Analogical proportions, multivalued dependencies and explanations

Sebastian Link[1] and Henri Prade[2] and Gilles Richard[2]

[1] School of Computer Science, The University of Auckland, Auckland, New Zealand
s.link@auckland.ac.nz
[2] IRIT, CNRS & Université Paul Sabatier, Toulouse, France
{prade, richard}@irit.fr

Abstract. Analogical proportions are statements of the form "a is to b as c is to d". They deal simultaneously with the similarities and differences between items, and they may be considered as a building block of analogical inference. This short paper establishes the existence of a close linkage between analogical proportions and (weak) multivalued dependencies in databases, thus providing an unexpected bridge between two distant areas of research: analogical reasoning and database design. (Weak) multivalued dependencies express a form of contextual logical independence. Besides, analogical proportions, which heavily rely on the comparison of items inside pairs and to the pairing of pairs exhibiting identical changes on attributes, are also a tool for providing adverse example-based explanations. Lastly, it is suggested that this may be applied to a data set reporting decisions in order to detect if some decision is unfair with respect to a sensitive variable (fairness being a matter of independence).

1 Introduction

It is always interesting to discover that the same concept has been introduced independently and for different purposes in two unrelated fields. It is the topic of the present paper where we establish that analogical proportions and analogical inference are at work in (weak) multivalued dependencies.

Analogical proportions are statements of the form a is to b as c is to d that relate two pairs of items on a comparative basis. Analogical proportions have proved to be instrumental in the formalization of analogical inference [13,2]. They started to receive a mathematical formalization about two decades ago [7,20,19,12], although analogical reasoning has long been regarded more as a heuristic way of making a plausible inference on the basis of a parallel between two situations deemed to be analogous. Recently, it has been shown that analogical proportions are a tool for building explanations about the value taken by some attribute of interest using examples and counterexamples.

A multivalued dependency [3] is a constraint between two sets of attributes in a relation in database theory. It states that when such a dependency holds if two tuples are equal on the first set of attributes then there exist two other tuples satisfying some particular constraints involving the two sets of attributes. It expresses that two sets of attributes take values that are logically independent of each other. When multivalued dependencies hold in a database, some redundancy takes place, which can be handled through an appropriate normalisation. In weak multivalued dependencies [6,4,5] the existence of three tuples satisfying some conditions entails the existence of a fourth tuple satisfying some other particular conditions.

The two fields of research, analogical reasoning and databases have in common to deal with data, still in very different perspectives: plausible inference on the one hand and design, updating and querying on the other hand. In the following, after bridging analogical proportions and multivalued dependencies, we discuss some synergy between the explanatory capabilities of analogical proportions and the idea of independence underlying multivalued dependencies, and we apply it to the evaluation of fairness.

The paper is organized as follows. First, a double background on analogical proportions and on multivalued dependencies is given in Section 2 and in Section 3 respectively. Then Section 4 establishes the bridge between the two concepts. Lastly, in Section 5, we make use of analogical proportions for providing adverse example-based explanations, and thanks to the independence semantics, we use it for discussing fairness. In Section 6, concluding remarks suggest lines for further research.

2 Analogical proportions

An analogical proportion (AP) is a statement of the form "a is to b as c is to d", linking four items a, b, c, d. It is denoted by $a : b :: c : d$. APs are supposed to satisfy the following properties:

1. $a : b :: c : d \Rightarrow c : d :: a : b$ (*symmetry*);
2. $a : b :: c : d \Rightarrow a : c :: b : d$ (*central permutation*).

which along with $a : b :: a : b$ (*reflexivity*), are the basic AP postulates (e.g., [14]). These properties mimic the behavior of arithmetic proportions (i.e., $a - b = c - d$) or geometric proportions (i.e., $\frac{a}{b} = \frac{c}{d}$) between numbers. Easy consequences of postulates are i) $a : a :: b : b$ (*identity*); ii) $a : b :: c : d \Rightarrow d : b :: c : a$ (*extreme permutation*); iii) $a : b :: c : d \Rightarrow b : a :: d : c$ (*internal reversal*); iv) $a : b :: c : d \Rightarrow d : c :: b : a$ (*complete reversal*).

