Distinguished curves and submanifolds in conformal geometry

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Abstract

Conformal geometry is a weakening of Riemannian geometry where one works with a smooth manifold equipped with an equivalence class of Riemannian metrics, where two metrics are equivalent if and only if they define the same angles between curves. Early interest in conformal equivalence included the question of biholomorphic equivalence of domains in the complex plane. Interest has also been driven by physics and general relativity, since light in spacetime follows null geodesics, and these only depend on the conformal structure. Recently there has been considerable progress in the study of codimension one submanifolds in conformal manifolds. At the other extreme, there has also been an increased understanding of the distinguished curves in conformal manifolds. In this thesis, we develop a complete basic tractor theory of conformal submanifolds of any codimension and use this to define a notion of distinguished conformal submanifolds. These distinguished submanifolds coincide with conformal circles and totally umbilic hypersurfaces in the extremal cases. We emphasize three conformal tractor objects which we show encode equivalent submanifold data. Our notion of distinguished submanifolds admits characterizations in terms of all three invariants. Our definition immediately leads to a procedure for proliferation of conserved quantities along these submanifolds. We also obtain a theorem which characterizes our distinguished conformal submanifolds in terms of an incidence relation and a parallel condition. We use this to show that zero loci of certain solutions to a conformally invariant equation are, if nonempty, distinguished submanifolds. These results extend existing results for conformal circles.

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Chapter 1

Introduction

This thesis has two main goals. Primarily, we seek to propose a notion of distinguished submanifolds in a conformal manifold which generalizes the two well-studied cases of conformal circles and totally totally umbilic hypersurfaces. We motivate this notion by showing that our proposed definition captures these two known classes of distinguished conformal submanifolds as extremal cases while simultaneously extending this notion to submanifolds of all codimension. Along the way we accomplish our second goal: the development and presentation of tractor calculus machinery for treating submanifolds in conformal submanifolds. While much of this theory is already known, our treatment focuses on three invariant tractor objects which contain equivalent information about the embedding. Previous treatments have not closely examined these objects and the relations between them, and we feel that this exactly provides the right framework for unifying and extending the aforementioned classes of conformal submanifolds. We note that this approach also has consequences in Riemannian geometry, since all these objects have Riemannian analogs, and moreover that they encode equivalent information follows from a Gauß-type equation and so is not limited to the conformal setting. We hope that the tools and machinery developed here will be of use to those studying further questions in conformal geometry and other related fields.

The study of sub-objects has repeatedly been a fruitful avenue in all areas of mathematics. Thus it is natural to study submanifolds in differential, Riemannian and conformal geometry. Riemannian submanifold theory is very well studied and one has many beautiful and useful results. As a famous example, a theorem of Nash states that for any Riemannian manifold (M, g) there is an embedding $F : M \to \mathbb{R}^N$ such that the pullback by F of the standard Euclidean metric is the Riemannian metric g, where the dimension N depends on $\dim M$ [55]. Namely, *any* Riemannian manifold may be regarded as a submanifold of a Euclidean space of sufficiently high dimension.

The main setting for this thesis will be that of conformal geometry. In conformal geometry, one relaxes the Riemannian condition to only requiring that angles between curves, and not lengths of curves, are defined. From the usual formula for angles defined in terms of a metric, we see that now one only has a metric *up to scale*. Consequently we may think of such as structure as being equipped with an equivalence class of metrics.

Conformal manifolds are examples of *parabolic geometries*, so named since they are curved versions of homogeneous spaces G/P where G is a semisimple Lie group and P is a parabolic subgroup. Projective geometry, Grassmanian structures, CR geometry and Lagrangean contact geometries are also examples of parabolic geometries for appropriate choices of the Lie group G and parabolic subgroup P. There is a rich representation theory of parabolic subgroups and subalgebras, and this turns out to strongly govern the possible behavior of these geometries, even in the curved setting. We will not explicitly require much of this representation theory, although its techniques can be used to prove in a general way many results which were previously proven in the setting of one specific class of parabolic geometry.

The tangent bundle of a conformal manifold does not admit a distinguished connection. Thus one does not immediately have an invariant machinery to approach questions in conformal geometry. One naïve method is to compute transformation laws under conformal rescalings, and try to identify invariants this way. We compute some basic transformation laws in Chapter 2, and even from this small selection one sees that the combinatorial explosion from iterated derivatives soon renders this approach intractable. One might instead hope for a calculus which resembles the familiar tensor calculus, and which builds conformal invariance into its construction. Luckily, it turns out that such a calculus does exist, and the situation is almost as nice as one could hope for. This elegant solution which was first introduced using modern terminology by Bailey, Eastwood and Gover in [4] is to show that a conformal manifold M of dimension n admits a distinguished connection on a vector bundle of rank $n+2$. This is the so-called *tractor bundle*, a portmanteau the first part of which comes from the name of Tracey Thomas (whose earlier works [65, 68, 67, 66] touched on many of these ideas and inspired the subsequent (re)discovery) and whose second part is named in the same style as vector, tensor, spinor, etc. The theory of tractors is also closely related to the work of Cartan (e.g. [23, 21, 22] or [60] for a modern summary of these ideas) and his notion of an *espace généralisé*, nowadays usually called a *Cartan geometry*. Indeed, the tractor (vector) bundles are certain associated bundles to the principal bundle of the Cartan geometry. The tractor theory aesthetically has much in common with the Riemannian theory. Namely, one has a distinguished connection on a vector bundle which preserves a non-degenerate metric. Thus one has a calculus for stating and solving problems in conformal geometry in a way that naturally builds in conformal invariance: any property stated in terms of tractors and the tractor connection is immediately a well-defined conformal property.

Distinguished curves have long been understood to be an important feature of any geometry. Using the Riemannian metric, there is a well-defined notion of the length of a curve

$\gamma : I \to M$. This is just the usual length formula for curves in Euclidean space but with the Riemannian metric replacing the Euclidean inner product. Thus one is naturally led to consider the curves joining two points which have minimal length among all such curves. These curves are called *geodesics*, and they are the distinguished curves of Riemannian geometry. Geodesics are also crucial to one characterization of Riemannian distinguished submanifolds: any geodesic of a totally geodesic submanifold is automatically a geodesic of the ambient Riemannian manifold.

There is a fairly comprehensive theory of hypersurfaces in conformal manifolds [4, 9, 62, 69]. This was then later expanded to include higher codimension submanifolds. Notable contributions in this area were made by Calderbank and Burstall [11], and in the PhD thesis of Curry [29]. The former is presented in a very different language from what we use. The latter closely aligns with our current treatment, and was an important source of inspiration for some ideas in this thesis. Despite this theory, there is not a generally agreed upon notion of distinguished submanifolds for general conformal manifolds in the same way that one has totally geodesic submanifolds in Riemannian and therefore also projective geometry. A totally geodesic submanifold may be characterized by several equivalent properties. We provide similar characterizations, in terms of conformal circles, for two distinct possible notions of distinguished conformal submanifold.

Importantly, one can construct scalar functions which are constant along geodesics. Conserved quantities are of interest to mathematicians and physicists in areas including dynamical systems, quantum mechanics and the Kerr, Kerr-NUT-(A)ds and Plebański-Demiański metrics and related questions of stability of black holes [1, 24, 35]. For example, given a sufficient number of first integrals along a curve, the trajectory of the curve is completely determined. In an extreme case, that of *superintegrable* geometries, the number of such first integrals along a curve exceeds the dimension of the ambient manifold [49].

The classical method of producing a conserved quantity along a curve is to pair the velocity field of the geodesic with a *Killing vector*. These are vector fields whose flows are continuous isometries of the Riemannian manifold. Formally, this is captured via the Lie derivative. If g is the metric of a Riemannian manifold, then X is a *Killing field* if it satisfies

$$\mathcal{L}_X g = 0, \tag{1.0.1}$$

which is called the *Killing equation*. This equation is not conformally invariant, and so not a good candidate for proliferating conserved quantities along conformal distinguished curves, but if one instead asks merely that $\mathcal{L}_X g$ is proportional to g, then this equation is indeed conformally invariant. A vector field X with this property is called a *conformal Killing field*.

Throughout this thesis, we employ Penrose's *abstract index notation* [58], which we describe briefly. We denote the tangent resp. cotangent bundle $\mathcal{E}^a$ resp. $\mathcal{E}_a$, and higher rank tensors are then denoted using multiple Latin letters. So ω_{ab} is a section of $\mathcal{E}_{ab} =$

$T^*M \otimes T^*M$, $T_a{}^b$ is a section of $\mathcal{E}_a \otimes \mathcal{E}^b = T^*M \otimes TM$ i.e. an endomorphism of TM and so on. Additionally, usual parentheses around indices denote symmetrization, and square brackets denote antisymmetrization. So, for example, for T_{ab} a section of $T^*M \otimes T^*M$, one has

$$T_{(ab)} = \frac{1}{2}\left(T_{ab} + T_{ba}\right),$$

which is a section of S^2T^*M, and

$$T_{[ab]} = \frac{1}{2}\left(T_{ab} - T_{ba}\right)$$

which is a section of $\Lambda^2 T^*M$. We will discuss this in more detail in Section 1.1. We may also sometimes abuse notation and use the same symbol for a bundle and its space of sections, writing, for example, $\omega_a \in \mathcal{E}_a$ when really we mean $\omega_a \in \Gamma(\mathcal{E}_a)$. We will also use abstract indices for the tensor calculus of submanifolds and for the tractor calculus. We use different sets of indices in each setting, so it should always be clear which bundles are involved in any situation. We will explain these conventions as we introduce the relevant settings.

As a more concrete example of this notation, using the properties of the Levi-Civita connection, it can be shown that (1.0.1) is equivalent to

$$\nabla_{(a}X_{b)} = 0, \tag{1.0.2}$$

where we have used the Riemannian metric to identify the vector field X with a 1-form (we have "lowered the index"). The conformally invariant *conformal Killing equation*

$$\mathcal{L}_X g = \lambda g, \tag{1.0.3}$$

where λ is a smooth function, is similarly equivalent to

$$\nabla_{(a}X_{b)_0} = 0, \tag{1.0.4}$$

where the subscript zero denotes the trace-free part. Such invariant geometric PDEs will play a very important role in our theory of conserved quantities. They provide a class of generalized symmetries which yield conserved quantities when contracted with suitable fields along a distinguished curve much as in the Riemannian manifold case.

We outline the structure of this thesis and the contents of each chapter. The remainder of Chapter 1 is devoted to establishing conventions for notation, linear algebra and differential geometry. We make extensive use of differential forms in this thesis, so we dedicate some time to them in particular.

Chapter 2 introduces conformal geometry. We calculate the transformations under a conformal rescaling of many of differential geometric objects introduced in Chapter 1 before introducing our main tool for studying conformal manifolds: the tractor calculus. We briefly discuss the Cartan-geometric view of conformal geometry and how this treatment is equivalent to the tractor bundle picture. Finally for this chapter we discuss conformally singular geometries. These are structures consist of a conformal manifold together with some additional distinguished object which has some singularity set.

In Chapter 3, we introduce the first well-known class of distinguished conformal submanifolds: conformal circles. We consider some history of conformal circles before reviewing our treatment of conformal circles from [39]. This is a tractor characterization of conformal circles which provides a large motivation for our later theory of more general distinguished conformal submanifolds.

Chapter 4 provides a complete basic treatment of submanifolds in conformal manifolds. One main purpose of this chapter is to introduce three submanifold tractor invariants which all contain equivalent information which we will later use to characterize our notion of *distinguished submanifolds*. We begin with the important notions of classical submanifolds in Riemannian manifolds, covering the orthogonal decomposition of the ambient tangent bundle into a subbundle isomorphic to the intrinsic tangent bundle and a normal bundle, the compatible decomposition of the ambient Levi-Civita connection which is given by the Gauß formula, and the Gauß, Codazzi, and Ricci equations. From there we give a tractor theory which follows essentially the same order. One has a tractor analog of the ambient tractor bundle decomposition and hence a tractor Gauß formula which defines a tractor second fundamental form. This tractor second fundamental form is the first of the aforementioned central objects in our treatment of conformal submanifolds. The decomposition of the ambient tractor bundle and the tractor Gauß formula immediately allow one to deduce tractor analogs of the Gauß, Codazzi and Ricci equations by arguing formally in the same way as for the Riemannian case. The decomposition of the ambient tractor bundle also has associated orthogonal projections, and the normal projector is the second special submanifold tractor invariant. (We could equally include the tangential projector as well, but its data is clearly equivalent to that of the normal projector and so we simply take one of these two.) We finish this chapter with a treatment of tractor differential forms. A conformal submanifold has a *tractor normal form* which is a tractor-valued differential form with rank equal to the dimension of the normal tractor bundle. This form is the third of the submanifold invariants, and we see that, as with the other two, there are explicit relations between it and the others. This chapter lays the foundation for our subsequent applications. The main results are Lemma 4.2.6, which relates the normal tractor projector and the tractor second fundamental form, Proposition 4.4.2 which relates the normal projector and the tractor normal form, and Theorem 4.4.5 which relates the tractor normal form and the tractor second fundamental form.

Chapter 5 is the first of our two applications chapters. We first revisit the subject of conformal circles, and show that a conformally circle may equally be described as a 1-dimensional submanifold whose tractor second fundamental form vanishes. This, together with a similar observation concerning totally umbilic hypersurfaces, provides motivation for our general definition of a distinguished conformal submanifold. A key result here is Theorem 5.2.1, which follows immediately from the main results of the previous chapter, and which shows that our three objects of interest encode equivalent data. We then consider some alternative ways that one could define distinguished for conformal submanifolds. These are phrased in terms of distinguished curves and come from considering characterizations of total geodesicity in Riemannian geometry.

In Chapter 6, our final chapter, we show some applications of our general theory of distinguished conformal submanifolds. These include conserved quantities, distinguished submanifolds as zero loci, and an incidence relation characterization which generalizes a result on conformal circles.

All manifolds and maps we work with are assumed to be smooth i.e. of class C^∞. We also assume that all manifolds M are oriented, although we will sometimes mention a modification that extends some result or construction to the non-oriented setting.

1.1 Linear algebra conventions

Throughout our work, we make extensive use of differential forms. We explicitly describe our notation and conventions here.

Definition 1.1.1 (Wedge product)**.** Let V be an inner product space, and let $\alpha \in \Lambda^p V$ and $\beta \in \Lambda^q V$ be a p- and q-form respectively. Then the $p+q-$form $\alpha \wedge \beta \in \Lambda^{p+q} V$ is defined by

$$\alpha \wedge \beta := \frac{(p+q)!}{p! \cdot q!} \operatorname{Alt}(\alpha \otimes \beta), \tag{1.1.1}$$

where

$$\operatorname{Alt}(\alpha)(x_1, \ldots, x_p) := \frac{1}{p!} \sum_{\sigma \in \mathfrak{S}_p} \operatorname{sgn}(\sigma) \alpha(x_{\sigma(1)}, \ldots, x_{\sigma(p)}). \tag{1.1.2}$$

The factor is chosen so that we have the following.

Proposition 1.1.2. *Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis for the inner product space V and let $\tau \in \mathfrak{S}_n$ be a fixed permutation. Then*

$$(v_1 \wedge \cdots \wedge v_n)(v_{\tau(1)}, \ldots, v_{\tau(n)}) = \operatorname{sgn}(\tau). \tag{1.1.3}$$

Proof. We prove this by induction. The base case $n = 1$ is trivial since our basis is assumed orthonormal and the only permutation is the identify. For the general case, let

$\omega := v_1 \wedge \cdots \wedge v_{n-1}$. Then

$$\begin{aligned}(\omega \wedge v_n)\left(v_{\tau(1)}, \ldots, v_{\tau(n-1)}, v_{\tau(n)}\right) &= \frac{[(n-1)+1]!}{(n-1)! \cdot 1!} \operatorname{Alt}(\omega \otimes v_n)\left(v_{\tau(1)}, \ldots, v_{\tau(n-1)}, v_{\tau(n)}\right) \\ &= \frac{n!}{(n-1)!} \cdot \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)(\omega \otimes v_n)\left(v_{\sigma(\tau(1))}, \ldots, v_{\sigma(\tau(n-1))}, v_{\sigma(\tau(n))}\right) \\ &= \frac{1}{(n-1)!} \sum_{\substack{\sigma \in \mathfrak{S}_n, \\ \sigma(n)=n}} \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}\right) \cdot v_n(v_n) \\ &= \frac{1}{(n-1)!} \sum_{\sigma \in \mathfrak{S}_{n-1}} \operatorname{sgn}(\tau) \operatorname{sgn}(\sigma) \omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(n-1)}\right) \\ &= \frac{1}{(n-1)!} \cdot \operatorname{sgn}(\tau) \sum_{\sigma \in \mathfrak{S}_{n-1}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\sigma) \\ &= \operatorname{sgn}(\tau).\end{aligned}$$

□

In particular, for the orthonormal basis $\{v_1, \ldots, v_n\}$, one has

$$(v_1 \wedge \cdots \wedge v_n)(v_1, \ldots, v_n) = 1. \tag{1.1.4}$$

In the abstract index notation, we denote antisymmetrization by enclosing indices in square brackets. Following Wald [70], our convention will be that

$$T_{[a_1 a_2 \cdots a_n]} := \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) T_{a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}}. \tag{1.1.5}$$

This is equal to $\operatorname{Alt}(T)$ defined in (1.1.2), namely

$$T_{[a_1 a_2 \cdots a_n]} v_1^{a_1} v_2^{a_2} \cdots v_n^{a_n} = \operatorname{Alt}(T)(v_1, v_2 \ldots, v_n).$$

The following is extremely useful when doing computations using forms.

Proposition 1.1.3. *The map* $\operatorname{Alt} : \otimes^* V \to \otimes^* V$ *is a projection onto the subspace* $\Lambda^* V$*, where* $\otimes^* V$ *and* $\Lambda^* V$ *denote the tensor and exterior algebras respectively.*

Proof. Clearly $\operatorname{im}(\operatorname{Alt}) \subset \Lambda^* V$. Let $T \in \Lambda^p V$. We need to show that $\operatorname{Alt}((\operatorname{Alt}(T)) = \operatorname{Alt}(T)$. Calculating

$$
\begin{aligned}
&\operatorname{Alt}\left((\operatorname{Alt}(T))\left((x_1,\dots,x_p\right)\right.\\
&= \frac{1}{p!}\sum_{\tau\in\mathfrak{S}_p}\operatorname{sgn}(\tau)(\operatorname{Alt}T)(x_{\tau(1)},\dots,x_{\tau(p)})\\
&= \frac{1}{p!}\sum_{\tau\in\mathfrak{S}_p}\left(\operatorname{sgn}(\tau)\frac{1}{p!}\sum_{\sigma\in\mathfrak{S}_p}\operatorname{sgn}(\sigma)T\left((x_{\sigma(\tau(1))},\dots,x_{\sigma(\tau(p))}\right)\right)\\
&= \frac{1}{(p!)^2}\sum_{\tau\in\mathfrak{S}_p}\left(\operatorname{sgn}(\tau)\sum_{\sigma\in\mathfrak{S}_p}\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau)T\left(x_{\sigma(1)},\dots,x_{\sigma(p)}\right)\right)\\
&= \frac{1}{(p!)^2}\sum_{\tau\in\mathfrak{S}_p}\sum_{\sigma\in\mathfrak{S}_p}\operatorname{sgn}(\sigma)T\left(x_{\sigma(1)},\dots,x_{\sigma(p)}\right)\\
&= \frac{1}{(p!)^2}\cdot p!\cdot\sum_{\sigma\in\mathfrak{S}_p}\operatorname{sgn}(\sigma)T\left(x_{\sigma(1)},\dots,x_{\sigma(p)}\right)\\
&= \operatorname{Alt}(T)(x_1,\dots,x_p).
\end{aligned}
$$

□

One practical consequence of this fact is that

$$S_{[a_1\cdots a_\ell]}T^{[a_1\cdots a_\ell]} = S_{[a_1\cdots a_\ell]}T^{a_1\cdots a_\ell} \tag{1.1.6}$$

for $S_{a_1\cdots a_\ell}, T_{a_1\cdots a_\ell} \in \mathcal{E}_{a_1\cdots a_\ell}$. This greatly simplifies many calculations. Note that Proposition 1.1.3 still holds replacing Alt with Sym, and hence (1.1.6) holds with antisymmetrization replaced with symmetrization.

Clearly $v^1_{[a_1}\cdots v^n_{a_n]}$ and $v^1\wedge\cdots\wedge v^n$ are equal up to a constant factor, and this is determined by the normalization condition (1.1.4).

We calculate

$$
\begin{aligned}
v^1_{[a_1}\cdots v^n_{a_n]}v_1^{a_1}\cdots v_n^{a_n} &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} v^1_{a_1}\cdots v^n_{a_n}v^{a_1}_{\sigma(1)}\cdots v^{a_n}_{\sigma(n)}\\
&= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\delta^1_{\sigma(1)}\cdots\delta^n_{\sigma(n)}\\
&= \frac{1}{n!}.
\end{aligned}
$$

Hence
$$n! \cdot v^1_{[a_1} \cdots v^n_{a_n]} v_1^{a_1} \cdots v_n^{a_n} = 1,$$
and so
$$v^1 \wedge \cdots \wedge v^n = n! \cdot v^1_{[a_1} \cdot v^n_{a_n]}. \tag{1.1.7}$$

1.2 Riemannian geometry

Let (M, g) be a Riemannian manifold. We will sometimes use the term "Riemannian manifolds" to also include pseudo-Riemannian manifolds, although for this thesis we will generally assume that (M, g) has Riemannian signature, i.e. signature $(n, 0)$. This is mainly to preclude the possibility of submanifolds that have induced a degenerate metric conformal structure. At times we will comment on how some results may be modified to hold for pseudo-Riemannian signatures. The most important result in Riemannian geometry is that the metric induces a distinguished connection on the tangent bundle.

Theorem 1.2.1. *Let (M, g) be a Riemannian manifold. Then there exists a unique connection ∇ on M which satisfies*

(1) $\nabla g = 0$, *and*

(2) $\nabla_X Y - \nabla_Y X = [X, Y]$.

A connection which satisfies (1) is said to be metric-preserving and a connection which satisfies (2) is called torsion free. This connection is called the Levi-Civita connection (associated to the metric g).

Proof. We show uniqueness. The metric-preserving condition (1) implies that
$$\begin{aligned} X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y(g(X,Z)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ -Z(g(X,Z) &= -g(\nabla_Z X, Y) - g(X, \nabla_Z Y). \end{aligned}$$
Adding the above three equations together and then using torsion-freeness (condition (2) above) shows that
$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2} &(X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &+ g([X,Y],Z) - g([X,Z],Y) - g([Y,Z],X)). \end{aligned} \tag{1.2.1}$$
The above is known as the *Koszul formula*, and since g is non-degenerate, (1.2.1) completely determines $\nabla_X Y$. It remains to show that the ∇ defined by (1.2.1) is a connection, which is a straightforward if tedious verification. □

The *Riemann curvature tensor* $R_{ab}{}^c{}_d$ is defined by

$$R_{ab}{}^c{}_d V^d := (\nabla_a \nabla_b - \nabla_b \nabla_a) V^c, \tag{1.2.2}$$

where $V^c \in \Gamma(\mathcal{E}^c)$. The *Ricci* tensor and *scalar curvature* are then defined by

$$R_{ab} := R_{ca}{}^c{}_b, \tag{1.2.3}$$

and

$$R := g^{ab} R_{ab} \tag{1.2.4}$$

respectively.

In dimensions $n \geq 3$, the curvature tensor admits the decomposition

$$R_{abcd} = W_{abcd} + 2g_{c[a}P_{b]d} + 2g_{d[b}P_{a]c}, \tag{1.2.5}$$

where W_{abcd} is the totally trace-free part of R_{abcd}, called the *Weyl tensor*, and P_{ab} is the *Schouten tensor*, which may be expressed in terms of the Ricci and scalar curvatures as

$$P_{ab} = \frac{1}{n-2}\left(R_{ab} - \frac{R}{2(n-1)} g_{ab} \right). \tag{1.2.6}$$

Note that the Schouten tensor is also frequently called the *Rho tensor*, and is sometimes defined to be the negative of the right-hand side of the above display. The Schouten tensor is an important object in conformal geometry. The decomposition (1.2.5) may also be written

$$R = W + g \owedge P, \tag{1.2.7}$$

where $\owedge$ denotes the *Kulkarni-Nomizu* product which maps $S^2T^*M \times S^2T^*M \to S^2\Lambda^2T^*M$ and whose definition can be read off (1.2.5). Finally, we let $J := g^{ab}P_{ab}$ be the trace of the Schouten tensor. This is related to the scalar curvature by

$$J = \frac{1}{2(n-1)} R. \tag{1.2.8}$$

In 2 dimensions, the Riemannian tensor has a single component. Hence

$$R_{abcd} = \kappa \left(g_{ac} g_{bd} - g_{bc} g_{ad} \right). \tag{1.2.9}$$

Tracing (1.2.9) appropriately, one sees that $\kappa = \frac{1}{2}R$, where as above R is the scalar curvature.

The Riemann curvature tensor also satisfies the *Bianchi* identities:

$$R_{ab}{}^c{}_d + R_{da}{}^c{}_b + R_{bd}{}^c{}_a = 0, \tag{1.2.10}$$

which, using the symmetries of the Riemann curvature tensor, may be written

$$R_{[ab}{}^{c}{}_{d]} = 0,$$

and

$$\nabla_a R_{bc}{}^{d}{}_{e} + \nabla_b R_{ca}{}^{d}{}_{e} + \nabla_c R_{ab}{}^{d}{}_{e} = 0, \tag{1.2.11}$$

equivalently

$$\nabla_{[a} R_{bc]}{}^{d}{}_{e}.$$

Contracting the above display with g^{ae} yields the *contracted Bianchi identity*:

$$\nabla^a R_{bcda} + \nabla_b R_{cd} - \nabla_c R_{bd} = 0. \tag{1.2.12}$$

Contracting again, this time with g^{bd}, yields another identity, sometimes called the *twice-contracted Bianchi identity*:

$$2\nabla^a R_{ab} = \nabla_b R. \tag{1.2.13}$$

Finally, using equations (1.2.6) and (1.2.8), this is seen to be equivalent to

$$\nabla^a P_{ab} = \nabla_b J. \tag{1.2.14}$$

Chapter 2

Conformal geometry

Conformal geometry is a weakening of pseudo-Riemannian geometry where one replaces the Riemannian metric with an equivalence class of metrics. Let M be a smooth manifold. We define an equivalence relation on metrics on M as follows. For $g, \widehat{g}$ metrics on M, we declare $g \sim \widehat{g}$ if, and only if

$$\widehat{g} = \Omega^2 g, \tag{2.0.1}$$

where Ω is a smooth, positive real-valued function on M. We denote by $\boldsymbol{c}$ an equivalence class of metrics related in this way and call the pair $(M, \boldsymbol{c})$ a *conformal manifold*. Note that all conformally related metrics necessarily have the same signature, so we may speak of the signature of the conformal manifold $(M, \boldsymbol{c})$.

We can equally view the conformal structure as a subbundle of S^2T^*M, where the fiber at $x \in M$ consists of all g_x for $g \in \boldsymbol{c}$. Note that for $g, \widehat{g} \in \boldsymbol{c}$ there is some $s \in \mathbb{R}_+$ such that $\widehat{g}_x = s^2 g_x$, and hence $\mathcal{Q}$ is an $\mathbb{R}_+$-subbundle. Sections of $\mathcal{Q}$ are in bijective correspondence with metrics in the conformal class, so $\boldsymbol{c} = \Gamma(\mathcal{Q})$.

2.1 Conformal densities

Before proceeding to our main treatment of conformal manifolds, we need an important tool. The bundles of conformal densities are a family of non-trivial line bundles indexed by their *weight* $w \in \mathbb{R}$. These are defined as associated bundles to the $\mathbb{R}_+$-principal bundle $\mathcal{Q}$. We have just seen that there is a principal $\mathbb{R}_+$-action on $\mathcal{Q}$ given by $s \cdot g_x = s^2 g_x$. For $w \in \mathbb{R}$, define an action ρ_w of $\mathbb{R}_+$ on $\mathbb{R}$ by $\rho_w(s)t := s^{-w}t$. Then define the *bundle of conformal w-densities* by

$$\mathcal{E}[w] := \mathcal{Q} \times_{\rho_w} \mathbb{R}, \tag{2.1.1}$$

where this notation denotes the associated bundle, i.e. the quotient bundle $(\mathcal{Q} \times \mathbb{R})/\sim$ with $(g, t) \sim (s^2 g, s^w t)$. It is a standard fact that sections of such an associated bundle are in bijective correspondence with suitably equivariant functions on the total space, see e.g. [51]. In this case, elements of $\Gamma(\mathcal{E}[w])$ are identified with functions $f : \mathcal{Q} \to \mathbb{R}$ such that $f(s^2 g_x) = s^w f(g_x)$. A choice of metric $g \in \boldsymbol{c}$ determines a trivialization of any of the bundles $\mathcal{E}[w]$, and the different trivializations corresponding to conformally related metrics depend on the weight w. Specifically, two metrics $g, \widehat{g} \in \boldsymbol{c}$ each determine a section of the bundle $\mathcal{Q} \to M$. Let $f : \mathcal{Q} \to \mathbb{R}$ be the homogeneous function corresponding to a section of $\mathcal{E}[w]$, as above. Then, pulling back f via the two sections determined by g and $\widehat{g}$ yields two functions on M related according to

$$f^{\widehat{g}} = \Omega^w f^g, \tag{2.1.2}$$

where as usual $\widehat{g} = \Omega^2 g$. Thus conformal densities of weight w may be thought of as real-valued functions on M that rescale by a factor of Ω^w when g is rescaled by Ω^2. It is clear from this (and the above associated bundle definition) that $\mathcal{E}[0] = \mathcal{E}$ the trivial bundle of smooth functions $M \to \mathbb{R}$. For any vector bundle $\mathcal{V}$, we let

$$\mathcal{V}[w] := \mathcal{V} \otimes \mathcal{E}[w], \tag{2.1.3}$$

and say that $\mathcal{V}[w]$ is a *weighted* bundle or *has weight* w. Such bundles are very natural in conformal geometry, and we will frequently encounter weighted versions of familiar bundles.

