta \sin \frac{4\pi}{3} \right). \tag{5.6.2}
\]

Equations 5.6.2 can be written in matrix form as

\[
\begin{pmatrix}
I_a \\
I_b \\
I_c
\end{pmatrix} =
I_0
\begin{pmatrix}
1 & 0 \\
-0.5 & 0.866 \\
-0.5 & -0.866
\end{pmatrix}
\begin{pmatrix}
\cos \theta \\
\sin \theta
\end{pmatrix}, \tag{5.6.3}
\]

and solved using the Moore-Penrose inverse to give

\[
\begin{pmatrix}
I_0 \cos \theta \\
I_0 \sin \theta
\end{pmatrix} =
\begin{pmatrix}
I_a \\
I_b \\
I_c
\end{pmatrix} =
\begin{pmatrix}
0.667 & -0.333 & -0.333 \\
0.577 & 0.577 & -0.577
\end{pmatrix}
\begin{pmatrix}
I_a \\
I_b \\
I_c
\end{pmatrix}. \tag{5.6.4}
\]

Furthermore, from equation 5.6.4 it can be shown that
\[ t_0^2 = \frac{1}{2.25} (I_a^2 + I_b^2 + I_c^2 - I_a I_b - I_a I_c - I_b I_c). \]  

In reality the intensity will not be uniform, and therefore the measured intensity components, \( I_a \), \( I_b \), and \( I_c \), will be components of different vectors. This, combined with measurement errors, will mean that the solution will not provide a perfect fit to each individual measurement. The mean square deviation of the individual measurements from the solution can be used as an indication of the validity of the result.
5.7 Incorporating the Effects of Near-Fields

The approaches described in this chapter are founded on the assumption of far field conditions. If this assumption is violated then a further error is introduced. The magnitude of this error is dependent on the field conditions present, and thus it is not possible to quote meaningful figures regarding the influence of near-fields in general plate intensity measurements. However, if the effects of near-fields could be incorporated in the measurements then the errors caused by their presence could be reduced.

Since the approaches described are based on there being two waves, propagating in orthogonal directions, present, a logical extension is to accommodate the presence of a third, evanescent, wave by using a third measurement when estimating the component of intensity in a particular direction. This requires that a direction, and hence rate, of decay be assumed, since only three amplitudes can be estimated from the three measurements. Numerical experiments have shown this approach to be effective in some field conditions but not in others, when a simple far field approximation gives better results.

The problem lies in the simplicity of the model used. If the measurements do not correspond to one-dimensional wave propagation then a part of the motion is attributed to a near-field with the specified direction of decay. If this closely matches the true field conditions then the intensity estimate will be accurate. However in many cases, such as two-dimensional propagation in the far field, there will be a large discrepancy between the true and assumed motions of the plate and hence a large error in the intensity estimate. It therefore does not appear possible to incorporate the effects of near-fields in 'simple' intensity estimates of this type.
Chapter 6: Intensity Measurement in Plates - Propagation in Two Dimensions Assumed

6.1 Introduction

The approaches described in the previous chapter are based on the assumption of far field conditions and one-dimensional propagation. These assumptions allow considerable simplification of the intensity expression. For more general usage such simplifications are not valid, however, and to obtain a more rigorous estimate of intensity most terms in the intensity expression must be evaluated or estimated individually. This may be achieved using finite difference approximations for the required spatial derivatives, as proposed by Pavic [2]. However the small transducer spacing necessary with the finite difference approximation means that poor conditioning is an inherent problem.

In this chapter an alternative approach to structural intensity measurement in plates is proposed. It is assumed that far field conditions exist, and that the displacement within a region can be described as the sum of a set of plane waves. The amplitude of these waves is considered to be a function of direction, and this function is approximated using a complex Fourier series as described in Section 6.2. In Section 6.3 the use of the Fourier series coefficients to determine the intensity is discussed. The principles of determining the coefficients of the Fourier series are given in Section 6.4, while different measurement systems and the associated conditioning are discussed in Section 6.5. Section 6.6 compares an intensity measurement using a Fourier series approach to one using a nine point finite difference approximation in a specific vibrational field.
6.2 Wave Amplitude as a Complex Fourier Series

It was seen in Chapter 2 that, within a vicinity of a point sufficiently removed from discontinuities or excitation, the displacement of a plate can be approximated as the sum of plane propagating waves. These waves may be of any direction, so the displacement can be written as

\[ w(x, y) = \int_{-\pi}^{\pi} A(\theta) e^{-ik_x(x \cos \theta + y \sin \theta)} d\theta , \]  

where \( A(\theta) \) is the complex wave amplitude as a function of propagation direction. Writing this amplitude as a complex Fourier series gives

\[ A(\theta) = \sum_{n=\infty}^{\infty} C_n e^{in\theta} . \]  

Equation 6.2.2 can then be substituted in equation 6.2.1 to give

\[ w(x, y) = \int_{-\pi}^{\pi} \left( \sum_{n=-\infty}^{\infty} C_n e^{in\theta} \right) e^{-ik_x(x \cos \theta + y \sin \theta)} d\theta , \]  

or

\[ w(x, y) = \sum_{n=-\infty}^{\infty} C_n \int_{-\pi}^{\pi} e^{in\theta} e^{-ik_x(x \cos \theta + y \sin \theta)} d\theta . \]

While there may be an infinite number of terms in the expansion, in practice the displacement will be estimated from a finite number of measurements, \( m \). It is therefore possible to estimate up to \( m \) terms in the Fourier series expansion. If \( m \) is odd then we may write \( m = 2q + 1 \) for some non-negative integer, \( q \), and the estimate of displacement given by the truncated Fourier series approximation is

\[ \tilde{w}(x, y) = \sum_{n=-q}^{q} C_n \int_{-\pi}^{\pi} e^{in\theta} e^{-ik_x(x \cos \theta + y \sin \theta)} d\theta . \]
6.3 Intensity in Terms of Complex Fourier Coefficients

Approximating the displacement of the plate to a complex Fourier series gives an analytical expression (equation 6.2.5) as an estimate of displacement. The spatial derivatives that are necessary to estimate the intensity in the far field may be evaluated by differentiating this expression as follows:

\[ w(x, y) = \sum_{n=-q}^{q} C_n \int_{-\pi}^{\pi} e^{i n \theta} e^{-ik_p \cos \theta (x \cos \theta + y \sin \theta)} d\theta \]

\[ \frac{\partial w}{\partial x}(x, y) = \sum_{n=-q}^{q} C_n \int_{-\pi}^{\pi} -ik_p \cos \theta e^{i n \theta} e^{-ik_p \cos \theta (x \cos \theta + y \sin \theta)} d\theta \]

\[ \frac{\partial w}{\partial y}(x, y) = \sum_{n=-q}^{q} C_n \int_{-\pi}^{\pi} -ik_p \sin \theta e^{i n \theta} e^{-ik_p \cos \theta (x \cos \theta + y \sin \theta)} d\theta \]

\[ \frac{\partial^2 w}{\partial x \partial y}(x, y) = \sum_{n=-q}^{q} C_n \int_{-\pi}^{\pi} -k_p^2 \cos \theta \sin \theta e^{i n \theta} e^{-ik_p \cos \theta (x \cos \theta + y \sin \theta)} d\theta \]

\[ \frac{\partial^2 w}{\partial x^2}(x, y) = \sum_{n=-q}^{q} C_n \int_{-\pi}^{\pi} -k_p^2 \cos^2 \theta e^{i n \theta} e^{-ik_p \cos \theta (x \cos \theta + y \sin \theta)} d\theta \]

\[ \frac{\partial^2 w}{\partial y^2}(x, y) = \sum_{n=-q}^{q} C_n \int_{-\pi}^{\pi} -k_p^2 \sin^2 \theta e^{i n \theta} e^{-ik_p \cos \theta (x \cos \theta + y \sin \theta)} d\theta \]

As the spatial derivatives given in equations 6.3.1 are functions of \(x\) and \(y\), it is possible to estimate the intensity for the entire region in which the proposed wave description given by equation 6.2.5 remains valid. Under these circumstances, when \(x\) and \(y\) are non-zero, the integrals in equations 6.3.1 are typically non-zero and thus all the calculated coefficients, \(C_n\), contribute to the intensity. However at the origin the position related exponential term simply equals unity, and the orthogonality of trigonometric functions means that the integrals are zero for most values of \(n\).
The relevant spatial derivatives given in equations 6.3.1 are therefore, at the coordinate origin, determined by a small number of coefficients. In fact

\[ w(0, 0) = 2\pi C_0 \]
\[ \frac{\partial w}{\partial x}(0, 0) = -ik_p \pi (C_{-1} + C_1) \]
\[ \frac{\partial w}{\partial y}(0, 0) = k_p \pi (-C_{-1} + C_1) \]
\[ \frac{\partial^2 w}{\partial x^2}(0, 0) = \frac{ik^2_p \pi}{2} (C_{-2} - C_2) \]
\[ \frac{\partial^2 w}{\partial y^2}(0, 0) = k^2_p \pi \left[ 0.5(C_{-2} + C_2) - C_0 \right]. \]

Substituting equations 6.3.2 into equations 3.3.4a and 3.3.4b shows the estimates of time-averaged far field intensity in the \( x \)- and \( y \)-directions to be

\[ \langle I_x(t) \rangle_{FS} = \frac{D\omega k_p^3}{2} \pi^2 \text{Re} \left[ 2C_0(C_{-1} + C_1)^* + (1-v)(C_{-2}^*C_{-1} + C_{-1}^*C_1) + (1+v)C_0^*(C_{-1} + C_1) \right] \]

6.3.3

and

\[ \langle I_y(t) \rangle_{FS} = \frac{D\omega k_p^3}{2} \pi^2 \text{Im} \left[ 2C_0(-C_{-1} + C_1)^* + (1-v)(C_{-2}^*C_{-1} + C_{-1}^*C_1) + (1+v)C_0^*(C_{-1} - C_1) \right] \].

6.3.4

It is therefore apparent that only five coefficients are necessary to determine the intensity at the origin. That five terms are sufficient to determine the intensity is more readily appreciated when one considers the relationship between the displacement and its five spatial derivatives given in equations 6.3.2. Since, in the far field,

\[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -k^2_p w, \]

6.3.5

only five of the field variables given in equations 6.3.2 are linearly independent, and the intensity may therefore be written in terms of these five variables.
6.4 Estimation of Fourier Series Coefficients

The Fourier coefficients may be estimated from measured parameters in the same way that wave amplitudes were estimated on the beam in Section 4.2. Equation 6.2.5 can be rewritten in matrix form as

\[ \mathbf{W} = \mathbf{T} \mathbf{C}, \]  

where

\[
\mathbf{W} = \begin{pmatrix} w(x_1, y_1) \\ \vdots \\ w(x_m, y_m) \end{pmatrix}, \quad \mathbf{T} = \begin{bmatrix}
\int_{-\pi}^{\pi} e^{-ip\theta} e^{-ik_x(x_1 \cos \theta + y_1 \sin \theta)} d\theta \\
\vdots \\
\int_{-\pi}^{\pi} e^{-ip\theta} e^{-ik_x(x_m \cos \theta + y_m \sin \theta)} d\theta
\end{bmatrix},
\]

\[
\mathbf{C} = \begin{pmatrix} C_{-q} \\ \vdots \\ C_q \end{pmatrix},
\]

and thus the Fourier coefficients may be estimated from

\[ \mathbf{C} = \mathbf{T}^{-1} \mathbf{W}. \]

Alternatively, if the number of measurements exceeds the number of coefficients being evaluated then the Moore-Penrose inverse may be used to obtain a least squares fit to the measured data as described in Section 4.2. However, for the remainder of the discussion it will be assumed that the number of measurements is equal to the number of Fourier coefficients unless specifically stated.
6.5 Measurement Systems and Conditioning

The estimation of intensity using equations 6.3.3 and 6.3.4 requires five Fourier series coefficients to be known, and these may be estimated from a minimum of five measurements. The method of evaluating the coefficients described in the previous section only provides an approximation of a truncated complex Fourier series, however. This means that calculated values of those five coefficients will be affected by the number of coefficients that are evaluated. It is therefore necessary to evaluate an adequate number of coefficients to ensure that the five coefficients required to estimate the intensity are known with sufficient accuracy. The implications of this are discussed later in this section.

The basis functions used in the complex Fourier series possess angular periodicity, and therefore the angular placement of the transducers used to estimate the coefficients must be appropriate for that purpose. This is analogous to the choice of a suitable linear spacing of transducers when evaluating wave amplitudes in a beam. In practice it is necessary to avoid angular spacings which coincide with a half-period of one of the basis functions, since this results in singularity of the matrix, $T$, used to evaluate the Fourier coefficients.

This behaviour is also apparent from the determinant of the matrix. Changing to polar coordinates, with

$$x = r \cos \phi, \quad y = r \sin \phi,$$

6.5.1

equation 6.2.5 becomes

$$w(r, \phi) = \sum_{n=-q}^{q} C_n \int_{-\pi}^{\pi} e^{-ik\nu r (\cos \phi \cos \theta + \sin \phi \sin \theta)} d\theta$$
Consider \( m \) measurements of displacement (or velocity or acceleration) taken at points on the circumference of a circle of radius, \( R \), with uniform angle, \( \phi \), between adjacent measurement points as shown in Figure 6.1.