The items a, b, c, d considered in the following are tuples of n attribute values. The values of the attributes may be Boolean (binary attributes) or nominal (discrete attributes with finite attribute domains having more than two values). The APs are defined component-wise. We first consider the case of one Boolean attribute applied to four items. Given a binary attribute applied to four items, described by Boolean variables a, b, c, d respectively, the following logical expression has been proposed for an AP [12]:

$$a : b :: c : d = ((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d))$$

This formula expresses that *a differs from b as c differs from d and b differs from a as d differs from c*. It is only true for 6 valuations, namely $0:0::0:0$; $1:1::1:1$; $0:1::0:1$; $1:0::1:0$; $0:0::1:1$; $1:1::0:0$. This is the minimal Boolean model agreeing with the three postulates of an AP [15]. Boolean APs enjoy a code independence property: $a:b::c:d \Rightarrow \neg a:\neg b::\neg c:\neg d$. In other words, encoding truth (resp. falsity) with 1 or with 0 (resp. with 0 and 1) is just a matter of convention, and does not impact the AP.

This easily extends to nominal or categorical values where a, b, c, d belong to a finite attribute domain $\mathcal{A}$. In that case, $a:b::c:d$ holds true only for the three following patterns $(a,b,c,d) \in \{(g,g,g,g),(g,h,g,h),(g,g,h,h)\}, g,h \in \mathcal{A}, g \neq h$. This generalizes the Boolean case where $\mathcal{A}=\{0,1\}$.

In the following, items are represented by tuples of n attribute values: e.g., $a=(a_1,\cdots,a_n)$, where a_i is the value of attribute i for the considered item, APs are defined componentwise:

$$a:b::c:d \text{ holds true if and only if } \forall i \in \{1,\cdots,n\}, a_i:b_i::c_i:d_i \text{ holds true.}$$

In the Boolean (and nominal) case, the equation $a:b::c:x$ where x is unknown does not always have a solution. Indeed neither $0:1::1:x$ nor $1:0::0:x$ have a solution (since 0111, 0110, 1000, 1001 are not valid patterns for an AP). Similarly, $g:h::h:x$ has no solution in the nominal case when $g \neq h$. The Boolean solution exists if and only if $(a \equiv b) \vee (a \equiv c)$ is true. If the solution exists, it is unique and given by $x=c\equiv(a\equiv b)$. In the nominal case, the solution exists (and is unique) if $a=b$ (then $d=c$) or if $a=c$ (then $d=b$) [13].

Table 1 provides an example of an AP with nominal attributes (with a database flavor!). Note that, assuming that the AP $a:b::c:d$ is true, one can indeed recalculate d from a, b, c. This corresponds to the case of a weak multivalued dependency as we shall see in the next section.

Table 1: AP: example with nominal attributes

	course	*teacher*	*time*
a	Maths	Peter	8 am
b	Maths	Peter	2 pm
c	Maths	Mary	8 am
d	Maths	Mary	2 pm

More generally, analogical inference amounts to an analogical jump stating that if an AP holds between four items for n attributes, an AP may also hold for an attribute $n+1$ (see [2] for the relation with the analogical jump: from $P(x), Q(x), P(y)$ infer $Q(y)$):

$$\frac{\forall i \in \{1,...,n\},\quad a_i:b_i::c_i:d_i \text{ holds}}{a_{n+1}:b_{n+1}::c_{n+1}:d_{n+1} \text{ holds}}$$

If a_{n+1}, b_{n+1}, c_{n+1} are known, this enables the prediction of d_{n+1}, provided that $a_{n+1}:b_{n+1}::c_{n+1}:x$ is solvable. This is the basis for analogical proportion-based classification [2].

3 Multivalued dependencies

In the following, we use standard database notations. Let R be a relation schema viewed as a set of attributes; X and Y denote subsets of attributes. A tuple t is a complete instantiation of the attributes in R describing some existing item. A relation $\mathtt{r}$ over R is a finite set of tuples over R. The restriction of a tuple t to the attributes in $X \subseteq R$ is denoted by $t[X]$. $t[XY]$ is short for $t[X \cup Y]$.

Functional dependencies and multivalued dependencies play an important role in the design of databases. A functional dependency $X \rightarrow Y$ ($X \subseteq R$ and $Y \subseteq R$) states that for any pair of tuples t_1 and t_2 obeying the relational schema R, if $t_1[X] = t_2[X]$ then $t_1[Y] = t_2[Y]$, which reads "X determines Y".