It appears that the bundles $\mathcal{E}[w]$ as defined depend on the conformal structure. We will show that this is not the case by showing that the conformal density bundles can be identified with the usual density bundles from differential geometry. Let $\alpha \in \mathbb{R}$. Recall that the bundle of α-densities is the line bundle associated to the linear frame bundle defined by the $\mathrm{GL}(n)$-representation $\rho_\alpha(A) := |\det(A)|^{-\alpha}$. Thus 1-densities are exactly the objects that may be integrated on a manifold in a coordinate-independent way. So, writing $\mathcal{F}$ for the $\mathrm{GL}(n)$-principal frame bundle of M, the bundle of α-densities is then defined as

$$\mathcal{D}[\alpha] := \mathcal{F} \times_{\rho_\alpha} \mathbb{R}. \tag{2.1.4}$$

Any 1-density then defines a conformal density of weight $-n$ as follows. Let φ be a 1-density. Then once again by the correspondence between sections of associated bundles and equivariant functions, we may view φ as a map $\mathcal{F} \to \mathbb{R}$ which is equivariant with respect to the $\mathrm{GL}(n)$-action on the frame bundle. Next, fix a Riemannian metric g on M. Then, since M is oriented, g determines a volume form which in local coordinates takes the form $\sqrt{|\det(g_{ij})|}$, and this induces a real-valued map on the frame bundle by calculating the volume of the frame according to the metric g. Thus we may define the map $\mathcal{F} \to \mathbb{R}$ by

$$u \mapsto \frac{\varphi(u)}{\mathrm{vol}(g)(u)}. \tag{2.1.5}$$

Moreover, since $\mathrm{vol}(g)(A \cdot u) = |\det(A)|\mathrm{vol}(g)$, this map in fact descends to a map $M \to \mathbb{R}$. Now, if we rescale g to $\widehat{g} = \Omega^2 g$, we see from the expression of the metric in local coordinates that $\mathrm{vol}(\widehat{g}) = \Omega^n \mathrm{vol}(g)$. Thus, fixing the 1-density φ, we may define a map $\mathcal{Q} \to \mathbb{R}$ by

$$g_x \mapsto \frac{\varphi(x)}{\mathrm{vol}(g)(x)}, \tag{2.1.6}$$

where we simply write that the function and the 1-density act on the point $x \in M$ since as we have seen above this quotient is independent of the frame chosen. From the transformation of the volume form under a conformal rescaling, this map is seen to be homogeneous of degree $-n$. Thus we have constructed an element of $\mathcal{E}[-n]$, and so shown that there is a correspondence between 1-densities and conformal densities of weight $-n$. More generally, one may show that $\left(-\frac{w}{n}\right)$-densities correspond to conformal densities of weight w by taking φ to be a $\left(-\frac{w}{n}\right)$-density and then defining the map $\mathcal{Q} \to \mathbb{R}$ by

$$g_x \mapsto \frac{\varphi(x)}{(\mathrm{vol}(g)(x))^{\frac{w}{n}}}. \tag{2.1.7}$$

One can check that as above for a fixed metric g this gives a well-defined map $M \to \mathbb{R}$, and is moreover homogeneous of degree w when viewed as a map on $\mathcal{Q}$, i.e. a section of $\mathcal{E}[w]$. Thus on any manifold we have the bundle of conformal densities of weight w available as the bundle of $\left(-\frac{w}{n}\right)$-densities, and fixing a conformal structure gives an identification with an associated bundle to $\mathcal{Q}$. This implies in particular that the isomorphisms $\mathcal{E}[-2n] \cong \mathcal{D}[2]$ and $\mathcal{E}[-n] \cong \mathcal{D}[1]$ hold, without any requirement that M be oriented. Moreover, note that the square of a line bundle is always oriented, since after squaring there is canonically a notion of positivity. Thus, even when M is not orientable, we will have $(\Lambda^n T^*M)^2 \cong \mathcal{E}[-2n]$ and, dually, $(\Lambda^n TM)^2 \cong \mathcal{E}[2n]$.

If M is in fact orientable, then a choice of orientation yields the additional isomorphism $\Lambda^n T^*M \cong \mathcal{E}[-n]$. To see this explicitly, note that a choice of metric $g \in \boldsymbol{c}$ results in a volume form $\mathrm{vol}^g \in \Gamma(\Lambda^n T^*M)$ associated to the metric g. Rescaling the metric conformally so that $\widehat{g} = \Omega^2 g$ results in the volume form transforming according to $\mathrm{vol}^{\widehat{g}} = \Omega^n \mathrm{vol}^g$. Comparing with (2.1.2), we see that there is a well-defined *weighted volume form* which we write $\mathrm{vol} \in \Gamma(\Lambda^n T^*M[n])$, i.e. this does not depend on the choice of metric. This weighted form then gives the isomorphism

$$\Lambda^n TM \xrightarrow{\cong} \mathcal{E}[n], \tag{2.1.8}$$

and dually an isomorphism $\Lambda^n T^*M \cong \mathcal{E}[-n]$. After choosing a metric $g \in \boldsymbol{c}$ and trivializing density bundles, the weighted form vol coincides with the unweighted volume form vol^g of the metric g.

Weighted bundles give yet another way to view a conformal structure. Tautologically, there is a $\mathcal{E}[2]$-valued non-degenerate bilinear form, which we denote $\boldsymbol{g}$. Choosing a metric $g \in \boldsymbol{c}$ and trivializing density bundles, $\boldsymbol{g}$ is simply given by g. Thus we may view $\boldsymbol{g}$ as a distinguished section of $S^2T^*M[2]$, which contains the full information of the conformal class. We call $\boldsymbol{g}$ the *conformal metric*.

A metric $g \in \boldsymbol{c}$ also determines a section $\sigma^g \in \Gamma(\mathcal{E}_+[1])$, where $\mathcal{E}_+[1]$ denotes the subbundle of the 1-density bundle consisting of those densities which are strictly positive. This is characterized by the property that the corresponding homogeneous function $\mathcal{Q} \to \mathbb{R}$ is equal to 1 along g. The conformal metric is then recovered by

$$\boldsymbol{g} = (\sigma_g)^2 g. \tag{2.1.9}$$

One sees that the right-hand side of the above display is indeed independent of the choice of metric g since rescaling g to $\widehat{g} = \Omega^2 g$ yields correspondingly $\sigma^{\widehat{g}} = \Omega^{-1}\sigma^g$. Conversely, given $\sigma \in \Gamma(\mathcal{E}_+[1])$,

$$g := \sigma^{-2}\boldsymbol{g} \tag{2.1.10}$$

is a metric in the conformal class. Thus there is a bijection between metrics $g \in \boldsymbol{c}$ and sections $\sigma \in \Gamma(\mathcal{E}_+[1])$. We call $\mathcal{E}_+[1]$ the bundle of *scales* and call the corresponding $g := \sigma^{-2}\boldsymbol{g} \in \boldsymbol{c}$ a *choice of scale*.

This conformal metric also realizes the isomorphism $(\Lambda^n TM)^2 \to \mathcal{E}[2n]$ from our discussion of conformal densities. Let $\omega^{a_1 a_2 \cdots a_n} \in \Gamma(\mathcal{E}^{[a_1 a_2 \cdots a_n]})$ be an n-vector, and define a map

$$\otimes^n \boldsymbol{g} : (\Lambda^n TM)^2 \longrightarrow \mathcal{E}[2n]$$

by

$$(\otimes^n \boldsymbol{g})(\omega) := \boldsymbol{g}_{a_1 b_1}\boldsymbol{g}_{a_2 b_2} \cdots \boldsymbol{g}_{a_n b_n}\omega^{a_1 a_2 \cdots a_n}\omega^{b_1 b_2 \cdots b_n}.$$

The isomorphism $(\Lambda^2 TM)^2 \xrightarrow{\cong} \mathcal{E}[2n]$ is this map, possibly up to a constant factor, depending on the multilinear algebra conventions. The image of this map clearly lies in $\mathcal{E}[2n]$, and it is invertible since $\boldsymbol{g}$ is invertible, with inverse essentially $\otimes^n \boldsymbol{g}^{-1}$. This isomorphism is equivalently realized by the weighted volume form, viewing vol $\otimes$ vol as a map $(\Lambda^n TM)^2 \to \mathcal{E}[2n]$ by contracting a section of $(\Lambda^n TM)^2$ with this squared weighted volume form in the obvious way.

We have already mentioned that a choice of scale trivializes the density bundles. Explicitly, a scale σ trivializes $\mathcal{E}[w]$ via the isomorphism $\mathcal{E}[w] \to \mathcal{E}[0] = \mathcal{E}$ defined by $\tau \mapsto \sigma^{-w}\tau$, where $\tau \in \Gamma(\mathcal{E}[w])$. This gives another way to see (2.1.2), namely by comparing the trivializations $\tau \mapsto \sigma^{-w}\tau$ and $\tau \mapsto \widehat{\sigma}^{-w}\tau$, and as we saw above, $\widehat{\sigma} = \Omega^{-1}\sigma$, where Ω is the function such that $\widehat{g} = \Omega^2 g$.

For $g \in \boldsymbol{c}$, we define a connection on $\mathcal{E}[w]$ by

$$\nabla_a \tau := \sigma^w \mathrm{d}\left(\sigma^{-w}\tau\right), \tag{2.1.11}$$

where $\sigma \in \Gamma(\mathcal{E}_+[1])$ is such that $\boldsymbol{g} = \sigma^2 g$. Note that $\sigma^{-w}\tau$ has weight 0 (i.e. it is a function), and the exterior derivative has a well-defined action on such objects.

The conformal density bundles may also be described as associated bundles to the conformal frame bundle $\mathcal{G}_0$, and it is hence clear that the Levi-Civita connection associated to a choice of metric $g \in \boldsymbol{c}$ will induce a connection on $\mathcal{E}[w]$, as a choice of Levi-Civita connection induces a connection on all bundles associated to the frame bundle. This induced connection coincides with the one defined in equation (2.1.11).

From the connection defined above, it immediately follows that $\nabla \boldsymbol{g} = 0$, where ∇ is the Levi-Civita connection of *any* metric in the conformal class. Thus we may use $\boldsymbol{g}$ (and its inverse $\boldsymbol{g}^{-1} \in \Gamma(S^2TM[-2])$) to raise and lower indices on a conformal manifold in the same way one uses the metric on a Riemannian manifold. Some caution is required however, since this now comes at the cost of changing the weight:

$$\begin{aligned} \boldsymbol{g}_{ab} : \mathcal{E}^a &\to \mathcal{E}_b[2], \\ v^a &\mapsto v_b := \boldsymbol{g}_{ab} v^a. \end{aligned}$$

We will henceforth always use the conformal metric and its inverse for all raising and lowering of indices. Moreover, we may retroactively replace any such uses of a choice of metric in Chapter 1. So for example, the Riemann curvature tensor with all indices lowered should be understood as $R_{abcd} = \boldsymbol{g}_{ce} R_{ab}{}^e{}_d$, and hence all instances of the metric g in (1.2.5) will be replaced with $\boldsymbol{g}$. Similarly, the scalar curvature should now be taken to be $R = \boldsymbol{g}^{ab} R_{ab}$. As mentioned above, this comes at the cost of changing conformal weights: whereas $R_{ab}{}^c{}_d \in \Gamma(\mathcal{E}_{[ab]}{}^c{}_d)$, when the index is lowered, $R_{abcd} \in \Gamma(\mathcal{E}_{[ab][cd]}[-2])$. The situation with the scalar curvature is similar. Writing $R^0 := g^{ab} R_{ab}$ for the usual Riemannian geometry scalar curvature, and $R := \boldsymbol{g}^{ab} R_{ab}$ for our modified scalar curvature defined with the conformal metric, we will have that $R^0 \in \Gamma(\mathcal{E}[0]) = C^\infty(M)$, while $R \in \Gamma(\mathcal{E}[-2])$. These weighted objects are more convenient to work with in the conformal setting.

2.2 Conformal transformations

Since $(M, \boldsymbol{c})$ is equipped with an equivalence class of conformally related metrics, it no longer possesses a distinguished connection on its tangent bundle TM. It is still very useful to record how the Levi-Civita connection transforms under a conformal rescaling of the metric. Throughout this section and indeed for the remainder of the thesis, we will always assume that $\widehat{g} = \Omega^2 g$ where Ω is a smooth positive function. Moreover, any symbol adorned with a caret should be understood to mean that symbol but associated to the metric $\widehat{g}$. So $\widehat{\nabla}$ denotes the Levi-Civita connection of $\widehat{g}$, $\widehat{P}$ its Schouten tensor, and so on. We also define $\Upsilon_a := \Omega^{-1} \nabla_a \Omega$. This expression arises frequently in conformal transformation formulae.

For the Levi-Civita connection on density bundles as defined in (2.1.11), we find that

$$\widehat{\nabla}_a \tau = \nabla_a \tau + w \Upsilon_a \tau. \tag{2.2.1}$$

Meanwhile for sections v^b and ω_b of $\mathcal{E}^b$ and $\mathcal{E}_b$ respectively, we find that

$$\widehat{\nabla}_a v^b = \nabla_a v^b + \Upsilon_a v^b - v_a \Upsilon^b + \Upsilon_c v^c \delta^b_a \tag{2.2.2}$$

and

$$\widehat{\nabla}_a \omega_b = \nabla_a \omega_b - \Upsilon_a \omega_b - \omega_a \Upsilon_b + \Upsilon^c \omega_c \boldsymbol{g}_{ab}. \tag{2.2.3}$$

These formulae can be deduced from the local formulae for the Christoffel symbols or the Koszul formula; we give a sketch using the later. Note that $\widehat{\nabla}$ is also defined by the Koszul formula (1.2.1), where all instances of g are replaced with $\widehat{g}$. Replacing $\widehat{g}$ with $\Omega^2 g$ and expanding using the Leibniz rule shows that

$$\begin{aligned} g(\widehat{\nabla}_X Y, Z) = g(\nabla_X Y, Z) + \Omega^{-1} X(\Omega) g(Y, Z) \\ + \Omega^{-1} Y(\Omega) g(X, Z), \end{aligned} \tag{2.2.4}$$

and rewriting this in abstract indices one arrives at (2.2.2). From there, (2.2.3) follows by raising lowering the b index, noting that $\widehat{\nabla} g = -2(\Omega^{-1}\nabla\Omega)g$. Also, since $\Upsilon^c = \boldsymbol{g}^{ac}\Upsilon_a \in \mathcal{E}^c[-2]$, the final term of (2.2.3) is in fact unweighted, despite the presence of the (weighted) conformal metric.

Using (2.2.1), (2.2.2) and (2.2.3) together, one can determine the transformation law for $\widehat{\nabla}$ acting on any weighted tensor bundle. For example for $V^b \in \mathcal{E}^b[w]$ and $\omega_b \in \mathcal{E}_b[w]$, one has

$$\widehat{\nabla}_a V^b = \nabla_a V^b + (w+1)\,\Upsilon_a V^b - V_a \Upsilon^b + \Upsilon_c V^c \delta^b_a \tag{2.2.5}$$

and

$$\widehat{\nabla}_a \omega_b = \nabla_a \omega_b + (w-1)\Upsilon_a \omega_b - \omega_a \Upsilon_b + \Upsilon^c \omega_c \boldsymbol{g}_{ab} \tag{2.2.6}$$

respectively. These also allow one to calculate the transformation law for the Riemann curvature tensor. Working with the curvature tensor with all indices lowered (using the conformal metric), we find that

$$\widehat{R}_{abcd} = R_{abcd} - 2\boldsymbol{g}_{c[a}\Lambda_{b]d} - 2\boldsymbol{g}_{d[b}\Lambda_{a]c}, \tag{2.2.7}$$

where

$$\Lambda_{ab} := \nabla_a \Upsilon_b - \Upsilon_a \Upsilon_b + \frac{1}{2}\Upsilon_c \Upsilon^c \boldsymbol{g}_{ab}. \tag{2.2.8}$$

In particular, from (2.2.7), one deduces that the Weyl tensor is conformally invariant:

$$\widehat{W}_{abcd} = W_{abcd}, \tag{2.2.9}$$

and therefore the failure of the curvature tensor to be conformally invariant is entirely due to the transformation of the Schouten tensor. Comparing with (1.2.5), we find

$$\widehat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2}\Upsilon_c \Upsilon^c \boldsymbol{g}_{ab}. \tag{2.2.10}$$

The same discussion around weights from the end of Section 2.1 applies here: namely, the Weyl tensor W_{abcd}, where the indices have been lowered using the conformal metric, has conformal weight -2, while the Weyl tensor $W_{ab}{}^c{}_d$ is a true (unweighted) tensor.

2.3 The tractor bundle

We next define the *(standard) tractor bundle*, $\mathcal{T}$. We fix a conformal manifold $(M, \boldsymbol{c})$, and we will assume that M has dimension $n \geq 3$. For a conformal manifold $(M, \boldsymbol{c})$, the conformal standard tractor bundle is a rank $(n+2)$-vector bundle which possesses a metric and a metric-preserving connection. Thus one obtains a *tractor calculus* which closely resembles the tensor calculus available on Riemannian manifolds.

The (conformal) tractor bundle has its origins in the work of Thomas [68, 67, 66], and was then rediscovered and presented in the modern language of vector bundles by Bailey, Eastwood and Gover [4]. Using abstract index notation, the standard tractor bundle will be denoted by $\mathcal{E}^A$, and for the remainder of this thesis we adopt the convention that upper case Latin indices will denote tractor bundles and sections thereof. There are several ways to define the tractor bundle. We will in fact first define the dual tractor bundle. The non-degenerate metric then allows us to identify the dual tractor bundle with the standard tractor bundle without further comment.

As we have seen in Section 2.1, on any smooth manifold, one has the bundle of conformal 1-densities that we call $\mathcal{E}[1]$. Its 2-jet bundle $J^2\mathcal{E}[1]$ admits the exact sequence at 2-jets [57], which takes the form

$$0 \to S^2T^*M[1] \to J^2\mathcal{E}[1] \to J^1\mathcal{E}[1] \to 0. \tag{2.3.1}$$

Recall that a conformal structure on M may equally be thought of as a conformal metric $\boldsymbol{g} \in \Gamma(S^2T^*M[2])$. The introduction of a conformal structure therefore determines a canonical splitting $S^2T^*M[1] = S^2_0T^*M[1] \oplus \mathcal{E}[-1]$ by mapping μ to its trace-free and trace parts, where traces are taken with the conformal metric:

$$\mu_{ab} \mapsto \left(\mu_{(ab)_0}, \frac{1}{n}\boldsymbol{g}^{cd}\mu_{cd}\right),$$

where $\mu_{(ab)_0} := \mu_{ab} - \frac{1}{n}\boldsymbol{g}^{cd}\mu_{cd}\boldsymbol{g}_{ab}$. The standard conformal cotractor bundle $\mathcal{T}^*$ is then defined as the quotient of $J^2\mathcal{E}[1]$ by the image of $S^2_0T^*M[1]$ and so has a filtration as given by the exact sequence

$$0 \to \mathcal{E}[-1] \xrightarrow{X} \mathcal{T}^* \to J^1\mathcal{E}[1] \to 0. \tag{2.3.2}$$

We will write the information of (2.3.2) as

$$\mathcal{T}^* = J^1\mathcal{E}[1] \mathbin{\rlap{+}\supset} \mathcal{E}[-1], \tag{2.3.3}$$

where we introduce the "semidirect sum" notation, $\mathbin{\rlap{+}\supset}$, to compactly represent such a short exact sequence. It may be shown that the semidirect sum notation is associative, i.e. $(A \mathbin{\rlap{+}\supset} B) \mathbin{\rlap{+}\supset} C = A \mathbin{\rlap{+}\supset} (B \mathbin{\rlap{+}\supset} C)$. Using this notation, the exact sequence at 1-jets

$$0 \to T^*M[1] \to J^1\mathcal{E}[1] \to \mathcal{E}[1] \to 0 \tag{2.3.4}$$

is represented by $J^1\mathcal{E}[1] = \mathcal{E}[1] \mathbin{\rlap{+}\supset} T^*M[1]$, and hence for the cotractor bundle $\mathcal{T}^*$ we have

$$\mathcal{T}^* = \mathcal{E}[1] \mathbin{\rlap{+}\supset} T^*M[1] \mathbin{\rlap{+}\supset} \mathcal{E}[-1]. \tag{2.3.5}$$

A choice of metric $g \in \boldsymbol{c}$ defines an isomorphism $\mathcal{E}_A \to \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ by mapping

$$J^2\mathcal{E}[1] \ni j^2_x\sigma \mapsto \begin{pmatrix} \sigma(x) \\ \nabla_a\sigma(x) \\ -\frac{1}{n}\boldsymbol{g}^{ab}\left(\nabla_a\nabla_b\sigma(x) + P_{ab}\sigma(x)\right) \end{pmatrix} \in \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1], \tag{2.3.6}$$

where ∇_a and P_{ab} are the Levi-Civita connection and Schouten tensor respectively associated to the metric g. To see that this map is well-defined, first note that it is clearly a map $J^2\mathcal{E}[1] \to \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$. To show that it descends to a well-defined map on $\mathcal{E}_A$, observe that if some $j^2_x\sigma$ is contained in the kernel, then in particular $j^1_x\sigma = 0$ (from the vanishing of the first two components), and hence the third component becomes $\boldsymbol{g}^{ab}\nabla_a\nabla_b\sigma(x)$, and this vanishes if, and only if, $\nabla_a\nabla_b\sigma(x)$ lies in $\mathcal{E}_{(ab)_0}[1] \subset J^2\mathcal{E}[1]$. Thus the map factors to a map $\mathcal{E}^A \to \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ which is injective on fibers. Moreover, since it may be seen that $\mathcal{E}_A$ and $\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ are both vector bundles of rank $n+2$, this is immediately seen to be an isomorphism. A choice of conformally related metric $\widehat{g} \in \boldsymbol{c}$ yields a different isomorphism $\mathcal{E}_A \to \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ where the Levi-Civita connection and Schouten tensor in (2.3.6) are replaced with their conformally related counterparts. Explicitly, this results in a section of $\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ which is related to the one corresponding to the metric $g \in \boldsymbol{c}$ by

$$\widehat{\begin{pmatrix} \sigma \\ \nabla_a\sigma \\ -\frac{1}{n}\boldsymbol{g}^{ab}\left(\nabla_a\nabla_b\sigma + P_{ab}\sigma\right) \end{pmatrix}} = \begin{pmatrix} \sigma \\ \nabla_a\sigma + \Upsilon_a\sigma \\ -\frac{1}{n}\boldsymbol{g}^{ab}\left(\nabla_a\nabla_b\sigma + P_{ab}\sigma\right) - \Upsilon^c\left(\nabla_c\sigma + \Upsilon_c\sigma\right) - \frac{1}{2}\Upsilon^c\Upsilon_c\sigma \end{pmatrix}. \tag{2.3.7}$$

Thus we have established that the tractor bundle is isomorphic to the direct sum bundle $\mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ *but not canonically*. So we may regard sections of $\mathcal{E}_A$ as triples

$(\sigma, \mu_a, \rho) \in \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ where we identify (σ, μ_a, ρ) with some other triple $(\widehat{\sigma}, \widehat{\mu}_a, \widehat{\rho})$ if, and only if,

$$\begin{pmatrix} \widehat{\sigma} \\ \widehat{\mu}_a \\ \widehat{\rho} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon_a & \delta_a^b & 0 \\ -\frac{1}{2}\Upsilon^c\Upsilon_c & -\Upsilon^b & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} \tag{2.3.8}$$

for some $\Upsilon_a = \Omega^{-1}\nabla_a\Omega$, where Ω is a positive function on M. This equivalence relation lets us easily test if such a triple is a well-defined tractor, since it must transform in this way.

We note that there is some choice in defining the map of (2.3.6). We follow the convention introduced in [4] where the tractor bundle was introduced. This construction is also used in [14] where it appears slightly different owing to a different sign convention for the Schouten tensor.

We introduce a new notation for representing tractors as opposed to the tuple/column vector notation we have used thus far. This notation will be much better suited to representing sections of tensor products of tractor bundles, and generally facilitates calculations with tractors. We have already seen in (2.3.2) that there is a conformally invariant map $X : \mathcal{E}[-1] \to \mathcal{T}^*$. We call this the *canonical tractor* (for the role it plays in the short exact sequence) or *position tractor* (since it turns out that it also invariantly encodes information about position on a conformal manifold). Moreover, a choice of metric $g \in \boldsymbol{c}$ determines maps $Y_A : \mathcal{E}[1] \to \mathcal{E}_A$ and $Z_A^a : \mathcal{E}_a[1] \to \mathcal{E}_A$, which we call the *tractor projectors* or *splitting tractors*. In this notation, we write

$$V_A \overset{g}{=} \sigma Y_A + \mu_a Z_A^a + \rho X_A, \tag{2.3.9}$$

where the g over the equality emphasizes that this splitting depends on the choice of metric $g \in \boldsymbol{c}$, although we will usually omit this in the sequel. We view Y_A and Z_A^a as sections of $\mathcal{E}_A[-1]$ and $\mathcal{E}_A^a[-1]$ respectively, and while X_A is conformally invariant, one sees from (2.3.8) that, for a conformally related choice of metric, Y_A and Z_A^a must transform according to

$$\widehat{Y}_A = Y_A - \Upsilon_a Z_A^a - \frac{1}{2}\Upsilon^a\Upsilon_a X_A \tag{2.3.10}$$

and

$$\widehat{Z}_A^a = Z_A^a + \Upsilon^a X_A \tag{2.3.11}$$

respectively.

Let U, V be sections of $\mathcal{T}^*$, with $U = (\sigma, \mu, \rho)$ and $V_A = (\sigma', \mu', \rho')$. One can easily check that

$$h(U, V) = \sigma\rho' + \boldsymbol{g}^{-1}(\mu, \mu') + \sigma'\rho. \tag{2.3.12}$$

defines a signature $(p+1, q+1)$ conformally invariant metric on $\mathcal{T}^*$, and we may henceforth identify $\mathcal{T}^*$ with its dual, which we write $\mathcal{T}$, or $\mathcal{E}^A$ in abstract index notation. Note that, having fixed a metric $g \in \boldsymbol{c}$, $\mathcal{E}^A \overset{g}{=} \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$, where we note that the conformal weight of the middle summand is different from (2.3.5) since the (weighted) conformal metric identifies $T^*M[1]$ and $TM[-1]$.

Written using the tractor projectors, the metric on $\mathcal{E}^A$ is

$$h_{AB} = 2X_{(A}Y_{B)} + \boldsymbol{g}_{ab}Z_A{}^a Z_B{}^b. \tag{2.3.13}$$

Hence $X^A Y_A = 1$, $Z_A{}^a Z^A{}_b = \delta^a{}_b$ and all other pairings of the splitting operators give zero.

The tractor bundle also admits a conformally invariant *tractor connection*, which is equivalent to the normal Cartan connection [15]. In a choice of scale,

$$\nabla^{\mathcal{T}}_a \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} = \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + \boldsymbol{g}_{ab}\rho + P_{ab}\sigma \\ \nabla_a \rho - P_{ab}\mu^b \end{pmatrix}. \tag{2.3.14}$$

If we wish to be explicit, we may write $\nabla^{\mathcal{T}}$ for the tractor connection, but usually we will just write ∇ and context will determine whether this denotes the tractor connection or a Levi-Civita connection. This implies that the action of the tractor connection on the tractor projectors is given by

$$\nabla_a X^A = Z^A{}_a\,, \qquad \nabla_a Z^A{}_b = -P_{ab}X^A - \boldsymbol{g}_{ab}Y^A\,, \qquad \nabla_a Y^A = P_a{}^b Z^A{}_b\,. \tag{2.3.15}$$

The general action on a section of a tractor bundle then follows from the Leibniz rule. In particular, one sees that $\nabla_a h_{AB} = 0$, so the tractor connection preserves the tractor metric. Therefore we will henceforth use the tractor metric to identify $\mathcal{T}$ and $\mathcal{T}^*$.

Finally, the *tractor curvature* $\Omega_{ab}{}^C{}_D$ of the tractor connection is defined by

$$\Omega_{ab}{}^C{}_D \Phi^D := 2\,\nabla_{[a}\nabla_{b]}\Phi^C$$

for $\Phi^A \in \Gamma(\mathcal{E}^A)$. Written using the tractor projectors takes, this takes the form

$$\Omega_{abCD} = W_{abcd}Z^c_C Z^d_D - 2C_{abc}X_{[C}Z^c_{D]}, \tag{2.3.16}$$

where W_{abcd} is the Weyl tensor, and C_{abc} is the *Cotton tensor*

$$C_{abc} := 2\nabla_{[a}P_{b]c}. \tag{2.3.17}$$

Recall that a Riemannian manifold (M, g) is said to be *(locally) conformally flat* if any point has a neighborhood where there exists a conformally related metric which is flat. The following well-known result, sometimes called the *Weyl-Schouten theorem*, characterizes this property.

Theorem 2.3.1. *Let (M, g) be a Riemannian n-manifold. Then M is locally conformally flat if, and only if*

- $W_{abcd} = 0$, *if* $n \geq 4$, *and*
- $C_{abc} = 0$, *if* $n = 3$.

See e.g. [2] for a proof.

We will say that $(M, \boldsymbol{c})$ is *flat* if there is a locally flat metric in the conformal class $\boldsymbol{c}$. Using the above theorem, we see that this notion of conformally flatness coincides exactly with the tractor connection being flat.

Theorem 2.3.2. *Let* $(M, \boldsymbol{c})$ *be a conformal manifold. Then* M *is flat if, and only if, the tractor curvature vanishes.*

Proof. If the connection is flat, then the Weyl and Cotton tensors both vanish, and hence by Theorem 2.3.1, $(M, \boldsymbol{c})$ is conformally flat in all dimensions $n \geq 3$.