![Figure 6.1: Transducer arrangement for Fourier series approximation.](image)

It can be shown from equation 6.5.2 that the elements of the transformation matrix, \( T \), are given by

\[
T_{j,n} = e^{in\phi} \int_{-\pi}^{\pi} e^{-in\phi e^{-ikRcos\theta}} d\theta
\]

\[
= e^{in(j-1)\phi} U_n,
\]

where

\[
U_n = \int_{-\pi}^{\pi} e^{in\phi e^{-ikRcos\theta}} d\theta
\]

and \( s_n \) is the column of matrix, \( T \), associated with Fourier index \( n \). As \( U_n \) is independent of the angular location of the measurement it is constant for any given column of the matrix. Furthermore, since multiplying a row or column of a matrix by a constant factor has the effect of
changing the determinant by that same factor, the *zeroes of the determinant* of the matrix, \( \mathbf{T} \), are the same as the zeroes for the auxiliary matrix, \( \overline{\mathbf{T}} \), where

\[
\overline{\mathbf{T}}_{j,r} = e^{i(n-1)\varphi}
\]

6.5.5

or, letting \( b = e^{i\varphi} \),

\[
\overline{\mathbf{T}} = \begin{bmatrix}
1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\
{b^{-q}} & \cdots & {b^{-1}} & 1 & b & \cdots & b^q \\
{b^{-2q}} & \cdots & {b^{-2}} & 1 & b^2 & \cdots & b^{2q} \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
{b^{-q(m-1)}} & \cdots & {b^{-(m-1)}} & 1 & b^{(m-1)} & \cdots & b^{q(m-1)}
\end{bmatrix}
\]

6.5.6

Zeroes of the determinant of matrix, \( \overline{\mathbf{T}} \), indicate angular spacings that will result in singularity of matrix \( \mathbf{T} \). It is apparent in this case that any uniform angular spacing may be used provided that it does not result in two measurements occurring at the same location, since singularity occurs only when

\[
\varphi = 2\pi, \cdots, \pm \frac{2\pi}{(m-2)}, \pm \frac{2\pi}{(m-1)}.
\]

6.5.7

A measurement system with all transducers evenly spaced around the circumference of a circle would therefore appear a practical proposition.

If, however, one measurement is taken at the centre, the elements of the row of matrix, \( \mathbf{T} \), associated with this measurement are of the form

\[
\int_{-\pi}^{\pi} e^{in\theta} d\theta = 0 \quad \forall n \neq 0
\]

\[
= 2\pi \quad n = 0.
\]

6.5.8
Thus only the central element of this row is non-zero, with the Fourier coefficient, \( C_0 \), being fully determined by this single measurement. The auxiliary matrix, \( \mathbf{T} \), now has the form

\[
\mathbf{T} = \begin{bmatrix}
0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\
b^{-q} & b^{-(q-1)} & \cdots & 1 & \cdots & b^{(q-1)} & b^q \\
b^{-2q} & b^{-2(q-1)} & \cdots & 1 & \cdots & b^{2(q-1)} & b^{2q} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
b^{-q(m-2)} & b^{-(q-1)(m-2)} & \cdots & 1 & \cdots & b^{(q-1)(m-2)} & b^{q(m-2)}
\end{bmatrix}
\]

6.5.9

Singularity occurs at the same angular spacings as in the previous case. In this situation, however, there are only \((m-1)\) transducers arranged circumferentially. Uniform distribution around the circumference therefore results in an angular spacing of \(\varphi = 2\pi/(m-1)\), which gives a singular matrix as may be seen from equation 6.5.7.

In summary, if the \(m\) measurements are uniformly distributed around the circumference the angular spacing is \(\varphi = 2\pi/m\) which makes the matrix \( \mathbf{T} \), in general, non-singular. If one of those measurements is taken at the centre of the circle and the remaining \((m-1)\) uniformly distributed around the circumference, the angular spacing is \(\varphi = 2\pi/(m-1)\), and the resulting matrix is always singular.

The Fourier coefficients may also be estimated solely from measurements of surface strain. The surface strain in the radial direction is

\[
\varepsilon_r = \frac{-h \frac{d^2w}{dr^2}}{2}.
\]

6.5.10

where \(h\) is the thickness of the plate. Substituting the expression for displacement given by equation 6.5.2 into equation 6.5.10 and differentiating to yield an expression for radial strain in terms of the Fourier series coefficients gives
\[ e_r \approx \frac{h k_p^2}{2} \sum_{n=-q}^{q} C_n e^{i n \phi} \int_{-\pi}^{\pi} \cos^2 \theta e^{i n \phi} e^{-i k_p R \cos \theta} d\theta. \quad 6.5.11 \]

If the strain gauges are placed with a uniform angular spacing, \( \phi \), on the circumference of a circle of radius \( R \) as shown in Figure 6.1, it can be shown from equation 6.5.11 that the elements of the transformation matrix, \( T \), are given by

\[
T_{j,s_n} = \frac{h k_p^2}{2} e^{i n(j-1)\phi} \int_{-\pi}^{\pi} \cos^2 \theta e^{i n \phi} e^{-i k_p R \cos \theta} d\theta
\]

\[
= e^{i n(j-1)\phi} V_n,
\]

where

\[
V_n = \frac{h k_p^2}{2} \int_{-\pi}^{\pi} \cos^2 \theta e^{i n \phi} e^{-i k_p R \cos \theta} d\theta
\]

6.5.13

and \( s_n \) is the column of matrix, \( T \), associated with Fourier index \( n \). The similarity to the case in which displacement was measured is readily apparent. Since \( V_n \) is constant for any given column, the zeroes of the determinant of matrix, \( T \), are the same as those of the auxiliary matrix, \( \bar{T} \), obtained by dividing each column of \( T \) by the appropriate \( V_n \). This auxiliary matrix is precisely the same as that determined for the displacement measurements, and therefore all the preceding comments regarding the angular spacing of displacement measurements and singularity are equally valid for measurements of radial strain.

Matrix singularity will also occur if one of the values of \( U_n \) (or \( V_n \) for strain measurements) is zero, since this will result in the elements of a column of the matrix, \( T \), being identically zero. As \( U_n \) and \( V_n \) are functions of the Fourier index, \( n \), and the radius of the circle (in radians) upon which the transducers are placed, \( k_p R \), they have roots at specific combinations of these variables. It is therefore necessary to avoid such placing radii for all the Fourier coefficients being evaluated.
While $U_n$ and $V_n$ are defined in terms of integrals in equations 6.5.4 and 6.5.13, they are closely related to Bessel functions, with

$$U_n(k_p R) = 2\pi (-i)^n J_n(k_p R),$$  \hspace{1cm} 6.5.14

$$V_n(k_p R) = -\frac{\pi hk_p^2}{4} (-i)^n \left( J_{n+1}(k_p R) - 2 J_n(k_p R) + J_{n-1}(k_p R) \right),$$  \hspace{1cm} 6.5.15

where $J_n$ is the Bessel function of the first kind of order $n$. Since the behaviour and properties of Bessel functions are well documented this simplifies the analysis of $U_n$ and $V_n$.

Transducer placing radii that result in zeroes of $U_n$ and $V_n$, and hence singularity of matrix $T$, when evaluating low order Fourier coefficients from displacement and strain measurements are shown in Table 6.1. Physically these radii may be interpreted as locations at which a particular Fourier coefficient makes no contribution to the measured variable. This is readily understood when considering the measurement of displacement. Each Fourier coefficient represents a reverberant wave field since the magnitude of the assumed wave remains constant with changing direction - only the phase varies. If that wave field were present in isolation then there would be nodes at which there was no motion. The roots of $U_n$ correspond to nodes of the assumed wave field, and thus if measurements are taken at those points then that particular Fourier coefficient cannot be determined.

<table>
<thead>
<tr>
<th>Fourier index $n$</th>
<th>Roots of $U_n$ (disp. or accel.)</th>
<th>Roots of $V_n$ (strain)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$k_p R = 2.40$</td>
<td>$k_p R = 1.84$</td>
</tr>
<tr>
<td>$\pm 1$</td>
<td>$k_p R = 3.84$</td>
<td>$k_p R = 3.52$</td>
</tr>
<tr>
<td>$\pm 2$</td>
<td>$k_p R = 5.14$</td>
<td>$k_p R = 1.56$</td>
</tr>
<tr>
<td>$\pm 3$</td>
<td>$k_p R = 6.39$</td>
<td>$k_p R = 2.65$</td>
</tr>
</tbody>
</table>

Table 6.1: Transducer placing radii $(k_p R)$ resulting in matrix singularity.
It is apparent that the use of identical transducers uniformly distributed on the circumference of a circle will allow the required Fourier coefficients to be estimated if specific placing radii are avoided. Other approaches to avoiding or overcoming matrix singularity include:

i. The use of an overdetermined system, in which the number of measurements exceeds the number of Fourier coefficients being estimated, utilising the Moore-Penrose inverse.

ii. The use of transducer arrays with a non-uniform angular spacing.

iii. Modification of the matrix, $T$, by equating the very small singular values and their corresponding elements in its inverse to zero, as proposed by Nash [32].

While the use of an overdetermined system does increase flexibility in transducer placing it means that additional transducers and measurement channels are required, increasing the cost of the measurement system. A more promising approach is the modification of matrix, $T$, which improves conditioning at the expense of theoretical accuracy.

The similarities of the estimation of Fourier coefficients to the wave decomposition in beams described in Section 4.2 are readily apparent. A vector of measured complex variables is transformed into a vector of complex coefficients through multiplication by a matrix, and the intensity may be estimated directly from these coefficients. The matrix is generated through the inversion of a matrix whose elements are determined by the particular variables being measured, the measurement locations, and the nature of the assumed vibrational field conditions. It therefore seems appropriate to consider the application of the array condition number to this approach. As was seen in Section 4.2, the 2-norm condition number of the matrix is influenced not only by the choice of measured variable but also by the chosen coordinate origin and the relative magnitude of each row. Thus for comparative purposes the use of a standardised origin at the array centre, and the normalising of the matrix rows by dividing each row by the constant (i.e. not $\theta$ or $n$ dependent)
terms associated with that row, is proposed. With this standardisation transducer arrays involving different transducer types can be compared in terms of their sensitivity to measurement error in the same manner as was possible on beams. However, in view of the practical difficulties in obtaining accurate phase matching between dissimilar transducer types only arrays involving one transducer type will be considered, therefore normalisation is unnecessary and we are simply interested in the 2-norm condition number of matrix T.

For a transducer system giving \( m \) measurements of displacement, at radius, \( R \), and uniform angular spacing, \( \varphi = \frac{2\pi}{m} \), the elements of matrix T are given by equation 6.5.3 as

\[
T_{j,n} = e^{\frac{j2\pi(n-1)}{m}} \quad U_n
\]

6.5.16

Since \( U_n \) is constant for a particular \( n \), the magnitudes of the elements in any given column, \( s_n \), are equal. Furthermore, if each column is normalised to have a magnitude of one by dividing by the appropriate \( |U_n| \) then the resulting matrix has a condition number of unity. The columns (and rows) of this matrix are therefore orthogonal. Since the matrix, T, may be found from this normalised matrix by multiplying each column by the appropriate \( |U_n| \) the condition number of T is given by the ratio of the largest and smallest \( |U_n| \), being

\[
\alpha \approx \frac{|U_n|_{\text{max}}}{|U_n|_{\text{min}}}
\]

6.5.17

It is clear that as the magnitude of Fourier index, \( n \), becomes large

\[
U_n = \int_{-\pi}^{\pi} e^{in\theta} e^{-ik_p R \cos \theta} d\theta \rightarrow \int_{-\pi}^{\pi} e^{in\theta} d\theta = 0,
\]

6.5.18
since the $e^{i\phi}$ term dominates $e^{-ik_p R \cos \theta}$ over most of the range of integration, particularly if the transducer placing radius, $k_p R$, is small. This can also be seen from the asymptotic expansion of the Bessel function for large positive real orders [27],

$$J_n(k_p R) \sim \frac{1}{\sqrt{2\pi n}} \left( \frac{ek_p R}{2n} \right)^n,$$

which decreases in proportion to $(1/n)^{n+0.5}$. For typical values of $k_p R$ in the range of 0.5 - 1.5, the maximum $|U_n|$ occurs for $n = 0$ or $n = \pm 1$, with subsequent increases in $|n|$ causing a decrease in $|U_n|$, the rate of decrease being faster for smaller values of $kR$. In view of the previous comments relating $|U_n|$ to the condition number it is apparent that conditioning deteriorates as more Fourier coefficients are estimated or, for typical spacings, as the transducer spacing is reduced. This is readily understood in physical terms by considering the influence that each Fourier coefficient has on the individual measurements. Having observed that the coefficient, $C_0$, is fully determined by the displacement at the origin it is apparent that if the placing radius is small all measurements are close to the origin and therefore dominated by $C_0$, the other coefficients having a relatively small influence. The problem of determining those coefficients from that particular set of measurements is thus poorly conditioned, and from this point of view a larger transducer spacing is desirable.

Similar comments apply to the measurement of radial strain. For a transducer system giving $m$ measurements of radial strain at radius, $R$, and uniform angular spacing, $\varphi = 2\pi/m$, the elements of matrix $T$ are given by equation 6.5.12 as

$$T_{i,s} = e^{-j(i-1)\varphi}$$

The condition number of the transformation matrix, $T$, is given by the ratio of the magnitudes of the largest and smallest $V_n$, being
\[ \alpha = \frac{|V_n|_{\text{max}}}{|V_n|_{\text{min}}} . \] 6.5.21

These characteristics are identical to those inherent with the accelerometer array, however the rate that \( |V_n| \) decreases with increasing \( n \) and decreasing \( k_p R \) is substantially slower than the rate of decrease of \( |U_n| \). This means that, for a given small \( k_p R \), the conditioning offered by strain measurement can be significantly better than that offered by acceleration measurement.

In the case of the beam optimum conditioning could be pursued without any inherent penalty, however on the plate improved conditioning is generally achieved at the expense of a loss of theoretical accuracy. This was seen in Chapter 5, when one-dimensional propagation was assumed, and also applies in this case.