Departing from a functional dependency, the definition of a multivalued dependency requires the existence of particular tuples in the database, under some conditions: The *multivalued dependency* [3,1] (see also [5]) $X \twoheadrightarrow Y$ (which can be read as "X multidetermines Y") holds on R if, for all pairs of tuples t_1 and t_2 in $\mathtt{r}$ such that $t_1[X] = t_2[X]$, there exists some tuple t_3 in $\mathtt{r}$ such that $t_3[XY] = t_1[XY]$ and $t_3[X(R \setminus Y)] = t_2[X(R \setminus Y)]$. Note that, as a consequence of the definition there also exists a tuple t_4 in $\mathtt{r}$ such as $t_4[XY] = t_2[XY]$ and $t_4[X(R \setminus Y)] = t_1[X(R \setminus Y)]$ (swapping the roles of t_1 and t_2).

Thus altogether, when $X \twoheadrightarrow Y$ holds, for all pairs of tuples t_1 and t_2 in $\mathtt{r}$ such that $t_1[X] = t_2[X]$, there exist tuples t_3 and t_4 in $\mathtt{r}$ such that

- $t_1[X] = t_2[X] = t_3[X] = t_4[X]$
- $t_1[Y] = t_3[Y]$
- $t_2[Y] = t_4[Y]$
- $t_1[R \setminus (X \cup Y)] = t_4[R \setminus (X \cup Y)]$
- $t_2[R \setminus (X \cup Y)] = t_3[R \setminus (X \cup Y)]$

A more simple, equivalent version of the above conditions can be expressed as follows: if we denote by (x, y, z) the tuple having values x, y, z for subsets X, Y, $R \setminus (X \cup Y)$ respectively, then whenever the tuples (p, q, r) and (p, s, u) exist in $\mathtt{r}$, the tuples (p, q, u) and (p, s, r) should also exist in $\mathtt{r}$. Note that in the definition of $X \twoheadrightarrow Y$, not only the attributes in X and in Y are involved, but also those in $R \setminus (X \cup Y)$, which departs from functional dependencies (where only the attributes in X and in Y are involved).

A multivalued dependency $X \twoheadrightarrow Y$ is trivial if Y is a subset of X, or if $X \cup Y$ is the whole set of attributes of the relation (then $R \setminus (X \cup Y)$ is empty).

In Table 2, the two multivalued dependencies $\{\text{course}\} \twoheadrightarrow \{\text{teacher}\}$ and $\{\text{course}\} \twoheadrightarrow \{\text{time}\}$ hold, as can be checked.

Note that Table 2 can be rewritten more compactly as in Table 3. This acknowledges the fact that $r = \{Maths\} \times \{Peter, Mary, Paul\} \times \{8am, 2pm\} \cup \{Comp.Sci.\} \times \{Peter, Mary\} \times \{8am\}$. Indeed the teachers attached to the course and the time attached to the course are *logically independent* of each other. Indeed, a multivalued dependency exists in a relation when there are at least three attributes, say X, Y and Z, and for a value of X there is a defined set of values of Y and a defined set of values of Z. Then, the set of values of Y is *independent* of set Z and vice versa.

Table 2: Multivalued dependencies: {course} $\twoheadrightarrow$ {teacher}; {course} $\twoheadrightarrow$ {time}

course	*teacher*	*time*
Maths	Peter	8 am
Maths	Peter	2 pm
Maths	Mary	8 am
Maths	Mary	2 pm
Maths	Paul	8 am
Maths	Paul	2 pm
Comp. Sci.	Peter	8 am
Comp. Sci.	Mary	8 am

Table 3: Compact writing of Table 2

course	*teacher*	*time*
Maths	{Peter, Mary, Paul }	{8 am, 2 pm }
Comp. Sci.	{Peter, Mary}	{8 am}

Moreover the following properties holds:
If $X \rightarrow Y$, then $X \twoheadrightarrow Y$.
If $X \twoheadrightarrow Y$, then $X \twoheadrightarrow R \setminus Y$
If $X \twoheadrightarrow Y$ and $Z \subseteq U$, then $XU \twoheadrightarrow YZ$
If $X \twoheadrightarrow Y$ and $Y \twoheadrightarrow Z$, then $X \twoheadrightarrow Z \setminus Y$.

Multivalued dependencies are of interest in databases since decomposition of a relation R into (X, Y) and $(X, R \setminus Y)$ is a lossless-join decomposition if and only if $X \twoheadrightarrow Y$ holds in R. Multivalued dependencies are involved in the 4th normal form in database normalization.