Conversely, suppose that $(M, \boldsymbol{c})$ is conformally flat. In dimension 3, the Weyl tensor vanishes identically. So the tractor connection is flat if, and only if, the Cotton tensor is zero. But according to Theorem 2.3.1, this is exactly equivalent to the conformal flatness of $(M, \boldsymbol{c})$. In dimensions $n \geq 4$, Theorem 2.3.1 gives that conformal flatness is equivalent to the vanishing of the Weyl tensor. On the other hand, combining the Bianchi identity (1.2.11) and its contracted versions (1.2.13) and (1.2.14), together with the decomposition of the Riemann curvature tensor (1.2.5) and equation (1.2.6) relating the Ricci and Schouten tensors, if follows that

$$\nabla^d W_{abcd} = (3 - n) C_{abc}.$$

Hence when $n \geq 4$, the vanishing of the Weyl tensor implies that the Cotton tensor vanishes also, and so the whole tractor curvature vanishes. Thus for all dimensions at least 3, the tractor curvature vanishes if, and only if, $(M, \boldsymbol{c})$ is conformally flat. □

Equivalently, one can think of (2.3.6) as defining a second-order linear differential operator $D : \mathcal{E}[1] \to \mathcal{E}_A$ by

$$D\sigma = \begin{pmatrix} n\sigma \\ n\nabla_a \sigma \\ -(\Delta\sigma + J\sigma) \end{pmatrix}, \tag{2.3.18}$$

where $\Delta := \boldsymbol{g}^{ab}\nabla_a\nabla_b$ and recall $J = \boldsymbol{g}^{ab}P_{ab}$ is the trace of the Schouten tensor (now taken with the conformal metric). We call this the *Thomas-D operator*. It is tautologically conformally invariant since the transformation of (2.3.6) *defines* conformal invariance. The factor of n here is merely a matter of convention. For some scale $\sigma \in \Gamma(\mathcal{E}[1])$, we define the *scale tractor*

$$I_A := \frac{1}{n} D_A \sigma. \tag{2.3.19}$$

Such

In fact, the above Thomas-D operator is merely a special case of a more general conformally invariant second order operator $D_A : \mathcal{E}_\Phi[w] \to \mathcal{E}_A \otimes \mathcal{E}_\Phi[w-1]$, where $\mathcal{E}_\Phi$ denotes *any* tractor bundle. This acts on $V \in \mathcal{E}_\Phi[w]$ by

$$D_A V \overset{g}{:=} \begin{pmatrix} (n+2w-2)wV \\ (n+2w-2)\nabla_a V \\ -(\Delta V + wJV) \end{pmatrix}, \tag{2.3.20}$$

where $\nabla_a = \nabla_a^{\mathcal{T}}$ is the induced tractor connection on $\mathcal{E}_\Phi$ and $\Delta := \boldsymbol{g}^{ab}\nabla_a^{\mathcal{T}}\nabla_b^{\mathcal{T}}$. It is clear that this recovers (2.3.18) when $\mathcal{E}_\Phi[w]$ is the bundle of conformal 1-densities, $\mathcal{E}[1]$.

Recall that a metric g is said to be *Einstein* if $R_{ab} = \lambda g_{ab}$ where R_{ab} is the Ricci tensor of the metric g and λ is a function. (In fact, the Bianchi identity implies that if such a λ exists, then it is necessarily constant.) One may ask if a given conformal manifold $(M, \boldsymbol{c})$ is "conformally Einstein", namely whether there is an Einstein metric in the conformal class. This question is closely related to the existence of a parallel standard tractor, and the Thomas-D operator gives the link between the two.

Theorem 2.3.3. *On a conformal manifold, there is a bijective correspondence between sections $\sigma \in \Gamma(\mathcal{E}[1])$ satisfying*

$$\nabla_{(a}\nabla_{b)_0}\sigma + P_{(ab)_0}\sigma = 0 \tag{2.3.21}$$

and parallel standard tractors I_A. The mapping from scales to parallel tractors is given by $\sigma \mapsto \frac{1}{n}D_A\sigma$, while the inverse map from parallel tractors to scales is $I_A \mapsto X^A I_A$. We call equation (2.3.21) the almost-Einstein equation.

Before proving the theorem, we briefly explain the reason for the *almost* in almost-Einstein equation. Using the formulae from Section 2.2, one can show that (2.3.21) is conformally invariant. Suppose that some $\sigma \in \Gamma(\mathcal{E}[1])$ solves this equation, and in addition, suppose for the time being that σ is nowhere-zero. Then, working in the scale σ, equation (2.3.21) becomes

$$P^g_{(ab)_0} = 0,$$

where $g := \sigma^{-2}\boldsymbol{g}$ is the metric determined by σ. One readily sees that $P^g_{(ab)_0} = 0$ if, and only if, the metric g is Einstein. Conversely, given some $g \in \boldsymbol{c}$ which is Einstein, then $g = \sigma^{-2}\boldsymbol{g}$ for some $\sigma \in \mathcal{E}_+[1]$, and this solves (2.3.21). So solutions σ to (2.3.21) which are nowhere-zero are in bijective correspondence with Einstein metrics in the conformal class.

However, note that one can still ask for solutions to (2.3.21) without requiring that they are non-vanishing. Such a $\sigma \in \Gamma(\mathcal{E}[1])$ determines an Einstein metric *where it is non-zero*, but this metric is no longer defined on the whole manifold, merely on $M\backslash\mathcal{Z}(\sigma)$, where $\mathcal{Z}(\sigma)$

is the zero locus of σ. Thus equation (2.3.21) gives a weakening of the notion of Einstein, hence the name *almost*-Einstein.

We now prove the theorem.

Proof of Theorem 2.3.3. Suppose that I_A is some standard tractor. Then I_A is parallel if, and only if,

$$\nabla_a \sigma - \mu_a = 0, \quad \nabla_a \mu_b + \boldsymbol{g}_{ab}\rho + P_{ab}\sigma, \quad \text{and} \quad \nabla_a \rho - P_{ab}\mu^b = 0.$$

Combining the first two of these equations, we see that I_A parallel implies that

$$\nabla_a \nabla_b \sigma + P_{ab}\sigma = -\rho \boldsymbol{g}_{ab},$$

and so σ solves (2.3.21). Conversely, one can check using (2.3.14) and (2.3.18) that $\nabla_a I_A = 0$, where $I_A := \frac{1}{n} D_A \sigma$ for σ solving (2.3.21). □

One can alternatively define the tractor bundle by *prolonging* the almost-Einstein equation. In this construction, one starts with (2.3.21) and introduces new variables for each derivative of σ until the derivatives of all variables are expressed in terms of the other variables and *their* derivatives. The Thomas-D operator then maps σ to its prolongation. This construction is done in detail in [28].

The almost-Einstein equation (2.3.21) is the simplest example of a *first BGG equation*. This family of overdetermined geometric PDEs will play an important role in our later treatment of conserved quantities (Chapter 6), and we will see that it is no coincidence that there is a bijective correspondence between solutions to the almost-Einstein equation and parallel sections of the standard tractor bundle.

We will call $(M, \boldsymbol{c}, \sigma)$ an *almost-Einstein manifold* if $\sigma \in \mathcal{E}[1]$ is a non-trivial solution to the Almost-Einstein equation (2.3.21). As we have already observed, such a solution defines an Einstein metric $g := \sigma^{-2}\boldsymbol{g}$ on $M \backslash \mathcal{Z}(\sigma)$. Note that solutions to (2.3.21) also correspond bijectively to parallel standard tractors even if σ has a non-empty zero locus. We henceforth assume that σ is not identically zero. Since $I_A := \frac{1}{n} D_A \sigma$ is parallel, it is everywhere non-zero, provided that $I_A \neq 0$, which holds as long as σ is not the zero section. Hence σ must be non-zero on an open, dense set. A parallel standard tractor corresponds to a special type of structure group reduction known as a *holonomy reduction*. By a theorem from [16], this stratifies M into a disjoint union of *curved orbits*. In the case of an almost-Einstein manifold, we have

Theorem 2.3.4. *Let $(M, \boldsymbol{c}, I)$ be an almost-Einstein manifold. Then stratification of M into curved orbits is according to the strict sign of $\sigma := X^A I_A$, and the zero locus $\mathcal{Z}(\sigma)$ satisfies*

- *If $I^2 \neq 0$, then $\mathcal{Z}(\sigma)$ is either empty or a smoothly embedded hypersurface.*

- *if* $I^2 = 0$*, then* $\mathcal{Z}(\sigma)$ *is either empty or, after removing isolated points, a smoothly embedded hypersurface.*

2.4 Parabolic geometries

Conformal geometry is an example of a *parabolic geometry*. These are Cartan geometries of type (G, P), where P is a parabolic subgroup of the semisimple Lie group G. In addition to conformal geometry, other examples of parabolic geometries include projective geometry and hypersurface type CR geometry, to name some of the most well-known. Our main tool for studying conformal geometries is the tractor calculus developed in the earlier sections of this chapter, however we here touch briefly on the more general Cartan-geometric viewpoint. We refer the reader to the excellent [17] for a comprehensive and thorough Cartan-geometric text on parabolic geometry, or [60] for a good introduction to Cartan geometry more generally. A *Cartan geometry* of type (G, H) on a manifold M consists of an H-principal fiber bundle $p : \mathcal{P} \to M$ together with a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ which is called the *Cartan connection*. The Cartan connection satisfies

1. $(r^h)^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega$ for all $h \in H$,
2. $\omega(\xi_X(u)) = X$ for each $X \in \mathfrak{h}$, and where ξ_X is the fundamental vector field corresponding to X, and
3. $\omega(u) : T_u\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}$.

A parabolic geometry is then a Cartan geometry of type (G, P), where P is a parabolic subgroup of G. Such a subgroup may be seen to correspond to a $|k|$*-grading* of the Lie algebra $\mathfrak{g}$ of G, that is, a decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$. Notable pioneering contributions in the theory and applications of these structures were made by Tanaka [63, 64]. Different classes of familiar parabolic geometries correspond to different choices of G and P. For example, in conformal geometry of signature (p, q), one takes $G = \mathrm{SO}(p + 1, q + 1)$ and P the stabilizer of a null line in $\mathbb{R}^{n+2}$, and projective geometry is given by $G = \mathrm{SL}(n + 1)$ and P the isotropy group of the line through the first vector in the standard basis of $\mathbb{R}^{n+1}$. These examples are both $|1|$-graded geometries, which have the simplest behavior. Contact geometries such as CR geometries and Lagrangean contact geometries are examples of $|2|$-graded geometries. Fixing the larger group G and choosing different parabolic subgroups also yields different geometries of interest. Generalizing the projective case, taking $G = \mathrm{SL}(n+1)$ and P to be the stabilizer of the subspace generated by the first k standard basis vectors gives the Grassmannian of all k-dimensional subspaces.

An important special example of a parabolic geometry of type (G, P) is the *(homogeneous) model*, namely taking M to be the homogeneous space G/P, with the Cartan connection being given by the Maurer-Cartan form. A recurring theme in the study of parabolic geometry is that properties on an arbitrary parabolic geometry of type (G, P) tend to closely follow the behavior of that property on the model. We will see this later in this thesis.

2.5 Conformally singular geometries

Geometries which are "singular in a conformal way", such as Poincaré-Einstein manifolds, have been shown to be useful in geometric scattering, the AdS/CFT-correspondence and other areas of mathematics and mathematical physics [1, 45, 48, 38]. We define here a very general class of structures which includes conformally singular geometries of interest, among other things.

Definition 2.5.1. We say that $(M, \boldsymbol{c}, \sigma)$ is an *almost pseudo-Riemannian manifold* if $I_A := \frac{1}{n} D_A \sigma$ is nowhere-zero and $j^1\sigma$ vanishes at most at isolated points.

We address this condition on the 1-jet of σ. If one only requires that $I_A \neq 0$, then it may be that $I_A = -\frac{1}{n}\Delta\sigma X_A$ at some non-isolated points if the 1-jet vanishes identically. This situation is quite unnatural, by which we mean it does not occur on the flat model.

To prove this, we will use the theory of holonomy reductions developed in [16]. We summarize the relevant results from that article. First, a parallel section of a tractor bundle induces a holonomy reduction, a special type of reduction of structure group on the G-principal bundle $\widehat{\mathcal{G}} := \mathcal{G} \times_P G$. Such a holonomy reduction induces a stratification of the underlying manifold M into a disjoint union of *curved orbits* which are parametrized by P-*type*; the various P-types may be thought of as corresponding to various possible relations between the given parallel tractor field and the canonical/position tractor X. (This stratification generalizes the stratification of almost-Einstein manifolds at the end of Section 2.3.) Finally, Theorem 2.6 of [16] asserts that even on a curved geometry, this stratification must be locally diffeomorphic to the model.

The almost Einstein case is particularly simple, since the parallel object is just a standard tractor. The condition $j^1\sigma$ vanishing at most at isolated points is equivalent to asking that $X \wedge I$ vanishes only at isolated points. In the language of the theory of holonomy reductions, the set of such points is a P-type, and therefore corresponds to a curved orbit on M.

On the model, X is the homogeneous coordinates of a point, and I is a parallel (therefore constant) vector. Therefore $X \wedge I = 0$ can only happen when X and I are parallel or anti-parallel modulo dilations, and hence this happens in at most two points, and none if $I^2 \neq 0$. Thus we see that on the model, a parallel tractor I can only satisfy $X \wedge I = 0$ in isolated points, and hence the machinery of [16] guarantees that the same will be true in the curved case. Thus we see that this 1-jet condition is in particular true of almost-Einstein structures.

Conformally compact manifolds are a special case of the almost pseudo-Riemannian manifolds defined above, and an important class of conformally singular geometries in their own right. We record the definition here.

Definition 2.5.2. Let M be a manifold with boundary with interior $\mathring{M}$. We say that M is *conformally compact* if $\mathring{g} = u^{-2}g$ where g is a metric on M and u is a defining function for ∂M, i.e.

1. $\mathcal{Z}(u) = \partial M$, and
2. $\mathrm{d}u$ is nowhere-zero on ∂M.

Chapter 3

Distinguished curves in conformal geometry

3.1 Conformal circles

For any geometry, it is natural to ask which curves are *distinguished* in some sense. We should mention from the outset that there are two classes of distinguished conformal curves. One notes that the usual geodesic equation is conformally invariant provided that the tangent vector of the curve in question is null. These curves are called *null geodesics*, but they will not be the main focus of our work. The non-null conformal distinguished curves are variously called *conformal circles* or *conformal geodesics*, and we will use these terms interchangeably to refer to this class of curves. Since we mostly work in Riemannian signature, these will be the only relevant class of distinguished curves for us, although null geodesics may be treated in a similar way to what we present here [39]. The notion of conformal circles has its origins in work of Fialkow, Yano and Schouten [71, 33, 59, 72]. The terminology *conformal circles* refers to the fact that in the homogeneous model with signature $(n, 0)$ these curves recover all circles on the sphere.

The conformal circle equation in taking the form usually seen in modern works appears in [72, Chapter VII, §2]. The equation is given therein as

$$\frac{d^3\xi^x}{ds^3} + \frac{d\xi^x}{ds}\left(g_{\mu\lambda}\frac{d^2\xi^\mu}{ds^2}\frac{d^2\xi^\lambda}{ds^2} - \frac{1}{n-2}L_{\mu\lambda}\frac{d\xi^\mu}{ds}\frac{d\xi^\lambda}{ds}\right) + \frac{1}{n-2}L_\lambda{}^x\frac{d\xi^\lambda}{ds} = 0,$$

where $L_{\mu\lambda}$ is defined by

$$L_{\mu\lambda} := -K_{\mu\lambda} + \frac{1}{2(n-1)}Kg_{\mu\lambda},$$

where $K_{\mu\lambda}$ and K are the Ricci tensor and scalar curvature respectively. Comparing the above display with (1.2.6), wee see that this is related to our Schouten tensor according to

$$L_{\mu\lambda} = -(n-2)P_{\mu\lambda}$$

Thus, Yano's equation rewritten with our notation and conventions is

$$u^c \nabla_c a^b - u^c P_c{}^b + a_c a^c u^b + u^c u^d P_{cd} u^b = 0. \tag{3.1.1}$$

We will explain shortly how this equation fits with the conformal circle equation as we will define it.

After tractor methods were first used in [4] to characterize conformal circles, further efforts were made to use such methods in [61, 39].

Notably, conformal manifolds are also a class of parabolic geometries, and thus inherit a notion of distinguished curve from in the sense of Cartan geometry. The class(es) of distinguished curves one obtains from this perspective agree with the more classical picture. We will discuss this briefly in Section 3.4.

We fix some notational conventions for this chapter. We will denote by γ a smooth curve in a conformal manifold $(M, \boldsymbol{c})$. For this chapter, we allow that $(M, \boldsymbol{c})$ have split signature. Unless otherwise stated, we will assume that γ is a non-null curve, namely its tangent vector always has non-zero length. If follows from continuity that the *sign* of the length of the tangent to γ must be constant, i.e. for all $u \in T\gamma$, one has $\boldsymbol{g}(u,u) > 0$ or $\boldsymbol{g}(u,u) < 0$ at every point along γ. The symbols u^b and a^b will always denote, respectively, the velocity and acceleration the curve γ. Recall that $a^b = u^a \nabla_a u^b$. We also define $u := \sqrt{|\boldsymbol{g}_{ab} u^a u^b|} \in \mathcal{E}[1]$. For some connection ∇, we will also use the notation $\frac{d^\nabla}{dt}$ to mean $u^a \nabla_a$. The connection ∇ may be a Levi-Civita connection or the standard tractor connection; this should be unambiguous from context. Finally, we define some important tractor fields associated to the curve γ. Recall that the canonical tractor X^B can be viewed as a section of $\mathcal{E}^B[1]$. Hence $u^{-1} X^B$ is an unweighted tractor. Define

$$U^B := u^a \nabla_a \left(u^{-1} X^B \right) \tag{3.1.2}$$

and

$$A^B := u^a \nabla_a U^B, \tag{3.1.3}$$

which we call the *velocity* and *acceleration tractors* respectively. Explicitly, one has

$$U^B = \begin{pmatrix} 0 \\ u^{-1} u^b \\ -u^{-3} \left(u_c a^c \right) \end{pmatrix} \tag{3.1.4}$$

and

$$A^B = \begin{pmatrix} -u \\ u^{-1} a^b - 2u^{-3} (u_c a^c) u^b \\ -u^{-3} \left(u_c \frac{da^c}{dt} \right) - u^{-3} a_c a^c + 3u^{-5} (u_c a^c)^2 - u^{-1} P_{cd} u^c u^d \end{pmatrix}. \tag{3.1.5}$$

3.1.1 Parametrized conformal circles

We begin by defining conformal circles to be solutions to a certain third order ODE. This is the point of view taken in [3], wherein a smooth curve γ is said to be a *conformal circle* if

$$u^c\nabla_c a^b = u^2 \cdot u^c P_c{}^b + 3u^{-2}\left(u_c a^c\right) a^b - \frac{3}{2}u^{-2}\left(a_c a^c\right) u^b - 2u^c u^d P_{cd} u^b, \tag{3.1.6}$$

where $u^2 = u \cdot u$ here should be understood to be *unweighted*. In that article it is stated that the conformal circle equation (3.1.6) is equivalent to a pair of equations:

$$\left(u^c\nabla_c a^{[b}\right) u^{a]} = 3\frac{u_c a^c}{u_d u^d} u^{[a} a^{b]} + \left(u_e u^e\right) u^c P_c{}^{[b} u^{a]}, \tag{3.1.7}$$

$$u_b u^a \nabla_a a^b = 3\frac{\left(u_c a^c\right)^2}{u_c u^c} - \frac{3}{2} a_c a^c - \left(u_e u^e\right) u^a u^b P_{ab}. \tag{3.1.8}$$

Clearly if the conformal circle equation holds then so too do (3.1.7) and (3.1.8). Conversely, supposing the above two equations hold, expanding the antisymmetrization of (3.1.7), contracting with u_a and then substituting (3.1.8) and rearranging gives the conformal circle equation.

The above are equations (7) and (6), respectively, from [3], and may be understood as

- a parametrization-independent equation, (3.1.7), and
- a choice of distinguished parametrization, (3.1.8).

Equation (3.1.8) may be rephrased in terms of the curve tractors defined at the beginning of this section. From (3.1.5) it follows that

$$A_B A^B = 2u^{-2} u_c \frac{da^c}{dt} + 3u^{-2} a_c a^c - 6u^{-4}(u_c a^c)^2 + 2P_{cd} u^c u^d, \tag{3.1.9}$$

and hence (3.1.8) holds if, and only if, $A^B A_B = 0$. The family of distinguished parametrizations specified by either (3.1.8) or the condition $A^B A_B = 0$ are the so-called *projective* parametrizations. Such parametrizations are in fact available for all (non-null) curves. The following proposition establishes this, as well as clarifying why such parametrizations are called projective. It also provides an alternative proof of Cartan's observation that any curve in a conformal manifold inherits a natural projective structure [21].

Proposition 3.1.1. *Any non-null curve γ in a conformal (or even Riemannian) manifold M may be reparametrized such that it obeys* (3.1.8)*. Moreover, the freedom in the choice of such a parametrization is* $\mathrm{PSL}(2, \mathbb{R})$*.*

Proof. Let

$$f := u_b u^a \nabla_a a^b - 3\frac{(u_c a^c)^2}{u^2} + \frac{3}{2} a_c a^c + u^2 u^c u^d P_{cd} \tag{3.1.10}$$

so that $f = 0$ is equation (3.1.8). Reparametrizing $\gamma = \gamma(t)$ with a new variable $s = g(t)$ also results in

$$u^a \mapsto \tilde{u}^a = \frac{d\gamma}{ds} = \frac{dt}{ds}\frac{d\gamma}{dt} = (g')^{-1} u^a \tag{3.1.11}$$

and

$$a^b \mapsto \tilde{a}^b = \frac{d^2\gamma}{ds^2} = \frac{dt}{ds}\frac{d}{dt}\left((g')^{-1} u^b\right) = (g')^{-2} a^b - (g')^{-3} g'' u^b. \tag{3.1.12}$$

(Note that equations (3.1.11) and (3.1.12) may be used to verify the parametrization-independence of (3.1.7).) Hence

$$\tilde{f} = (g')^{-4} f - u^2 (g')^{-4} \left(\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2\right).$$

We can always solve the ODE $\tilde{f} = 0$, and hence any curve γ may be locally reparametrized such that (3.1.8) is satisfied. To prove the second statement of the theorem, we note that there is freedom in doing so corresponding to solutions of

$$\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 = 0, \tag{3.1.13}$$

which is the Schwarzian differential equation, whose solutions are of the form

$$g(t) = \frac{at + b}{ct + d}, \qquad \text{where } ad - bc = 1.$$

□

Since we have already seen that equation (3.1.6) is equivalent to the pair of equations (3.1.7) and (3.1.8), and we have just established that a curve always admits a projective parametrization, equivalently a reparametrization always exists such that (3.1.8) is satisfied, we have hence proved

Proposition 3.1.2. *A curve γ satisfies* (3.1.7) *if, and only if, it admits a reparametrization such that the conformal circle equation* (3.1.6) *holds.*

Thus the conformal circle equation as given in (3.1.6) implies that the curve is projectively parametrized. We will thus refer to equation (3.1.6) as the *projectively parametrized conformal circle equation* and, in light of the previous proposition, we call (3.1.7) the *unparametrized conformal circle equation.* We will say that a curve is an *unparametrized* *conformal circle* if it solves (3.1.7) or, equivalently (by the previous proposition), if it admits a projective reparametrization solving (3.1.6).

Yano's conformal circle equation (3.1.1) also involves an implicit choice of parameter. Specifically, one has that (3.1.1) is equivalent to (3.1.7) together with the curve having a unit speed parametrization (according to the metric in which one is working). To see this, note first that the unit speed condition implies that $u_c a^c = 0$. Substituting this and the unit speed condition $u_e u^e = 1$ into (3.1.7) and expanding the antisymmetrizations yields

$$(u^c \nabla_c a^b) u^a - (u^c \nabla_c a^a) u^b = u^c P_c{}^b u^a - u^c P_c{}^a u^b.$$

Contracting the above display with u_a yields

$$u^c \nabla_c a^b - u^c P_c{}^b - (u_a u^c \nabla_c a^a) u^b + u^c u^d P_{cd} u^b = 0,$$

which is almost (3.1.1). Finally, since $u_a a^a = 0$, it follows that also $u^c \nabla_c (u_a a^a) = 0$, and hence $u_a u^c \nabla_c a^a = -a_c a^c$. Substituting this into the above display then exactly gives (3.1.1).

We have the following characterization of unparametrized conformal circles [38]. This is [3, Proposition 3.3] but for *un*parametrized curves.

Proposition 3.1.3. *Let $(M, \boldsymbol{c})$ be a conformal manifold and γ a non-null, oriented curve in M. Then γ is an unparametrized conformal circle if, and only if, γ is an unparametrized geodesic for some metric in the conformal class and $u^c P_c{}^b \propto u^b$, where P is the Schouten tensor for the given metric.*

Proof. Suppose first that γ is an unparametrized conformal circle. In [3] it is claimed without proof that one may find a scale making the acceleration zero. (In fact, it may also be shown using results on submanifolds from later in this thesis, c.f. Proposition 5.1.2.) Another proof of this may be found in [30], and a slight modification of the argument therein shows that one can instead find a new scale such that $\widehat{a}^b \propto u^b$, where $\widehat{a}^b := u^c \widehat{\nabla}_c u^b$, the acceleration computed in the new scale. Thus locally there exists a metric in the conformal class for which γ has $a^b = f u^b$ for some smooth function f. Working in this scale and substituting this expression for the acceleration into equation (3.1.7), we see that

$$0 = (u_e u^e) u^c P_c{}^{[b} u^{a]},$$

whence it must be that $u^c P_c{}^b \propto u^b$, since recall we assume that the curve γ is non-null.

Conversely, suppose that such a scale exists, and that $u^c P_c{}^b \propto u^b$, where P_{ab} is the Schouten tensor of this special scale. Then, since we have that $a^b \propto u^b$, the right-hand side of (3.1.7) is clearly zero. Writing $a^b = f u^b$ and expanding out the left-hand side, one sees that this is also zero. So equation (3.1.7) is satisfied, and hence γ is an unparametrized conformal circle. □

In Chapter 5, we will see that multiple potential notions of conformal distinguished submanifold admit characterizations in terms of conformal circles. These characterizations were studied by Belgun [6] without the use of tractors. In light of this, it seems prudent to confirm that Belgun's notion of conformal circle is the same as ours. Given a non-null curve γ, Belgun defines the *conformal acceleration* to be

$$\begin{aligned} a(\gamma)^b &:= u^2 u^c P_c{}^b - u^c \nabla_c a^b + 3u^{-2}\left(u_c a^c\right) a^b \\ &\quad + \left(-6u^{-4}\left(u_c a^c\right) + \frac{3}{2}u^{-2}\left(a_c a^c\right) + 2u^{-2}\left(u_c u^d \nabla_d a^c\right)\right) u^b, \end{aligned} \tag{3.1.14}$$

and defines conformal circles to be those non-null curves with vanishing conformal acceleration. Substituting the $u_c u^d \nabla_d a^c$ term of the above display with (3.1.8), one calculates

$$a(\gamma)^b = u^2 \cdot u^c P_c{}^b + 3u^{-2}(u_c a^c) - \frac{3}{2}u^{-2}(a_c a^c)u^b - 2u^c u^d P_{cd} u^b - u^c \nabla_c a^b,$$

and hence $a(\gamma)^b = 0$ is exactly the projectively parametrized conformal circle equation (3.1.6).

Proposition 3.1.4. *The curve γ is a projectively parametrized conformal circle if, and only if, $A_B A^B = 0$ and $dA^B/dt = 0$.*

Proof. We have already seen that $A^B A_B = 0$ is equivalent to the projective parametrization condition (3.1.8). When $A_B A^B = 0$, the acceleration tractor takes the simpler form

$$A^B = \begin{pmatrix} -u \\ u^{-1}a^b - 2u^{-3}\left(u_c a^c\right) u^b \\ \frac{1}{2}u^{-3} a_c a^c \end{pmatrix}.$$

Hence,

$$\begin{aligned} \frac{dA^B}{dt} &= \left[-u^{-1} \cdot u_c a^c - u^{-1} \cdot a^c u_c + 2u^{-3}\left(u_c a^c\right) u_d u^d\right] Y^B \\ &\quad \left[-u \cdot u^c P_c{}^b - u^{-2}\left(u^{-1} u_c a^c\right) a^b + u^{-1}\frac{da^b}{dt} + 6u^{-5}(u_c a^c)^2 u^b - 2u^{-3}\left(a_c a^c\right) u^b \right. \\ &\qquad \left. -2u^{-3}\left(u_c \frac{da^c}{dt}\right) u^b - 2u^{-3}\left(u_c a^c\right) a^b + \frac{1}{2}u^{-3}\left(a_c a^c\right) u^b\right] Z_b^B \\ &\quad + \left[-\frac{3}{2}u^{-5}\left(u_d a^d\right)\left(a_c a^c\right) + u^{-3}\left(a_c \frac{da^b}{dt}\right) - u^c P_{cd}\left(u^{-1}a^d - 2u^{-3}\left(u_e a^e\right) u^d\right)\right] X^B \\ &= \begin{pmatrix} 0 \\ u^{-1}\frac{da^b}{dt} - u \cdot u^c P_c{}^b - 3u^{-3}\left(u_c a^c\right) a^b + \frac{3}{2}u^{-3}\left(a_c a^c\right) u^b + 2u^{-1} u^c u^d P_{cd} u^b \\ u^{-3} a_c \frac{da^c}{dt} - u^{-1} u^c a^d P_{cd} - \frac{3}{2}u^{-5}\left(u_d a^d\right) a_c a^c + 2u^{-3} u^c u^d P_{cd}\left(u_e a^e\right) \end{pmatrix} \end{aligned} \tag{3.1.15}$$

$$= \begin{pmatrix} 0 \\ u^{-1}\frac{da^b}{dt} - u \cdot u^c P_c{}^b - 3u^{-3}\left(u_c a^c\right) a^b + \frac{3}{2}u^{-3}\left(a_c a^c\right) u^b + 2u^{-1}u^c u^d P_{cd} u^b \\ u^{-2}a_b\left(u^{-1}\frac{da^b}{dt} - u \cdot u^c P_c{}^b - 3u^{-3}\left(u_c a^c\right) a^b + \frac{3}{2}u^{-3}\left(a_c a^c\right) u^b + 2u^{-1}u^c u^d P_{cd} u^b\right) \end{pmatrix},$$

and one sees that the middle slot vanishes precisely when

$$\frac{da^b}{dt} - u^2 \cdot u^c P_c{}^b - 3u^{-2}\left(u_c a^c\right) a^b + \frac{3}{2}u^{-2}\left(a_c a^c\right) u^b + 2u^c u^d P_{cd} u^b = 0,$$

which is exactly (3.1.6). Moreover, if $A_B A^B = 0$, then also $A_B \frac{dA^B}{dt} = 0$, and therefore if both the top and middle slots of $\frac{dA^B}{dt}$ vanish, then so must the third (since the top slot of A^B is non-zero). Note that the equality below (3.1.15) also shows this explicitly. This completes the proof. □

3.1.2 Unparametrized conformal circles

Our goal in this section is to develop a parametrization-independent tractor theory of conformal circles. We have seen a tractor characterization of conformal circles in Proposition 3.1.4, but this is exclusively for projectively parametrized curves. While such parametrizations are always available, it would nevertheless be preferable to state results in a parametrization-independent way. On the other hand, equation (3.1.7) is parametrization-independent, but it is not phrased in the language of tractors. This philosophy underpins the treatment of conformal circles of [39]. As a first step towards the stated goal, we introduce weighted versions of the velocity and acceleration vectors.

A nowhere-null curve γ with velocity $u^a \in \Gamma(\mathcal{E}^a|_\gamma)$ determines a scale $u \in \Gamma(\mathcal{E}_+[1]|_\gamma)$ along the curve according to

$$u := \sqrt{|\boldsymbol{g}_{ab} u^a u^b|}, \tag{3.1.16}$$

where the absolute value allows for the case that γ is timelike. Using the *conformal* metric and the unweighted velocity vector results in an overall weight.