The conditioning is, in essence, a measure of the relative importance of the individual coefficients in the Fourier series expansions of the measured variables. Good conditioning indicates that all the coefficients make a significant contribution to the expansions, while poor conditioning occurs if one or more coefficients are insignificant in the expansions. If acceleration is measured at points on a circle centred on the coordinate origin, as discussed earlier in this section, and the radius of that circle is very small then the Fourier series expansions of those measurements will be dominated by the \( C_0 \) coefficient. This is because the displacement, and hence the acceleration, at the origin is completely defined by \( C_0 \). The Fourier series expansions of the measurements therefore converge rapidly, and the conditioning is poor. As the placing radius is increased the higher order terms make a larger contribution to the Fourier series expansions of the measurements, which thus converge more slowly, and conditioning improves (up to a point). Increasing the radius beyond this point causes the condition to deteriorate owing to the reducing importance of \( C_0 \), the higher order terms still being significant, so while the conditioning is deteriorating the Fourier series expansions of the measured variables still converge only slowly.
When certain Fourier coefficients are ignored, as they are when only \( m \) terms are found, those coefficients that are evaluated must deviate from their true value in compensation. If the Fourier series expansions of the measured variables converge slowly the ignored higher order coefficients are of more significance, and this will result in larger errors in the coefficients that are evaluated. Since, for typical transducer placing radii, improved conditioning and slower convergence of the Fourier series expansions both result from increasing the placing radius, improved conditioning is achieved at the expense of a systematic error.

In practice the error that results from ignoring the higher order Fourier coefficients appears to affect predominantly the highest order coefficient, and thus it is advantageous to evaluate more than the five coefficients necessary to calculate the intensity. The error is then largely restricted to coefficients that have no direct influence on the intensity estimate. This is apparent from Table 6.2, which gives the coefficients for a wave of unit amplitude propagating in the \( x \)-direction, calculated from five, seven and nine acceleration measurements. The true values of the coefficients are given by

\[
C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(\theta) e^{-in\theta} d\theta, \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \delta(\theta) e^{-in\theta} d\theta \\
= 0.1592, \quad n = 0, \pm 1, \pm 2, \ldots \quad 6.5.22
\]

It can be seen that the evaluation of a larger number of coefficients reduces the error in the coefficients, \( C_{-2} - C_2 \), used to calculate the intensity, however this results in poorer conditioning, more expensive measurement systems, and greater computational requirements. A measurement system in which seven Fourier coefficients are estimated therefore seems appropriate, although, depending on the accuracy required, satisfactory results may be achieved with the evaluation of five coefficients.
<table>
<thead>
<tr>
<th>Placing radius</th>
<th>$k_r R = 0.1$</th>
<th>$k_r R = 0.5$</th>
<th>$k_r R = 1$</th>
<th>$k_r R = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of accel.</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>Cond. no.</td>
<td>799</td>
<td>47910</td>
<td>$3.83 \times 10^8$</td>
<td>30.7</td>
</tr>
<tr>
<td>$C_{-4}$</td>
<td>-</td>
<td>-</td>
<td>0.1592-0.00161i</td>
<td>-</td>
</tr>
<tr>
<td>$C_{-3}$</td>
<td>-</td>
<td>0.1592-0.00200i</td>
<td>0.1592</td>
<td>-</td>
</tr>
<tr>
<td>$C_{-2}$</td>
<td>0.1592-0.00271i</td>
<td>0.1592</td>
<td>0.1592</td>
<td>0.1592-0.01331i</td>
</tr>
<tr>
<td>$C_{-1}$</td>
<td>0.1592</td>
<td>0.1592</td>
<td>0.1592+0.00001i</td>
<td>0.1592</td>
</tr>
<tr>
<td>$C_0$</td>
<td>0.1592</td>
<td>0.1592</td>
<td>0.1592</td>
<td>0.1592</td>
</tr>
<tr>
<td>$C_1$</td>
<td>0.1592</td>
<td>0.1592</td>
<td>0.1592</td>
<td>0.1592+0.00011i</td>
</tr>
<tr>
<td>$C_2$</td>
<td>0.1592-0.00271i</td>
<td>0.1592</td>
<td>0.1592</td>
<td>0.1592+0.01331i</td>
</tr>
<tr>
<td>$C_3$</td>
<td>-</td>
<td>0.1592-0.00200i</td>
<td>0.1592</td>
<td>-</td>
</tr>
<tr>
<td>$C_4$</td>
<td>-</td>
<td>-</td>
<td>0.1592-0.00161i</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 6.2: Fourier coefficients evaluated for a plane wave of unit amplitude propagating in the x-direction, using 5, 7, and 9 accelerometers.
The pursuit of optimal conditioning can also conflict with the assumption that the displacement within the region in which the measurements are taken can be approximated as the sum of plane waves, as expressed in equation 6.1.1. While this form of displacement is possible in a plate, it is not the only solution, even in the far field. It was stated in Chapter 2 that, in general, the assumption of purely plane propagating waves being present is dependent not only on the region being remote from discontinuities, but also on the extent of that region being small relative to the distance from all discontinuities and sources of excitation. As in the case of the finite difference approximation, if the assumed displacement does not match the true displacement there will be an error in the intensity estimate. It is thus desirable, in terms of minimising systematic errors, to have the measurement points close together.
6.6 Comparison of Fourier Series and Finite Difference Approaches

It is difficult to obtain a valid comparison of different techniques for plate intensity measurements. Essentially, the 'best' system will have the optimum compromise between low systematic error and low sensitivity to a number of possible random errors. The simplest way of comparing sensitivities is to apply a random error to the measured variable and to look at the effect on the intensity measurement. However, as these sensitivities are often dependent on the specific field conditions, this requires that the field conditions be assumed. The comparison may therefore not be valid under other conditions.

In this case it is desired to compare an intensity estimate given using the Fourier series approach with that given by a finite difference based approach suitable for use in the same field conditions (ie. far field). Seven measurements of acceleration are used to determine seven Fourier coefficients, and these are then used to estimate the intensity as described in the previous sections. As a comparison a finite difference approach is used in which far field conditions are assumed, and the necessary spatial derivatives estimated from nine acceleration measurements as described in Appendix C. This finite difference approximation was considered by Shibata et al. [24] to be the most suitable for plate intensity measurements.

A complication arising from this comparison is the different transducer arrangements required. The Fourier series approach uses accelerometers arranged in a circle, while they are placed in a square for the finite difference approach. The term 'placing radius' is therefore used to describe the size of the Fourier series array, and 'separation' to describe that of the finite difference approach. While the terminology differs, when these terms are equal the overall size of the arrays, and the distances between adjacent accelerometers, are very similar.

For the measurements shown it is assumed that there are plane waves propagating in the positive and negative x-direction, with amplitudes 1 and 0.4 at the origin respectively. The component of intensity in the x-direction is measured, relative errors in this direction being more critical than in
others since this is the direction of energy flow. In all cases the intensity measurements are normalised, the systematic error being indicated by normalising with respect to the true intensity, while random errors are isolated from systematic errors by normalising with respect to the intensity that would be measured in the absence of errors in the measured variables.

The systematic errors for the Fourier series approach and the finite difference are shown in Figure 6.2. As expected, it is apparent that the systematic error associated with the finite difference approach increases with increasing transducer spacing. This is because the polynomials that are fitted to the measurements do not exactly describe the plate behaviour. The systematic error associated with the Fourier series approach is very small and almost independent of transducer spacing. It should be noted, however, that this is because the deformation assumed in this approach matches the field conditions. If some other field conditions, such as radial propagation, were present, then there would be an increase in the systematic error. The increase in systematic error that is apparent with this approach when the placing radius \( k_p R = 2.4 \) coincides with near singularity of the matrix used to evaluate the Fourier coefficients, the 2-norm condition number of this matrix being shown as a function of placing radius in Figure 6.3. Note that, by expressing the finite difference approach in matrix form, it would be possible to determine a condition number for that method, however the relationship between the two condition numbers is not obvious.

Figures 6.4 and 6.5 show the error that occurs when using the Fourier series and finite difference approaches if the measured variables are subjected to random phase errors uniformly distributed on the interval \([-2',+2']\). Both approaches show a high sensitivity to phase error at small spacings which decreases as spacing increases. The sensitivity of the Fourier series approach also increases at a placing radius of \( k_p R = 2.4 \) due to poor conditioning.

Figures 6.6 and 6.7 show the error in the intensity estimate if random errors, uniformly distributed on the interval \([-2\%,+2\%]\), are applied to the magnitude of the measured variables. Again small spacings result in a high sensitivity, which reduces with increasing transducer separation, and the sensitivity of the Fourier series approach becomes large when approaching matrix singularity.
Figure 6.2: Systematic error: — Fourier series approach (vs transducer placing radius, $k_p R$), -- 9 point finite difference approximation (vs transducer separation, $k_p \Delta$).

Figure 6.3: Variation of 2-norm condition number of matrix, $T$, used in evaluation of Fourier series coefficients.
Figure 6.4: Effect of random phase errors on intensity estimate - Fourier series approach.

Figure 6.5: Effect of random phase errors on intensity estimate - 9 point finite difference approach.
The sensitivity to amplitude and phase errors also gives an indication of sensitivity to error in transducer placement. If a transducer is displaced by a small amount then the measured variable will change. The size of this change will depend on the field conditions, but the effect on the intensity estimate will be determined by the sensitivity of the calculation to changes in the measured variable.

The effect of a random variation in the wavenumber, uniformly distributed on the interval [-2\%,+2\%], on the Fourier series and finite difference approaches is shown in Figures 6.8 and 6.9 respectively. The sensitivity of the Fourier series approach to this error increases with increasing transducer placing radius, reaching a maximum at $k_pR=2.4$ when the transformation matrix, $T$, is singular. It is also consistently higher than that of the finite difference approach, which is almost independent of transducer spacing. The greater sensitivity of the Fourier series approach may be explained by the use of the wavenumber both in the evaluation of the Fourier coefficients and as a factor to the third power in the entire intensity expression. In contrast it is only used as a factor to the second power in the estimation of the shear component of intensity in the finite difference approach.

While these results are specific to the particular field conditions, it is apparent that the low systematic error of the Fourier series approach means that a relatively large transducer separation can be used without causing large systematic errors. As a consequence, the transducer placing radius can be chosen to give a well conditioned problem and hence low sensitivity to measurement errors. This is in contrast to the finite difference approach, in which large systematic errors occur if the transducer spacing is chosen on the basis of sensitivity to measurement error.

It is also apparent that there is a close relationship between sensitivity to errors and the 2-norm condition number of the matrix used to determine the Fourier coefficients. The use of the condition number to judge how suitable a particular transducer arrangement is for intensity measurement is therefore justified.
Figure 6.6: Effect of random amplitude errors on intensity estimate - Fourier series approach.

Figure 6.7: Effect of random amplitude errors on intensity estimate - 9 point finite difference approach.
Figure 6.8: Effect of random errors in wavenumber estimate - Fourier series approach.

Figure 6.9: Effect of random errors in wavenumber estimate - 9 point finite difference approach.
Part C

Experimental Intensity Measurement
This part of the thesis describes the application of the proposed techniques in experimental measurement. In Chapter 7 experimental measurements on a beam are described. Intensity measurements are taken in the far field, in a region where a single near-field is present, and in the presence of two near-fields, and compared with the power supplied to the beam.

Chapter 8 describes far field intensity measurements on a plate using two, three and seven accelerometers. These techniques are used to calculate the total energy flow down the plate, which is then compared to the total power supplied to the plate, and also to measure the net energy flow into a region bounded by an irregular contour.
Chapter 7: Experimental Intensity Measurement - Beam

7.1 Experimental Apparatus

In unmodified form the experimental apparatus was intended to approximate an infinite beam. A length of steel strip 6000 x 50 x 6 mm was suspended from piano wire at four points along its length, with each end embedded in sand to approximate anechoic terminations. The beam was excited using a 'coil and magnet' non-contacting exciter, driven through a Brüel and Kjær Type 8200 force transducer. All acceleration measurements were taken using a Brüel and Kjær Type 4374 accelerometer, which with a mass of 0.65 grams caused minimal mass loading of the beam. Charge amplifiers were Brüel and Kjær Type 2624, 2635 or 2651, depending on availability.

Figure 7.1: Experimental apparatus - beam.
7.2 Measurement Principles

As the losses in the beam due to structural damping and radiation in the form of sound are very small relative to the losses in the sandboxes, virtually all the energy supplied to the beam will be dissipated in the sandboxes. As such it will form part of the net energy flow past a given point in the beam, and the measured energy flow should equal the input power. In all measurements on the beam the total measured energy flow is compared with the input power as an indication of the accuracy of the measurement.

The excitation of the beam was provided by the analyser's source using a 'burst chirp', or rapid sine sweep, with the duration of the sweep as a fraction of the measurement period controlled by the user. This gives a repeatable signal, band-limited to the frequencies of interest and, if the frequency bands are chosen suitably, avoids the need for windowing.

Initial processing of all signals was performed using a Hewlett Packard HP 35665A Dynamic Signal Analyser. However even for the simplest measurements, the two channels of the analyser were insufficient to take the required number of simultaneous measurements to calculate the input power and the energy flow in each arm of the beam. This problem was overcome through the use of the frequency response technique, as proposed by Linjama and Lahti [7]. The transfer functions relating the response of the beam at each of the required points to the applied force at the driving point were determined prior to the intensity measurement. The responses at the measurement points could then be calculated from the measured force spectrum, rather than measured directly. Input power was calculated from measurements of exciting force and driving point acceleration, and intensity from the calculated responses. MATLAB software was used for the necessary manipulation of linear spectra and transfer functions.

The transfer functions were determined by averaging several individual measurements. The use of a coherent average, in which both amplitude and phase are averaged, has allowed the effects of noise to be reduced since there is a fixed phase relationship between excitation and response, while most noise will be of random phase.
7.3 Beam Properties and Behaviour

The nominal material properties of the beam were given as

- Modulus \( E = 200 \text{ GPa} \)
- Linear density \( \rho A = 2.44 \text{ kg/m} \)
- Flexural stiffness \( EI = 192 \text{ Nm}^2 \).