A multivalued dependency, given two particular tuples, requires the existence of other tuples. Its weak form only requires the existence of one tuple given three particular tuples.

A *weak multivalued dependency* [4] $X \twoheadrightarrow_w Y$ holds on R if, for all tuples t_1, t_2, t_3 in $\mathtt{r}$ such that $t_1[XY] = t_2[XY]$ and $t_1[X(R \setminus Y)] = t_3[X(R \setminus Y)]$ there is some tuple t_4 in $\mathtt{r}$ such that $t_4[XY] = t_3[XY]$ and $t_4[X(R \setminus Y)] = t_2[X(R \setminus Y)]$. It can be checked that if $X \twoheadrightarrow Y$ then $X \twoheadrightarrow_w Y$.

The existence of the weakmultivalued dependency $X \twoheadrightarrow_w Y \ / \ Z$ is sufficient for ensuring the commutativity of Y and Z in the nesting process that enables us to rewrite Table 2 into Table 3 [5].

4 Analogical proportion & multivalued dependency:the link

Let us go back to analogical proportions. As for multivalued dependencies they involved four tuples, which are taken by pairs. Let us consider the Boolean case first. When one considers a pair of tuples (a, b), one can distinguish between the attributes where the two tuples are equal and the attributes where the two tuples disagree. If we take two pairs (a, b) and (c, d) whose tuples are equal on the same attributes and which disagree *in the same way* on the other attributes (when $(a_i, b_i) = (1, 0)$ (resp. $(0, 1)$), $(c_i, d_i) = (1, 0)$ (resp. $(0, 1)$)), these two pairs form an AP; see Table 4 where attributes A_1 to A_n have been suitably ordered and where all the possible situations are exhibited. As can be seen, we recognize the six valuations, vertically, which make a Boolean AP true. Conversely, any AP can be put under this form (with possibly some empty columns) [18].

Table 4: AP: Pairing pairs

	A_1-A_{i-1}	A_i-A_{j-1}	A_j-A_{k-1}	A_k-$A_{\ell-1}$	A_ℓ-A_{m-1}	A_m-A_n
a	1	0	1	0	1	0
b	1	0	1	0	0	1
c	1	0	0	1	1	0
d	1	0	0	1	0	1

This can be easily generalized to nominal attributes, as shown in Table 5, where a, b, c, d are equal on the subset of attributes X, where $a = b \neq c = d$ on the subset of attributes Y, and where the same change take place between a and b and between c and d for attributes in Z. Note that by central permutation, we can exchange the roles of Y and Z.

Table 5: AP: the nominal case

	X (full identity)	Y (pair identity)	Z (change)
a	s	t	v
b	s	t	w
c	s	u	v
d	s	u	w

Let us now first examine the weak multivalued dependency: for all tuples t_1, t_2, t_3 in $\mathtt{r}$ such that $t_1[XY] = t_2[XY]$ and $t_1[X(R \setminus Y)] = t_3[X(R \setminus Y)]$ there is some tuple t_4 in $\mathtt{r}$ such that $t_4[XY] = t_3[XY]$ and $t_4[X(R \setminus Y)] = t_2[X(R \setminus Y)]$.

Then if $t_1[XY] = t_2[XY] = (s, t)$ and $t_1[X(R \setminus Y)] = t_3[X(R \setminus Y)] = (s, v)$ there exists a tuple t_4 in $\mathtt{r}$ such that $t_4[XY] = t_3[XY] = (s, u)$ and $t_4[X(R \setminus Y)] = t_2[X(R \setminus Y)] = (s, w)$. We recognize Table 5 with $t_1 = a$, $t_2 = b$, $t_3 = c$, $t_4 = d$. Thus there is a perfect match between a weak multivalued dependency and an analogical proportion. In fact, the existence of t_4 in $\mathtt{r}$ amounts to the existence of a (unique) solution for $a : b :: c : x$ in Table 5.

The case of a multi-valued dependency is slightly different, as we are going to see. Indeed $X \twoheadrightarrow Y$ holds as soon as whenever the tuples (p, q, r) and (p, s, u) exist in $\mathtt{r}$ on subsets X, Y, $R \setminus (X \cup Y)$, the tuples (p, q, u) and (p, s, r) also exist in $\mathtt{r}$ on subsets X, Y, $R \setminus (X \cup Y)$. This corresponds to Table 6 where (r, u, u,

r) is not a valid valuation for an AP. So $t_1 : t_2 :: t_3 : t_4$ does not hold (in fact, this corresponds to another logical proportion called "paralogy" [14]).