Lemma 3.1.5. *Let γ be an oriented non-null curve. Then there exists a unique weighted vector field $\mathbf{u}^a \in \Gamma(T\gamma[-1])$ along the curve that is compatible with the orientation and satisfies*

$$\mathbf{u}^a \mathbf{u}_a = \begin{cases} 1, & \text{if } \gamma \text{ is spacelike,} \\ -1, & \text{if } \gamma \text{ is timelike.} \end{cases} \tag{3.1.17}$$

Proof. Let $u^a \in \Gamma(T\gamma)$ be an **unweighted** velocity field which is compatible with the orientation, and let $u \in \Gamma(\mathcal{E}_+[1]|_\gamma)$ be defined as in (3.1.16). Then $\mathbf{u}^a := u^{-1}u^a$ satisfies (3.1.17) and is independent of the choice of velocity vector u^a. □

Definition 3.1.6. Let γ be an oriented nowhere-null curve on $(M, \boldsymbol{c})$. We call the canonical weighted vector field $\mathbf{u}^a$ of Lemma 3.1.5 the *weighted velocity* of γ. Given $g \in \boldsymbol{c}$, define the *weighted acceleration* $\mathbf{a}^b \in \Gamma(\mathcal{E}^b[-2]|_\gamma)$ of γ by

$$\mathbf{a}^b := \mathbf{u}^c \nabla_c \mathbf{u}^b. \tag{3.1.18}$$

From the defining property of $\mathbf{u}^b$ (equation (3.1.17)) it is clear that

$$\mathbf{u}^b \mathbf{a}_b = 0. \tag{3.1.19}$$

Under a conformal rescaling, one calculates

$$\widehat{\mathbf{a}}^b = \widehat{\mathbf{a}}^b + \mathbf{u}^a \Upsilon_a \mathbf{u}^b \mp \Upsilon^b, \tag{3.1.20}$$

where the sign is such that $\mathbf{u}^a \mathbf{u}_a = \pm 1$.

For future reference, it is also useful to note the relationship between the weighted and unweighted acceleration vectors. We have

$$\mathbf{a}^c = u^{-2} a^c - u^{-3} \left(u^b \nabla_b u \right) u^c, \tag{3.1.21}$$

and

$$a^c = u^2 \mathbf{a}^c + u \left(\mathbf{u}^b \nabla_b u \right) \mathbf{u}^c. \tag{3.1.22}$$

Viewing γ as a submanifold in $(M, \boldsymbol{c})$, in turns out that the weighted velocity and acceleration recover known submanifold invariants. This will be treated in more detail in Section 5.1, but for now we simply note that the weighted velocity is the weighted volume form of the submanifold and the weighted acceleration is the weighted mean curvature.

We can now state a weighted version of (3.1.7).

Lemma 3.1.7. *Let γ be an oriented nowhere-null curve on $(M, \boldsymbol{c})$. Then γ is a conformal circle if, and only if its weighted velocity and acceleration satisfy the conformally invariant equation*

$$\left(\mathbf{u}^c \nabla_c \mathbf{a}^{[a} \right) \mathbf{u}^{b]} = \pm \mathbf{u}^c P_c{}^{[a} \mathbf{u}^{b]}, \qquad \textit{whenever } \mathbf{u}^a \mathbf{u}_a = \pm 1, \tag{3.1.23}$$

for any $g \in \boldsymbol{c}$ with Levi-Civita connection ∇.

Proof. Recall that γ possesses a canonical weighted velocity as defined in Lemma 3.1.5. Choose a scale $\sigma \in \Gamma(\mathcal{E}_+[1]|_\gamma)$ along γ and define $u^a := \sigma \mathbf{u}^a$. This and equations (3.1.21) and (3.1.22) allow one to convert between unweighted and weighted quantities. Substituting these as appropriate into one of (3.1.7) or (3.1.23) yields the other. □

We now have the necessary results to state and prove our parametrization-independent tractor characterizations of conformal circles. There are several related results all of which are phrased in terms of a certain 3-tractor field along the curve. The main object of import here is a certain 3-tractor field along a curve.

Theorem 3.1.8. *On a pseudo-Riemannian or conformal manifold a nowhere null curve γ is an oriented conformal circle if and only if along γ there is a parallel 3-tractor $0 \neq \Sigma \in \Gamma(\Lambda^3 \mathcal{T}|_\gamma)$ such that*

$$X \wedge \Sigma = 0. \tag{3.1.24}$$

For a given oriented conformal circle γ the 3-tractor Σ_γ satisfying (3.1.24) *is unique up to multiplication by a positive constant, and unique if we specify $|\Sigma_\gamma|^2 = -1$ when γ is spacelike, or $|\Sigma_\gamma|^2 = 1$ when γ is timelike.*

Moreover, it turns out that the unique 3-tractor of the theorem has a simple expression in terms of tractors that we have already seen. Choosing $\sigma \in \Gamma(\mathcal{E}_+[1]|_\gamma)$ and with the velocity and acceleration tractors as above, define the 3-tractor $\Sigma^{ABC} \in \Gamma(\mathcal{E}^{[ABC]})$ by

$$\Sigma^{ABC} := 6\sigma^{-1} X^{[A} U^B A^{C]}. \tag{3.1.25}$$

We claim that this object has the properties required of the 3-tractor Σ from the theorem. To verify this, we must prove several things. While it appears that this expression depends on σ (and therefore on the parametrization of γ), it turns out that this is not the case.

Lemma 3.1.9. *An unparametrized oriented nowhere-null curve γ canonically determines a 3-tractor $\Sigma \in \Gamma(\Lambda^3 \mathcal{T}|_\gamma)$ by* (3.1.25).

Proof. Choose a scale $\sigma \in \Gamma(\mathcal{E}_+[1]|_\gamma)$ along the curve γ. Then since X^A, U^B and A^C are defined via conformal tractors and the conformal tractor connection, Σ^{ABC} can only depend on γ and σ. Choosing an ambient scale to split the tractor bundles, it follows from (3.1.4) and (3.1.5) that

$$\Sigma^{ABC} = \pm 6 \mathbf{u}^c X^{[A} Y^B Z_c^{C]} + 6 \mathbf{u}^b \mathbf{a}^c X^{[A} Z_b^B Z_c^{C]}, \tag{3.1.26}$$

and note in particular that this expression does not depend on σ, i.e. Σ^{ABC} depends only on the curve γ. □

Finally one verifies that this Σ^{ABC} has the properties required by Theorem 3.1.8.

Proposition 3.1.10. *Let γ be a nowhere-null oriented curve on $(M, \boldsymbol{c})$ with associated 3-tractor Σ^{ABC} as defined by equation* (3.1.25). *Then γ is a conformal circle if and only if Σ^{ABC} is constant along γ.*

Proof. Let $\mathbf{u}^a$ be the weighted velocity of γ. Then differentiating equation (3.1.26) gives

$$\mathbf{u}^d \nabla_d \Sigma^{ABC} = 6\left(\mathbf{u}^d \nabla_d \mathbf{a}^c \mp \mathbf{u}^d P_d{}^c\right) \mathbf{u}^b\, X^{[A} Z^B{}_b Z^{C]}{}_c\,, \qquad \text{whenever } \mathbf{u}^a \mathbf{u}_a = \pm 1,$$

and the result follows immediately from Lemma 3.1.7. □

3.2 Conserved quantities

We defer a full discussion of this until Chapter 6, wherein our discussion of conserved quantities on distinguished conformal submanifolds of arbitrary codimension will subsume the technical discussion we would present here. We instead here give some examples which illustrate how the tractor approach to conformal circles allows one to proliferate conserved quantities. To state explicitly what we mean, we say that a scalar function $Q : M \to \mathbb{R}$ is a *conserved quantity* (along a curve γ) if $u^a \nabla_a Q = 0$, where u^a is the velocity of γ.

While our ultimate goal is to be able to generate a quantity and verify that it is conserved in a more elegant way, for the time being we will use more elementary methods to prove that our example quantities are conserved. We will contrast this with the tractor framework we will introduce in Chapter 6 and show how much more effectively this perspective allows one to both come up with such a quantity in the first place, and moreover prove that it is conserved. The important point here is that distinguished curves are characterized in terms of parallel conformal tractors. Thus, roughly speaking, if we can find a parallel (at least along the curve) tractor field that pairs with Σ to yield a scalar function, then that function will be constant along the curve. In fact, we may construct a field which is polynomial in Σ and the conformal metric rather than exactly Σ itself. Actually, the field need not even be parallel, but at a minimum its derivative must be annihilated by Σ (or the tractor field which is polynomial in Σ and the tractor metric). This significantly increases the number of tractor fields we may use together with Σ and the conformal tractor metric to produce candidate conserved quantities. The difficulty with this method is finding suitable parallel tractor fields. It will turn out that a certain class of overdetermined PDEs provide a source of such tractors. These are the *first BGG equations* and include many well-known and studied geometric PDEs. They admit a very elegant tractor theory, and it turns out that, at least in the flat case, the tractor sections they provide are always parallel.

Theorem 3.2.1. *Let M be a smooth manifold equipped with either a pseudo-Riemannian or conformal structure, and let γ be a conformal circle of M with weighted velocity and acceleration $\mathbf{u}^a$ and $\mathbf{a}^b$ respectively. Suppose that $k_{ab} \in \Gamma(\mathcal{E}_{ab}[3])$ is a conformal Killing-Yano 2-form, i.e. k_{ab} satisfies the conformally invariant equation*

$$\nabla_a k_{bc} = \nabla_{[a} k_{bc]} - \frac{2}{n-1} \boldsymbol{g}_{a[b} \nabla^p k_{c]p}. \tag{3.2.1}$$

Then

$$Q := \mathbf{u}^b \mathbf{a}^c k_{bc} \mp \frac{1}{n-1} \mathbf{u}^b \nabla^p k_{pb} \tag{3.2.2}$$

is a first integral of γ.

Proof. We calculate $\mathbf{u}^a \nabla_a Q$ explicitly. We emphasize that after we have developed our general theory of conserved quantities for distinguished submanifolds, verifying that objects such as Q are conserved will be *significantly* simpler.

Note that the weighted conformal circle equation (3.1.23) is equivalent to

$$\mathbf{u}^b \nabla_b \mathbf{a}^c = \pm \mathbf{u}^b P_b{}^c - (P_{bd} \mathbf{u}^b \mathbf{u}^d \pm \mathbf{a}^d \mathbf{a}_d) \mathbf{u}^c. \tag{3.2.3}$$

Hence

$$\begin{aligned} \mathbf{u}^a \nabla_a Q &= \pm \mathbf{u}^a \mathbf{u}^b P_a{}^c k_{bc} + \mathbf{u}^a \mathbf{u}^b \mathbf{a}^c \nabla_a k_{bc} \\ &\qquad \mp \frac{1}{n-1} \mathbf{a}^b \nabla^p k_{pb} \mp \frac{1}{n-1} \mathbf{u}^c \mathbf{u}^a \nabla_a \nabla^p k_{pc} \\ &= \pm \mathbf{u}^a \mathbf{u}^b P_a{}^c k_{bc} \mp \frac{1}{n-1} \mathbf{u}^c \mathbf{u}^a \nabla_a \nabla^p k_{pc} \end{aligned} \tag{3.2.4}$$

where we have used the skew-symmetry of k_{bc}, equation (3.2.3) and that $\mathbf{u}_a \mathbf{u}^a = \pm 1$ and $\mathbf{u}_b \mathbf{a}^b = 0$. Commuting covariant derivatives on the second term and again using the k_{bc} satisfies (3.2.1) shows that for the second term

$$\begin{aligned} \mathbf{u}^c \mathbf{u}^a \nabla_a \nabla^p k_{pc} &= \mathbf{u}^c \mathbf{u}^a \nabla^p \nabla_a k_{pc} - (n-2) \mathbf{u}^c \mathbf{u}^a P_a{}^p k_{pc} \\ &= \frac{1}{n-1} \mathbf{u}^c \mathbf{u}^a \nabla_a \nabla^p k_{pa} - (n-2) \mathbf{u}^c \mathbf{u}^a P_a{}^p k_{pc}. \end{aligned}$$

Hence

$$\mathbf{u}^c \mathbf{u}^a \nabla_a \nabla^p k_{pc} = (n-1) \mathbf{u}^c \mathbf{u}^a P_a{}^p k_{cp},$$

and substituting this into (3.2.4) shows that $\mathbf{u}^a \nabla_a Q = 0$. □

We will see conformal Killing-Yano forms again in the general discussion of conserved quantities on submanifolds of arbitrary codimension, as conformal Killing-Yano forms (of the appropriate rank) turn out to be the most natural candidates for constructing conserved quantities.

3.3 A note on null geodesics

Let $(M, \boldsymbol{c})$ be a conformal manifold. Suppose now that γ is a geodesic for some metric $g \in \boldsymbol{c}$ with tangent vector everywhere-null. Rescaling the metric g conformally, one has sees that

$$u^a \widehat{\nabla}_a u^b = 2(\Upsilon_a u^a) u^b,$$

and hence there is a reparametrization of γ with velocity $\tilde{u}^b$ such that

$$\tilde{u}^a \nabla_a \tilde{u}^b = 0.$$

Thus the curve γ viewed as an unparametrized curve (equivalently as a submanifold of M) is conformally invariant. This verifies that the null geodesics of a conformal manifold form a well-defined class of conformally invariant distinguished curves. The class of null geodesics as a distinct class of conformal distinguished curves can also be seen from the parabolic geometry perspective; see Section 3.4 for more details.

Null geodesics may also be characterized in terms of conformal tractors, albeit differently from conformal circles. Recall that the velocity and acceleration tractors defined in equations (3.1.2) and (3.1.3) involve u^{-1}, and so are obviously not defined when γ is null. First, one verifies the following "weighted null geodesic equation", the null geodesic analog of Lemma 3.1.7.

Lemma 3.3.1. *Let γ be an oriented curve on $(M, \boldsymbol{c})$. Then γ is an unparametrized null geodesic if, and only if, there exists a non-vanishing null vector field $\mathbf{u}^a \in \Gamma(T\gamma[-2])$ along γ satisfying the conformally invariant equation*

$$\mathbf{u}^a \nabla_a \mathbf{u}^b = 0, \tag{3.3.1}$$

where ∇ is the Levi-Civita connection of any $g \in \boldsymbol{c}$. The weighted velocity field is unique up to a positive factor that is constant along γ.

Proof. The conformal invariance of (3.3.1) follows from (2.2.2) and (2.2.1). Suppose that γ is an unparametrized null geodesic for $(M, \boldsymbol{c})$, and $u^a \in \Gamma(T\gamma)$ is a smooth non-zero vector field which is consistent with the orientation. Since γ is an unparametrized geodesic, it follows that $u^a \nabla_a u^b = f u^b$ for some smooth function f along γ. Locally, we may find a solution $\sigma \in \Gamma(\mathcal{E}_+[1]|_\gamma)$ to the ODE $2\sigma^{-1} u^a \nabla_a \sigma = f$. Then $\mathbf{u}^a := \sigma^{-2} u^a \in \Gamma(\mathcal{E}^a[-2]|_\gamma)$ solves (3.3.1), and is seen to be independent of the parametrization of γ initially used.

Conversely, suppose $\mathbf{u}^a \in \Gamma(T\gamma[-2])$ is null and solves (3.3.1). Choosing a density $\sigma \in \Gamma(\mathcal{E}_+[1]|_\gamma)$. Defining $u^a := \sigma^2 \mathbf{u}^a$, one sees that u^a satisfies an equation of the form $u^a \nabla_a u^b = f u^b$, where f is a smooth function whose explicit form is unimportant. In particular, γ may be reparametrized such that $\tilde{u}^a \nabla_a \tilde{u}^b = 0$, where $\tilde{u}^a$ denotes the velocity vector in the new parametrization. Thus γ is an unparametrized null geodesic. □

It follows from the transformation laws of the tractor projectors that $\mathbf{u}^b \mapsto 2X^{[A}Z_b^{B]}\mathbf{u}^b$ defines a conformally invariant map $\mathcal{E}^b[-2] \to \mathcal{E}^{[AB]}$. We denote this by $\mathbb{X}_b^{AB}$. The null geodesic analog of Prop 3.1.10 is then

Proposition 3.3.2. *An oriented curve γ on $(M, \boldsymbol{c})$ is an unparametrized null geodesic if, and only if, it admits a non-vanishing section $\mathbf{u}^b \in \Gamma(T\gamma[-2])$ such that $\Sigma^{AB} := \mathbb{X}_b^{AB}\mathbf{u}^b$ is parallel along γ.*

Proof. From equation (2.3.14),

$$\mathbf{u}^a\nabla_a\Sigma^{AB} = \mathbb{X}_b^{AB}\mathbf{u}^a\nabla_a\mathbf{u}^b - 2X^{[A}Y^{B]}\boldsymbol{g}_{ab}\mathbf{u}^a\mathbf{u}^b, \tag{3.3.2}$$

and since the two terms on the right-hand side are linearly independent if non-zero, it must be that γ satisfies the conditions of Lemma 3.3.1. □

One may also proliferate conserved quantities along null geodesics using the same approach as for conformal circles and more general conformal submanifolds. Since null geodesics correspond to 2-tractors rather than 3-tractors, the precise sections that one uses to manufacture scalar functions will be different, but the method is exactly the same. Namely, given a null geodesic γ, one constructs a scalar function which is polynomial in the 2-tractor Σ^{AB} associated to γ, the tractor metric and some tractor field which is parallel along the curve.

3.4 Conformal circles as distinguished curves in parabolic geometry

A general theory of distinguished curves in parabolic geometries is developed in [18]. While our methods here are not those of general parabolic geometry theory, we briefly remark on this here. For a general parabolic geometry of type (G, P), it is a recurring theme that the situation in the curved case closely mirrors what happens in the homogeneous model. The stratification of parabolic geometries into curved orbits provides a strong example of this [16]. One therefore reasonably expects that any classes of distinguished curves on the homogeneous model G/P would determine classes of distinguished curves in the curved geometry. This turns out to be the case. The homogeneous model (G, P) has a natural class of distinguished curves, namely those coming from 1-parameter subgroups in G. These are further divided into different types according to the orbits of $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \mathfrak{g}_{-1}$ under the action of G_0. Recall that the tangent bundle of a parabolic geometry is $\mathcal{G} \times_P \mathfrak{g}/\mathfrak{p}$, so $\mathfrak{g}_- \cong \mathfrak{g}/\mathfrak{p}$ gives all possible initial directions for a distinguished curve. In Riemannian signature conformal geometry, there is a single orbit which is the conformal circles. In pseudo-Riemannian signature, there are three orbits, classified by the sign of the length of the velocity of the curve; curves with null velocity are the null geodesics, and the other two classes are conformal circles. Parabolic geometries with longer gradings yield more interesting classes still. For example, in CR geometry, which is $|2|$-graded, the *chains* which generalize the chains of Chern and Moser [27] correspond to elements of $\mathfrak{g}_{-2}$.

In [39] it was shown that the distinguished curves of projective geometry, which are unparametrized geodesics of connections in the projective class, admit a similar characterization via tractors, namely they are characterized by a tractor form which is parallel along the curve for the appropriate tractor connection, and moreover this tractor form has the property that wedging with the (projective) canonical tractor yields zero. We strongly suspect that the same will hold for all parabolic geometries, and have even seen some evidence for this in the CR case in some work in progress. Additionally, the characterization of a distinguished curve in terms of a special scale given in Proposition 3.1.3 should have versions in other parabolic geometries, see [30] for more details including an example in contact Lagrangean geometry.

Chapter 4

Submanifolds in Riemannian and conformal geometry

In this chapter, we begin in earnest our discussion of submanifolds in conformal geometry. We develop general machinery for studying conformal submanifolds which we will turn to application in the later chapters of this thesis. The general theory of Riemannian submanifolds is well studied and much has been written on the subject. We briefly review the results that are relevant for our applications, before turning our attention to the less studied submanifolds of conformal manifolds. In particular, our approach will be one that utilizes the tractor calculus introduced in Chapter 2. The conformal structure of the ambient manifold induces a conformal structure on any submanifold. Thus a submanifold Γ in a conformal manifold $(M, \boldsymbol{c})$ possesses its own *intrinsic* tractor bundle, namely the tractor bundle of the conformal manifold Γ with this induced conformal structure. Analogously to the intrinsic submanifold tangent bundle in pseudo-Riemannian submanifold geometry, this intrinsic tractor bundle is isomorphic to a subbundle of the ambient tractor bundle. Moreover, this subbundle admits a complement which is isomorphic to a *tractor normal bundle*. Similarly, one obtains tractor analogs of the Gauß equation, and thus a *tractor second fundamental form*. Other notable equations and invariants from the pseudo-Riemannian setting also admit conformal analogs.

Hypersurfaces in conformal manifolds have been studied to a much greater extent than submanifolds of higher codimension. In [4], where the tractor calculus (in its modern form) was first written down, the authors define the *normal tractor* for a hypersurface and then use this to give tractor characterizations of some conformally invariant properties. A pivotal result in the theory of submanifolds in conformal manifolds is that the ambient tractor bundle has a *normal* subbundle, and the orthogonal complement of this bundle is isomorphic to the intrinsic tractor bundle. This was first observed for hypersurfaces [9], and then later for submanifolds of all codimensions [29, 11]. These two works develop and summarize much of the theory of conformal submanifolds of arbitrary codimension, albeit using different techniques and presentation. We feel that the approach of Calderbank and Burstall [11] is less intuitive than that of Curry [29], and since Curry works in terms of tractors, the results have much aesthetically in common with Riemannian submanifold theory. Our conventions and techniques largely mirror those of Curry's work. Using these, we give a unified treatment of hypersurfaces and higher codimension submanifolds, but make remarks where the hypersurface case yields some simplification. Our approach emphasizes the role of the tractor normal projector and the tractor normal form as fundamental objects from which all (in a suitable sense) other submanifold invariants may be recovered.

We take a moment here to summarize the main new results of this chapter. Proposition 4.2.4 (resp. 4.4.6) shows how the tractor second fundamental form is recovered from the tractor normal projector (resp. tractor normal form). In Proposition 4.4.4 and Theorem 4.4.5, we compute explicit expressions for the derivative of the tractor normal form; the corresponding expression for the tractor normal projector is the content of Lemma 4.2.6. Among the new results in this chapter, the main ones are Theorems 4.2.7 (resp. 4.4.7) where we prove the tractor projector (resp. tractor normal) form being parallel is a necessary and sufficient condition for the tractor second fundamental form to vanish.

We regard the tractor normal form as the best generalization to higher codimension of the hypersurface normal tractor. We illustrate this approach by showing how the tractor analog of the second fundamental form may be expressed in terms of either the normal projector of the normal form. These relations will form the basis for our discussion of distinguished conformal submanifolds in the subsequent chapter.

4.1 Submanifolds in Riemannian manifolds

First, we state some conventions. We will denote by Γ an embedded submanifold of the ambient manifold M. (In the case of a curve, we will usually use the lower case γ.) Where necessary, ι will denote the embedding, so one has $\iota : \Gamma \to M$. We will also freely identify Γ with $\iota(\Gamma)$. We will generally explicitly state any additional structure with which M is endowed (e.g. Riemannian, conformal). We also reserve $m := \dim \Gamma$ and $d := \operatorname{codim} \Gamma$. Recall that n is already reserved to denote $\dim M$, and therefore $n = m + d$.

In general, we adopt the convention that Latin letters from the start of the alphabet $(a, b, c, \ldots)$ will denote ambient indices, while indices from later in the alphabet $(i, j, k, \ldots)$ will denote tangential indices. So, for example, $\mathcal{E}^a$ is the usual tangent bundle TM, $\mathcal{E}^i$ is the tangent bundle of the submanifold $T\Gamma$, and $\mathcal{E}^a{}_i$ denotes the bundle $TM \otimes T^*\Gamma$. Note that indices alone will not distinguish sections of TM and $TM|_\Gamma$ (technically $TM|_{\iota(\Gamma)}$), so v^a could be a section of either $\mathcal{E}^a$ or a section of $\mathcal{E}^a|_\Gamma$, where $\mathcal{E}^a|_\Gamma \to \Gamma$ is the pullback bundle ι^*TM.

Definition 4.1.1. In abstract indices, the canonical map $T\iota : T\Gamma \to TM$ will be written Π_i^a and viewed as a section of $T^*\Gamma \otimes TM|_\Gamma$.

We will frequently identify $T\Gamma$ with $T\iota(T\Gamma) \subset TM|_\Gamma$ without comment.

Definition 4.1.2 (Pullback metric)**.** Choosing a metric g on M determines a metric on Γ by pullback (since the map ι is an immersion). Denoting this pullback metric by g_Γ, one has $g_\Gamma = \iota^* g$.

Explicitly, for $U, V \in \mathfrak{X}(\Gamma)$,

$$(\iota^* g)(U, V) = g(T\iota(U), T\iota(V)) = g(\Pi(U), \Pi(V)). \tag{4.1.1}$$

Writing g_{ij} for g_Γ, the above display may be rewritten in abstract indices as

$$g_{ij} U^i U^j = g_{ab} \Pi_i^a U^i \Pi_j^b V^j, \tag{4.1.2}$$

whence

$$g_{ij} = \Pi_i^a \Pi_j^b g_{ab}, \tag{4.1.3}$$

and one sees that the map Π_i^a is equivalently restriction to the submanifold Γ when acting on ambient form indices.

For any inclusion of manifolds $\iota : \Gamma \hookrightarrow M$, there is a short exact sequence of vector bundles on Γ

$$0 \longrightarrow T\Gamma \xrightarrow{T\iota} TM|_\Gamma \xrightarrow{p} T_{M/\Gamma} := TM|_\Gamma / T\iota(T\Gamma) \longrightarrow 0 \tag{4.1.4}$$

and in this, $T\iota = \Pi_i^a : \mathcal{E}^i \to \mathcal{E}^a|_\Gamma$ is the map which pushes forward vector fields.

Dualizing the above sequence,

$$0 \longrightarrow \left(T_{M/\Gamma}\right)^* \xrightarrow{p^*} T^*M|_\Gamma \xrightarrow{(T\iota)^*} T^*\Gamma \longrightarrow 0 \tag{4.1.5}$$

and here the map $(T\iota)^* : T^*M|_\Gamma \to T^*\Gamma$ is the restriction of forms on M to the submanifold Γ; in abstract indices, this again the map Π_i^a as in e.g. (4.1.3).

If moreover M is equipped with a Riemannian metric g or a conformal metric $\boldsymbol{g}$, one may use such a metric to split the above sequences. We shall split (4.1.4) by constructing a map $TM|_\Gamma \to T\Gamma$ which is a left inverse to $T\iota$; this map is essentially a tangential projection. We can realize this via the following composition, which uses only $T\iota$, the ambient metric g and the pullback metric g_Γ:

$$TM|_\Gamma \xrightarrow{g_{ab}} T^*M|_\Gamma \xrightarrow{\Pi^b_j} T^*\Gamma \xrightarrow{g^{ij}} T\Gamma$$

$$v^a \longmapsto g_{ab}v^a \longmapsto \Pi^b_j g_{ab} v^a \longmapsto g^{ij}\Pi^b_j g_{ab} v^a$$

Denote this composition by $\Pi^i_a : TM|_\Gamma \to T\Gamma$:

$$\Pi^i_a := g^{ij}\Pi^b_j g_{ab}. \tag{4.1.6}$$

Note that here we are assuming that the induced metric g_{ij} is invertible. We shall henceforth assume that this holds. In particular, this means that the techniques developed throughout the remainder of this thesis will not apply to null submanifolds, for example, the null geodesics discussed in Section 3.3.

The notation Π^i_a is therefore also consistent with the practice of raising and lowering indices via a metric. It remains to verify that this is a left inverse of $T\iota$, namely that the composition $\Pi^i_a \Pi^a_j$ is the identity on $T\Gamma$.

From equations (4.1.3) and (4.1.6), we see that

$$\Pi^i_a \Pi^a_j = \left(g^{ik}\Pi^b_k g_{ab}\right)\Pi^a_j = g^{ik} g_{ab} \Pi^a_j \Pi^b_k = g^{ik} g_{jk} = \delta^i_j.$$

Since we know that the sequence (4.1.4) splits, $T\iota(T\Gamma)$ admits an orthogonal complement inside $TM|_\Gamma$.

To describe this complement explicitly, let

$$N_p\Gamma := \{n \in T_pM : g_p(n, v) = 0 \text{ for all } v \in T\iota(T\Gamma)_p\}. \tag{4.1.7}$$

The *normal bundle* $N\Gamma$ is then defined as the disjoint union over Γ of these fibers:

$$N\Gamma := \coprod_{p \in \iota(\Gamma)} N_p\Gamma. \tag{4.1.8}$$

Note that by equation (4.1.7), the normal bundle (and therefore also the above decomposition) is invariant under a conformal rescaling of the metric.

The normal bundle is clearly an orthogonal complement to the subbundle $T\iota(T\Gamma) \subset TM$:

$$TM|_\Gamma = T\iota(T\Gamma) \oplus N\Gamma. \tag{4.1.9}$$

Moreover, the composition $\Pi^a_b := \Pi^a_i \Pi^i_b : TM|_\Gamma \to T\iota(T\Gamma)$, is the identity on $T\iota(T\Gamma)$ and zero on $N\Gamma$. Thus Π^a_b is orthogonal projection onto the first factor of (4.1.9). It then follows that $\mathrm{N}_b^a := \delta_b^a - \Pi_b^a : TM|_\Gamma \to N\Gamma \subset TM|_\Gamma$ is projection onto the second factor of (4.1.9).

This also shows that $T_{M/\Gamma} = TM|_\Gamma / T\iota(T\Gamma)$ may be identified with the normal bundle $\mathrm{N}\Gamma$, since $\mathrm{im}\,(\mathrm{N}_b^a) = N\Gamma$ and $\ker\,(\mathrm{N}_b^a) = T\iota(T\Gamma) \cong T\Gamma$.

Summarizing, we have

$$0 \longrightarrow \mathcal{E}^i \underset{\Pi_a^i}{\overset{\Pi_i^a}{\rightleftarrows}} \mathcal{E}^a|_\Gamma \xrightarrow{\mathrm{N}_a^b} N\Gamma \longrightarrow 0 \tag{4.1.10}$$

Write ∇_i for the pullback connection $\iota^*\nabla$ on $\iota^*TM \cong TM|_\Gamma$, where ∇ is the Levi-Civita connection of (M, g). Equivalently,

$$\nabla_i V^b = \Pi_i^a \nabla_a \widetilde{V}^b, \tag{4.1.11}$$

where V^b is a section of $\mathcal{E}^b|_\Gamma$, and $\widetilde{V}^b$ is any extension of V^b to the whole of M. It can be seen that $\nabla_i V^b$ as defined here does not depend on the choice of extension.