7.3.1 Evaluation of Wavenumber

It was shown in Chapter 2 that the wavenumber for flexural vibrations in a beam is given by

\[
k = \sqrt{\frac{\rho A}{EI}}.
\]

7.3.1

Substituting the given material properties in equation 7.3.1 shows the theoretical relationship between wavenumber and frequency to be

\[
k = 0.842\sqrt{f}.
\]

7.3.2

In practice, however, there is often a considerable degree of uncertainty in the nominal material properties, and under these circumstances a more reliable estimate of the wavenumber may be derived from dynamic measurements. An estimate of the true wavenumber can be obtained by including an extra measurement point in the measurement system, as described by Meyer et al [29] and Wagstaff et al [30]. Alternatively, by assuming that the predicted relationship between wavenumber and frequency (i.e. the wavenumber is proportional to the square root of the frequency) is borne out in practice, it is possible to determine the constant of proportionality experimentally as described by Shoavi [33]. In this approach far field conditions are assumed, so the displacement of the beam is given by
\[ w(x) = A^+ e^{-ikx} + A^- e^{ikx}. \]  

The displacement at two locations a distance, \( \Delta \), apart can therefore be written

\[ w_1 = A^+ + A^-, \quad w_2 = A^+ e^{-i k \Delta} + A^- e^{i k \Delta}, \]

and thus

\[ w_1 - w_2 = A^+ (1 - \cos k \Delta + i \sin k \Delta) + A^- (1 - \cos k \Delta - i \sin k \Delta). \]

If \( k \Delta = 2n\pi, \ n=1,2,3, \ldots \) (i.e. the spacing of the points is an integral number of wavelengths) then equation 7.3.5 is zero, irrespective of the wave amplitudes. Thus, by finding the first frequency, \( f_s \), at which the expression \( w_1/(w_1 - w_2) \), or alternatively \( w_i^* w_2/(w_i^* w_2 - w_i^* w_2) \) which has benefits in terms of signal processing, becomes singular we also determine the wavelength at that frequency.

If the first singularity occurs at frequency \( f_s \), and it is assumed that

\[ k = \alpha \sqrt{f}, \]

then, since the points are one wavelength apart at this singularity, we may write

\[ \alpha = \frac{2\pi}{\Delta \sqrt{f_s}}. \]

Table 7.1 gives the results of evaluating equation 7.3.7 using a variety of transducer spacings to determine a mean representative value of \( \alpha \) within the frequency range of interest. It can be seen that in this case the predicted and measured values of \( \alpha \) are in close agreement.
<table>
<thead>
<tr>
<th>Separation $\Delta$ (m)</th>
<th>Frequency $f_\alpha$ (Hz)</th>
<th>$\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.179</td>
<td>1748.5</td>
<td>0.8394</td>
</tr>
<tr>
<td>0.199</td>
<td>1400.5</td>
<td>0.8437</td>
</tr>
<tr>
<td>0.216</td>
<td>1189.5</td>
<td>0.8434</td>
</tr>
<tr>
<td>0.243</td>
<td>941.0</td>
<td>0.8429</td>
</tr>
<tr>
<td>0.275</td>
<td>736.5</td>
<td>0.8419</td>
</tr>
<tr>
<td>0.325</td>
<td>526.5</td>
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</tr>
<tr>
<td>0.398</td>
<td>353.5</td>
<td>0.8397</td>
</tr>
<tr>
<td>0.488</td>
<td>231.5</td>
<td>0.8462</td>
</tr>
<tr>
<td>0.599</td>
<td>156.0</td>
<td>0.8398</td>
</tr>
<tr>
<td>0.741</td>
<td>101.5</td>
<td>0.8416</td>
</tr>
<tr>
<td>1.006</td>
<td>55.0</td>
<td>0.8422</td>
</tr>
</tbody>
</table>

$\bar{\alpha} = 0.8421$

Table 7.1: Estimation of variable, $\alpha$, relating frequency and wavenumber.

### 7.3.2 Performance of Sandboxes as Anechoic Terminations

While it was not essential that the beam terminations be perfectly anechoic, low reflections provide a favourable environment for initial intensity measurements since reverberant conditions are particularly demanding.

A typical driving point acceleration of the 'infinite' beam is shown in Figure 7.2, and it is apparent that the sandboxes (900 mm long) were not effective in damping the low frequency vibrations. This is probably largely due to the sandbox being relatively short in terms of wavelengths at these frequencies. At higher frequencies (above 200 Hz), reflection coefficients of relatively low magnitude ($<0.2$) could be obtained. These results were heavily dependent on how the sand was distributed in the sandbox, the optimum distribution appearing to be with the sand ramped in such

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a way that the depth of sand increases to the full depth of the beam at its end, thus providing a gradual change in impedance.

Figure 7.2: Driving point acceleration of experimental 'infinite' beam.
7.4 Intensity Measurement in the Far Field

7.4.1 Procedure

The acceleration at two points in each arm of the beam was used to estimate the amplitudes of the waves present. These points were midway between the driving point and the entry to the sandboxes, and thus approximately one metre from discontinuities. This distance corresponded to 0.75 wavelengths for the lowest frequency used, allowing any near-field to decay to less than 1% of its original value and justifying the assumption of far field conditions. The energy flow was calculated from the wave amplitudes using equation 3.2.10.

Response points were chosen to ensure adequate matrix conditioning within the frequency range under examination. Frequencies from 30 Hz to 2130 Hz were examined in narrow bands, 30-130 Hz and 200 Hz bands thereafter.

7.4.2 Results

Figure 7.3 shows the measured energy flow and transmitted power in the frequency range 530-2130 Hz, the results of the measurements in narrow bands being given in Appendix D.

It can be seen that the two measurements show generally good agreement, although the effects of noise are apparent, particularly in the input power spectrum. This is due to a poor signal to noise ratio, which could be improved through the use of more suitable transducers, increased excitation levels, or averaging.
Figure 7.3: Measured input and transmitted power - far field:

- - - input power, - - - transmitted power.
7.5 Intensity Measurement in a Single Near-Field

7.5.1 Procedure

Blocking masses were added to each arm of the beam, located 1.3 m from the driving point. These provided, in their vicinity, two significant propagating waves and one significant near-field component. The blocking masses clamped the beam along a length of \( d = 75 \) mm, and were of mass \( m = 1.147 \) kg and rotational inertia \( J = 0.00058 \) kgm\(^2\). Accelerations at three points in each arm were used to evaluate wave component amplitudes, with the closest being 9 mm from the blocking mass, the remaining two being placed to ensure adequate matrix conditioning within the frequency range under examination. Frequencies from 30 Hz to 2130 Hz were examined in narrow bands, 30-130 Hz and 200 Hz bands thereafter. As a comparison, wave amplitudes were also estimated using the two measurement points closest to the blocking mass, assuming far field conditions. Transmitted power was calculated from both sets of wave component data.

7.5.2 Results

Figure 7.4 shows the magnitudes of wave amplitudes in one arm of the beam at the measurement point closest to the blocking mass, over the frequency range 530-2130 Hz. The near-field magnitude is generally greater than half the principal propagating wave magnitude, verifying the existence of a significant near-field. Experimental measurements of the reflection coefficients relating the two propagating wave amplitudes were found to correspond well in terms of magnitude with those predicted using the approach described in Appendix E over this frequency range (Figure 7.5). Predictions of phase and near-field amplitude were less good, however an arbitrary 6 mm shift of the assumed point of action of the blocking mass improved results considerably. This indicates that the difference between theory and experiment may largely be due to difficulty in predicting the point of action when the blocking mass is attached along a finite length of beam, rather than at a point. Similar phase errors were observed by Shoavi [33], and attributed to local inaccuracies in the thin beam model at discontinuities. It is worth noting that
the intensity measurement is independent of these errors in this case, being solely dependent on the magnitudes of the propagating waves, so the transmission of vibrational energy through the blocking mass can still be predicted with reasonable accuracy. Figure 7.6 shows input power, along with transmitted power measured by assuming a near-field wave component (using a three accelerometer system) and assuming far field conditions (using two accelerometers), with the results of the narrow band measurements being given in Appendix F. The near-field assumption provides results generally in close agreement with input power, while the far field assumption, which in effect assumes that all near-fields have zero amplitude, exhibits significant errors, especially at lower frequencies where the near-field extends furthest.

![Graph showing wave magnitudes](image)

**Figure 7.4:** Wave magnitudes:
— incident propagating; - - - reflected propagating; ...... reflected near-field.

In each of the frequency bands examined the total discrepancy between input power and transmitted power calculated under the near-field assumption is less than 14%, and generally less than 7%. Localised frequencies of high error are associated with anomalies in the driving point frequency response, and appear to be due to resonances in the driving system. This may be observed in Figure 7.6, where there is a significant discrepancy between input and transmitted power.
power around 1900 Hz, which coincides with a small anomaly in the driving point accelerance shown in Figure 7.7.

Time-averaged energy density within 500 mm of the blocking mass at 530 Hz, calculated using equations 3.2.11 and 3.2.12, is shown in Figure 7.8. The equality of the kinetic and potential energy components in the far field is apparent, as is the spatial variation associated with the presence of waves propagating in both directions (see equations 3.2.13 and 3.2.14). At this frequency the effect of the blocking mass is to restrict both rotational and translational motion in its immediate vicinity. The resultant higher strains and reduced velocities are evident in the increased potential energy density and reduced kinetic energy density close to the blocking mass.

Figure 7.5: Reflection coefficient (propagating-propagating) of blocking mass:
- - - theoretical; - - - measured.
Figure 7.6: Measured power within a single near-field: —— input power; - - - transmitted power (using three accelerometers); ...... transmitted power (using two accelerometers).

Figure 7.7: Driving point acceleration of beam with blocking mass.
Figure 7.8: Time averaged energy density in vicinity of blocking mass at 530 Hz: 
— total energy density;  - - - kinetic energy density;  ....... potential energy density.
7.6 Intensity Measurement Within Two Near-Fields

7.6.1 Procedure

The blocking masses were placed in each arm 100 mm from the driving point to provide two significant near-fields within the regions of measurement. A frequency band of 130-930 Hz was investigated. Wave component amplitudes in each arm were calculated both using four equally spaced acceleration measurements, and using two acceleration measurements in conjunction with two measurements of strain using piezo-electric film strain gauges of length 30 mm. The accelerometer system used a 27 mm spacing, while the hybrid measurement system retained approximately the same overall dimension, with the maximum possible separation for both the accelerometers and the strain gauges.

To allow the use of both strain and acceleration measurements in the wave amplitude calculation, it was necessary to perform some form of calibration. In this case wave amplitudes in the far field were evaluated using two acceleration measurements and using two strain measurements. The strain gauge sensitivity was determined to be that which minimised the difference between the magnitudes of the outgoing wave estimated by the two systems, over the range 130-330 Hz. This makes no allowance for phase mismatch, and an improved calibration method is desirable.

7.6.2 Results

The measurement of transmitted power taken using four measurements of acceleration at the maximum uniform spacing possible within the region is shown in Figure 7.9a. The measured power displays large fluctuations at low frequencies that are not seen in the input power spectrum. This is a consequence of poor matrix conditioning and measurement noise, together with the effects of (small) errors in assumed transducer spacing and wavenumber. At higher frequencies array condition improves to provide reasonable correlation between input and transmitted power.
Figure 7.9: Measured power in region where two near-fields exist:
(a) — input power; - - - transmitted power (using four accelerometers).
(b) — input power; - - - transmitted power (using a hybrid system); ..... transmitted power (using two accelerometers).
The same measurement performed using a hybrid measurement system, comprising of two acceleration and two strain measurements, results in close agreement between input and transmitted power over the entire frequency range (see Figure 7.9b), and the 'noisy' characteristics exhibited by the method using four acceleration measurements are absent. Figure 7.9b also shows an intensity estimate using two acceleration measurements, the effects of near-fields being neglected and resulting in a substantial error.

![Array condition number graph](image)

Figure 7.10: Array condition number as used for experimental intensity measurements in the presence of two near-fields: —— hybrid system; - - - four accelerometer system.

The improved intensity measurement using the hybrid system can be attributed to the improvement in the conditioning of the problem offered by the hybrid measurement system over the four accelerometer array, which results in a reduced sensitivity to experimental inaccuracy and error. The array condition numbers of the two measurement systems are shown as a function of frequency in Figure 7.10, in which the relatively poor conditioning of the accelerometer array at low frequencies is readily apparent.
Chapter 8: Experimental Intensity Measurement - Plate

8.1 EXPERIMENTAL APPARATUS

A brass sheet 1800 x 850 x 3 mm was suspended in a frame to approximate simple supports on three edges, with the lower (850 mm) edge free. The apparatus is shown in Figure 8.1. Damping was applied to the lower region of the plate, ensuring that most of the energy supplied by excitation will be dissipated in this region, and thus that there will be a significant net flow of energy down the plate.

Figure 8.1: Experimental apparatus - plate.
Simple supports, which are not simple to achieve in practice, were desirable because of their reflection characteristics - namely that a reflected near-field is produced only if a near-field is incident on the boundary. Therefore, if the plate is sufficiently wide and the intensity measurements are taken in a region sufficiently distant from discontinuities such as excitation, damping, and the free edge, the effects of near-fields on the measurement can be ignored. This means that an intensity measurement system suitable for far field measurements can be used right to the edge of the plate, allowing the input power to be compared with a line integral of intensity. In other circumstances this would require either a more elaborate measurement system or the perhaps false assumption that the presence of near-fields has a negligible effect on the intensity measurement.

Any attempt at reproducing the characteristics of a simple support for experimental purposes will inevitably result in a large but finite translational stiffness and a small but finite rotational stiffness. This deviation from the ideal support properties will have an effect on the dynamic characteristics of the apparatus.

![Diagram of the plate support](image_url)

**Figure 8.2: Exploded detail of plate support.**

The simple support in this case was approximated by attaching the edge of the plate, using epoxy glue and screws, to a thin metal diaphragm as shown in Figure 8.2, the diaphragm being lightly tensioned through the use of shims between it and the frame. Translational location was provided
by the in-plane stiffness of the diaphragm, while rotational freedom was permitted by its low flexural stiffness.