Table 6: Multivalued dependency and the failure of the AP

	X	Y	$R \setminus (X \cup Y)$
t_1	p	q	r
t_2	p	s	u
t_3	p	q	u
t_4	p	s	r

Fortunately, it is possible to reorder the tuples for obtaining a valid AP. Indeed $t_1 : t_4 :: t_3 : t_2$ does hold. See Table 7.

Table 7: Multivalued dependency and the AP recovered

	X	Y	$R \setminus (X \cup Y)$
t_1	p	q	r
t_4	p	s	r
t_3	p	q	u
t_2	p	s	u

5 Explanations and fairness

We now go back to Table 5 which describes what is an AP in the nominal case. Moreover, we have singled out a (nominal) attribute called *Result*, supposed to depend on the other attributes, it may be the class to which the tuple belongs, or the result of an evaluation / selection for each tuple. We have also identified the roles plaid by each subset of attributes: attributes X having the same values for the four tuples, attributes Y stating the different contexts of pairs (a, b) and (c, d), attributes Z describing the change(s) inside the pairs, which may be associated or not with a change on the value of *Result*. This is Table 8.

Table 8: Results associated with tuples

	X (shared values)	Y (context)	Z (change)	Result
a	s	t	v	p
b	s	t	w	q
c	s	u	v	p
d	s	u	w	?

We recognize the schema of analogical inference in Table 8 ($p : q :: p : x$ always has a (unique) solution $x = q$; see the end of Section 2; we do not consider the case $p : p :: q : x$ which can be obtained by central permutation, exchanging Y and Z). We leave aside the case $p = q$, which suggests that the rule $X = s \to Result = p$ may hold, and even that a functional dependency $X \to Result$ might hold. Referring to notations in Table 8, if for all a, b, c in $\mathtt{r}$

there exists d in $\mathtt{r}$, $X \twoheadrightarrow_w Y$ and $X \twoheadrightarrow_w Z$ hold in $\mathtt{r}$; if for all a, d in $\mathtt{r}$ there exist b, c in $\mathtt{r}$, $X \twoheadrightarrow Y$ and $X \twoheadrightarrow Z$ hold in $\mathtt{r}$. It is also true changing Z in *Result*.

Table 8 is also a basis for presenting analogical proportion-based explanations [17,8]. Indeed the answer to the question "why $Result(d)$ is not p?" is to be found in the values taken by the *change* attributes for d. Note that when c is a close neighbor of d, the number of *change* attributes is small. This looks like the definition of a *contrastive* explanation [10], namely we have

$$\exists x = c \in r.\ [\bigwedge_{j \in R \setminus change} (x_j = c_j = d_j)] \wedge (Result(d) \neq p)$$

where c_j is the value of tuple c for attribute j. Such a c could be termed as an *adverse example*. But the analogy-based explanation is richer, we know at least another pair (here (a, b)), with another *context* value, where the same change of attribute values leads to the same change of *Result* value as in pair (c, d), which suggests the possibility of the following rule (with an abductive flavor)

$$\forall\ t, (context = t) \wedge (change = w) \rightarrow Result((s, t, w)) = q$$

However, nothing forbids that $\exists\ a' \in r$, $\exists\ b' \in r$ such that $a' = (s, t', v)$, $b' = (s, t', w)$ with $Result(a') = Result(b') = p$, which would provide an exception to the rule. The strength of the explanation would depend on the relative cardinalities of pairs such as (a, b) and (a', b').

Table 9: Is *Result* for d fair (and if no, why)?

	X (shared values)	Y (diploma)	Z (sex)	Result
a	s	*yes*	M	P
b	s	*yes*	F	P
c	s	*no*	M	N
d	s	*no*	F	P

Estimating fairness is a matter of conditional stochastic independence [11]. However we have seen that multi-valued dependencies and thus analogical proportions exhibit logical independence relations. Thus the violation of an AP (and thus of a multivalued dependency) in Table 9 suggests that the value of $Result(d)$ is unfair.

6 Concluding remarks

The link established in this paper between analogical proportions and multivalued dependencies should lead to further developments, besides explanation and fairness. One may wonder if the axiomatic characterization of dependencies may bring some new light on analogical proportions. Other questions worth of interest are: What might be the impact on explanation capabilities [8] on analogical querying [16]? Can we handle uncertain data with analogical proportions, as in database design [9]?

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