In a slight abuse of notation, we will sometimes not distinguish between the section $V^b \in \Gamma(\mathcal{E}^b|_\Gamma)$ and the extension $\tilde{V}^b \in \Gamma(\mathcal{E}^b)$ when switching between the pullback connection and the ambient connection. Since ultimately we will restrict to along the submanifold, this inaccuracy will not affect the final formula.

Clearly conformally related metrics on M will pullback to conformally related metrics on Γ. Thus the conformal structure $\boldsymbol{c}$ on M induces a conformal structure on Γ, which we denote by $\boldsymbol{c}_\Gamma$. Then

$$\boldsymbol{c}_\Gamma = \{g_\Gamma := \iota^* g : g \in \boldsymbol{c}\}.$$

Note that $(\Gamma, \boldsymbol{c}_\Gamma)$ is a conformal manifold in its own right; we will refer to objects defined solely on Γ without considering its embedding in the ambient manifold as *intrinsic*.

As a conformal manifold, it possesses its own conformal metric $\boldsymbol{g}_{ij} \in \mathcal{E}_{(ij)}[2]$, which we call the *intrinsic conformal metric*.

Any choice of ambient metric $g \in \boldsymbol{c}$ induces a Levi-Civita connection on the Riemannian manifold $(\Gamma, \iota^* g)$. This connection is very closely related to the pullback of the ambient Levi-Civita connection determined by g:

Theorem 4.1.3. *Let $\iota : \Gamma \hookrightarrow M$ be a submanifold of a conformal manifold $(M, \boldsymbol{c})$. Choose a metric $g \in \boldsymbol{c}$, and let ∇ and D denote the Levi-Civita connections of g and g_Γ respectively, where $g_\Gamma := \iota^* g$ is the pullback metric on Γ. Then*

$$D_i V^j = \Pi_b^j \nabla_i \left(\Pi_k^b V^k \right), \tag{4.1.12}$$

where $V^j \in \mathcal{E}^j$.

Proof. The right-hand side of (4.1.12) defines a connection on $T\Gamma$. Moreover, one can check that it is torsion-free and preserves the intrinsic metric g_Γ, and thus must be the Levi-Civita for the metric g_Γ by uniqueness. For details, see e.g. [56]. □

Thus the intrinsic Levi-Civita connection of the submanifold Γ is completely described by the tangential part of a choice of Levi-Civita connection for an ambient metric which restricts to the given metric on the submanifold.

The ambient connection also induces a connection on the normal bundle $N\Gamma$:

Definition 4.1.4 (Normal connection)**.** Let $\nu \in \Gamma(N\Gamma)$, and define

$$\nabla_i^\perp \nu^a := \mathrm{N}_b^a \nabla_i \nu^b, \tag{4.1.13}$$

where ∇_i is the pullback connection. The above formula clearly defines a connection on $N\Gamma$ which we call the *normal connection*.

Suppose now that $\nu \in \Gamma(N\Gamma[w])$, so now we allow ν to have a weight. From (2.2.1) and (2.2.2), one sees that

$$\widehat{\nabla}_i^\perp \nu^a = \nabla_i^\perp \nu^a + (w+1)\Upsilon_i \nu^a. \tag{4.1.14}$$

In particular, coupling the normal connection with the Levi-Civita connection on $\mathcal{E}[-1]$ yields a conformally invariant connection on $N\Gamma[-1]$. Recall that the conformal metric has conformal weight 2 and therefore restricts to a genuine metric on $N\Gamma[-1]$, since, for any $\mu, \nu \in \Gamma(N\Gamma[-1])$, we have $\boldsymbol{g}_{ab}\mu^a\nu^b$ is a section of the trivial bundle $\mathcal{E}$. Thus one may locally construct orthonormal bases for $N\Gamma[-1]$. The restriction of the ambient conformal metric to the normal bundle is also clearly preserved by this normal connection.

4.1.1 The Gauß formula

The normal part of the ambient Levi-Civita connection acting on tangent vectors is characterized by the *second fundamental form*.

Definition 4.1.5 (Second fundamental form)**.** Let $\Gamma \hookrightarrow M$ be a submanifold of a Riemannian manifold (M, g). The *second fundamental form*, denoted $I\!I_{ij}{}^c$, is defined by the Gauß formula:

$$\nabla_i u^c = \Pi_j^c D_i u^j + I\!I_{ij}{}^c u^j, \tag{4.1.15}$$

where ∇_i is the pullback connection $\iota^*\nabla$ of the Levi-Civita connection ∇ of g, $u^j \in \mathcal{E}^j$, and $u^c := \Pi_j^c u^j$. Note that the bilinear form $I\!I_{ij}{}^c$ defined by this equation is symmetric, since both the ambient and intrinsic Levi-Civita connections are torsion-free, and the Lie bracket of vector fields tangent to the submanifold remains tangential.

Remark 4.1.6 (A note on conventions)**.** Another common way to define the second fundamental form, particularly in the case of hypersurfaces, is to define the second fundamental form to be the derivative of the unit conormal (or the normal projection in the case of non-hypersurface submanifolds). This agrees with our convention *up to sign*:

$$II^{\text{Gauß}}{}_{ij}{}^{c} = -II^{\nabla \text{N}}{}_{ij}{}^{c}. \tag{4.1.16}$$

This means that some of our later formulae will be slightly different from formulae appearing in other literature in the field (usually only different up to a sign). Our convention will be to define the second fundamental form by the Gauß formula, namely $II^{\text{Gauß}}{}_{ij}{}^{c}$ is the section $II_{ij}{}^{c}$ from (4.1.15).

It is also common in the hypersurface case to view the second fundamental form as a scalar-valued symmetric 2-form on Γ, denoted II_{ij} rather than the $N\Gamma$-valued form defined here. In this case, one recovers the normal-valued form by declaring

$$II_{ij}{}^{c} := II_{ij} N^{c},$$

where N^c is the submanifold unit normal vector field.

Definition 4.1.7 (Mean curvature vector)**.** Let $g \in \boldsymbol{c}$, $II_{ij}{}^{c}$ be the second fundamental form defined by the Levi-Civita connection of g. The *mean curvature vector* is the $\boldsymbol{g}$-trace of the second fundamental form:

$$H^{c} := \frac{1}{\dim \Gamma} \boldsymbol{g}^{ij} II_{ij}{}^{c}. \tag{4.1.17}$$

Division by the dimension of Γ means that

$$II_{ij}{}^{c} = \mathring{II}_{ij}{}^{c} + \boldsymbol{g}_{ij} H^{c}, \tag{4.1.18}$$

where $\mathring{II}_{ij}{}^{c} := II_{(ij)_0}{}^{c}$ is the trace-free (with respect to the conformal metric) part of the second fundamental form.

Remark 4.1.8 (A note on weights.)**.** In equation (4.1.17), we take the trace with the conformal metric. Therefore, the mean curvature will have a conformal weight: $H^c \in \mathcal{E}^c[-2]$. To recover the usual (unweighted) mean curvature, we choose a metric in the conformal class to trivialize the density bundles. Recall that a choice of metric $g \in \boldsymbol{c}$ determines a scale $\sigma \in \mathcal{E}_+[1]$ by $g = \sigma^{-2}\boldsymbol{g}$; the unweighted mean curvature is then given by $\tilde{H}^c := \sigma^2 H^c$, where σ is this same scale. This is the usual mean curvature vector of Γ, viewed as a submanifold of the Riemannian manifold (M, g). We will not make this distinction explicitly in what follows, since "working in the scale σ" corresponds to $\sigma = 1$ and hence in the scale the weighted and unweighted mean curvatures agree. However one can always infer which mean curvature vector is being used from the surrounding objects. Since all formulae are implicitly written relative to a chosen metric, we always have a trivialization of the density bundles available.

Although the second fundamental form is not conformally invariant, it has a simple conformal transformation law, which we calculate. Choose an ambient metric $g \in \boldsymbol{c}$ with pullback g_Σ. Let $\widehat{g} \in \boldsymbol{c}$ be a conformally related ambient metric and $\widehat{g_\Gamma} := \iota^* \widehat{g}$ be its pullback. We make three observations which aid us in computing the conformal transformation of the second fundamental form. First, the ambient Levi-Civita connection of the conformally related metric must also satisfy its own Gauß formula for the transformed rescaled Levi-Civita connection and the rescaled second fundamental form:

$$\widehat{\nabla}_i u^c = \Pi^c_j \widehat{D}_i u^j + \widehat{I\!I}_{ij}{}^c u^j. \tag{4.1.19}$$

Second, we have already seen in (2.2.2) how the ambient Levi-Civita connection transforms under a conformal rescaling:

$$\widehat{\nabla}_i u^c = \nabla_i u^c + \Upsilon_i u^c - u_i \Upsilon^c + \Upsilon_d u^d \Pi^c_i.$$

Note that the i index in the above display is tangential and therefore for the pullback connection, $\delta^c_i = \Pi^c_i$.

Finally, the submanifold intrinsic Levi-Civita connection will transform similarly, since $(\Gamma, \boldsymbol{c}_\Gamma)$ is itself a conformal manifold and $D \in \boldsymbol{c}_\Gamma$:

$$\widehat{D}_i u^j = D_i u^j + \Upsilon_i u^j - u_i \Upsilon^j + \Upsilon_k u^k \delta^j_i. \tag{4.1.20}$$

We can combine the above to determine the transformation of the second fundamental form.

$$\begin{aligned}
\widehat{\nabla}_i u^c &= \nabla_i u^c + \Upsilon_i u^c - u_i \Upsilon^c + \Upsilon_d u^d \Pi^c_i \\
&= \left(\Pi^c_j D_i u^j + I\!I_{ij}{}^c u^j\right) + \Upsilon_i u^c - u_i \Upsilon^c + \Upsilon_d u^d \Pi^c_i \\
&= \Pi^c_j D_i u^j + I\!I_{ij}{}^c u^j + \Upsilon_i \Pi^c_j u^j - u_i \left(\Pi^c_d + \mathrm{N}^c_d\right) \Upsilon^d + \Upsilon_k u^k \Pi^c_i \\
&= \Pi^c_j \left(D_i u^j + \Upsilon_i u^j - u_i \Upsilon^j + \Upsilon_k u^k \delta^j_i\right) + \left(I\!I_{ij}{}^c u^j - u_i \mathrm{N}^c_d \Upsilon^d\right) \\
&= \Pi^c_j \widehat{D}_i u^j + \left(I\!I_{ij}{}^c - \boldsymbol{g}_{ij} \mathrm{N}^c_d \Upsilon^d\right) u^j,
\end{aligned}$$

so comparing with (4.1.19), the term in brackets must be $\widehat{I\!I}_{ij}{}^c$.

We conclude that under a conformal transformation,

$$\widehat{I\!I}_{ij}{}^c = I\!I_{ij}{}^c - \boldsymbol{g}_{ij} \mathrm{N}^c_d \Upsilon^d. \tag{4.1.21}$$

Since this transformation is by pure trace, it follows immediately that $\mathring{I\!I}_{ij}{}^c$ is conformally invariant:

$$\widehat{\mathring{I\!I}}_{ij}{}^c = \mathring{I\!I}_{ij}{}^c.$$

Thus the transformation of (4.1.21) is entirely due to the transformed mean curvature, whence

$$\widehat{H}^c = H^c - \mathrm{N}^c_d \Upsilon^d. \tag{4.1.22}$$

We will be interested in various special classes of conformal submanifolds. We introduce the first of these here. Recall that a submanifold Γ in a Riemannian manifold (M, g) is called *totally geodesic* if its second fundamental form vanishes. Since the second fundamental form is not a conformal invariant, this notion does not make sense on a conformal manifold. However, we have seen that the *trace-free* second fundamental form is conformally invariant. This gives our first example of a class of special conformal submanifolds.

Definition 4.1.9 (Totally umbilic submanifold)**.** Let M be a Riemannian or conformal manifold, $\Gamma \hookrightarrow M$ a submanifold. A point $p \in \Gamma$ where $\mathring{I\!I}_{ij}{}^c$ vanishes is called *umbilic*. Γ is said to be *totally umbilic* if every point of Γ is umbilic.

We will often omit the word "totally" when discussing a totally umbilic submanifold and simply say that a submanifold Γ satisfying Definition 4.1.9 is *umbilic*. Umbilic submanifolds are a well-studied class of submanifolds, of interest in mathematics and physics [26, 25].

Having defined the second fundamental form, we can state the Gauß, Codazzi-Mainardi and Ricci equations along the submanifold Γ:

$$R_{ijkl} = R^{\Gamma}_{ijkl} + 2\boldsymbol{g}_{cd} I\!I_{l[i}{}^c I\!I_{j]k}{}^d, \tag{4.1.23}$$

$$R_{ij}{}^c{}_k \mathrm{N}^d_c = 2D_{[i} I\!I_{j]k}{}^d, \tag{4.1.24}$$

$$R_{ij}{}^a{}_b \mathrm{N}^c_a \mathrm{N}^b_d = R^{\perp}{}_{ij}{}^c{}_d + 2\boldsymbol{g}^{kl} I\!I_{l[i}{}^c I\!I_{j]kd}, \tag{4.1.25}$$

where $R_{ijkl} := \Pi^a_i \Pi^b_j \Pi^c_k \Pi^d_l R_{abcd}$ is the curvature of the ambient Levi-Civita connection restricted to Γ, R^{Γ}_{ijkl} is the intrinsic Riemann curvature tensor (i.e. the curvature of the connection D), D is the intrinsic Levi-Civita connection coupled to the normal connection and $R^{\perp}{}_{ij}{}^c{}_d$ is the curvature of the normal connection $\nabla^{\perp}_i$. These are standard equations in Riemannian submanifold geometry. Proofs may be found in e.g. [56], but we sketch the idea here. These formulae are all derived by substituting the Gauß formula (4.1.15) into equation (1.2.2) which defines the curvature of the pullback connection ∇_i. Using the decomposition $TM = T\Gamma \oplus N\Gamma$, we may write a section $v^c \in \Gamma(\mathcal{E}^c)$ as a tuple $(\Pi^c_d v^d, \mathrm{N}^c_d v^d)$. Note also that $\Pi^c_d v^d = \Pi^c_k u^k$ for a unique $u^k \in \Gamma(\mathcal{E}^k)$. In a slight abuse of notation, we will simply write v^k for this unique section u^k. From the Gauß formula (4.1.15), we have that

$$\nabla_j \begin{pmatrix} V^k \\ \mathrm{N}^c_d V^d \end{pmatrix} = \begin{pmatrix} D_j V^k - I\!I_j{}^k{}_d V^d \\ I\!I_{jk}{}^c V^k + \nabla^{\perp}_j \left(\mathrm{N}^c_d V^d\right) \end{pmatrix}.$$

Acting again with the connection on such a tuple gives the action of the Riemann curvature $R_{ij}{}^c{}_d$:

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) \begin{pmatrix} V^k \\ \mathrm{N}^c_d V^d \end{pmatrix}$$

$$= \begin{pmatrix} R^{\Gamma}{}_{ij}{}^{k}{}_{l}V^{l} - 2D_{[i}\left(\mathit{II}_{j]}{}^{k}{}_{d}V^{d}\right) - 2\boldsymbol{g}_{ef}\mathit{II}^{k}{}_{[i}{}^{f}\mathit{II}_{j]l}{}^{e}V^{l} - 2\mathit{II}_{[i}{}^{k}{}_{|e|}\nabla^{\perp}_{j]}\left(\mathrm{N}^{e}_{d}V^{d}\right) \\ 2\mathit{II}_{k[i}{}^{c}\left(D_{j]}V^{k}\right) - 2\boldsymbol{g}^{kl}\mathit{II}_{k[i}{}^{c}\mathit{II}_{j]ld}V^{d} + 2\nabla^{\perp}_{[i}\left(\mathit{II}_{j]k}{}^{c}V^{k}\right) + R^{\perp}{}_{ij}{}^{c}{}_{d}V^{d} \end{pmatrix}$$

$$= \begin{pmatrix} R^{\Gamma}{}_{ij}{}^{k}{}_{l} + 2\boldsymbol{g}_{cd}\mathit{II}_{l[i}{}^{c}\mathit{II}^{k}{}_{j]}{}^{d} & -2D_{[i}\mathit{II}_{j]}{}^{k}{}_{d} \\ 2D_{[i}\mathit{II}_{j]l}{}^{c} & R^{\perp}_{ij}{}^{c}{}_{d} + 2\boldsymbol{g}^{kl}\mathit{II}_{l[i}{}^{c}\mathit{II}_{j]kd} \end{pmatrix} \begin{pmatrix} V^{l} \\ \mathrm{N}^{d}_{e}V^{e} \end{pmatrix},$$

where we have used

$$\nabla^{\perp}_{j}\left(\mathit{II}_{ik}{}^{c}V^{k}\right) = \left(D_{j}\mathit{II}_{ik}{}^{c}\right)V^{k} + \mathit{II}_{ik}{}^{c}\left(D_{j}V^{k}\right)$$

and the corresponding

$$D_{i}\left(\mathit{II}_{j}{}^{k}{}_{c}V^{c}\right) = \left(D_{i}\mathit{II}_{j}{}^{k}{}_{c}\right)\mathrm{N}^{c}_{d}V^{d} + \mathit{II}_{j}{}^{k}{}_{c}\nabla^{\perp}_{i}\left(\mathrm{N}^{c}_{d}V^{d}\right)$$

for the adjoint second fundamental form to simplify the off-diagonal entries. (Note that while we write D or $\nabla^{\perp}$ here, in reality the connections written will be coupled to at least one other connection in order to act on the illustrated section.) Note also that we have interchanged i and j in the second term of both diagonal entries, resulting in a sign change. One can check that the resulting expressions really are equal to those in the previous line by noting which of the two indices i and j is on the copy of the second fundamental forms which acts on the appropriate element of the tuple. The formulae (4.1.23), (4.1.24) and (4.1.25) now follow by projecting out various elements of this matrix.

4.1.2 Submanifolds and densities; minimal scales

For any $w \in \mathbb{R}$, recall $\mathcal{E}[w]$ denotes the line bundle of conformal w-densities (on M), defined in Section 2.1. Write $\mathcal{E}_{\Gamma}[w]$ for the intrinsic bundle of w-densities on Γ and $\mathcal{E}[w]|_{\Gamma}$ for the restriction of ambient w-densities to Γ. Recall that a choice of metric $g \in \boldsymbol{c}$ trivializes the density bundle $\mathcal{E}[w]$. One sees that, trivializing $\mathcal{E}[w]|_{\Gamma}$ via $g \in \boldsymbol{c}$ yields an isomorphic bundle to trivializing $\mathcal{E}_{\Gamma}[w]$ via $g_{\Gamma} := \iota^{*}g$. Thus the bundles $\mathcal{E}[w]|_{\Gamma}$ and $\mathcal{E}_{\Gamma}[w]$ are isomorphic. We also extend our notation for weighted ambient bundles to bundles over Γ. If $\mathcal{B}$ is a bundle over Γ, then we set $\mathcal{B}[w] := \mathcal{B} \otimes \mathcal{E}_{\Gamma}[w]$.

Recall that the Levi-Civita connection of a metric $g \in \boldsymbol{c}$ acts on a density $\tau \in \mathcal{E}[w]$ according to

$$\nabla\tau = \sigma^{w}\mathrm{d}\left(\sigma^{-w}\tau\right), \tag{4.1.26}$$

where $\sigma \in \mathcal{E}[1]$ is the scale determined by the metric g and d is the exterior derivative. Any connection acts on smooth functions by the exterior derivative. Therefore

$$\nabla_{i}\tau = D_{i}\tau,$$

for $\tau \in \mathcal{E}[w]|_{\Gamma} \cong \mathcal{E}_{\Gamma}[w]$.

When working with submanifolds, it will often be convenient to work in an ambient scale that is adapted to that submanifold. The right notion of adapted here is that the mean curvature of the scale should be zero.

Definition 4.1.10 (Minimal scale)**.** Let $\Gamma \hookrightarrow M$ be a submanifold. A scale $\sigma \in \mathcal{E}_+[1]$ for which the mean curvature vector H^c of Γ vanishes identically is a *minimal scale.*

It was observed in e.g. [9, 36] that such scales exist for hypersurfaces in conformal submanifolds. Curry [29] proves that such scales exist for submanifolds of all codimension. Calderbank and Burstall [11] similarly arrive at a minimal scale for a submanifold, which in their setting they call the *canonical Möbius reduction.* Given a smooth curve in a conformal manifold, one may always choose a scale in the conformal class for which the curve is a geodesic for the corresponding Levi-Civita connection (c.f. Proposition 5.1.2 and see e.g. [30, Lemma 2] for a more elementary proof). One can view a minimal scale as a generalization of such an adapted curve scale. Indeed, we will see in Section 5.1 that a scale for which a given curve is an affine geodesic is exactly a minimal scale for the curve viewed as a 1-dimensional submanifold $\gamma \hookrightarrow M$, and thus the existence of minimal scales gives an alternative proof of the mentioned result about special scales for curves.

Just as we refer to both a scale and the metric it determines as a scale, given a minimal scale σ, we will sometimes refer to the metric $g := \sigma^{-2}\boldsymbol{g}$ as a minimal scale. Thus we have

Theorem 4.1.11. *Let $\Gamma \hookrightarrow M$ be a submanifold in a conformal manifold $(M, \boldsymbol{c})$, and $g_\Gamma \in \boldsymbol{c}_\Gamma$ a metric on Γ from the induced conformal structure. Then there exists a metric $g \in \boldsymbol{c}$ on M such that $g_\Gamma = \iota^* g$ and the mean curvature vector of Γ in the scale g vanishes.*

4.1.3 Volume and normal forms

Recall that the normal bundle $N^*\Gamma$ is a vector bundle with d-dimensional fibers. Thus $\Lambda^d N^*\Gamma$ is a line bundle. There is thus a distinguished section $N_{a_1 a_2 \cdots a_d} \in \Gamma(\Lambda^d N^*\Gamma)$ of this line bundle characterized by

1. $N_{a_1 a_2 \cdots a_d} v^{a_1} = 0$ for all $v \in \Gamma(T\Gamma)$; and
2. $N_{a_1 a_2 \cdots a_d} N^{a_1 a_2 \cdots a_d} = d!$.

It follows from these properties that, given a local orthonormal basis $\{n^1, n^2, \cdots, n^d\}$ for $N^*\Gamma$, one has

$$N_{a_1 a_2 \cdots a_d} = d! \cdot n^1_{[a_1} n^2_{a_2} \cdots n^d_{a_d]}. \tag{4.1.27}$$

Much of the information about the embedding $(\Gamma, g_\Gamma) \hookrightarrow (M, g)$ is encoded in this section; many of the Riemannian invariants may be expressed in terms of it. We will see in subsequent sections that there is a tractor analog of this object which similarly encodes the conformal invariants in the case where M is instead equipped with a conformal structure.

It is also useful for future applications to record the relations between the ambient and submanifold volume forms and this normal form.

Proposition 4.1.12.

$$\mathrm{vol}_M^{a_1 a_2 \cdots a_{n-d} a_{n-d+1} \cdots a_{n-1} a_n} N_{a_{n-d+1} \cdots a_n} = d! \cdot \mathrm{vol}_\Gamma^{a_1 a_2 \cdots a_m}. \tag{4.1.28}$$

Proof. Fix local orthonormal bases for the tangent bundle $\{u_1, \ldots, u_m\}$ and the normal bundle $\{n_1, \ldots, n_d\}$. With our conventions,

$$\mathrm{vol}_M^{a_1 a_2 \cdots a_m a_{m+1} \cdots a_n} = n! \cdot u_1^{[a_1} u_2^{a_2} \cdots u_m^{a_m} n_1^{a_{m+1}} \cdots n_d^{a_n]}.$$

Hence

$$\begin{aligned}
&\mathrm{vol}_M^{a_1 a_2 \cdots a_m a_{m+1} \cdots a_{n-1} a_n} N_{a_{m+1} \cdots a_n} \\
&= n! \cdot u_1^{[a_1} u_2^{a_2} \cdots u_m^{a_m} n_1^{a_{m+1}} \cdots n_d^{a_n]} \cdot N_{a_{m+1} \cdots a_n} \\
&= n! \cdot \frac{m! \cdot (n-m)!}{n!} \cdot u_1^{[a_1} u_2^{a_2} \cdots u_m^{a_m]} \cdot n_1^{[a_{m+1}} \cdots n_d^{a_n]} \cdot N_{a_{m+1} \cdots a_n} \\
&= m! \cdot u_1^{[a_1} u_2^{a_2} \cdots u_m^{a_m]} \cdot \left(d! \cdot n_1^{[a_{m+1}} \cdots n_d^{a_n]} \cdot N_{a_{m+1} \cdots a_n} \right) \\
&= m! \cdot u_1^{[a_1} u_2^{a_2} \cdots u_m^{a_m]} \cdot \left(N^{a_{m+1} \cdots a_n} N_{a_{m+1} \cdots a_n} \right) \\
&= \left(m! \cdot u_1^{[a_1} u_2^{a_2} \cdots u_m^{a_m]} \right) \cdot d! \\
&= d! \cdot \mathrm{vol}_\Gamma^{a_1 a_2 \cdots a_m},
\end{aligned}$$

where, in the third line, we partition the tangent and normal vectors into groups of size $n-d$ and d, and only the partition that does not have tangent and normal vectors mixed will yield a nonzero contraction. □

As a corollary, we see the relationship between the intrinsic volume form, the ambient volume form and the normal form.

Corollary 4.1.13.

$$\mathrm{vol}^M_{a_1 \cdots a_m a_{m+1} \cdots a_n} = \mathrm{vol}^\Gamma_{a_1 \cdots a_m} \wedge N_{a_{m+1} \cdots a_n} \tag{4.1.29}$$

Proof. Note that further contracting the right-hand side of (4.1.28) with $\mathrm{vol}^\Gamma_{a_1 \cdots a_m}$ yields

$$\mathrm{vol}^\Gamma_{a_1 \cdots a_m} \left(d! \cdot \mathrm{vol}_\Gamma^{a_1 a_2 \cdots a_m} \right) = d! \cdot m!.$$

Now, $\mathrm{vol}^{\Gamma}_{[a_1\cdots a_m}N_{a_{m+1}\cdots a_n]} \in \mathcal{E}_{[a_1\cdots a_n]}[n]$, which is a line bundle, and hence $\mathrm{vol}^{\Gamma}_{[a_1\cdots a_m}N_{a_{m+1}\cdots a_n]} = f\mathrm{vol}^{M}_{a_1\cdots a_n}$ for some f a real-valued function on M.

Since

$$\mathrm{vol}^{M}_{a_1\cdots a_n}\mathrm{vol}_{M}^{a_1\cdots a_n} = n!,$$

and

$$\mathrm{vol}_{M}^{a_1\cdots a_n}\mathrm{vol}^{\Gamma}_{a_1\cdots a_m}N_{a_{m+1}\cdots a_n} = m!\cdot d!,$$

it follows that

$$f = \frac{m!\cdot d!}{n!}.$$

With our conventions,

$$\mathrm{vol}^{\Gamma}_{a_1\cdots a_m}\wedge N_{a_{m+1}\cdots a_d} = \frac{(m+d)!}{m!\cdot d!}\cdot \mathrm{vol}^{\Gamma}_{[a_1\cdots a_m}N_{a_{m+1}\cdots a_n]}.$$

and since $n = m + d$, it follows that

$$\mathrm{vol}^{M}_{a_1\cdots a_m a_{m+1}\cdots a_n} = \mathrm{vol}^{\Gamma}_{a_1\cdots a_m}\wedge N_{a_{m+1}\cdots a_n} \tag{4.1.30}$$

□

The results of this section still hold replacing all objects with their weighted counterparts. There is a weighted submanifold volume form $\mathrm{vol}^{\Gamma} \in \Gamma(\Lambda^m T^*\Gamma[m])$ constructed as in Section 2.1, and there is a weighted normal form $N \in \Gamma(\Lambda^d T^*M[d])$ defined by (4.1.27), with the orthogonal basis of normals replaced by an orthonormal (according to the weighted conformal metric) basis for $N^*\Gamma[1]$. The wedge of these weighted forms recovers the weighted volume form of M.

4.2 Submanifold tractors

Having reviewed the relevant Riemannian submanifold geometry theory, we turn our attention to developing tractor calculus for submanifolds. The theory here turns out to be almost as nice as one could hope, with tractor analogs of many of the standard tools and equations of Riemannian submanifold geometry. We remind the reader that we assume the ambient conformal structure has Riemannian signature, and thus the induced conformal structure on the submanifold is guaranteed to be non-degenerate.

4.2.1 The intrinsic and normal tractor bundles

Much as is the case in Riemannian geometry, the intrinsic tractor bundle of the submanifold is isomorphic to a subbundle of the ambient tractor bundle, and moreover this subbundle admits an orthogonal (with respect to the tractor metric) complement which is isomorphic to the normal bundle of the submanifold. This decomposition also induces a decomposition of the ambient tractor connection which in turn leads to a tractor Gauß formula and a tractor second fundamental form. This tractor second fundamental form will play a central role in the theory of distinguished conformal submanifolds we present in later chapters.

Definition 4.2.1 (Normal tractor bundle)**.** Let $\Gamma \hookrightarrow M$ be a submanifold in a conformal manifold $(M, \boldsymbol{c})$. Let $n_a \in N^*\Gamma[1]$ be a normal covector of conformal weight 1 and consider the map $N^*\Gamma[1] \to \mathcal{T}^*M|_\Gamma$ given by

$$n_a \mapsto N_A \stackrel{g}{=} \begin{pmatrix} 0 \\ n_a \\ n_a H^a \end{pmatrix} \tag{4.2.1}$$

where the above is written with respect to some $g \in \boldsymbol{c}$ and H^a is the mean curvature vector of Γ in that scale.

Using the transformation law for the mean curvature (4.1.22), one easily verifies that the above map is conformally invariant, and hence its image is a well-defined subbundle of the ambient tractor bundle:

$$\begin{aligned}
\widehat{N}_A &= \widehat{n}_a \widehat{Z}_A^a + \widehat{n}_a \widehat{H}^a \widehat{X}_A \\
&= n_a(Z_A^a + \Upsilon^a X_A) + n_a(H^a - \Upsilon^b \mathrm{N}_b^a) X_A \\
&= n_a Z_A^a + n_a \Upsilon^a X_A + n_a H^a X_A - n_b \Upsilon^b X_A \\
&= n_a Z_A^a + n_a H^a X_A \\
&= N_A.
\end{aligned}$$

We call the image of the map (4.2.1) the (dual) *normal tractor bundle* and denote it by $\mathcal{N}^*$ or $\mathcal{N}_A$ if we wish to explicitly show indices.