The diaphragm is considered clamped at the frame and pinned at its mid-point for flexural motion, and that a state of plane strain exists when translational motion occurs. Under these circumstances it can be shown that the translational and rotational stiffnesses of the support may be expressed in terms of the diaphragm properties as

\[ K_r = \frac{2E_d t}{(1 - \nu^2)l}, \quad K_p = \frac{8D_d}{l}. \]  \hspace{1cm} 8.1.1

where \( E_d \) is Young's modulus of the diaphragm and \( D_d \) is its flexural stiffness per unit width. If the support is represented by springs, as shown in Figure 8.3, then equating the bending moment and shear force at the plate boundary (from equations 2.3.6 and 2.3.13a) to those applied by the diaphragm gives

\[-D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) = K_p \frac{\partial w}{\partial x} \]  \hspace{1cm} 8.1.2a

\[ D \left( \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} \right) = K_r w. \]  \hspace{1cm} 8.1.2b

Figure 8.3: Representation of experimental 'simple support'.
Assuming that a plane propagating wave, travelling at an angle, $\theta$, to the $x$-axis, is incident on a boundary lying parallel to the $y$-axis then the displacement of the plate is given by

$$w(x,y) = \left( A_p e^{-ik_p x \cos \theta} + B_p e^{ik_p x \cos \theta} + C_p e^{ik_p y \sqrt{1 + \sin^2 \theta}} \right) e^{-ik_p y \sin \theta}.$$ \hspace{1cm} (8.1.3)

Equation 8.1.3 may be substituted into equations 8.1.2a and b to give two simultaneous equations in three variables. These can be solved in terms of the incident wave amplitude, $A_p$, to give the relevant reflection coefficients of the boundary.

![Graph](image)

Figure 8.4: Normalised near-field amplitude $C_p^n/A_p$ (normal incidence).

In this case the aim was to choose dimensions and materials for the supporting diaphragm that would minimise the magnitude of the reflection coefficient that relates near-field amplitude to incident propagating wave amplitude. It is immediately apparent that, even ignoring the restricted range of material dimensions available from stock, the final solution will be a compromise. In order to increase translational stiffness it is necessary to use a material of higher modulus, increase the diaphragm thickness or decrease its width. All of these approaches will also increase the
rotational stiffness, which ideally would be zero. Conversely reducing rotational stiffness inevitably also reduces translational stiffness. Furthermore, the optimum ratio of the two stiffnesses is a function of frequency so it is necessary to determine a suitable compromise for the frequency range of interest.

On the basis of this analysis, a brass diaphragm, with $E=112$ GPa, $l=38$ mm, $t=0.2$ mm, was predicted to give small near-field amplitudes in the frequency range 100-500 Hz while being relatively easy to mount in a frame constructed of stock steel sections. The predicted near-field amplitude resulting from a propagating wave at normal incidence is shown in Figure 8.4.
8.2 Measurement Principles

The plate was excited through a stinger by a Ling V201 electrodynamic shaker. A burst chirp, or rapid sine sweep, generated by a Hewlett Packard HP35665A Dynamic Signal Analyser was used to drive the shaker, allowing excitation to be confined to the required frequency range and avoiding the need for windowing. The exciting force and driving point acceleration were measured using a Brüel and Kjær Type 8200 force transducer and PCB 321A03 accelerometer respectively. Intensity measurements were derived from a number of simultaneous acceleration measurements taken using PCB 353B65 accelerometers. Signal conditioning was provided for the force transducer via Brüel and Kjær Type 2635 charge amplifier, while the PCB accelerometers were supplied using a PCB F483 B03 12 channel power unit.

For commissioning tests the Dynamic Signal Analyser was used for both excitation and data acquisition, while all subsequent data acquisition and processing was performed using an IBM PC equipped with a National Instruments AT-MIO-16L-9 Multifunction I/O Board. LabVIEW software was used to control the data acquisition as well as for subsequent processing. The analog signals were passed through a TechFilter 16 channel board-mounted 75 dB/octave low-pass filter prior to conversion to digital form to eliminate aliasing problems.

Intensity measurements were taken using a sampling rate of 2048 Hz in conjunction with 2048 samples to give 1 Hz resolution. The A-D conversion, having 12 bit resolution, has a 72 dB dynamic range. It was therefore desirable that any high frequency signal should have been attenuated by 72 dB before it could appear as an alias in the measurement band. A filter cutoff frequency of 800 Hz ensured this for frequencies up to 500 Hz, while keeping the cutoff frequency and measurement band well separated to minimise any phase mismatch between channels.
8.3 Plate Properties and Behaviour

The nominal material properties of the plate were given as

Modulus $E=112$ GPa
Density $\rho=8530$ kg/m$^3$
Thickness $h=3.0$ mm
Flexural stiffness / unit width $D=276.9$ Nm
Poisson's ratio $\nu=0.3$

8.3.1 Natural Frequencies

<table>
<thead>
<tr>
<th>Mode</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
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<td>Natural Frequency</td>
<td>Measured</td>
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<td>9.9</td>
<td>15.7</td>
<td>24.3</td>
<td>29.1</td>
<td>32.3</td>
<td>35.4</td>
<td>37.8</td>
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<tr>
<td>Predicted (Anal)</td>
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<td>9.2</td>
<td>14.8</td>
<td>24.3</td>
<td>28.6</td>
<td>31.1</td>
<td>36.3</td>
<td>37.0</td>
<td>45.3</td>
</tr>
<tr>
<td>Predicted (FE)</td>
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<td>10.4</td>
<td>16.1</td>
<td>24.8</td>
<td>28.9</td>
<td>31.9</td>
<td>36.7</td>
<td>38.0</td>
<td>47.1</td>
</tr>
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<table>
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<tr>
<th>Mode</th>
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<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19</th>
<th>20</th>
</tr>
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<td>Measured</td>
<td>49.?</td>
<td>58.2</td>
<td>64.9</td>
<td>68.7</td>
<td>?</td>
<td>72.5</td>
<td>73.3</td>
<td>81.8</td>
<td>89.6</td>
<td>91.5</td>
<td>93.9</td>
</tr>
<tr>
<td>Predicted (Anal)</td>
<td>51.1</td>
<td>57.9</td>
<td>64.4</td>
<td>66.3</td>
<td>69.5</td>
<td>72.6</td>
<td>72.8</td>
<td>81.4</td>
<td>91.1</td>
<td>91.6</td>
<td>93.4</td>
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<tr>
<td>Predicted (FE)</td>
<td>51.8</td>
<td>59.4</td>
<td>64.6</td>
<td>67.7</td>
<td>70.2</td>
<td>74.0</td>
<td>74.8</td>
<td>83.4</td>
<td>91.9</td>
<td>93.6</td>
<td>96.2</td>
</tr>
</tbody>
</table>

Table 8.1: Natural frequencies of experimental apparatus (undamped).
While the experimental apparatus was designed to minimise the amplitudes of near-fields, this particular property is difficult to verify experimentally. Therefore, in order to verify that the apparatus was performing approximately as expected, the natural frequencies of the undamped plate were compared with those predicted both analytically and using a finite element model. While this does not indicate whether the predicted performance is being achieved, it is likely to indicate any substantial problems in the apparatus. The measured and predicted natural frequencies were found to be in close agreement up to 100 Hz, as shown in Table 8.1.

Above 100 Hz the discrepancy between measured and predicted values is similar to the modal separation, and comparison becomes meaningless. However, close agreement over the first twenty modes appears to indicate that the supports are performing approximately as predicted.

8.3.2 Damping

Loss factors in the plate were estimated from the driving point inerance. Measured loss factors prior to damping treatment being applied were less than $3.5 \times 10^{-3}$, and generally less than $1 \times 10^{-3}$, over the frequency range 100-600 Hz. Since the loss factor for brass is expected to be less than $1 \times 10^{-3}$ [26], it would appear that the supports and their attachments had not contributed greatly to the losses.

For the experimental measurements, damping was applied at the lower end of the plate in the form of sand and carpet underfelt. While the effect differed at each resonance, this typically increased the average of the measured loss factor by a factor of twenty to thirty, giving loss factors of 2-3%.

8.3.3 Estimation of Plate Wavenumber

The relationship between plate wavenumber and frequency can be calculated from the material properties given at the start of this section, using equation 2.3.40 or 2.3.52 to give
\[ k_p = \sqrt{\omega} \sqrt{\frac{\rho h}{D}} \]

8.3.1

\[ \Rightarrow k_p = 1.382 \sqrt{f} . \]

8.3.2

In order to verify this relationship experimentally the wavenumber in a 50 mm wide offcut of the plate was determined using the procedure described in Section 7.3.1. The frequencies at which singularities occurred for particular transducer spacings are given in Table 8.2.

<table>
<thead>
<tr>
<th>Separation Δ (m)</th>
<th>Frequency ( f_\alpha ) (Hz)</th>
<th>( \alpha )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.199</td>
<td>490.5</td>
<td>1.426</td>
</tr>
<tr>
<td>0.216</td>
<td>426</td>
<td>1.409</td>
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<td>0.242</td>
<td>330.5</td>
<td>1.428</td>
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<tr>
<td>0.263</td>
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<td>1.443</td>
</tr>
<tr>
<td>0.290</td>
<td>240.5</td>
<td>1.397</td>
</tr>
<tr>
<td>0.300</td>
<td>228.5</td>
<td>1.386</td>
</tr>
<tr>
<td>0.313</td>
<td>212</td>
<td>1.379</td>
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<tr>
<td>0.346</td>
<td>166</td>
<td>1.409</td>
</tr>
<tr>
<td>0.378</td>
<td>138</td>
<td>1.415</td>
</tr>
</tbody>
</table>

Table 8.2: Estimation of variable, \( \alpha \), relating frequency and wavenumber.

The resultant wavenumber relationship must then be corrected for the extra stiffness endowed by Poisson's effect. Comparing equations 7.3.1 and 8.3.1 reveals that the relationship between beam and plate wavenumbers is given by

\[ k_{\text{plate}} = \sqrt[4]{1 - v^2} k_{\text{beam}} . \]

8.3.3
\[ \alpha_{\text{plate}} = \sqrt{1 - v^2} \alpha_{\text{beam}}. \] 8.3.4

Therefore, substituting Poisson's ratio in equation 8.3.4 gives

\[ \alpha_{\text{plate}} = 0.9767 \alpha_{\text{beam}} \] 8.3.5

\[ = 1.377, \] 8.3.6

and therefore

\[ k_p = 1.377 \sqrt{f}. \] 8.3.7

It can be seen that this is in close agreement with the predicted relationship given in equation 8.3.2.
8.4 Performance of Data Acquisition Equipment

The estimation of structural intensity requires simultaneous measurement on several channels, and the sensitivity of the estimate to errors in the measured variables depends on the technique being used and on the vibrational field conditions that are present. However in all cases good matching between measurement channels is important, particularly in terms of phase.

Whenever an electrical signal is processed it is subjected to transformations which may distort it from its original form. In the case of the transducers used in these experiments the polarisation of a piezo-electric element is amplified either at the transducer (in the case of the PCB accelerometers) or at a separate charge amplifier (for the Brüel and Kjær force transducer). This signal is then amplified again and low-pass filtered using a seven pole elliptical filter before analog-to-digital conversion. Minor differences between nominally identical components mean that even when the same type of equipment is used on all channels there will be some phase mismatch.

An additional source of phase mismatch is the sequential rather than simultaneous sampling of the measurement channels. If the interchannel delay is known it is possible to correct each spectral line with the appropriate phase shift. For this reason the interchannel delay will be considered first, and all other phase errors considered after correction for interchannel delay.

In the LabVIEW manuals it was stated that a minimum safe interchannel delay was calculated on the basis of the required settling time for the instrumentation, 10 μs being added to this to give the interchannel delay. However interrogating LabVIEW while operating indicated an interchannel delay of 9 μs. In view of this conflicting information an identical signal was connected to channels 0 and 15, with all channels being sampled, and the frequency response calculated. The phase mismatch at a given angular frequency, ω, due to sequential sampling is

\[ \phi = \omega \tau, \] 8.4.1
where \( \tau \) is the total delay between the two channels. In this case, because all channels are sampled, the total delay is fifteen times the interchannel delay. The interchannel delay can thus be estimated from the phase of the frequency response, shown in Figure 8.5, and is found to be approximately 20 \( \mu s \). It therefore appears likely that the 9 \( \mu s \) stated is actually the minimum safe interchannel delay, (particularly since the AT-MIO-16L-9 A-D board has a maximum sampling rate of 100 kHz and thus requires 10 \( \mu s \) to settle and measure) and the 10 \( \mu s \) safety factor is then added to this to give an interchannel delay of 19 \( \mu s \). However, as experimental results indicate a 20 \( \mu s \) interchannel delay this has been used to correct all measurements.

![Figure 8.5: Phase mismatch due to interchannel delay between channels 0 and 15.](image)

Since the precise sources of any other phase errors that are present are unknown each measurement channel should ideally be treated as a unit, with the same transducer, amplifier and leads, and the same amplifier and filter settings. With one channel designated as the reference the frequency response of each channel relative to the reference was found. The principal difficulty with this approach is ensuring that the two accelerometers are subjected to the same excitation, since a typical electrodynamic shaker will generate a rotational motion of the table as well as the
intended translation. While there are mounting schemes that can reduce the consequences of this effect they generally introduce other problems [34]. In this case the accelerometers were mounted side-by-side. By moving the accelerometers around on the shaker table it was possible to determine a placing which minimised the effects of the rotational resonances. Frequency responses were determined from the average of a number of measurements with the transducers placed in this fashion, and then a further average with the positions of the transducers reversed.

![Graph showing phase mismatch between channels 2 and 7.](image)

**Figure 8.6:** Phase mismatch between channels 2 and 7.

As a further check on matching between measurement channels, the individual components were compared. The matching of the accelerometers was checked using the Dynamic Signal Analyser, while the data acquisition system (filter and A-D conversion) was tested by determining the frequency response between channels. When these effects were combined the trends agreed closely with those determined by testing each measurement channel as a complete unit.

Matching between channels was generally quite good, the phase mismatch between channel 2 (reference channel) and channel 7, which exhibited the greatest discrepancy, being shown in
Figure 8.6. While the observed phase errors were relatively low, small phase corrections ($< 0.3^\circ$) to channels 3, 7 and 8, as linear functions of frequency, were considered worthwhile. Amplitude matching was within 1%, and no correction was considered necessary.

Also apparent in some measurements was an increase in the magnitude of fluctuations in relative phase at higher frequencies (above 400 Hz). This effect will tend to increase errors in the intensity estimates at these frequencies.
8.5 Comparison of Finite Difference and Wave-Based Approaches to Intensity Measurement Using Two Transducers

8.5.1 Procedure

Intensity measurements were made using two accelerometers, utilising the finite difference based and wave-based approaches described in Sections 5.3.2 and 5.3.3 respectively. Measurements of the component of intensity directed down the plate were taken every 25 mm along two lines, 350 mm apart, across the plate. See Figure 8.7. At each location three individual intensity estimates were taken and averaged. A transducer separation of 50 mm was used, being approximately a tenth of a wavelength at 100 Hz and a quarter-wavelength at 500 Hz.

![Diagram of experimental apparatus](image)

**Figure 8.7:** Schematic diagram of experimental apparatus.

The measurements were processed in two ways. In the first the average measured intensity across the upper line (A) was calculated to give an estimate of the energy flow. This was compared with the power supplied to the plate. However, since knowledge of the wavenumber and flexural stiffness are required to estimate the intensity but not the input power, imperfect knowledge of these quantities can lead to an apparent bias in the results that is not a consequence of the
measurement system. In the second approach the measured energy flows across the two lines were compared. This disguises any bias introduced by imperfect knowledge of the material properties, and also gives an indication as to whether a significant amount of energy is being lost to the supports.

8.5.2 Results

A comparison of the measured energy flow down the plate with the measured power input is shown in Figure 8.8. As was predicted in Section 5.4, the finite difference approach results in the energy flow being underestimated while the wave approach results in overestimation. This effect is most evident at higher frequencies owing to the relatively large transducer spacing in terms of wavelengths and is more apparent in Figure 8.9, which shows the difference between these two quantities relative to the input power. The systematic errors that are evident in these intensity measurements would be reduced if a smaller transducer spacing was used, however this would result in poorer conditioning and therefore increased sensitivity to measurement errors. It is thus necessary to strike a balance between theoretical accuracy and adequate conditioning, and this may mean that a particular spacing is only suitable for a narrow frequency range.

It was shown in Section 5.4 that the systematic error associated with estimating intensity using two accelerometers is dependent on the direction of wave propagation. In the experimental apparatus the losses within the plate and at the simply supported boundaries are small relative to the losses in the damped region, and thus there is a significant net energy flow down the plate. There are, however, a large number of modes within the frequency range of interest, each with its own directions of wave propagation. This means that there is no 'typical' propagation direction, and the degree of the systematic error is therefore dependent on the particular mode under investigation. For this reason it is more informative to consider the total power input and total measured energy flow over the frequency range (i.e. the sum of all spectral lines) for each approach. The relative difference between these two measurements, as given by
\[ E_{ip} = \frac{\sum_{n=1}^{401} P_{\text{flow}}(n) - \sum_{n=1}^{401} P_{in}(n)}{\sum_{n=1}^{401} P_{in}(n)}, \] 8.5.1

gives an indication of the overall bias of the intensity estimate, was found to be 6.3% for the wave-based approach and -14.1% for the finite difference approach. The mean, over all spectral lines, of the magnitude of the relative difference at each spectral line, as defined by

\[ \overline{E_{|n|}} = \frac{1}{401} \sum_{n=1}^{401} \left| \frac{P_{\text{flow}}(n) - P_{in}(n)}{P_{in}(n)} \right|, \] 8.5.2

gives an indication of the typical magnitude of the difference at any spectral line. This was found to be 11.3% for the wave-based approach and 15.6% for the finite difference approach. These measurements show the wave-based approach to give slightly better results than the finite difference approach under the experimental conditions.

![Graph](https://via.placeholder.com/150)

**Figure 8.8:** Input power and measured energy flow - finite difference and wave-based 2-accelerometer measurements: —— input power; - - - energy flow (wave approach); ····· energy flow (finite difference approach).
Figure 8.9: Relative difference between measured power input and energy flow measured using two accelerometers: --- wave approach; - - - finite difference approach.

Figure 8.10: Normalised difference in transmitted power across two lines on plate.
Large localised differences are found at approximately 260 Hz and 460 Hz. These frequencies correspond closely to the predicted natural frequencies of the undamped plate when there are five and seven nodal lines respectively running down the plate and none across it. Furthermore, tests of the experimental apparatus (with damping treatment in place), show damping to be low at these frequencies. It is assumed that the direction of wave propagation at these frequencies, which is almost directly across the plate, results in poor damping performance in the experimental apparatus, with a large proportion of the losses occurring within the plate itself or at the supports. The more resonant behaviour of the plate at these frequencies results in high sensitivity to phase errors in the measurements, and thus a poor estimate of the intensity.

The normalised difference in energy flows across the two lines, which is identical for the wave-based and finite difference approaches, is shown in Figure 8.10. It can be seen that the two measured energy flows are in reasonable agreement, with variation about an approximately zero mean throughout the frequency range. This indicates that the two-accelerometer techniques are giving consistent results, and that the losses in the plate and at the boundary are small relative to the net energy flow down the plate, and thus small relative to the losses in the damped region of the plate.
8.6 The Use of Three Accelerometers to Determine the Intensity Vector

8.6.1 Procedure

Intensity measurements were performed using the wave-based three accelerometer approach described in Section 5.6. In a similar procedure to that described in Section 8.5.1, the net energy flow across a line on the plate was compared with the input power. The mean energy flow was estimated from three individual measurements of intensity at each of 32 locations, 25 mm apart, on the line. Less locations were used than in the two accelerometer case because the size of the three accelerometer array prevented measurements at points very close to the plate edge.

A transducer separation of 50 mm was used, with the coordinate system oriented in such a way that the normal to the line on the plate lay at 45° to both the x- and y-axes, as shown in Figure 8.11. Both components of the intensity vector are therefore required to estimate the energy flow across the line, and the comparison of measured energy flow to input power can give an indication of the validity of the estimate of the intensity vector, rather than of one of its components.

![Diagram](image_url)

**Figure 8.11:** Coordinate system for intensity measurements using three accelerometers.
8.6.2 Results

The input power and the energy flow as measured using the three accelerometer approach are shown in Figure 8.12, and the relative difference between the two measurements in Figure 8.13. It is apparent that they are generally in good agreement, with the energy flow appearing to be slightly overestimated. This is borne out by the relative difference between the input power and the measured energy flow, $E_p$, as defined in equation 8.5.1, which is 3%, and is to be expected since the three accelerometer approach being used is based on the two accelerometer wave-based approach which has an inherent bias. The mean magnitude of the relative difference at a spectral line, $E_p$, as defined in equation 8.5.2, is 7.1%. From these measurements it would appear that the three accelerometer approach is, under the experimental conditions, giving better results than the wave-based two accelerometer technique upon which it is based. This may be explained by the fact that the use of the least squares solution in the three accelerometer approach reduces not only the effects of measurement noise but also the maximum magnitude of the error due to the use of a finite transducer spacing.

![Figure 8.12: Input power and measured energy flow in plate: —— input power; - - - energy flow (three accelerometer approach).](image)

150
Figure 8.13: Relative difference between measured power input and energy flow measured using three accelerometer approach.
8.7 Intensity Measurement Using a Complex Fourier Series

8.7.1 Procedure

The Fourier series approach described in Chapter 6 was used to estimate the net energy flow across Line A as shown in Figure 8.7. For each intensity measurement, seven Fourier series coefficients were estimated from seven measurements of acceleration. These measurements were taken at a uniform angular spacing on a circle of 50 mm radius. As in the previous cases the intensity at each location was calculated as the average of three individual estimates, and measurement locations were separated by 25 mm giving, in this case, 31 locations.

8.7.2 Results

Power input to the plate and the net energy flow as measured using the Fourier series approach are shown in Figure 8.14, and the relative difference between the two measurements in Figure 8.15.

![Graph showing power and frequency]

Figure 8.14: Input power and measured energy flow: —— input power; - - - energy flow (Fourier series approach).
Again, the measurements show generally good agreement. The relative difference between the two measurements, $E_p$ (equation 8.5.1) is -1.7%, and the mean magnitude of the difference at any spectral line, $\overline{E_{|\cdot|}}$ (equation 8.5.2) is 6.2%.

![Graph](image)

**Figure 8.15:** Relative difference between measured power input and energy flow measured using Fourier series approach.
8.8 Summary of Performance of Plate Intensity Measurement Systems

The performance of the four plate intensity measurement systems used for experimental measurements is summarised in Table 8.3 in terms of the difference measurements used in the previous three sections. As expected, of the two accelerometer techniques the finite difference approach has resulted in an underestimation of the energy flow, while the wave-based approach has resulted in an overestimation. It can also be seen that the wave-based approach has given slightly better results than that utilising a finite difference approximation. This is due to the systematic errors associated with the wave-based approach being more favourably related to propagation direction than those of the finite difference approach, giving lower absolute errors.

<table>
<thead>
<tr>
<th>Difference</th>
<th>2 acc. finite diff.</th>
<th>2 acc. wave</th>
<th>3 acc. wave</th>
<th>7 acc. Fourier</th>
</tr>
</thead>
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<tr>
<td>Difference in total power, $E_{\text{wp}}$</td>
<td>-14.1%</td>
<td>6.3%</td>
<td>3.0%</td>
<td>-1.7%</td>
</tr>
<tr>
<td>Mean mag. difference, $E_{</td>
<td></td>
<td>}$</td>
<td>15.6%</td>
<td>11.3%</td>
</tr>
</tbody>
</table>

Table 8.3: Differences between input power and net energy flow as estimated using two, three and seven accelerometers.

Using three accelerometers to determine the intensity vector has given results superior to those of the approaches using two accelerometers. Being founded on the two accelerometer wave-based technique, it also results in the energy flow being overestimated. However, in this case the system is overdetermined, and a ‘best fit’ to the measured data is found. This has the effect of reducing the maximum levels of error, giving an improved intensity estimate.

The best results have been achieved with the Fourier series approach. Unlike the techniques based on the assumption of one-dimensional wave propagation the Fourier series approach provides,
within the limitations of the measurement equipment, an accurate estimate of local intensity. It therefore does not rely on spatial averaging to reduce the effects of systematic errors in the individual measurements, and thus gives a more reliable estimate of the energy flow.
8.9 Intensity Measurement on an Irregular Contour

8.9.1 Procedure

The net energy flow into a triangular region on the plate, as shown in Figure 8.16, was estimated using the two and three accelerometer approaches and also the Fourier series approach to intensity measurement. A 50 mm accelerometer separation was used for the two and three accelerometer approaches, while seven accelerometers with a uniform angular spacing on a circle of 50 mm radius were used for the Fourier series approach. In all cases the average of three intensity estimates was obtained at each location, with 25 mm between locations.

The contour was chosen in an attempt to illustrate the limitations of the assumption of one-dimensional wave propagation, as is inherent in the two and three accelerometer approaches. It was hoped that the irregular contour would not allow an accurate estimate spatial averaged intensity to be determined using these techniques. The more rigorous Fourier series approach, in which the systematic error in each measurement is small, should give a better performance under these circumstances.

![Diagram of triangular contour](image)

**Figure 8.16:** Placing and size of triangular contour.
8.9.2 Results

The measured net energy flow into the triangle across Sides A, B, and C, is shown in Figures 8.17-8.19 respectively, normalised with respect to the power input to the plate. It is intuitive that the orientation of the triangle means that the dominant proportion of the energy entering the region does so across Side A. Side B, being both close to and parallel to the simply supported plate boundary, has a relatively small energy flow across it. Since there should be virtually no net energy flow into the region the energy flow across Side C should therefore be very similar to that across Side A, but in the opposite direction (i.e. out of the triangular region). This general trend is seen in Figures 8.17-8.19, however more notable is that all three approaches echo the same characteristics. Thus where absolute accuracy is not critical the simple techniques involving two or three accelerometers may be adequate, justifying their use in many practical situations.

The total measured energy flow into the triangular region (normalised with respect to the power input to the plate) is shown in Figure 8.20. It is apparent in Figure 8.20 that the Fourier series approach has given better results over the frequency range than the other techniques. The poorer performance of the simpler techniques is believed due to relatively large errors in the individual measurements, owing to the assumption of one-dimensional wave propagation, and there being insufficient space to reduce these by spatial averaging.
Figure 8.17: Measured power into region across Side A: —— Fourier series approach; --- three accelerometer approach; ··········· two accelerometer approach.

Figure 8.18: Measured power into region across Side B: —— Fourier series approach; --- three accelerometer approach; ··········· two accelerometer approach.
Figure 8.19: Measured power into region across Side C: —— Fourier series approach; 
- - - three accelerometer approach; ······· two accelerometer approach.

Figure 8.20: Measured power into region: —— Fourier series approach; 
- - - three accelerometer approach; ······· two accelerometer approach.
Chapter 9: Concluding Remarks

The theory relating to structural intensity measurement is considerably in advance of what is achievable in practice. Expressions for intensity in beams and plates in terms of displacement have been known for many years [2], yet intensity measurements of this type are still only practical in the simplest situations. The calculation of intensity in this way, requiring the computation of the difference of a number of near-identical measured quantities, is prone to ill-conditioning and the final result can therefore be extremely sensitive to errors and noise in the measured data. Structural intensity measurement thus requires measured data of very high quality, particularly in terms of phase matching between the measurement channels. This level of accuracy appears to be at the limit of, or even exceed, that which is possible with current transducers and signal conditioning (amplifying, filtering, and A-D conversion). There is, therefore, considerable merit in the modification of the intensity calculation to reduce its sensitivity to measurement errors. Achieving an optimal balance between systematic errors (if any) and conditioning will allow the best possible intensity measurement with the available equipment.

For most structural intensity measurements, finite difference approximations are used to estimate the necessary spatial derivatives of displacement. A fundamental problem with this approach is the underlying assumption that the deformation can be described using low order polynomial functions. Since this assumption is not strictly valid a systematic error is introduced. If the measurement points are close together then the discrepancy between the true and assumed deformations is small and therefore so is the error. If, however, the distance between the measurement points is large then the discrepancy between the true and assumed deformations is also large, leading to large errors in the intensity estimate. It is thus necessary to use closely spaced measurement points in order to minimise systematic errors when implementing the finite difference approximations. However under these circumstances the individual measurements are
nearly identical, and the calculation of spatial derivatives from the measurements is poorly conditioned.