The map $N^*\Gamma[1] \to \mathcal{N}^*$ defined in (4.2.1) is clearly injective and hence defines an isomorphism $N^*\Gamma[1] \cong \mathcal{N}^*$. Raising indices with the tractor metric, one sees that the same formula defines an isomorphism $N\Gamma[-1]$ to a subbundle of $\mathcal{T}M|_\Gamma$, which we call the *normal tractor bundle*, and denote $\mathcal{N}$ or $\mathcal{N}^A$. Note that the restriction of the tractor metric to $\mathcal{N}$ coincides with the restriction of the ambient conformal metric to $N\Gamma[-1]$. So we may construct a local orthonormal frame for the normal tractor bundle by simply mapping such a frame for $N\Gamma[-1]$ to $\mathcal{N}$ under the isomorphism (4.2.1).

Let $\mathcal{N}^\perp$ denote the orthogonal complement of $\mathcal{N}$ inside $\mathcal{T}M|_\Gamma$. Then one has a decomposition

$$\mathcal{T}M|_\Gamma = \mathcal{N}^\perp \oplus \mathcal{N} \tag{4.2.2}$$

with corresponding projection maps $\mathrm{N}^A_B : \mathcal{E}^B|_\Gamma \to \mathcal{N}^A$ and $\Pi^A_B : \mathcal{E}^B|_\Gamma \to \mathcal{N}^\perp$, where

$$\Pi^A_B := \delta^A_B - \mathrm{N}^A_B. \tag{4.2.3}$$

In fact, we shall soon see that the bundle $\mathcal{N}^\perp$ is isomorphic to the standard tractor bundle $\mathcal{T}\Gamma$ of the submanifold Γ. For now, we conclude by computing an explicit expression for this normal projector in the tractor projector notation.

Lemma 4.2.2. *For a choice of scale, the tractor normal projector is given by*

$$\mathrm{N}^A_B = \mathrm{N}^a_b Z^A_a Z^b_B + H^a Z^A_a X_B + H_b X^A Z^b_B + (H^d H_d) X^A X_B, \tag{4.2.4}$$

where the H^c is the mean curvature vector in the chosen scale.

Proof. The right-hand side of (4.2.4) defines a conformally invariant bundle map $\mathcal{E}^A \to \mathcal{N}^B$ which moreover acts as the identity on sections of $\mathcal{N}^A$ as defined in Definition 4.2.1. □

Our convention is to use the word "intrinsic" to refer to objects of the submanifold Γ viewed as a conformal/Riemannian manifold without any embedding in an ambient space. Thus the *intrinsic tractor bundle* refers to the standard tractor bundle $\mathcal{T}\Gamma$ of the conformal manifold $(\Gamma, \boldsymbol{c}_\Gamma)$.

We are now ready to prove another conformal analog of a Riemannian theorem, namely that the intrinsic tractor bundle of a submanifold is isomorphic to the orthogonal complement of the normal tractor bundle. For hypersurfaces, this was first observed in [9], and then further developed in [44, 62, 69]. The same result for higher codimension conformal submanifolds was shown in [29, 11].

Theorem 4.2.3. *The intrinsic tractor bundle $\mathcal{T}\Gamma$ is canonically isomorphic to the orthogonal complement $\mathcal{N}^\perp$ of the normal tractor bundle via a bundle isomorphism which preserves both the metric and the filtration. We denote this isomorphism Π^A_I. Explicitly, in a general ambient scale $g \in \boldsymbol{c}$, it is given by*

$$V^I \stackrel{g_\Gamma}{=} \begin{pmatrix} \sigma \\ \mu^i \\ \rho \end{pmatrix} \xmapsto{\Pi^A_I} V^A \stackrel{g}{=} \begin{pmatrix} \sigma \\ \mu^a - H^a\sigma \\ \rho - \frac{1}{2}H^a H_a \sigma \end{pmatrix} \tag{4.2.5}$$

Proof. Fix a scale $g_\Gamma \in \boldsymbol{c}_\Gamma$, and let $g \in \boldsymbol{c}$ be a scale that satisfies $\iota^* g = g_\Gamma$. We need to show that the map (4.2.5) is unchanged if we replace g by some conformally related $\widehat{g} = \Omega^2 g$ and

g_Γ by $\widehat{g_\Gamma} = \Omega^2 g_\Gamma$. Equivalently, we need to show that the following diagram commutes

$$\begin{array}{ccc} [\mathcal{E}^I]_{g_\Gamma} & \xrightarrow{\Pi_I^A} & [\mathcal{E}^A]_g|_\Gamma \\ \downarrow & & \downarrow \\ [\mathcal{E}^J]_{\widehat{g_\Gamma}} & \xrightarrow[\Pi_J^B]{} & [\mathcal{E}^B]_{\widehat{g}}|_\Gamma \end{array} \tag{4.2.6}$$

where the vertical maps are conformal rescaling, and the horizontal maps are (4.2.5) in the appropriate scale. Write $\Upsilon_a = \Omega^{-1}\nabla_a\Omega$ and $\Upsilon_i = \Omega^{-1}D_i\Omega$. Note that $\Upsilon_i = \Pi_i^a\Upsilon_a$. Applying Π_I^A and then rescaling is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^b & \delta_a^b & 0 \\ -\frac{1}{2}\Upsilon^c\Upsilon_c & -\Upsilon_a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -H^a & \Pi_i^a & 0 \\ -\frac{1}{2}H^cH_c & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^b - H^b & \Pi_i^b & 0 \\ -\frac{1}{2}\Upsilon^c\Upsilon_c + H^a\Upsilon_a - \frac{1}{2}H^cH_c & -\Upsilon_i & 1 \end{pmatrix},$$

while first rescaling and then applying Π_J^B corresponds to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -\widehat{H}^b & \Pi_j^b & 0 \\ -\frac{1}{2}\widehat{H}^c\widehat{H}_c & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \Upsilon^j & \delta_i^j & 0 \\ -\frac{1}{2}\Upsilon^k\Upsilon_k & -\Upsilon_i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\widehat{H}^b + \Pi_j^b\Upsilon^j & \Pi_i^b & 0 \\ -\frac{1}{2}\widehat{H}^c\widehat{H}_c - \frac{1}{2}\Upsilon^k\Upsilon_k & -\Upsilon_i & 1 \end{pmatrix}.$$

Using equation (4.1.22), we see that

$$-\widehat{H}^b + \Pi_j^b\Upsilon^j = -H^b + \mathrm{N}_c^b\Upsilon^c + \Pi_j^b\Upsilon^j = -H^b + \Upsilon^b$$

and

$$\begin{aligned} -\frac{1}{2}\widehat{H}^c\widehat{H}_c - \frac{1}{2}\Upsilon^k\Upsilon_k &= -\frac{1}{2}H^cH_c + H^e\Upsilon_e - \frac{1}{2}\left(\Upsilon^k\Upsilon_k + \mathrm{N}^{cd}\Upsilon_c\Upsilon_d\right) \\ &= -\frac{1}{2}H^cH_c + H^e\Upsilon_e - \frac{1}{2}\Upsilon^c\Upsilon_c, \end{aligned}$$

whence the above two matrix products are equal. Hence the map Π_I^A is conformally invariant. Moreover, the map is clearly injective, and the image is also easily seen to be annihilated by any section of $\mathcal{N}$.

In the minimal scale case, the map Π_I^A clearly preserves the metric and filtration, and since we have verified conformal invariance, this is sufficient to complete the proof. □

The inverse isomorphism of (4.2.5) is the map $\mathcal{N}^\perp \to \mathcal{T}\Gamma$ given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -H^a & \Pi_i^a & 0 \\ -\frac{1}{2}H^cH_c & 0 & 1 \end{pmatrix}. \tag{4.2.7}$$

Just as the intrinsic and ambient tractor bundles are related, so are the intrinsic and ambient tractor connections.

Define a connection $\check{\nabla}$ on $\mathcal{T}\Gamma$ by

$$\check{\nabla}_i V^J := \Pi^J_B \nabla_i \left(\Pi^B_K V^K \right), \tag{4.2.8}$$

where $V^J \in \mathcal{E}^J$ and ∇ denotes the usual ambient tractor connection. (When required, we will refer to $\check{\nabla}$ as the *checked connection.*) We denote by p_{ij} the *intrinsic Schouten tensor*, namely the Schouten tensor of a metric $g_{ij} \in \boldsymbol{c}_\Gamma$. We will also encounter the restriction of the ambient Schouten tensor to $T\Gamma$, which we denote by $P_{ij} = \Pi^a_i \Pi^b_j P_{ab}$, where recall P_{ab} is the Schouten tensor of $g \in \boldsymbol{c}$. Note that in general these two objects are not equal, and this is the reason that we do not have verbatim a tractor version of Theorem 4.1.3.

Fix metrics $g \in \boldsymbol{c}$ and $g_\Gamma \in \boldsymbol{c}_\Gamma$ such that $\iota^* g = g_\Gamma$ to facilitate calculation. Then explicitly we see that

$$\begin{aligned}
\check{\nabla}_i V^J &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi^j_b & 0 \\ -\frac{1}{2}H^cH_c & -H_b & 1 \end{pmatrix} \nabla_i \left[\begin{pmatrix} 1 & 0 & 0 \\ -H^b & \Pi^b_k & 0 \\ -\frac{1}{2}H^cH_c & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma \\ \mu^k \\ \rho \end{pmatrix} \right] \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi^j_b & 0 \\ -\frac{1}{2}H^cH_c & -H_b & 1 \end{pmatrix} \nabla_i \begin{pmatrix} \sigma \\ \mu^b - H^b\sigma \\ \rho - \frac{1}{2}H^cH_c\sigma \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi^j_b & 0 \\ -\frac{1}{2}H^cH_c & -H_b & 1 \end{pmatrix} \begin{pmatrix} \nabla_i\sigma - \mu_i \\ \nabla_i \left(\mu^b - H^b\sigma\right) + P_i{}^b\sigma + \Pi^b_i \left(\rho - \frac{1}{2}H^cH_c\sigma\right) \\ \nabla_i \left(\rho - \frac{1}{2}H^cH_c\sigma\right) - P_{ic}\left(\mu^c - H^c\sigma\right) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Pi^j_b & 0 \\ -\frac{1}{2}H^cH_c & -H_b & 1 \end{pmatrix} \begin{pmatrix} \nabla_i\sigma - \mu_i \\ \nabla_i\mu^b - (\nabla_iH^b)\sigma - H^b\nabla_i\sigma + P_i{}^b\sigma + \Pi^b_i\rho - \frac{1}{2}\Pi^b_iH^cH_c\sigma \\ \nabla_i\rho - (H^c\nabla_iH_c)\sigma - \frac{1}{2}H^cH_c\nabla_i\sigma - P_{ic}\mu^c + P_{ic}H^c\sigma \end{pmatrix} \\
&= \begin{pmatrix} \nabla_i\sigma - \mu_i \\ \Pi^j_b\left(\nabla_i\mu^b - (\nabla_iH^b)\sigma - H^b\nabla_i\sigma + P_i{}^b\sigma + \Pi^b_i\rho - \frac{1}{2}\Pi^b_iH^cH_c\sigma\right) \\ -\frac{1}{2}H^cH_c(\nabla_i\sigma - \mu_i) - H_b\left(\nabla_i\mu^b - (\nabla_iH^b)\sigma - H^b\nabla_i\sigma + P_i{}^b\sigma + \Pi^b_i\rho - \frac{1}{2}\Pi^b_iH^cH_c\sigma\right) \\ +\nabla_i\rho - (H^c\nabla_iH_c)\sigma - \frac{1}{2}H^cH_c\nabla_i\sigma - P_{ic}\mu^c + P_{ic}H^c\sigma \end{pmatrix} \\
&= \begin{pmatrix} D_i\sigma - \mu_i \\ D_i\mu^j + P_i{}^j\sigma + \delta^j_i\rho - (\Pi^j_b\nabla_iH^b)\sigma - \frac{1}{2}\delta^j_iH^cH_c\sigma \\ D_i\rho - P_{ic}\mu^c + \frac{1}{2}H^cH_c\mu_i - H_b\nabla_i\mu^b \end{pmatrix} \\
&= \begin{pmatrix} D_i\sigma - \mu_i \\ D_i\mu^j + p_i{}^j\sigma + \delta^j_i\rho - (-\mathring{I\!I}_i{}^j{}_b - \delta^j_iH_b)H^b\sigma - \frac{1}{2}\delta^j_iH^cH_c\sigma + (P_i{}^j - p_i{}^j)\sigma \\ D_i\rho - p_{ic}\mu^c + \frac{1}{2}H^cH_c\mu_i - H_b(\mathring{I\!I}_{ij}{}^b + \boldsymbol{g}_{ij}H^b)\mu^j + (P_i{}^j - p_i{}^j)\mu^j \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} D_i\sigma - \mu_i \\ D_i\mu^j + p_i{}^j\sigma + \delta_i^j\rho \\ D_i\rho - p_{ic}\mu^c \end{pmatrix} + \begin{pmatrix} 0 \\ \left(P_i{}^j - p_i{}^j + H_b \mathring{I\!I}_i{}^{jb} + \tfrac{1}{2}H^cH_c\delta_i^j\right)\sigma \\ -\left(P_{ij} - p_{ij} + H_b\mathring{I\!I}_{ij}{}^b + \tfrac{1}{2}H_bH^b\boldsymbol{g}_{ij}\right)\mu^j \end{pmatrix}$$

$$= D_i^{\mathcal{T}\Gamma}\begin{pmatrix} \sigma \\ \mu^j \\ \rho \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ P_i{}^j - p_i{}^j - H_b\mathring{I\!I}_i{}^{jb} + \tfrac{1}{2}H^cH_c\delta_i^j & 0 & 0 \\ 0 & -\left(P_{ij} - p_{ij} + H_b\mathring{I\!I}_{ij}{}^b + \tfrac{1}{2}H_bH^b\boldsymbol{g}_{ij}\right) & 0 \end{pmatrix}\begin{pmatrix} \sigma \\ \mu^j \\ \rho \end{pmatrix}.$$

Defining

$$\mathsf{S}_{iJK} := 2\left(P_{ij} - p_{ij} + H_b\mathring{I\!I}_{ij}{}^b + \frac{1}{2}H_bH^b\boldsymbol{g}_{ij}\right) Z^j_{[J}X_{K]}, \tag{4.2.9}$$

we thus have

$$\check{\nabla}_i V^J = D_iV^J + \mathsf{S}_i{}^J{}_K V^K. \tag{4.2.10}$$

Since $\check{\nabla}$ and D are conformally invariant objects, it follows that S_{iJK} must also be. We call S_{iJK} the *difference tractor*. Equation (2.3.11) together with the invariance of the canonical tractor X show that $Z^j_{[J}X_{K]}$ is conformally invariant, and therefore the symmetric tensor

$$\mathcal{F}_{ij} := P_{ij} - p_{ij} + H_b\mathring{I\!I}_{ij}{}^b + \frac{1}{2}H_bH^b\boldsymbol{g}_{ij} \tag{4.2.11}$$

must be invariant as well. This tensor is called the *Fialkow tensor*, seemingly having first been studied by Aaron Fialkow in [32] under the name *deviation tensor* (where it takes a different yet equivalent form, c.f. Proposition 4.2.9). It was calculated in this current form as the discrepancy between the projection of the ambient tractor connection and the intrinsic tractor connection in [62]. Note that this tensor essentially measures the difference between the restricted ambient Schouten and the intrinsic Schouten: if one works in a minimal scale, equation (4.2.11) becomes

$$\mathcal{F}_{ij} = P_{ij} - p_{ij}.$$

Having thus defined the Fialkow tensor, the S tractor may be rewritten

$$\mathsf{S}_{iJK} = 2\mathcal{F}_{ij}\, Z^j_{[J}X_{K]}. \tag{4.2.12}$$

The ambient tractor connection also induces a connection on the normal tractor bundle. For $N^A \in \Gamma(\mathcal{N}^A)$ a section of the normal bundle, define

$$\nabla_i^{\mathcal{N}} N^A := \mathrm{N}^A_B \nabla_i N^B. \tag{4.2.13}$$

We call this the *normal tractor connection*. Note that sections of the normal bundle always have zero in their top slot. Since the middle slot of such a tractor is necessarily conformally invariant (as can be seen from equation (2.3.11)), it follows that the middle slot of $\nabla_i^{\mathcal{N}} N^A$ is conformally invariant. This is easily seen to be the normal connection (4.1.13) acting on sections of $N\Gamma[-1]$, and confirms our prior observation that the normal connection is invariant when acting on sections of this bundle.

Using (2.3.14) and (4.2.4), one sees that

$$\nabla_i^{\mathcal{N}} \begin{pmatrix} 0 \\ n^a \\ H_c n^c \end{pmatrix} = \begin{pmatrix} 0 \\ \nabla_i^{\perp} n^a \\ H_c \nabla_i^{\perp} n^c \end{pmatrix}, \tag{4.2.14}$$

which confirms that this is nothing but the normal connection acting on $N\Gamma[-1]$, and then mapped invariantly into the ambient tractor bundle via (4.2.1).

4.2.2 A tractor Gauß formula

The checked connection defined in (4.2.8) describes the tangential part of the ambient tractor connection acting on tractors that are tangent to the submanifold. Hence we may define a tractor second fundamental via a Gauß type formula. Much as in the Riemannian case (4.1.15), this gives the normal part of the ambient tractor connection acting on a tractor tangent to the submanifold:

$$\nabla_i V^B = \Pi_J^B \check{\nabla}_i V^J + \mathbb{L}_{iK}{}^B V^K, \tag{4.2.15}$$

or, using (4.2.10),

$$\nabla_i V^B = \Pi_J^B \left(D_i V^J + \mathsf{S}_i{}^J{}_K V^K \right) + \mathbb{L}_{iK}{}^B V^K, \tag{4.2.16}$$

where $V^B := \Pi_J^B V^J \in \mathcal{E}^B|_\Gamma$ is the image of the section $V^J \in \mathcal{E}^J$ under the isomorphism $\mathcal{T}\Gamma \to \mathcal{N}^\perp$. The tractor second fundamental form is then a 1-form on Γ valued in $\mathcal{T}^*\Gamma \otimes \mathcal{N}$. This tractor second fundamental form appears in early works on hypersurfaces in conformal submanifolds [44, 62, 69], with the general codimension version found in [29, 11].

By Theorem 4.2.3, we have $\mathcal{N}^\perp \cong \mathcal{T}\Gamma$, and hence (4.2.2) reads

$$\mathcal{T}M|_\Gamma = \mathcal{T}\Gamma \oplus \mathcal{N}. \tag{4.2.17}$$

We shall compute explicit expressions for this tractor second fundamental form in Theorem 4.2.8 and Proposition 4.2.11. We display the first of these here to give a preview. In the tractor projector notation, the tractor second fundamental form is given by

$$\begin{aligned} \mathbb{L}_{iK}{}^B = {} & \mathring{I\!I}_{ij}{}^c Z_K^j Z_c^B + \mathrm{N}_a^c \left(P_i{}^a - \nabla_i H^a \right) X_K Z_c^B \\ & + H_c \mathring{I\!I}_{ij}{}^c Z_K^j X^B + H_c \left(P_i{}^c - \nabla_i H^c \right) X_K X^B. \end{aligned}$$

The Gauß formula describes the action of the ambient connection on an ambient tractor which is tangent to the submanifold. Using (4.2.17), any section of the ambient tractor bundle $V^B \in \mathcal{E}^B$ may be written as a pair $(V^\top, V^\perp) \in \mathcal{T}\Gamma \oplus \mathcal{N}$. From the tractor Gauß formula, it then follows that the action of the ambient tractor connection on such a pair is then given by

$$\nabla_i \begin{pmatrix} V^\top \\ V^\perp \end{pmatrix} = \begin{pmatrix} D_i + \mathsf{S}_i & -\mathbb{L}_i{}^T \\ \mathbb{L}_i & \nabla_i^{\mathcal{N}} \end{pmatrix} \begin{pmatrix} V^\top \\ V^\perp \end{pmatrix}, \tag{4.2.18}$$

where

- D_i is the intrinsic submanifold tractor connection,
- S_i is the difference tractor of (4.2.9),
- $\mathbb{L}_i$ is the tractor second fundamental form and $\mathbb{L}_i{}^T$ is its adjoint (with respect to the tractor metric) of the tractor second fundamental form, which may be viewed as a section of $\mathcal{T}\Gamma \otimes \mathcal{N}^*$, and
- $\nabla^{\mathcal{N}}$ is the normal tractor connection (4.2.13).

In particular, note that if $V^\perp = 0$ (i.e. V is really tangent to the submanifold Γ) then (4.2.18) simply reads as the Gauß formula (4.2.15).

4.2.3 The tractor normal projector

We have already defined the projection $\mathrm{N}^A_B : \mathcal{E}^B|_\Gamma \to \mathcal{N}^A$ which is the orthogonal projection onto the second factor in the decomposition $\mathcal{T}M|_\Gamma = \mathcal{T}\Gamma \oplus \mathcal{N}$. This normal projector plays a major role in our theory of conformal submanifolds. The tractor normal projector encodes much of the information about the conformal embedding $\Gamma \hookrightarrow M$, and many conformal invariants of this embedding may be recovered from formulae involving this normal projector. In this section, we prove the first such relation: we relate the tractor normal projector to the tractor second fundamental form.

Proposition 4.2.4. *The tractor second fundamental form is given by*

$$\mathbb{L}_{iK}{}^B = \Pi^C_K \mathrm{N}^B_A \nabla_i \Pi^A_C. \tag{4.2.19}$$

Proof. Let N^A be a section of the normal tractor bundle $\mathcal{N}$. Note that $\Pi^A_C N_A = 0$, and hence

$$0 = \nabla_i(\Pi^A_C N_A) = (\nabla_i \Pi^A_C) N_A + \Pi^A_B \nabla_i N_A,$$

whence

$$\Pi^A_C \nabla_i N_A = -N_A \nabla_i \Pi^A_C. \tag{$*$}$$

As a consequence of the tractor Gauß formula (4.2.16),

$$N_B \mathbb{L}_{iK}{}^B V^K = N_B \nabla_i V^B = -V^B \nabla_i N_B = -V^K \Pi^B_K \nabla_i N_B$$

for all $V^K \in \mathcal{E}^K$, and therefore

$$N_B \mathbb{L}_{iK}{}^B = -\Pi^B_K \nabla_i N_B = -\Pi^C_K \Pi^A_C \nabla_i N_A.$$

Combining this with $(*)$, we have that

$$N_B \mathbb{L}_{iK}{}^B = -\Pi^C_K \left(-N_A \nabla_i \Pi^A_C\right) = N_B \Pi^C_K \mathrm{N}^B_A \nabla_i \Pi^A_C,$$

and this must hold for any section N^B of the normal tractor bundle, whence the result follows. □

Equation (4.2.19) gives several other equivalent formulae for the tractor second fundamental form. These all stem from using $\delta^A_B = \Pi^A_B + \mathrm{N}^A_B$ together with the fact that $\nabla_i \delta^A_B = 0$ to switch between tangential and normal projectors.

For example,

$$\mathbb{L}_{iK}{}^B = \Pi^C_K \mathrm{N}^B_A \nabla_i \Pi^A_C = \Pi^C_K \mathrm{N}^B_A \nabla_i \left(\delta^A_C - \mathrm{N}^A_C\right) = -\Pi^C_K \mathrm{N}^B_A \nabla_i \mathrm{N}^A_C. \tag{4.2.20}$$

We will use this formula to compute the tractor second fundamental form explicitly.

It may be verified that the C index of $\mathrm{N}^B_A \nabla_i \mathrm{N}^A_C$ is annihilated by any normal section $N^C \in \Gamma(\mathcal{N}^C)$. Therefore $\mathrm{N}^B_A \nabla_i \mathrm{N}^A_C$ is a section of $\mathcal{E}_i \otimes (\mathcal{N}^\perp)_C \otimes \mathcal{N}^B$. We have seen in Theorem 4.2.3 that the map Π^C_K restricts to an isomorphism $\mathcal{N}^* \xrightarrow{\cong} \mathcal{T}^*\Gamma$, and so identifies $\mathrm{N}^B_A \nabla_i \mathrm{N}^A_C$ with $\mathbb{L}_{iK}{}^B$. In the sequel, we will often make use of this identification without explicitly saying that we are doing so. Where we do wish to make the distinction, we will use $\overline{\mathbb{L}}_{iC}{}^B$ to mean $-\mathrm{N}^B_A \nabla_i \mathrm{N}^A_C$, and this is then related to the "true" tractor second fundamental form by

$$\mathbb{L}_{iK}{}^B = \Pi^C_K \overline{\mathbb{L}}_{iC}{}^B. \tag{4.2.21}$$

Alternatively, substituting for the normal projector in (4.2.19),

$$\mathbb{L}_{iK}{}^B = \Pi^C_K \left(\delta^B_A - \Pi^B_A\right) \nabla_i \Pi^A_C = \Pi^C_K \nabla_i \Pi^B_C - \Pi^C_K \Pi^B_A \nabla_i \Pi^A_C,$$

and in fact, the second term in the final equality must be zero since the left hand side requires that the B index is normal (this also implies that the B index of $\Pi^C_K \nabla_i \Pi^B_C$ is normal).

Thus

$$\mathbb{L}_{iK}{}^B = \Pi^C_K \nabla_i \Pi^B_C. \tag{4.2.22}$$

Remark 4.2.5. Note that the derivation of the above relations uses nothing uniquely tractorial; it is merely a consequence of a Gauß-type formula and compatible orthogonal projectors. This means that those same relations will hold replacing the tractor projectors and tractor second fundamental form with the tangential and normal projectors, and usual second fundamental form of Riemannian submanifold geometry. Thus for example,

$$II_{ij}{}^{c} = -\Pi^{b}_{j}\mathrm{N}^{c}_{a}\nabla_i\mathrm{N}^{a}_{b}, \tag{4.2.23}$$

where ∇_i is now the pullback Levi-Civita connection. While the tangential and normal projectors are conformally invariant, however the Levi-Civita connection *is not*. Therefore, together with (2.2.2) and (2.2.3), equation (4.2.23) gives another way to derive (4.1.21) (the transformation law for the Riemannian second fundamental form).

Using equation (4.2.18) with a purely normal section of $\mathcal{T}\Gamma \oplus \mathcal{N}$, one may repeat the argument of Proposition 4.2.4 with a slight modification (or simply by observing that the ambient tractor metric is preserved by the ambient tractor connection, and hence also the pullback connection) to obtain a similar formula for the transpose of the tractor second fundamental form:

$$\mathbb{L}_i{}^{K}{}_{B} = -\Pi^{K}_{C}\mathrm{N}^{A}_{B}\nabla_i\mathrm{N}^{C}_{A}. \tag{4.2.24}$$

The above relations allow us to prove

Lemma 4.2.6. *Let* N^{C}_{B} *be the normal tractor projector. Then*

$$\nabla_i\mathrm{N}^{C}_{B} = -\overline{\mathbb{L}}_i{}^{C}{}_{B} - \overline{\mathbb{L}}_{iB}{}^{C}. \tag{4.2.25}$$

Proof. Noting that $\mathrm{N}^{C}_{B} = \mathrm{N}^{C}_{A}\mathrm{N}^{A}_{B}$, we have

$$\nabla_i\mathrm{N}^{C}_{B} = \mathrm{N}^{A}_{B}\nabla_i\mathrm{N}^{C}_{A} + \mathrm{N}^{C}_{A}\nabla_i\mathrm{N}^{A}_{B}.$$

The first and second terms on the right-hand side are equations (4.2.20) and (4.2.24) respectively without the tangential projectors. But from equation (4.2.21) these are exactly (negative) $\overline{\mathbb{L}}$ and its adjoint. □

From this formula, the various expressions above for the tractor second fundamental form and its adjoint are now clear: the B and C indices on the right-hand side are normal and tangential (resp. tangential and normal) and hence by applying the appropriate projector one extracts one or the other. On the other hand, from the tractor Gauß formula (4.2.18), one can see that these such projections will correspond to either the tractor second fundamental form or its adjoint.

The lemma shows the relationship between the normal projector and the tractor second fundamental form.

Theorem 4.2.7. *The tractor normal projector is parallel if, and only if, the tractor second fundamental form vanishes, i.e.,* $\nabla_i \mathrm{N}_B^C = 0$ *if, and only if,* $\mathbb{L}_{iJ}{}^C = 0$.

Proof. Recall that Π_J^B gives an isomorphism $\mathcal{T}^*\Gamma \to (\mathcal{N})^{\perp *}$, and hence $\mathbb{L} = 0$ if, and only if, $\overline{\mathbb{L}} = 0$. Since

$$\overline{\mathbb{L}}_i{}^C{}_B = h^{CF} h_{BE} \overline{\mathbb{L}}_{iE}{}^F,$$

if one of the terms of the right-hand side of (4.2.25) vanishes, then so does the other. So the vanishing of the tractor second fundamental form implies that the tractor normal projector is parallel. Equation (4.2.20) shows that the converse also holds. □

We now turn to the main result of this section: using (4.2.20) to derive an explicit formula for the tractor second fundamental form in terms of splitting tractors.

Theorem 4.2.8. *The tractor second fundamental form is given by*

$$\begin{aligned} \mathbb{L}_{iJ}{}^C = \mathring{I\!I}_{ij}{}^c Z_J^j Z_c^C + \mathrm{N}_a^c \left(P_i{}^a - \nabla_i H^a\right) X_J Z_c^C \\ + H_c \mathring{I\!I}_{ij}{}^c Z_J^j X^C + H_a \left(P_i{}^a - \nabla_i H^a\right) X_J X^C. \end{aligned} \tag{4.2.26}$$

Proof. We compute $\mathrm{N}_A^C \nabla_i \mathrm{N}_B^A$ using the formula from Lemma 4.2.2. We then apply Π_J^B, the formula for which is given in Theorem 4.2.3 to complete the proof.