This problem can be reduced if the assumed deformation matches the true deformation more closely. The measurement points can then be placed further apart without introducing excessive errors in the estimation of spatial derivatives. Use of a wave-based approach assumes that the deformation can be described in terms of trigonometric functions and, if near-fields are incorporated, exponential functions. Within the limits of simple bending theory this assumption is exact for beams, and in many circumstances is a valid approximation for plates.

Since the wave description is exact, the intensity in a beam can be calculated exactly using a wave-based approach. Unlike the finite difference approach there are no systematic errors resulting from the use of a finite transducer spacing, and the spacing may therefore be chosen to optimise the conditioning of the intensity calculation.

The conditioning of the intensity calculation is dependent both on the vibrational field present (ie. the degree of reverberance) and on the measurement system used (ie. measured variables, measurement locations). By expressing the intensity in terms of wave amplitudes the influence of the vibrational field on the conditioning is readily apparent. The conditioning of the estimation of wave amplitudes from measured variables can be assessed using the 2-norm condition number of a matrix. Since the matrix is determined by the assumed field conditions, the measured variables, and the measurement locations, and is independent of the specific vibrational field present, this condition number may be considered a property of the measurement system. It may therefore be used for comparing different measurement systems, and is termed the *array condition number* in this thesis.

Use of the array condition number reveals that the combined measurement of different variables in one measurement system (particularly acceleration and strain) can offer much improved conditioning for intensity measurement between closely spaced discontinuities. Experimental
results showed such a measurement system to give a much better intensity estimate than a conventional four accelerometer array.

While the wave-based approach is exact for beams, in general it approximates the behaviour of plates. There is thus potentially a systematic error associated with applying wave-based techniques to plates, just as there is for the finite difference approach. The simplest intensity measurements on plates involve assuming far field conditions and one-dimensional wave propagation, and allow the component of the intensity vector in a particular direction to be estimated from the displacement and slope at a point. The intensity vector itself may be estimated from two such measurements in orthogonal directions. If the assumption of one-dimensional wave propagation is violated then an error will result, however in a uniform wave field that error has a zero mean. A number of such measurements in a region can therefore give a good indication of the net energy flow, even though the error in each measurement may be substantial.

Traditionally the above technique is implemented with two accelerometers, using a finite difference approximation to estimate the displacement and slope. As an alternative a wave-based approximation has been proposed. Both approaches are subject to systematic errors due to the use of a finite, rather than infinitely small, transducer spacing, however the nature of these errors is such that the wave-based approach can give a better estimate of the intensity vector.

The wave-based technique using two accelerometers has been extended to incorporate a third accelerometer, the components of the intensity vector in three directions being estimated. These overdetermine the intensity vector, allowing it to be estimated in such a way as to best fit the measured data and reducing the effects of measurement errors and noise. This technique provides an estimate of the intensity vector using the minimum number of measurement channels.

If two-dimensional wave propagation is assumed then it is necessary to evaluate a larger number of spatial derivatives than in the case of one-dimensional propagation. This necessitates the use of a larger number of measurements, and conditioning becomes more of a problem.
It has been shown that, remote from discontinuities, the motion of a plate can be approximated as the sum of plane waves propagating in different directions. A technique has been proposed in which the wave amplitude, as a function of propagation direction, is approximated as a truncated complex Fourier series. The intensity can then be written in terms of the Fourier coefficients, and furthermore only five coefficients are necessary to determine the intensity at the coordinate origin. The Fourier series coefficients can be estimated in a similar fashion to wave amplitudes on a beam, and the previously described array condition number can therefore be used to compare the conditioning offered by different measurement systems. On the basis of inherent systematic errors and conditioning an array of seven identical transducers, uniformly spaced in a circle and used to evaluate seven Fourier coefficients, is considered the most suitable for the Fourier series approach.

The Fourier series approach has been shown numerically to offer considerable benefits over a finite difference approximation when plane waves are present. Systematic errors are low even at relatively large transducer spacings and thus improved conditioning can be achieved at a lesser cost in terms of theoretical accuracy.

Experimental measurements taken using the proposed wave-based plate intensity measurement techniques have shown all to be viable. It would appear that, under these experimental conditions, the Fourier series approach gives a better intensity estimate than those techniques based on the assumption of one-dimensional wave propagation. These simpler techniques, requiring fewer transducers and lesser computing capacity, have been shown to give relatively good results, however, justifying their use in applications where absolute accuracy is not imperative.

The work described in this thesis illustrates some of the ways in which wave-based techniques may be used to estimate intensity in beams and plates. The principle motivation for this is to achieve a better compromise between conditioning and systematic errors than that offered by those approaches based on a finite difference approximation. By improving the conditioning of the intensity calculation the sensitivity to measurement errors is reduced. The range of
applications where intensity measurements can be taken is thus extended, and the limitations of the available measurement equipment become less critical.
Chapter 10: Suggestions for Further Research

One of the most severe limitations to structural intensity measurement is the performance of currently available transducers and necessary signal processing. With ideal measurement equipment the finite difference approximations proposed by Pavic [2] in 1976 could be implemented without problems. Measurements of flexural wave intensity could then be made on beams and plates within the limits of simple bending theory. It is the fact that the measurement systems are not ideal that inhibits these measurements. As a consequence, two worthwhile directions for further research are therefore the improvement of transducers and signal processing, and the developing of approaches that may reduce sensitivity to measurement error.

The use of piezo-electric strain gauges in conjunction with conventional accelerometers for intensity measurements between closely spaced discontinuities on a beam was shown in this thesis to give promising results. However, it would be necessary to investigate the consistency of gauge properties, together with the possible methods of calibrating such transducer arrays, before their use could be recommended for practical measurements.

The combined measurement of linear and angular acceleration at a point was shown in Chapter 5 to have significant benefits for simple plate intensity measurements. Investigation as to how this might be achieved with adequate accuracy and low noise, without excessive mass or rotational inertia, would make a worthwhile contribution to intensity measurement.

A wave-based approach to intensity measurement in plates may offer still further potential. The Fourier series approximation proposed in this thesis opens up a range of possible measurement systems which could be investigated in conjunction with the evaluation of different numbers of coefficients. In particular, the use of a measurement system composed entirely of strain gauges is
worthy of further investigation. The ability to incorporate the presence of near-fields in wave-based plate intensity measurements would also be a worthwhile goal.

While the measurement of structural intensity is a worthwhile goal, the challenge does not end there. The interpretation of those measurements is not necessarily simple. Frequency domain intensity measurements give a vast quantity of information (an intensity vector for each spectral line at each location) which is difficult to evaluate and display. Investigation of the relationship between spatial and frequency averages could reveal ways in which the number of measurements and quantity of data might be reduced, aiding interpretation.
Chapter 11: References


Appendices
Appendix A:

Finite Difference Approximations for Beam Intensity Measurements

The general expression for flexural wave intensity in a beam may be found by adding equations 3.2.3 to give

\[ \langle P(t) \rangle = EJ \left( \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x \partial t} \right) \]  

A.1

The spatial derivatives may be replaced with the following finite difference approximations,

\[ w = \frac{w_2 + w_3}{2} \quad \frac{\partial w}{\partial x} = \frac{w_3 - w_2}{\Delta} \]

\[ \frac{\partial^2 w}{\partial x^2} = \frac{w_1 - w_2 - w_3 + w_4}{2\Delta^2} \quad \frac{\partial^3 w}{\partial x^3} = \frac{-w_1 + 3w_2 - 3w_3 + w_4}{\Delta^3} \]  

A.2

using the measurement locations shown in Figure A.1, as proposed by Pavic [2]. Note, however, that the numbering of the measurement points is the reverse of that used in reference [2], with the location number increasing with increasing \( x \)-coordinate.

Figure A.1: Transducer arrangement for beam intensity measurement.

If processing is to be performed in the frequency domain the intensity can then be written in terms of the imaginary parts of the cross-spectra of the measurements [3] as
\[ \langle P(t) \rangle_{FD} = \frac{EI\omega}{2\Delta^3} \text{Im}(4w_2w_3^* - w_1w_3^* - w_2w_4^*). \] \hspace{1cm} A.3

It was noted in Chapter 3 that in the absence of near-fields the components of intensity due to the shear force and due to the bending moment are equal. This means that, under far field conditions, it is only necessary to measure one component of intensity (either shear force or bending moment), which can then be doubled to estimate the true intensity. The displacement of a beam under flexural vibration is given in equation 2.2.10 as

\[ w(x,t) = (A^+ e^{-ikx} + A^- e^{ikx} + A_n^+ e^{-kx} + A_n^- e^{kx}) e^{i\omega t}, \] \hspace{1cm} A.4

and in regions remote from discontinuities this reduces to

\[ w(x,t) = (A^+ e^{-ikx} + A^- e^{ikx}) e^{i\omega t}, \] \hspace{1cm} A.5

since the near-field terms (those associated with real exponentials) decay exponentially with distance from the point of origin. We may therefore write

\[ \frac{\partial^2 w}{\partial x^2} = -k^2 w \] \hspace{1cm} A.6

and the component of intensity due to the shear force is

\[ \langle P_s(t) \rangle = EI \left( \frac{\partial}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} \right) \right) \left( \frac{\partial w}{\partial t} \right) \] \hspace{1cm} A.7

\[ = EI \left( -k^2 \frac{\partial w}{\partial x} - i\omega \frac{\partial w}{\partial t} \right) \] \hspace{1cm} A.8

\[ = -\frac{EI\omega k^2}{2} \text{Im}(w^* \frac{\partial w}{\partial x}). \] \hspace{1cm} A.9
The total intensity is therefore

\[ \langle P(t) \rangle_{FF} = 2 \langle P_0(t) \rangle \]

\[ = -EI\omega k^2 \Im(w^* \frac{\partial w}{\partial x}). \]

Using the finite difference approximation given in equation A.2 this becomes

\[ \langle P(t) \rangle_{FD} = \frac{-EI\omega k^2}{\Delta} \Im(w_2^* w_3), \]

so the far field flexural wave intensity in a beam can be estimated from the cross-spectrum of two measurements of displacement or its temporal derivatives.

If far field conditions are present the displacements at the measurement points can be simulated as

\[ w_2 = A^* e^{ikx/2} + A^- e^{-ikx/2} \]

\[ w_3 = A^* e^{-ikx/2} + A^- e^{ikx/2} \]

and substituting these expressions in equation A.12 shows the intensity estimate provided by the finite difference approximation to be

\[ \langle P(t) \rangle_{FD} \approx \frac{EI\omega k^2 \sin(k\Delta)}{\Delta} \left( |A^*|^2 - |A^-|^2 \right). \]

Comparing this to the exact result given by equation 3.2.10 reveals that the relationship between the true intensity and the estimate given by the finite difference approximation is

\[ \frac{\langle P(t) \rangle_{FD}}{\langle P(t) \rangle} = \frac{\sin(k\Delta)}{k\Delta}. \]
It is therefore apparent that the calculation of intensity using equation A.12 results in an underestimation owing to the error in the finite difference approximation.

Equation A.12 is only valid in the far field, so for more general measurement it will be necessary to use equation A.3. It can be shown that if far field conditions are simulated and equation A.3 used to estimate the intensity then the accuracy of the estimate is given by

$$\frac{\langle P(t) \rangle_{FD}}{\langle P(t) \rangle} = \frac{\sin(k\Delta) 2(1 - \cos(k\Delta))}{k\Delta} \frac{k^2}{(k\Delta)^2}. \quad A.16$$

In the same manner, if purely evanescent waves are simulated the accuracy of the finite difference approach is given by

$$\frac{\langle P(t) \rangle_{FD}}{\langle P(t) \rangle} = 8\sinh^3(k\Delta/2)\frac{\cosh(k\Delta/2)}{(k\Delta)^3}. \quad A.17$$

If only one wave type exists then any errors due to the use of a finite difference approximation may be corrected using the appropriate factor. If, however, both propagating and evanescent waves are present then there is no simple correction for finite difference error, and it is then desirable to minimise that error. This may be achieved by using a small transducer spacing, $k\Delta$, since as $k\Delta$ approaches zero equations A.15-A.17 approach unity, indicating no error in the approximation.
Appendix B:

Intensity Estimation from the Combined Measurement of Acceleration and Strain - Finite Difference Approach

It was noted by Pavic [15] that the simplest procedures for estimating flexural wave intensity in beams resulted from the combined use of accelerometers and strain gauges. This is illustrated in the following derivation, however calculations are expressed in terms of displacement, rather than acceleration, for clarity.

If the transducers are placed as shown in Figure B.1, finite difference approximations can be used to express the required spatial derivatives in terms of displacement and strain, giving

\[
\begin{align*}
    w &= \frac{w_1 + w_2}{2}, \\
    \frac{\partial w}{\partial x} &= \frac{w_2 - w_1}{\Delta}, \\
    \frac{\partial^2 w}{\partial x^2} &= \frac{-\left(\varepsilon_1 + \varepsilon_2\right)}{2z}, \\
    \frac{\partial^3 w}{\partial x^3} &= \frac{\left(\varepsilon_1 - \varepsilon_2\right)}{z\Delta},
\end{align*}
\]

B.1

where \( z \) is the separation of the strain gauge from the neutral plane of bending.

![Diagram](Image)

Figure B.1: Transducer placing - combined strain/displacement measurement.