First, differentiating (4.2.4) gives

$$\begin{aligned} \nabla_i \mathrm{N}_B^A &= (\nabla_i \mathrm{N}_b^a) Z_a^A Z_B^b + \mathrm{N}_b^a \left(-P_{ia} X^A - \boldsymbol{g}_{ia} Y^A\right) Z_B^b + \mathrm{N}_b^a Z_a^A \left(-P_i^b X_B - \delta_i^b Y_B\right) \\ &\quad + (\nabla_i H^a) Z_a^A X_B + H^a \left(-P_{ia} X^A - \boldsymbol{g}_{ia} Y^A\right) X_B + H^a Z_a^A Z_{Bi} \\ &\quad + (\nabla_i H_b) X^A Z_B^b + H_b Z_i^A Z_B^b + H_b X^A \left(-P_i{}^b X_B - \delta_i^b Y_B\right) \\ &\quad + 2(H^d \nabla_i H_d) X^A X_B + H^d H_d Z_i^A X_B + H^d H_d X^A Z_{Bi} \\ &= \left(\nabla_i \mathrm{N}_b^a + H^a \boldsymbol{g}_{ib} + H_b \delta_i^a\right) Z_a^A Z_B^b \\ &\quad + \left(-\mathrm{N}_b^a P_{ia} + \nabla_i H_b + H_d H^d \boldsymbol{g}_{ib}\right) X^A Z_B^b \\ &\quad + \left(-\mathrm{N}_b^a P_i{}^b + \nabla_i H^a + H_d H^d \delta_i^a\right) Z_a^A X_B \\ &\quad + \left(-H^a P_{ia} - H_b P_i{}^b + 2H^d \nabla_i H_d\right) X^A X_B. \end{aligned}$$

From (4.2.4), it follows that

$$\mathrm{N}_A^C Z_a^A = \mathrm{N}_a^c Z_c^C + H_a X^C \quad \text{and} \quad \mathrm{N}_A^C X^A = 0.$$

Hence

$$\mathrm{N}_A^C \nabla_i \mathrm{N}_B^A = \left(\nabla_i \mathrm{N}_b^a + H^a \boldsymbol{g}_{ib} + H_b \delta_i^a\right) \left(\mathrm{N}_a^c Z_c^C + H_a X^C\right) Z_B^b$$

$$
\begin{aligned}
&+ \left(-\mathrm{N}_b^a P_i{}^b + \nabla_i H^a + H_d H^d \delta_i^a\right) \left(\mathrm{N}_a^c Z_c^C + H_a X^C\right) X_B \\
&= \left(\mathrm{N}_a^c \nabla_i \mathrm{N}_b^a + H^c \boldsymbol{g}_{ib}\right) Z_B^b Z_c^C + \mathrm{N}_a^c \left(\nabla_i H^a - P_i{}^a\right) X_B Z_c^C \\
&\quad + H_a \left(\nabla_i \mathrm{N}_b^a + H^a \boldsymbol{g}_{ib}\right) Z_B^b X^C + H_a \left(\nabla_i H^a - P_i{}^a\right) X_B X^C.
\end{aligned}
$$

All that remains is to apply the tangential tractor projector Π_J^B. According to (4.2.5),

$$
\Pi_J^B Z_B^b = \Pi_j^b Z_J^j \quad \text{and} \quad \Pi_J^B X_B = X_J.
$$

Therefore

$$
\begin{aligned}
\Pi_J^B \mathrm{N}_A^C \nabla_i \mathrm{N}_B^A &= \Pi_j^b \left(\mathrm{N}_a^c \nabla_i \mathrm{N}_b^a + H^c \boldsymbol{g}_{ib}\right) Z_J^j Z_c^C + \mathrm{N}_a^c \left(\nabla_i H^a - P_i{}^a\right) X_J Z_c^C \\
&\quad + H_a \Pi_j^b \left(\nabla_i \mathrm{N}_b^a + H^a \boldsymbol{g}_{ib}\right) Z_J^j X^C + H_a \left(\nabla_i H^a - P_i{}^a\right) X_J X^C \\
&= \left(-I\!I_{ij}{}^c + H^c \boldsymbol{g}_{ij}\right) Z_J^j Z_c^C + \mathrm{N}_a^c \left(\nabla_i H^a - P_i{}^a\right) X_J Z_c^C \\
&\quad + H_c \left(-I\!I_{ij}{}^c + H^c \boldsymbol{g}_{ib}\right) Z_J^j X^C + H_a \left(\nabla_i H^a - P_i{}^a\right) X_J X^C \\
&= -\mathring{I\!I}_{ij}{}^c Z_J^j Z_c^C + \mathrm{N}_a^c \left(\nabla_i H^a - P_i{}^a\right) X_J Z_c^C \\
&\quad - H_c \mathring{I\!I}_{ij}{}^c Z_J^j X^C + H_a \left(\nabla_i H^a - P_i{}^a\right) X_J X^C,
\end{aligned}
$$

where we note that $H_a \nabla_i \mathrm{N}_b^a = H_c \mathrm{N}_a^c \nabla_i \mathrm{N}_b^a$, and we have used the observation from Remark 4.2.5 to replace $\Pi_j^b \mathrm{N}_a^c \nabla_i \mathrm{N}_b^a$ with $I\!I_{ij}{}^c$.

Finally, equation (4.2.20) shows that $\mathbb{L}_{iJ}{}^C$ is equal to negative of the above, which is exactly the formula claimed in the theorem. □

4.2.4 Some alternative formulae

It is useful to have some alternative formulae for some of the invariant objects we have seen in this chapter. Here we compute such formulae for the Fialkow tensor (4.2.11) and the tractor second fundamental form (4.2.26). These formulae will facilitate the proofs of certain results later in this thesis.

This formula for the Fialkow tensor has appeared in several places. Fialkow's original deviation tensor [32] is a mix of equation (4.2.11) and the following. It was also calculated as in the following proposition in [29]. This expression for the Fialkow is better suited to several applications than (4.2.11), and has the benefit of being manifestly conformally invariant.

Proposition 4.2.9. *The Fialkow tensor is given by*

$$
\mathcal{F}_{ij} = \frac{1}{m-2} \left(W_{icjd} \mathrm{N}^{cd} + \frac{W_{abcd} \mathrm{N}^{ac} \mathrm{N}^{bd}}{2(m-1)} \boldsymbol{g}_{ij} + \mathring{I\!I}_i{}^{kc} \mathring{I\!I}_{jkc} - \frac{\mathring{I\!I}^{klc} \mathring{I\!I}_{klc}}{2(m-1)} \boldsymbol{g}_{ij} \right), \qquad (4.2.27)
$$

where $W_{icjd} = \Pi_i^a \Pi_b^i W_{acbd}$ *is the ambient Weyl with two indices restricted to* Γ, *and* $m = \dim \Gamma \geq 3$.

Proof. We use the Ricci decomposition of the Riemann curvature tensor in the Gauß formula (4.1.23):

$$\begin{aligned} W_{ijkl} + \boldsymbol{g}_{ki}P_{jl} - \boldsymbol{g}_{kj}P_{il} + \boldsymbol{g}_{lj}P_{ik} - \boldsymbol{g}_{li}P_{jk} = {W^\Gamma}_{ijkl} + \boldsymbol{g}_{ki}p_{jl} - \boldsymbol{g}_{kj}p_{il} + \boldsymbol{g}_{lj}p_{ik} \\ - \boldsymbol{g}_{li}p_{jk} + I\!I_{ilc}{I\!I_{jk}}^c - I\!I_{jlc}{I\!I_{ik}}^c, \end{aligned}$$

and then apply the map $T_{ijkl} \mapsto \frac{1}{m-2}\left({T_{ikj}}^k - \frac{{T_{kl}}^{kl}}{2(m-1)}\boldsymbol{g}_{ij}\right)$ to both sides:

$$\begin{aligned} &\frac{1}{m-2}\Big[\Big({W_{ikj}}^k + \boldsymbol{g}_{ji}{P_k}^k - \boldsymbol{g}_{jk}{P_i}^k + {\boldsymbol{g}^k}_k P_{ij} - {\boldsymbol{g}^k}_i P_{kj}\Big) \\ &\quad -\frac{1}{2(m-1)}\Big({W_{kl}}^{kl} + {\boldsymbol{g}^k}_k {P_l}^l - {\boldsymbol{g}^k}_l {P_k}^l + {\boldsymbol{g}^l}_l {P_k}^k - {\boldsymbol{g}^l}_k {P_l}^k\Big)\boldsymbol{g}_{ij}\Big] \\ &= \frac{1}{m-2}\Big[\Big({W^\Gamma}_{ikj}{}^k + \boldsymbol{g}_{ji}{p_k}^k - \boldsymbol{g}_{jk}{p_i}^k + {\boldsymbol{g}^k}_k p_{ij} - {\boldsymbol{g}^k}_i p_{kj} + I\!I_{ikc}{I\!I_j}^{kc} - I\!I_{ijc}{I\!I_k}^{kc}\Big) \\ &\quad -\frac{1}{2(m-1)}\Big({W^\Gamma}_{kl}{}^{kl} + {\boldsymbol{g}^k}_k {p_l}^l - {\boldsymbol{g}^k}_l {p_k}^l + {\boldsymbol{g}^l}_l {p_k}^k - {\boldsymbol{g}^l}_k {p_l}^k + {I\!I_k}^l{}_c {I\!I_l}^{kc} - {I\!I_k}^k{}_c {I\!I_l}^{lc}\Big)\boldsymbol{g}_{ij}\Big]. \end{aligned}$$

After some simplification, one arrives at

$$\begin{aligned} &\frac{1}{m-2}{W_{ikj}}^k + P_{ij} - \frac{{W_{kl}}^{kl}}{2(m-1)(m-2)}\boldsymbol{g}_{ij} \\ &\quad = \frac{1}{m-2}W^\Gamma_{ikj}{}^k + p_{ij} + \frac{1}{m-2}\Big[I\!I_{ikc}{I\!I_j}^{kc} - m \cdot H_c {I\!I_{ij}}^c\Big] \\ &\qquad - \frac{{W^\Gamma}_{kl}{}^{kl}}{2(m-1)(m-2)}\boldsymbol{g}_{ij} - \frac{1}{2(m-1)(m-2)}\Big({I\!I_k}^l{}_c{I\!I_l}^{kc} - m^2 \cdot H_c H^c\Big)\boldsymbol{g}_{ij}. \end{aligned} \tag{4.2.28}$$

Now,

$${W_{ikj}}^k = \boldsymbol{g}^{kl}W_{ikjl} = \boldsymbol{g}^{kl}\Pi_k^c\Pi_l^d W_{icjd} = \Pi^{cd}W_{icjd} = (g^{cd} - \mathrm{N}^{cd})W_{icjd} = -W_{icjd}\mathrm{N}^{cd},$$

where the term involving the metric vanishes since the Weyl tensor is totally trace-free.

Similarly,

$${W_{kl}}^{kl} = W_{abcd}\mathrm{N}^{ac}\mathrm{N}^{bd}.$$

Both terms involving the intrinsic Weyl tensor will also vanish since the intrinsic Weyl tensor is also totally trace-free.

After accounting for these observations, (4.2.28) becomes

$$-\frac{1}{m-2}W_{icjd}\mathrm{N}^{cd} + P_{ij} - \frac{W_{abcd}\mathrm{N}^{ac}\mathrm{N}^{bd}}{2(m-1)(m-2)}\boldsymbol{g}_{ij} = p_{ij} + \frac{1}{m-2}\Big({I\!I_i}^{kc}I\!I_{jkc} - m \cdot H_c{I\!I_{ij}}^c\Big)$$

$$- \frac{1}{2(m-1)(m-2)} \left(I\!I_{klc} I\!I^{klc} - m^2 \cdot H_c H^c \right) \boldsymbol{g}_{ij}.$$

To obtain the final formula, we split the second fundamental form into trace-free and trace parts:

$$\begin{aligned} I\!I_i{}^{kc} I\!I_{jkc} &= \left(\mathring{I\!I}_i{}^{kc} + \delta_i^k H^c \right) \left(\mathring{I\!I}_{jkc} + \boldsymbol{g}_{jk} H_c \right) \\ &= \mathring{I\!I}_i{}^{kc} I\!I_{jkc} + 2H_c \mathring{I\!I}_{ij}{}^c + \boldsymbol{g}_{ij} H_c H^c, \end{aligned}$$

and

$$\begin{aligned} I\!I_{klc} I\!I^{klc} &= \left(\mathring{I\!I}_{klc} + \boldsymbol{g}_{kl} H_c \right) \left(\mathring{I\!I}^{klc} + \boldsymbol{g}^{kl} H^c \right) \\ &= \mathring{I\!I}_{klc} \mathring{I\!I}^{klc} + \boldsymbol{g}^{kl} \mathring{I\!I}_{klc} H^c + \boldsymbol{g}_{kl} \mathring{I\!I}^{klc} H_c + \boldsymbol{g}_{kl} \boldsymbol{g}^{kl} H_c H^c \\ &= \mathring{I\!I}_{klc} \mathring{I\!I}^{klc} + m \cdot H_c H^c. \end{aligned}$$

Hence

$$\begin{aligned} &- \frac{1}{m-2} W_{icjd} \mathrm{N}^{cd} + P_{ij} - \frac{W_{abcd} \mathrm{N}^{ac} \mathrm{N}^{bd}}{2(m-1)(m-2)} \boldsymbol{g}_{ij} \\ &= p_{ij} + \frac{1}{m-2} \left(\mathring{I\!I}_i{}^{kc} I\!I_{jkc} + 2H_c \mathring{I\!I}_{ij}{}^c + \boldsymbol{g}_{ij} H_c H^c - m \cdot H_c \left(\mathring{I\!I}_{ij}{}^c + \boldsymbol{g}_{ij} H^c \right) \right) \\ &\qquad - \frac{1}{2(m-1)(m-2)} \left(\mathring{I\!I}_{klc} \mathring{I\!I}^{klc} + m \cdot H_c H^c - m^2 \cdot H_c H^c \right) \boldsymbol{g}_{ij} \\ &= p_{ij} + \frac{1}{m-2} \left(\mathring{I\!I}_i{}^{kc} \mathring{I\!I}_{jkc} - (m-2) \cdot H_c \mathring{I\!I}_{ij}{}^c - (m-1) \cdot H_c H^c \boldsymbol{g}_{ij} \right) \\ &\qquad - \frac{1}{2(m-1)(m-2)} \left(\mathring{I\!I}_{klc} \mathring{I\!I}^{klc} - m(m-1) \cdot H_c H^c \right) \boldsymbol{g}_{ij}. \end{aligned}$$

Therefore

$$\begin{aligned} P_{ij} - p_{ij} &= \frac{1}{m-2} \left(W_{icjd} \mathrm{N}^{cd} + \frac{W_{abcd} \mathrm{N}^{ac} \mathrm{N}^{bd}}{2(m-1)} + \mathring{I\!I}_i{}^{kc} \mathring{I\!I}_{jkc} - \frac{\mathring{I\!I}_{klc} \mathring{I\!I}^{klc}}{2(m-1)} \boldsymbol{g}_{ij} \right) \\ &\qquad - \frac{1}{m-2} (m-2) \cdot H_c \mathring{I\!I}_{ij}{}^c - \left(- \left(\frac{m-1}{m-2} \right) + \frac{m(m-1)}{2(m-1)(m-2)} \right) \boldsymbol{g}_{ij} H_c H^c \\ &= \frac{1}{m-2} \left(W_{icjd} \mathrm{N}^{cd} + \frac{W_{abcd} \mathrm{N}^{ac} \mathrm{N}^{bd}}{2(m-1)} + \mathring{I\!I}_i{}^{kc} \mathring{I\!I}_{jkc} - \frac{\mathring{I\!I}_{klc} \mathring{I\!I}^{klc}}{2(m-1)} \boldsymbol{g}_{ij} \right) - H_c \mathring{I\!I}_{ij}{}^c - \frac{1}{2} \boldsymbol{g}_{ij} H_c H^c, \end{aligned}$$

which implies

$$\begin{aligned}P_{ij} - p_{ij} + H_c \mathring{I\!I}_{ij}{}^c + \frac{1}{2} H_c H^c \boldsymbol{g}_{ij} \\ = \frac{1}{m-2} \left(W_{icjd} \mathrm{N}^{cd} + \frac{W_{abcd} \mathrm{N}^{ac} \mathrm{N}^{bd}}{2(m-1)} + \mathring{I\!I}_i{}^{kc} \mathring{I\!I}_{jkc} - \frac{\mathring{I\!I}_{klc} \mathring{I\!I}^{klc}}{2(m-1)} \boldsymbol{g}_{ij} \right).\end{aligned}$$

The left-hand side of this is exactly equal to the Fialkow tensor (4.2.11), while the right-hand side is the expression from the statement of the proposition. □

Remark 4.2.10. If Γ is a hypersurface, the above formula may be simplified further. In this case, $\mathrm{N}^{ac} = N^a N^c$, where N^a is the unit normal to the hypersurface. Then one sees that $W_{abcd} \mathrm{N}^{ac} \mathrm{N}^{bd} = 0$ by the symmetries of the Weyl tensor. Moreover, for a hypersurface, one may write $I\!I_{ij}{}^c = I\!I_{ij} N^c$ and $H^c = H N^c$, viewing the second fundamental form as a symmetric 2-tensor and the mean curvature as a scalar.

With these observations, the formula of the proposition simplifies to

$$\mathcal{F}_{ij} = \frac{1}{m-2} \left(W_{icjd} N^c N^d + \mathring{I\!I}_i{}^k \mathring{I\!I}_{jk} - \frac{\mathring{I\!I}^{kl} \mathring{I\!I}_{kl}}{2(m-1)} \boldsymbol{g}_{ij} \right). \tag{4.2.29}$$

Next, we use the Codazzi-Mainardi equation (4.1.24) to give an alternative expression for the tractor second fundamental form (equation (4.2.26)).

Proposition 4.2.11. *We have*

$$\begin{aligned}\mathbb{L}_{iJ}{}^C = \mathring{I\!I}_{ij}{}^c Z^j_J Z^C_c - \frac{1}{m-1} \left(D^j \mathring{I\!I}_{ij}{}^d + \Pi^b_i W_{abce} \mathrm{N}^{ae} \mathrm{N}^{cd} \right) X_J Z^C_d \\ + H_c \mathring{I\!I}_{ij}{}^c Z^j_J X^C - \frac{1}{m-1} H_d \left(D^j \mathring{I\!I}_{ij}{}^d + \Pi^b_i W_{ab}{}^d{}_e \mathrm{N}^{ae} \right) X_J X^C,\end{aligned} \tag{4.2.30}$$

where the submanifold intrinsic Levi-Civita connection D_i is coupled to the normal connection.

Proof. We use the Codazzi-Mainardi equation to re-express the term involving the Schouten tensor. Substituting the Ricci decomposition of the Riemann curvature into (4.1.24) gives

$$\Pi^a_i \Pi^b_j \Pi^e_k \left(W_{abce} + P_{ac} \boldsymbol{g}_{be} - P_{bc} \boldsymbol{g}_{ae} + P_{be} \boldsymbol{g}_{ac} - P_{ae} \boldsymbol{g}_{bc} \right) \mathrm{N}^{cd} = D_i I\!I_{jk}{}^d - D_j I\!I_{ik}{}^d,$$

and after contracting with $\boldsymbol{g}^{ik}$, we arrive at

$$W_{ij}{}^{ci} \mathrm{N}^d_c - (m-1) P_j{}^c \mathrm{N}^d_c = D^i I\!I_{ij}{}^d - D_j I\!I_i{}^{id} = D^i I\!I_{ij}{}^d - m \nabla^\perp_j H^d.$$

Now substituting the decomposition of the second fundamental form into trace and trace-free parts:

$$W_{ij}{}^{ci}\mathrm{N}_c^d - (m-1)P_j{}^c\mathrm{N}_c^d = D^i\left(\mathring{I\!I}_{ij}{}^d + \boldsymbol{g}_{ij}H^d\right) - m\nabla_j^\perp H^d$$
$$W_{ij}{}^{ci}\mathrm{N}_c^d - (m-1)P_j{}^c\mathrm{N}_c^d = D^i\mathring{I\!I}_{ij}{}^d - (m-1)\nabla_j^\perp H^d.$$

Hence

$$\mathrm{N}_c^d\left(P_j{}^c - \nabla_j H^d\right) = \frac{1}{m-1}\left(W_{ij}{}^{ci}\mathrm{N}_c^d - D^i\mathring{I\!I}_{ij}{}^d\right), \tag{4.2.31}$$

the left-hand side of which is exactly a term appearing in our formula for the tractor second fundamental form (4.2.26).

To complete the proof, we observe that

$$W_{ij}{}^{ci} = \boldsymbol{g}^{ik}\Pi_i^a\Pi_j^b\Pi_k^e W_{ab}{}^c{}_e = \Pi_j^b\Pi^{ae}W_{ab}{}^c{}_e = -\Pi_j^b\mathrm{N}^{ae}W_{ab}{}^c{}_e.$$

□

Once again, the formula simplifies for hypersurfaces. This gives a well-known tractor characterization of umbilic submanifolds.

Corollary 4.2.12. *Let $\Gamma \hookrightarrow M$ be a hypersurface in a conformal manifold $(M, \boldsymbol{c})$. Then the tractor second fundamental form of Γ is zero if, and only, Γ is totally umbilic.*

Proof. If Γ is a hypersurface, then $\mathrm{N}^{ae} = N^aN^e$, where N^a is the unit normal to Γ. Therefore

$$W_{abce}\mathrm{N}^{ae}\mathrm{N}^{cd} = W_{abce}N^aN^eN^cN^d = 0,$$

and

$$H_dW_{ab}{}^d{}_e\mathrm{N}^{ae} = HN_dW_{ab}{}^d{}_eN^aN^e = 0,$$

where H is the (scalar) mean curvature of Γ.

Hence equation (4.2.30) becomes

$$\begin{aligned}\mathbb{L}_{iJ}{}^C = \mathring{I\!I}_{ij}{}^cZ_J^jZ_c^C - \frac{1}{m-1}D^j\mathring{I\!I}_{ij}{}^dX_JZ_c^C \\ + H_c\mathring{I\!I}_{ij}{}^cZ_J^jX^C - \frac{1}{m-1}H_dD^j\mathring{I\!I}_{ij}{}^dX_JX^C.\end{aligned} \tag{4.2.32}$$

From this it is clear that $\mathbb{L}_{iJ}{}^C = 0$ if, and only if, $\mathring{I\!I}_{ij}{}^c = 0$, i.e. if Γ is umbilic. □

4.3 Low-dimensional conformal submanifolds

Thus far, when studying conformal manifolds, we have restricted to the case where the conformal manifold has dimension at least 3. For dimensions 1 and 2, various parts of the construction of Section 2.3 no longer hold. In dimension 2, we can still define the tractor bundle $\mathcal{T}\Sigma$ as a quotient of the 2-jet bundle $J^2\mathcal{E}[1]$ by the subbundle isomorphic to $S_0^2T^*M[1]$, but there is no longer a well-defined Schouten tensor so we do not immediately get an intrinsic tractor connection on $\mathcal{T}\Sigma$. In dimension 1, the trace-free part of $S^2T^*M[1]$ is trivial, so defining the tractor bundle as a quotient of the $J^2\mathcal{E}[1]$ just yields the full 2-jet bundle. While we will still require that our ambient conformal manifold has dimension at least 3, we may use the ambient structure to define an intrinsic tractor bundle and an intrinsic tractor connection for low-dimensional conformal submanifolds.

4.3.1 2-dimensional conformal submanifolds

We first note that, for $\Gamma \subset M$ with $\dim\Gamma = 2$, there is an obvious way to assign a Schouten tensor to each metric $g_\Gamma \in \boldsymbol{c}_\Gamma$: namely, take the Schouten tensor of some metric $g \in \boldsymbol{c}$ which extends g_Γ, and then restrict to $T\Gamma$. We verify that this defines an intrinsic "Schouten" tensor, i.e. a section of $S^2T^*\Gamma$ which transforms by (2.2.10). Let $g_\Gamma, \widehat{g}_\Gamma \in \boldsymbol{c}_\Gamma$ be two conformally related submanifold metrics, and let $g, \widehat{g} \in \boldsymbol{c}$ be *minimal* ambient metrics which extend g_Γ and $\widehat{g}_\Gamma$ respectively. Let P_{ab} be the ambient Schouten tensor of g, and define $p_{ij} := \Pi_i^a\Pi_j^b P_{ab}$, i.e. the restriction of P_{ab} to the submanifold. Then the Schouten tensor of $\widehat{g}_\Gamma$ is $\widehat{p}_{ij} = \Pi_i^a\Pi_j^b\widehat{P}_{ab}$, and hence

$$\widehat{p}_{ij} = \Pi_i^a\Pi_j^b\left(P_{ab} - \nabla_a\Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}\Upsilon_c\Upsilon^c\boldsymbol{g}_{ab}\right) = p_{ij} - D_i\Upsilon_j + \Upsilon_i\Upsilon_j - \frac{1}{2}\Upsilon_k\Upsilon^k\boldsymbol{g}_{ij}$$

where we have used that if g and $\widehat{g}$ are both minimal, then (4.1.22) implies that $\mathrm{N}_d^c\Upsilon^d = 0$, so Υ_b can be identified with some submanifold form $\Upsilon_j \in \Gamma(\mathcal{E}_j)$. Note that this also shows that this assignment of Schouten tensor does not depend on the choice of minimal scale extending g: if g and $\widehat{g}$ are both minimal scales which extend g_Γ, (i.e. $\widehat{g}|_\Gamma = g|_\Gamma$), then $\Omega|_\Gamma = 1$ and we see that $\widehat{p}_{ij} = p_{ij}$. The above display shows that p_{ij} satisfies the required transformation law, and thus we have an assignment of a "Schouten" tensor for each metric $g_\Gamma \in \boldsymbol{c}_\Gamma$, and hence we have a well-defined tractor connection on $\mathcal{T}\Gamma$. The tractor connection determined by this assignment of a Schouten tensor amounts to declaring that the intrinsic tractor connection is the checked connection of (4.2.8):

$$D_i^{\mathcal{T}\Gamma}V^J := \check{\nabla}_i V^J = \Pi_B^J\nabla_i\left(\Pi_K^B V^K\right), \tag{4.3.1}$$

where ∇_i on the right-hand side is the ambient tractor connection. This approach, while straightforward, does have some problems. Recall that the Fialkow tensor (4.2.11) is essentially the difference between the checked connection and the intrinsic tractor connection, and so this assignment of an intrinsic Schouten tensor is equivalent to prescribing that the Fialkow tensor vanishes identically. In our discussion of distinguished submanifolds in Chapter 5, the Fialkow tensor will be an obstruction to the submanifold having a certain property, and so if we wish for the theorem to say something non-trivial even in the $m = 2$ case, the Fialkow tensor being identically zero is undesirable.

We instead proceed by first showing that there is a natural Fialkow tensor for a 2-dimensional submanifold, and then defining the intrinsic Schouten in terms of this Fialkow so that (4.2.11) still holds. Consider equation (4.2.27). On first glance, the right-hand side of this equation appears to be singular when $m = 2$. We claim however that this singularity is removable.

First, we note that, when $m = 2$,

$$W_{icjd}\mathrm{N}^{cd} = -W_{icjd}\Pi^{cd} = -W_{ikjl}\Pi^{kl} = -W_{ikjl}\boldsymbol{g}^{kl},$$

and W_{ikjl} inherits Riemann tensor symmetries from the ambient Weyl tensor. But $S^2\Lambda^2T^*\Gamma$ is a line bundle, and hence $W_{ikjl} = \kappa\left(\boldsymbol{g}_{ij}\boldsymbol{g}_{kl} - \boldsymbol{g}_{kj}\boldsymbol{g}_{il}\right)$ for some scalar function κ. Hence $W_{icjd}\mathrm{N}^{cd} = -W_{ikjl}\boldsymbol{g}^{kl} = -\kappa\boldsymbol{g}_{ij}$. So on the one hand, $\boldsymbol{g}^{ij}W_{icjd}\mathrm{N}^{cd} = -2\kappa$. On the other hand,

$$\boldsymbol{g}^{ij}W_{icjd}\mathrm{N}^{cd} = \boldsymbol{g}^{ij}\Pi_i^a\Pi_j^bW_{acbd} = \Pi^{ab}W_{acbd}\mathrm{N}^{cd} = -\mathrm{N}^{ab}W_{acbd}\mathrm{N}^{cd},$$

so we see that

$$W_{icjd}\mathrm{N}^{cd} = -\frac{W_{acbd}\mathrm{N}^{ab}\mathrm{N}^{cd}}{2}\boldsymbol{g}_{ij}.$$

Thus when $\dim\Gamma = 2$, the first two terms in parentheses of equation (4.2.27) cancel.

Secondly, we treat the remaining terms. By the Cayley-Hamilton theorem, the square of any trace-free endomorphism of a 2-dimensional vector space is proportional to the identity map. Thus we see that, when $m = 2$,

$$\mathring{I\!I}_i{}^{kc}\mathring{I\!I}_{jkc} = \lambda\boldsymbol{g}_{ij}, \tag{4.3.2}$$

and tracing it follows that

$$\lambda = \frac{1}{2}\mathring{I\!I}^{klc}\mathring{I\!I}_{klc}. \tag{4.3.3}$$

Thus the third and fourth terms of (4.2.27) combine to give

$$\mathring{I\!I}_i{}^{kc}\mathring{I\!I}_{jkc} - \frac{\mathring{I\!I}^{klc}\mathring{I\!I}_{klc}}{2(m-1)}\boldsymbol{g}_{ij} = \frac{1}{2}\left(1 - \frac{1}{m-1}\right)\mathring{I\!I}^{klc}\mathring{I\!I}_{klc}\boldsymbol{g}_{ij} = \frac{m-2}{2(m-1)}\mathring{I\!I}^{klc}\mathring{I\!I}_{klc}\boldsymbol{g}_{ij},$$

and that factor of $(m-2)$ will exactly cancel with the prefactor in (4.2.27). Hence when $\dim\Gamma = 2$, we have a well-defined Fialkow tensor:

$$\mathcal{F}_{ij} = \frac{1}{2}\mathring{I\!I}^{klc}\mathring{I\!I}_{klc}\,\boldsymbol{g}_{ij}. \tag{4.3.4}$$

We are now ready to define our assignment of a Schouten tensor for each intrinsic metric in the submanifold conformal class. Fix $g_\Gamma \in \boldsymbol{c}_\Gamma$. Then we define the Schouten tensor for g_Γ by

$$p_{ij} := P_{ij} + H_b \mathring{I\!I}_{ij}{}^b + \frac{1}{2} H_b H^b \boldsymbol{g}_{ij} - \frac{1}{2} \mathring{I\!I}^{klc} \mathring{I\!I}_{klc} \boldsymbol{g}_{ij}, \tag{4.3.5}$$

where $P_{ij} = \Pi_i^a \Pi_j^b$ for P_{ab} the Schouten tensor of any metric $g \in \boldsymbol{c}$ such that $\iota^* g = g_\Gamma$. We have

$$\widehat{p}_{ij} = p_{ij} - D_i \Upsilon_j + \Upsilon_i \Upsilon_j - \frac{1}{2} \Upsilon_k \Upsilon^k \boldsymbol{g}_{ij}, \tag{4.3.6}$$

and so this p_{ij} satisfies the required transformation law for a Schouten tensor. In fact, $P_{ij} + H_b \mathring{I\!I}_{ij}{}^b + \frac{1}{2} H_b H^b \boldsymbol{g}_{ij}$ already transforms correctly, there is the freedom to add any conformally invariant term. As already discussed, we choose $-\frac{1}{2} \mathring{I\!I}^{klc} \mathring{I\!I}_{klc} \boldsymbol{g}_{ij}$ so that the right-hand side of (4.2.11) agrees with that of (4.2.27) in the $m = 2$ case. We may then define an intrinsic conformal tractor connection on the tractor bundle $\mathcal{T}\Sigma$ by

$$D_i^{\mathcal{T}\Sigma} \begin{pmatrix} \sigma \\ \mu_j \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu_j + p_{ij} \sigma + \boldsymbol{g}_{ij} \rho \\ \nabla_i \rho - p_{ij} \mu^j \end{pmatrix}, \tag{4.3.7}$$

where p_{ij} is the Schouten tensor associated to the choice of scale by (4.3.5). Since p_{ij} transforms as a Schouten tensor, this connection is invariant and one readily sees that it preserves the induced submanifold metric.