Substituting the spatial derivatives given in equations B.1 into the expression for time-averaged intensity (equation A.1) gives
\[
\langle P(t) \rangle = EI \left\{ \frac{1}{2} \left( \varepsilon_1 - \varepsilon_2 \right) \frac{\partial (w_1 + w_2)}{\partial t} \right\} + \left\{ \frac{1}{\Delta} \left( \varepsilon_1 + \varepsilon_2 \right) \frac{\partial (w_2 - w_1)}{\partial t} \right\},
\]
\[
= -\frac{EI\omega}{2} \text{Im} \left[ \left( \varepsilon_1 - \varepsilon_2 \right)^* \frac{(w_1 + w_2)}{2} + \left( \varepsilon_1 + \varepsilon_2 \right)^* \frac{(w_2 - w_1)}{2\Delta} \right],
\]
\[
= \frac{EI\omega}{2z\Delta} \text{Im} \left( \varepsilon_2^* w_1 - \varepsilon_1^* w_2 \right). 
\]
B.2
Appendix C:

Finite Difference Approximations for Far Field Intensity Measurement in Plates

![Figure C.1: Measurement locations for finite difference approximation - far field intensity.](image)

If displacement is measured at a set of locations as shown in Figure C.1 then the displacement and its spatial derivatives that are required to estimate far field intensity may be approximated as

\[
  w = w_s, \quad \frac{\partial w}{\partial x} \approx \frac{w_{s+1} - w_{s-1}}{2\Delta}, \quad \frac{\partial w}{\partial y} \approx \frac{w_{s+1} - w_{s-1}}{2\Delta}, \quad \frac{\partial^2 w}{\partial x^2} \approx \frac{w_{s+2} - 2w_{s+1} + w_{s+2}}{\Delta^2},
\]

\[
  \frac{\partial^2 w}{\partial y^2} \approx \frac{w_{s+1} - 2w_{s+1} + w_{s-1}}{\Delta^2}, \quad \frac{\partial^2 w}{\partial x \partial y} \approx \frac{w_{s+1} - w_{s+1} - w_{s-1} + w_{s-1}}{4\Delta^2}.
\]  

C.1

If the plate wavenumber, \( k_p \), is known then these are sufficient to calculate the intensity since in the far field the higher order derivatives in the intensity expressions (equations 3.3.4a and b) can be written in terms of the lower orders as

\[
  \frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2} = -k_p^2 \frac{\partial w}{\partial x}, \quad \frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3} = -k_p^2 \frac{\partial w}{\partial y}.
\]  

C.2

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Appendix D

Results of Beam Intensity Measurement - Far Field

The following graphs show the results of the measurements described in Section 7.4 in narrow frequency bands. Power input to the beam apparatus is compared to the energy flow measured assuming far field conditions.

Figure D1: Measurement of input power and energy flow in beam (far field):
          —— input power;  —— measured energy flow.
Figure D2: Measurement of input power and energy flow in beam (far field):
— input power; - - - - measured energy flow.

Figure D3: Measurement of input power and energy flow in beam (far field):
— input power; - - - - measured energy flow.
Figure D4: Measurement of input power and energy flow in beam (far field):
--- input power; - - - - measured energy flow.

Figure D5: Measurement of input power and energy flow in beam (far field):
--- input power; - - - - measured energy flow.
Figure D6: Measurement of input power and energy flow in beam (far field):
--- input power; - - - - measured energy flow.

Figure D7: Measurement of input power and energy flow in beam (far field):
--- input power; - - - - measured energy flow.
Figure D8: Measurement of input power and energy flow in beam (far field):
--------- input power; - - - - measured energy flow.

Figure D9: Measurement of input power and energy flow in beam (far field):
--------- input power; - - - - measured energy flow.
Figure D10: Measurement of input power and energy flow in beam (far field):
— input power; - - - - measured energy flow.

Figure D11: Measurement of input power and energy flow in beam (far field)
— input power; - - - - measured energy flow.
Appendix E

Calculation of Blocking Mass Characteristics

The following analysis follows that proposed by Mace [35]. The block has mass, \( m \), and rotational inertia, \( J \), about the \( y \)-axis. It is assumed that the blocking mass is perfectly rigid, and that no bending of the beam occurs where it is clamped by the blocking mass.

![Diagram of coordinate system and sign conventions for calculation of blocking mass characteristics.](image)

Figure E.1: Coordinate system and sign conventions for calculation of blocking mass characteristics.

If propagating and evanescent waves are incident on the blocking mass from region 1 then the displacement in this region is given by

\[
w_1(x) = A^* e^{-ikt} + A e^{ikt} + A_\nu e^{-\kappa t} + A_\nu^* e^{\kappa t}.
\]  

If there are no waves incident on the blocking mass from region 2 then the displacement in this region is solely the result of waves transmitted by the blocking mass, and is given by

\[
w_2(x') = B^* e^{-ikt'} + B e^{-\kappa t'}.
\]
Since there is no bending of the beam where clamped by the blocking mass the displacements at the edges of the blocking mass are related by

\[ w_2(0) = w_1(0) + d \left( \frac{\partial w_1}{\partial x}(0) \right). \]  

E.3

Substituting equations E.1 and E.2 into equation E.3 gives

\[ B^* + B_N^* = \left( A^* + A^- + A_N^* + A_N^- \right) + d \left( -ikA^* + ikA^- - kA_N^* + kA_N^- \right). \]  

E.4

Furthermore, the slopes at the edges of the blocking mass are equal so

\[ \frac{\partial w_2}{\partial x}(0) = \frac{\partial w_1}{\partial x}(0), \]  

E.5

and if the wave expressions given in equations E.1 and E.2 are substituted equation E.5 becomes

\[ \left( -ikB^* - kB_N^* \right) = \left( -ikA^* + ikA^- - kA_N^* + kA_N^- \right). \]  

E.6

The requirement of translational equilibrium in the z-direction means that

\[ Q_2 - Q_1 = m \frac{\partial^2 w}{\partial t^2}, \]  

E.7

and substituting the expression for shear force given by equations 2.2.5 and 2.2.7 into equation E.7 gives

\[ -EI \left( \frac{\partial^3 w_2}{\partial x^3}(0) - \frac{\partial^3 w_1}{\partial x^3}(0) \right) = \frac{m}{2} \left( \frac{\partial^3 w_1}{\partial t^2}(0) + \frac{\partial^3 w_2}{\partial t^2}(0) \right). \]  

E.8

Substituting equations E.1 and E.2 into equation E.8 and rearranging yields
\[ \frac{-m\omega^2}{2} (B^* + B_N^*) + EI (ik^3 B^* - k^3 B_N^*) = \]
\[ \frac{m\omega^2}{2} (A^* + A_N^* + A_N^-) + EI (ik^3 A^* - ik^3 A^- - k^3 A_N^* + k^3 A_N^-). \]  
E.9

Similarly, the requirement of rotational equilibrium means that

\[ (M_1 - M_2) + \frac{Q_1}{2} \frac{d}{dx} + \frac{Q_2}{2} \frac{d}{dx} = J \frac{\partial^3 w}{\partial x^2 \partial t^2}(0). \]  
E.10

Substituting the expressions for shear force and bending moment given in equations 2.2.5 and 2.2.7 into equation E.10 gives

\[ -EI \left( \frac{\partial^2 w_1}{\partial x^2}(0) - \frac{\partial^2 w_2}{\partial x^2}(0) \right) - \frac{EId}{2} \left( \frac{\partial^3 w_1}{\partial x^3}(0) + \frac{\partial^3 w_2}{\partial x^3}(0) \right) = -\omega^2 J \frac{\partial w_2}{\partial x}(0), \]  
E.11

and substituting equations E.1 and E.2 into equation E.11 and rearranging yields

\[ -\omega^2 J (-ikB^* - kB_N^*) + \frac{EId}{2} (ik^3 B^* - k^3 B_N^*) - EI (-k^2 B^* + k^2 B_N^*) = \]
\[ -EI (-k^2 A^* - k^2 A^- + k^2 A_N^* + k^2 A_N^-) \frac{EId}{2} (ik^3 A^* - ik^3 A^- - k^3 A_N^* + k^3 A_N^-). \]  
E.12

Equations E.4, E.6, E.9, and E.12 may be written in matrix form as

\[ YB = XA, \]  
E.13

where

\[ A^T = (A^*, A_N^*, A^-, A_N^-), \quad B^T = (B^*, B_N^*, 0, 0), \]
\[
X = \begin{bmatrix}
1 - i dk & 1 - dk & 1 + idk & 1 + dk \\
-ik & \frac{-m \omega^2 - iElk^3}{2} & \frac{m \omega^2 - iElk^3}{2} & \frac{m \omega^2 + Elk^3}{2} \\
Eik^2 - \frac{iElk^3}{2} & -Eik^2 + \frac{Elk^3}{2} & Eik^2 + i \frac{Elk^3}{2} & -Eik^2 - \frac{Elk^3}{2}
\end{bmatrix},
\]

\[
Y = \begin{bmatrix}
1 & 1 & 0 & 0 \\
-ik & \frac{-m \omega^2 + iElk^3}{2} & \frac{-m \omega^2 - Elk^3}{2} & 0 \\
\frac{ik \omega^2 J + iElk^3}{2} + Elk^2 & \frac{k \omega^2 J - Elk^3}{2} - Eik^2 & 0 & 0
\end{bmatrix}.
\]

E.14

We may therefore write

\[
A = X^{-1}YB,
\]

E.15

\[
\begin{pmatrix}
A^* \\
A_N^* \\
A^* \\
A_N^*
\end{pmatrix}
= C
\begin{pmatrix}
B^* \\
B_N^* \\
0 \\
0
\end{pmatrix},
\]

E.16

where

\[
C = X^{-1}Y.
\]

E.17

Partitioning matrix C, equation E.16 becomes

\[
\begin{pmatrix}
A^* \\
A_N^* \\
A^* \\
A_N^*
\end{pmatrix}
= \begin{bmatrix} [C_{11}] & [C_{12}] \end{bmatrix}
\begin{pmatrix}
B^* \\
B_N^* \\
0 \\
0
\end{pmatrix},
\]

E.18

and therefore
\[
\begin{pmatrix}
A^+ \\
A^- \\
\end{pmatrix}
= [C_{11}]
\begin{pmatrix}
B^+ \\
B^- \\
\end{pmatrix}
\]  
E.19

\[
\Rightarrow
\begin{pmatrix}
B^+ \\
B^- \\
\end{pmatrix}
= [C_{11}]^{-1}
\begin{pmatrix}
A^+ \\
A^- \\
\end{pmatrix}
. 
E.20
\]

The matrix \([C_{11}]^{-1}\) is thus the transmission coefficient matrix, the individual elements relating the wave amplitudes as follows:

\[
C_{11}^{-1}(1,1) \quad \text{incident propagating - transmitted propagating}
\]

\[
C_{11}^{-1}(1,2) \quad \text{incident near-field - transmitted propagating}
\]

\[
C_{11}^{-1}(2,1) \quad \text{incident propagating - transmitted near-field}
\]

\[
C_{11}^{-1}(2,2) \quad \text{incident near-field - transmitted near-field.}
\]

It is also apparent from equation E.18 that

\[
\begin{pmatrix}
A^+ \\
A^- \\
\end{pmatrix}
= [C_{21}]
\begin{pmatrix}
B^+ \\
B^- \\
\end{pmatrix}
, 
E.21
\]

and substituting equation E.20 into equation E.21 gives

\[
\begin{pmatrix}
A^+ \\
A^- \\
\end{pmatrix}
= [C_{21}][C_{11}]^{-1}
\begin{pmatrix}
A^+ \\
A^- \\
\end{pmatrix}
. 
E.22
\]

The matrix \([C_{21}][C_{11}]^{-1}\) is therefore the matrix of reflection coefficients. Letting

\[
R = [C_{21}][C_{11}]^{-1}
, 
E.23
\]

the individual elements relate the wave amplitudes as follows:
\( R(1,1) \)  incident propagating - reflected propagating
\( R(1,2) \)  incident near-field - reflected propagating
\( R(2,1) \)  incident propagating - reflected near-field
\( R(2,2) \)  incident near-field - reflected near-field.
Appendix F

Results of Beam Intensity Measurement - Single Near-Field

The following graphs show the results of the measurements described in Section 7.5 in narrow frequency bands. Power input to the beam apparatus is compared to the energy flow as measured in the vicinity of the blocking masses when the presence of a single near-field is assumed, and when far field conditions are assumed.

Figure F1: Measurement of input power and energy flow in the presence of a single near-field:

— input power; - - - - measured energy flow (assuming near-field present);

------- measured energy flow (assuming far field conditions).
Figure F2: Measurement of input power and energy flow in the presence of a single near-field:

- --- input power; - - - - measured energy flow (assuming near-field present);
- ----- measured energy flow (assuming far field conditions).

Figure F3: Measurement of input power and energy flow in the presence of a single near-field:

- --- input power; - - - - measured energy flow (assuming near-field present);
- ----- measured energy flow (assuming far field conditions).
Figure F4: Measurement of input power and energy flow in the presence of a single near-field:

- input power;
- - - measured energy flow (assuming near-field present);
- - - - measured energy flow (assuming far field conditions).

Figure F5: Measurement of input power and energy flow in the presence of a single near-field:

- input power;
- - - measured energy flow (assuming near-field present);
- - - - measured energy flow (assuming far field conditions).
Figure F6: Measurement of input power and energy flow in the presence of a single near-field:
— input power; - - - - measured energy flow (assuming near-field present);
- - - - - - measured energy flow (assuming far field conditions).

Figure F7: Measurement of input power and energy flow in the presence of a single near-field:
— input power; - - - - measured energy flow (assuming near-field present);
- - - - - - measured energy flow (assuming far field conditions).
Figure F8: Measurement of input power and energy flow in the presence of a single near-field:
- - - - input power; - - - - measured energy flow (assuming near-field present);
- - - - - - measured energy flow (assuming far field conditions).

Figure F9: Measurement of input power and energy flow in the presence of a single near-field:
- - - - input power; - - - - measured energy flow (assuming near-field present);
- - - - - - measured energy flow (assuming far field conditions).
Figure F10: Measurement of input power and energy flow in the presence of a single near-field:
--- input power; - - - - measured energy flow (assuming near-field present);
............... measured energy flow (assuming far field conditions).

Figure F11: Measurement of input power and energy flow in the presence of a single near-field:
--- input power; - - - - measured energy flow (assuming near-field present);
............... measured energy flow (assuming far field conditions).