4.3.2 1-dimensional conformal submanifolds

The case $\Gamma \subset M$ with $\dim \Gamma = 1$ is considerably simpler. As we have already observed, if $\dim \Gamma = 1$, then $S_0^2 T^*\Gamma$ is trivial, and so the tractor bundle constructed as a quotient is simply the full 2-jet bundle. In this case, we define $T\Gamma$ to be the image of $\mathcal{N}^\perp$ under the map Π_A^I, where this map is just defined to be (4.2.7), noting that this is well-defined even when $\dim \Gamma = 1$. As in the 2-dimensional case, equation (4.2.27) again appears to be singular. Since $\mathring{I\!I}_{ij}{}^c$ vanishes identically for a curve, we need only consider the first two terms. We see that $W_{icjd} \mathrm{N}^{cd} = \kappa \mathbf{u}_i \mathbf{u}_j = \kappa \boldsymbol{g}_{ij}$, where $\mathbf{u}_i \in \Gamma(\mathcal{E}_i[1])$ is a (weighted) unit length tangent vector to the curve Γ. Now, κ is obtained by tracing, and since the ambient Weyl tensor is totally trace-free, we will have

$$\kappa = \boldsymbol{g}^{ij} W_{icjd} \mathrm{N}^{cd} = -\boldsymbol{g}^{ij} W_{icjd} \Pi^{cd} = -\boldsymbol{g}^{ij} W_{ikjl} \boldsymbol{g}^{kl}. \tag{4.3.8}$$

But once again W_{ikjl} inherits Riemann tensor symmetries from the ambient Weyl tensor, and so is a section of $S^2 \Lambda^2 T^*\Gamma[2]$ where $\dim \Gamma = 1$, and so must vanish identically. The same holds for $W_{abcd} \mathrm{N}^{ac} \mathrm{N}^{bd}$. Thus in the 1-dimensional case, the first two terms of (4.2.27) are both zero, and combined with our observation about the trace-free second fundamental form, we see that the entire right-hand side of (4.2.27) becomes zero.

We mention now that the vanishing of the Fialkow tensor when $\dim\Gamma = 1$ does not create any problems in our later treatment of distinguished conformal submanifolds, so there is no reason to define the Fialkow tensor otherwise.

The observation concerning the vanishing of the Fialkow tensor from Section 4.3.1 still holds: namely that the Fialkow tensor vanishing is equivalent to the intrinsic tractor connection being exactly the checked connection. Thus in the 1-dimensional case we now have a tractor bundle and a connection on that bundle.

Our conventions for the Fialkow tensor are thus summarized as follows:

$$\mathcal{F}_{ij} = \begin{cases} P_{ij} - p_{ij} + H_b \mathring{I\!I}_{ij}{}^b + \frac{1}{2} H_b H^b \boldsymbol{g}_{ij} & \text{if } m \geq 3, \\ \frac{1}{2} \mathring{I\!I}^{klc} \mathring{I\!I}_{klc} \boldsymbol{g}_{ij} & \text{if } m = 2, \\ 0 & \text{if } m = 1. \end{cases} \tag{4.3.9}$$

4.4 Tractor differential forms

It is useful to introduce some notation for tractor forms in general before proceeding to the specific case of the normal form. From the composition series for the standard tractor bundle, one sees that

$$\mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]} = \mathcal{E}_{[a_2 \cdots a_k]}[k] \mathbin{+\!\!\!\!\supset} \begin{array}{c} \mathcal{E}_{[a_1 a_2 \cdots a_{k-1} a_k]}[k] \\ \oplus \\ \mathcal{E}_{[a_3 \cdots a_k]}[k-2] \end{array} \mathbin{+\!\!\!\!\supset} \mathcal{E}_{[a_2 \cdots a_k]}[k-2]. \tag{4.4.1}$$

The tractor projectors for the standard tractor bundle induce tractor projectors on the bundles of tractor forms. Since these will be very important for us, we introduce dedicated notation for these.

$$\begin{aligned} \mathbb{Y}_{A_1 A_2 \cdots A_{k-1} A_k}{}^{a_2 \cdots a_{k-1} a_k} &:= Y_{[A_1} Z_{A_2}^{a_2} \cdots Z_{A_{k-1}}^{a_{k-1}} Z_{A_k]}^{a_k} && \in \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]}{}^{a_2 \cdots a_{k-1} a_k}[-k] \\ \mathbb{Z}_{A_1 A_2 \cdots A_{k-1} A_k}^{a_1 a_2 \cdots a_{k-1} a_k} &:= Z_{[A_1}^{a_1} Z_{A_2}^{a_2} \cdots Z_{A_{k-1}}^{a_{k-1}} Z_{A_k]}^{a_k} && \in \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]}^{a_1 a_2 \cdots a_{k-1} a_k}[-k] \\ \mathbb{W}_{A_1 A_2 A_3 \cdots A_{k-1} A_k}{}^{a_3 \cdots a_{k-1} a_k} &:= X_{[A_1} Y_{A_2} Z_{A_3}^{a_3} \cdots Z_{A_{k-1}}^{a_{k-1}} Z_{A_k]}^{a_k} && \in \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]}{}^{a_3 \cdots a_{k-1} a_k}[-k+2] \\ \mathbb{X}_{A_1 A_2 \cdots A_{k-1} A_k}{}^{a_2 \cdots a_{k-1} a_k} &:= X_{[A_1} Z_{A_2}^{a_2} \cdots Z_{A_{k-1}}^{a_{k-1}} Z_{A_k]}^{a_k} && \in \mathcal{E}_{[A_1 A_2 \cdots A_{k-1} A_k]}{}^{a_2 \cdots a_{k-1} a_k}[-k+2] \end{aligned} \tag{4.4.2}$$

It is also useful to record the derivatives (with the ambient tractor connection) of these splitting operators for later use.

$$\begin{aligned} \nabla_b \mathbb{Y}_{A_1 A_2 A_3 \cdots A_k}{}^{a_2 a_3 \cdots a_k} &= P_{b a_1} \mathbb{Z}_{A_1 A_2 A_3 \cdots A_k}^{a_1 a_2 a_3 \cdots a_k} + (k-1) + P_b{}^{a_2} \mathbb{W}_{A_1 A_2 A_3 \cdots A_k}{}^{a_3 \cdots a_k} \\ \nabla_b \mathbb{Z}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k} &= -k \cdot P_b{}^{a_1} \mathbb{X}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k} - k \cdot \delta_b{}^{a_1} \mathbb{Y}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k} \\ \nabla_b \mathbb{W}_{A_1 A_2 A_3 \cdots A_k}{}^{a_3 \cdots a_k} &= -\boldsymbol{g}_{b a_2} \mathbb{Y}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k} + P_{b a_2} \mathbb{X}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k} \\ \nabla_b \mathbb{X}_{A_1 A_2 A_3 \cdots A_k}{}^{a_2 a_3 \cdots a_k} &= \boldsymbol{g}_{b a_1} \mathbb{Z}_{A_1 A_2 A_3 \cdots A_k}^{a_1 a_2 a_3 \cdots a_k} - (k-1) \delta_b{}^{a_2} \mathbb{W}_{A_1 A_2 A_3 \cdots A_k}{}^{a_3 \cdots a_k}. \end{aligned} \tag{4.4.3}$$

In general, there is an isomorphism $\left(\Lambda^k N^*\Gamma\right)[k] \cong \Lambda^k \mathcal{N}^*$ for $k \leq d = \operatorname{codim}\Gamma$. For $\nu_{a_1 a_2 \cdots a_k} \in \left(\Lambda^k N^*\Gamma\right)[k]$, the isomorphism is given explicitly by

$$\nu_{a_1 a_2 \cdots a_k} \mapsto \nu_{a_1 a_2 \cdots a_k} \mathbb{Z}^{a_1 a_2 \cdots a_k}_{A_1 A_2 \cdots A_k} + k \cdot \nu_{b a_2 \cdots a_k} H^b \mathbb{X}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k}. \tag{4.4.4}$$

Invariance of this map may be checked via the transformation formulae for the tractor form projectors and the mean curvature:

$$\begin{aligned}
&\widehat{\nu}_{a_1 a_2 \cdots a_k} \widehat{\mathbb{Z}}^{a_1 a_2 \cdots a_k}_{A_1 A_2 \cdots A_k} + k \cdot \widehat{\nu}_{b a_2 \cdots a_k} \widehat{H}^b \widehat{\mathbb{X}}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k} \\
&= \nu_{a_1 a_2 \cdots a_k} \left(\mathbb{Z}^{a_1 a_2 \cdots a_k}_{A_1 A_2 \cdots A_k} + k \cdot \Upsilon^{a_1} \mathbb{X}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k} \right) \\
&\quad + k \cdot \nu_{b a_2 \cdots a_k} \left(H^b - \mathrm{N}^b_{a_1} \Upsilon^{a_1} \right) \mathbb{X}_{A_1 A_2 \cdots A_d}{}^{a_2 \cdots a_k} \\
&= \nu_{a_1 a_2 \cdots a_k} \mathbb{Z}^{a_1 a_2 \cdots a_k}_{A_1 A_2 \cdots A_k} \\
&\quad + k \cdot \left(\nu_{b a_2 \cdots a_k} H^b + \nu_{a_1 a_2 \cdots a_k} \Upsilon^{a_1} - \nu_{b a_2 \cdots a_k} \mathrm{N}^b_{a_1} \Upsilon^{a_1} \right) \mathbb{X}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k} \\
&= \nu_{a_1 a_2 \cdots a_k} \mathbb{Z}^{a_1 a_2 \cdots a_k}_{A_1 A_2 \cdots A_k} + k \cdot \nu_{b a_2 \cdots a_k} H^b \mathbb{X}_{A_1 A_2 \cdots A_k}{}^{a_2 \cdots a_k}.
\end{aligned}$$

Note that in the case $k = 1$ this map is simply the map $N^*\Gamma[1] \to \mathcal{N}^*$ of (4.2.1).

4.4.1 The tractor volume form

The top exterior power of the tractor bundle possesses a distinguished section, the *tractor volume form*, defined by

$$\epsilon_{A_1 A_2 A_3 \cdots A_{n+2}} := \mathrm{vol}^M_{a_3 \cdots a_{n+2}} \mathbb{W}_{A_1 A_2 A_3 \cdots A_{n+2}}{}^{a_3 \cdots a_{n+2}}, \tag{4.4.5}$$

where $\mathrm{vol}^M_{a_3 \cdots a_{n+2}} \in \left(\Lambda^n T^* M\right)[n]$ is the (weighted) Riemannian volume form of M. While not a *true* volume form, it nonetheless has much in common with such objects which makes many similar definitions and operations possible.

The tractor normal form has two important properties, both of which will be useful for our purposes:

1. the tractor volume form is parallel for the tractor connection; and
2. the tractor volume form induces an isomorphism $\Lambda^k \mathcal{T} \to \Lambda^{n+2-k} \mathcal{T}^*$ defined by the mapping $\nu^{A_{n+3-k} \cdots A_{n+2}} \mapsto \epsilon_{A_1 A_2 \cdots A_{n+2-k} A_{n+3-k} \cdots A_{n+2}} \nu^{A_{n+3-k} \cdots A_{n+2}}$.

4.4.2 The tractor normal form

As a special case of (4.4.4), we get an isomorphism of normal top forms with the top exterior power of the normal tractor bundle. The image of the usual Riemannian normal form under this map gives a distinguished section of this bundle, and this section is closely related to the tractor second fundamental form and the tractor normal projector.

Definition 4.4.1 (Tractor normal form)**.** Let $\Gamma \hookrightarrow M$ be a submanifold in a conformal manifold $(M, \boldsymbol{c})$, and let $N_{a_1a_2\cdots a_d} \in \Lambda^d N^*\Gamma[d]$ be the (weighted) Riemannian normal form. Define the *tractor normal form* to be the section $N_{A_1A_2\cdots A_d} \in \Gamma(\Lambda^d \mathcal{N}^*)$

$$N_{A_1A_2\cdots A_d} := N_{a_1a_2\cdots a_d}\mathbb{Z}^{a_1a_2\cdots a_d}_{A_1A_2\cdots A_d} + d \cdot N_{ba_2\cdots a_d}H^b\mathbb{X}_{A_1\cdots A_d}{}^{a_2\cdots a_d}. \tag{4.4.6}$$

One readily sees that

$$\begin{aligned} N_{A_1A_2\cdots A_d}N^{A_1A_2\cdots A_d} &= N_{a_1a_2\cdots a_d}N^{b_1b_2\cdots b_d}\mathbb{Z}^{a_1a_2\cdots a_d}_{A_1A_2\cdots A_d}\mathbb{Z}^{A_1A_2\cdots A_d}_{b_1b_2\cdots b_d} \\ &= N_{a_1a_2\cdots a_d}N^{a_1a_2\cdots a_d} \\ &= d!, \end{aligned}$$

since all other contractions of the $\mathbb{X}$ and $\mathbb{Z}$ projectors are zero.

In light of this, the tractor normal form may equivalently be characterized as the unique section $N_{A_1\cdots A_d}$ of $\Lambda^d\mathcal{N}^*$ such that

1. $N_{A_1A_2\cdots A_d}v^{A_1} = 0$ for all $v \in \mathcal{N}^\perp$, and
2. $N_{A_1A_2\cdots A_d}N^{A_1A_2\cdots A_d} = d!$.

Given an orthonormal basis $\{N^1_A, \ldots, N^d_A\}$ for the normal tractor bundle, one sees that

$$d! \cdot N_{[A_1} \cdots N_{A_d]} = N^1_{A_1} \wedge \cdots \wedge N^d_{A_d} \tag{4.4.7}$$

is clearly orthogonal to all sections of $\mathcal{N}^\perp$ and satisfies the above normalization condition.

Our task is now to relate the tractor normal form to the other objects introduced, namely, the tractor normal projector and the tractor second fundamental form. These relationships will lay the foundation for the notion of distinguished submanifold that we will introduce in the following chapter.

First, the tractor normal projector.

Proposition 4.4.2. *The tractor projector* N^A_B *is equal to*

$$\mathrm{N}^{A_1}_{A_2} = \frac{1}{(d-1)!}N^{A_1B_2\cdots B_d}N_{A_2B_2\cdots B_d}. \tag{4.4.8}$$

Proof. Let $\{N^A_1, N^A_2, \ldots, N^A_d\}$ be an orthonormal basis for the normal tractor bundle. Then by equation (4.4.7),

$$N^{A_1B_2\cdots B_d} = d! \cdot N^{[A_1}_1 N^{B_2}_2 \cdots N^{B_d]}_d.$$

The contraction $N^{A_1 B_2 \cdots B_d} N_{A_2 B_2 \cdots B_d}$ is a sum of terms of the form

$$\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) N^{A_1}_{\sigma(1)} N^{B_2}_{\sigma(2)} \cdots N^{B_d}_{\sigma(d)} N^{\tau(1)}_{A_2} N^{\tau(2)}_{B_2} \cdots N^{\tau(d)}_{B_d},$$

where $\sigma, \tau \in \mathfrak{S}_d$. Now, we claim that such a term will be non-zero if, and only if $\sigma = \tau$.

Clearly, if $\sigma = \tau$, then

$$\operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) N^{A_1}_{\sigma(1)} N^{B_2}_{\sigma(2)} \cdots N^{B_d}_{\sigma(d)} N^{\tau(1)}_{A_2} N^{\tau(2)}_{B_2} \cdots N^{\tau(d)}_{B_d} = N^{A_1}_{\sigma(1)} N^{\sigma(1)}_{A_2}.$$

Conversely, if $\sigma \neq \tau$, then there is some $i \in \{1, 2, \ldots, d\}$ such that $\sigma(i) \neq \tau(i)$. Then the contraction will contain

$$N^{B_i}_{\sigma(i)} N^{\tau(i)}_{B_i} = \delta^{\tau(i)}_{\sigma(i)},$$

by orthogonality, which is zero since we are assuming that $\sigma(i) \neq \tau(i)$. Hence the only non-zero terms in the contraction are those where the same permutation is applied to both sets of indices.

Finally, we need only count how many such terms there are. We have just established that there are $d!$ non-zero terms. Fixing $\sigma(1)$, one sees that there are $(d-1)!$ remaining possibilities for σ, all of which will lead to $N^{A}_{\sigma(1)} N^{\tau(1)}_{B}$. Thus as $\sigma(1)$ ranges over $\{1, 2, \ldots, d\}$, we have that

$$\begin{aligned} N^{A_1 B_2 \cdots B_d} N_{A_2 B_2 \cdots B_d} &= \sum_{\sigma \in \mathfrak{S}_d} N^{A_1}_{\sigma(1)} N^{\sigma(1)}_{A_2} \\ &= (d-1)! \cdot \left(N^{A_1}_1 N^1_{A_2} + N^{A_1}_2 N^2_{A_2} + \cdots + N^{A_1}_d N^d_{A_2} \right) \\ &= (d-1)! \cdot N^{A_1}_{A_2}. \end{aligned}$$

□

Remark 4.4.3. The factor here looks slightly strange, but the following shows that this factor is consistent with our conventions. We compute N^A_A in two different ways.

First from N^A_B expressed in terms of an orthonormal basis for the normal bundle:

$$\begin{aligned} \mathrm{N}^A_A &= \sum_{i=1}^{d} N_i^A N^i_A \\ &= d, \end{aligned}$$

and secondly using the relationship derived in the previous proposition:

$$\mathrm{N}^A_A = \frac{1}{(d-1)!} N^{A B_2 \cdots B_d} N_{A B_2 \cdots B_d}$$

$$
\begin{aligned}
&= \frac{1}{(d-1)!} N^{ab_2\cdots b_d} N_{ab_2\cdots b_d} \\
&= \frac{1}{(d-1)!} \cdot d! \\
&= d.
\end{aligned}
$$

We compute two expressions for $\nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d}$; both are useful for different applications. First, we differentiate (4.4.3) directly using the derivatives of the tractor form projectors from (4.4.3).

Proposition 4.4.4. *The derivative of the tractor normal form expressed in the tractor projector notation is*

$$
\begin{aligned}
\nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d} &= \left[\nabla_i N_{a_1 a_2 \cdots a_{d-1} a_d} + d \cdot N_{b a_2 \cdots a_{d-1} a_d} H^b \boldsymbol{g}_{i a_1} \right] \mathbb{Z}^{a_1 a_2 \cdots a_d}_{A_1 A_2 \cdots A_d} \\
&+ d \cdot \left[\nabla_i \left(N_{b a_2 \cdots a_{d-1} a_d} H^b \right) - N_{a_1 a_2 \cdots a_{d-1} a_d} P_i^{\ a_1} \right] \mathbb{X}_{A_1 A_2 \cdots A_d}^{\quad\ a_2 \cdots a_d}.
\end{aligned} \tag{4.4.9}
$$

Proof. Differentiating equation (4.4.6):

$$
\begin{aligned}
\nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d} &= \left(\nabla_i N_{a_1 a_2 \cdots a_d} \right) \mathbb{Z}^{a_1 a_2 \cdots a_d}_{A_1 A_2 \cdots A_d} \\
&+ N_{a_1 a_2 \cdots a_d} \left(-d \cdot P_i^{\ a_1} \mathbb{X}_{A_1 A_2 \cdots A_d}^{\quad\ a_2 \cdots a_d} - d \cdot \delta_i^{\ a_1} \mathbb{Y}_{A_1 A_2 \cdots A_d}^{\quad\ a_2 \cdots a_d} \right) \\
&+ d \cdot \nabla_i \left(N_{b a_2 \cdots a_d} H^b \right) \mathbb{X}_{A_1 A_2 \cdots A_d}^{\quad\ a_2 \cdots a_d} \\
&+ d \cdot N_{b a_2 \cdots a_d} H^b \left(\boldsymbol{g}_{i a_1} \mathbb{Z}^{a_1 a_2 \cdots a_d}_{A_1 A_2 \cdots A_d} - (d-1) \cdot \delta_i^{\ a_2} \mathbb{W}_{a_1 a_2 a_3 \cdots a_d}^{\quad\quad A_3 \cdots A_d} \right) \\
&= \left[\nabla_i N_{a_1 a_2 \cdots a_{d-1} a_d} + d \cdot N_{b a_2 \cdots a_{d-1} a_d} H^b \boldsymbol{g}_{i a_1} \right] \mathbb{Z}^{a_1 a_2 \cdots a_d}_{A_1 A_2 \cdots A_d} \\
&+ d \cdot \left[\nabla_i \left(N_{b a_2 \cdots a_{d-1} a_d} H^b \right) - N_{a_1 a_2 \cdots a_{d-1} a_d} P_i^{\ a_1} \right] \mathbb{X}_{A_1 A_2 \cdots A_d}^{\quad\ a_2 \cdots a_d},
\end{aligned}
$$

where we use the fact that any terms where the i index is contracted into the normal form will vanish, since i is tangential. □

The derivative of the tractor normal form is also closely related to the tractor second fundamental form; the second of our two expressions for $\nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d}$ makes precise this relationship.

Theorem 4.4.5. *The tractor normal form is related to the tractor second fundamental form by*

$$
\nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d} = -d \cdot \mathbb{L}_{i[A_d}^{\quad\ A_0} N_{A_1 A_2 \cdots A_{d-1}] A_0}, \tag{4.4.10}
$$

Proof. Fix an orthonormal basis of normal tractors $\{N^1_A, \dots, N^d_A\}$. (Such a basis is simply the image of an orthonormal basis for $N^*\Gamma[1]$ under the isomorphism (4.2.1).) Recall equation (4.2.20):

$$-\mathbb{L}_{iA_d}{}^{A_0} = \mathrm{N}^{A_0}_B \nabla_i \mathrm{N}^B_{A_d}.$$

We have already made use of this to compute one explicit expression for the tractor second fundamental form. We make use of it again to compute $\mathrm{N}^{A_0}_B \nabla_i \mathrm{N}^B_{A_d}$ in a different way.

Working with the same orthonormal basis of normal tractors,

$$\begin{aligned}
\mathrm{N}^{A_0}_B \nabla_i \mathrm{N}^B_{A_d} &= \left(N^{A_0}_1 N^1_B + \dots + N^{A_0}_d N^d_B\right) \nabla_i \left(N^B_1 N^1_{A_d} + \dots + N^B_d N^d_{A_d}\right) \\
&= \left(N^{A_0}_1 N^1_B + \dots + N^{A_0}_d N^d_B\right) \left(N^1_{A_d} \nabla_i N^B_1 + N^B_1 \nabla_i N^1_{A_d} + \cdots \right. \\
&\qquad \left. + N^d_{A_d} \nabla_i N^B_d + N^B_d \nabla_i N^d_{A_d}\right)
\end{aligned}$$

Expanding the final line of the above will yield two types of terms:

- terms of the form $N^{A_0}_k N^k_B \left(\nabla_i N^\ell_{A_d}\right) N^B_\ell$, which are zero unless $k = \ell$, in which case it simplifies to $N^{A_0}_k \nabla_i N^k_{A_d}$; and
- terms of the form $N^{A_0}_k N^k_B N^\ell_{A_d} \nabla_i N^B_\ell$, which are zero unless $k \neq \ell$.

Thus

$$-\mathbb{L}_{iA_d}{}^{A_0} = \sum_{k=1}^{d} N^{A_0}_k \nabla_i N^k_{A_d} + \sum_{k \neq \ell} N^{A_0}_k N^\ell_{A_d} \left(N^B_k \nabla_i N^\ell_B\right). \tag{4.4.11}$$

We now use this formula for to compute the right-hand side of the equation (4.4.10).

$$\begin{aligned}
-\mathbb{L}_{i[A_d}{}^{A_0} N_{A_1 A_2 \cdots A_{d-1}]A_0} &= \left(\sum_{k=1}^{d} N^{A_0}_k \nabla_i N^k_{[A_d}\right) N_{A_1 A_2 \cdots A_{d-1}]A_0} \\
&\quad + \left(\sum_{k \neq \ell} \left(N^B_k \nabla_i N^\ell_B\right) N^{A_0}_k N^\ell_{[A_d}\right) N_{A_1 A_2 \cdots A_{d-1}]A_0}.
\end{aligned} \tag{4.4.12}$$

We deal with each of these sums separately. Note that $N^{A_0}_k N_{A_1 A_2 \cdots A_{d-1} A_0}$ appears in both terms, so we compute an expression for this as an intermediate calculation.

It is convenient to interchange A_k and A_0, so that the two copies of N^k have the same tractor index before expanding the antisymmetrization. This will incur a factor of -1

unless $d = k$; in this case those indices are already the same. This is the reason for the Kronecker delta term in the following:

$$\begin{aligned}
N_k^{A_0} N_{A_1 A_2 \cdots A_{d-1} A_0} &= d! \cdot N_k^{A_0} N^1_{[A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} N^k_{A_k} N^{k+1}_{A_{k+1}} \cdots N^d_{A_0]} \\
&= (-1)^{1-\delta_{k,d}} \cdot d! \cdot N_k^{A_0} N^1_{[A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} N^k_{A_0} N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}} N^d_{A_k]} \\
&= (-1)^{1-\delta_{k,d}} \cdot \sum_{\sigma \in \mathrm{Sym}\{0,1,2,\ldots,d-1\}} \mathrm{sgn}(\sigma) N_k^{A_0} N^1_{A_{\sigma(1)}} N^2_{A_{\sigma(2)}} \cdots N^{k-1}_{A_{\sigma(k-1)}} \\
&\qquad \cdot N^k_{A_{\sigma(0)}} N^{k+1}_{A_{\sigma(k+1)}} \cdots N^{d-1}_{A_{\sigma(d-1)}} N^d_{A_{\sigma(k)}} \\
&= (-1)^{1-\delta_{k,d}} \cdot \sum_{\substack{\sigma \in \mathrm{Sym}\{0,1,2,\ldots,d-1\}, \\ \sigma(0)=0}} \mathrm{sgn}(\sigma) N^1_{A_{\sigma(1)}} N^2_{A_{\sigma(2)}} \cdots N^{k-1}_{A_{\sigma(k-1)}} \\
&\qquad \cdot (N_k^{A_0} N^k_{A_{\sigma(0)}}) N^{k+1}_{A_{\sigma(k+1)}} \cdots N^{d-1}_{A_{\sigma(d-1)}} N^d_{A_{\sigma(k)}} \\
&= (-1)^{1-\delta_{k,d}} \cdot \sum_{\sigma \in \mathfrak{S}_{d-1}} \mathrm{sgn}(\sigma) N^1_{A_{\sigma(1)}} N^2_{A_{\sigma(2)}} \cdots N^{k-1}_{A_{\sigma(k-1)}} \\
&\qquad \cdot N^{k+1}_{A_{\sigma(k+1)}} \cdots N^{d-1}_{A_{\sigma(d-1)}} N^d_{A_{\sigma(k)}} \\
&= (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot N^1_{[A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}} N^d_{A_k]}. \qquad (4.4.13)
\end{aligned}$$

After interchanging A_k and A_0 as described above, the contracted term will be zero unless $\sigma(0) = 0$. Fixing $\sigma(0) = 0$, the sum effectively runs over $\sigma \in \mathrm{Sym}\{1, 2, \ldots, d-1\} = \mathfrak{S}_{d-1}$, and therefore one sees that the sum is just the full antisymmetrization over the uncontracted indices, with a factor of $(d-1)!$ so that there are no fractions when that is expanded.

The salient observations from this calculation are:

- N^k does not appear; and
- the expression is (up to a factor) the antisymmetrization of all remaining elements of the orthonormal basis.

We are now ready to proceed with the proof. We claim that the second sum of (4.4.12) vanishes. In fact, we claim that *each summand* of the second sum vanishes.

We see this via the above formula for $N_k^{A_0} N^\ell_{[A_d} N_{A_1 A_2 \cdots A_{d-1}] A_0}$. The reason for this is the following: $N_k^{A_0} N^\ell_{[A_d} N_{A_1 A_2 \cdots A_{d-1}] A_0}$ is equal to a constant multiplied by the wedge product of the N^j_A for $j \neq k$. In particular, this contains N^ℓ, since in the second sum, $k \neq \ell$. Therefore $N_k^{A_0} N^\ell_{[A_d} N_{A_1 A_2 \cdots A_{d-1}] A_0}$ is skew over the indices $A_1, \ldots, A_d$, and contains two copies of N^ℓ_A.

More formally,

$$\begin{aligned}
&N_k^{A_0} N^{\ell}_{[A_d} N_{A_1 A_2 \cdots A_{d-1}] A_0} \\
&\quad = (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot N^{\ell}_{[A_d} N^1_{[A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} N^d_{A_k} N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}]]} \\
&\quad = (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot N^{\ell}_{[A_d} N^1_{A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} N^d_{A_k} N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}]} \\
&\quad = (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot N^{\ell}_{[A_d} N^1_{A_1} N^2_{A_2} \cdots N^{\ell}_{A_\ell} \cdots N^{k-1}_{A_{k-1}} N^d_{A_k} N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}]} \\
&\quad = 0.
\end{aligned}$$

The above is written with $\ell < k$, but the result clearly still holds if $\ell > k$.

We use equation (4.4.13) again in computing the other sum from (4.4.12).

For a general term,

$$\begin{aligned}
&N_k^{A_0} \nabla_i N^k_{[A_d} N_{A_1 A_2 \cdots A_{d-1}] A_0} \\
&\quad = (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot \left(\nabla_i N^k_{[A_d}\right) N^1_{[A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} N^d_{A_k} N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}]]} \\
&\quad = (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot \left(\nabla_i N^k_{[A_d}\right) N^1_{A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} N^d_{A_k} N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}]} \\
&\quad = (-1)^{1-\delta_{k,d}} \cdot (-1)^{1-\delta_{k,d}} \cdot (d-1)! \cdot \left(\nabla_i N^k_{[A_k}\right) N^1_{A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} N^d_{A_d} N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}]} \\
&\quad = (d-1)! \cdot \left(\nabla_i N^k_{[A_k}\right) N^1_{A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} N^d_{A_d} N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}]} \\
&\quad = (d-1)! \cdot N^1_{[A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} \left(\nabla_i N^k_{A_k}\right) N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}} N^d_{A_d]}.
\end{aligned}$$

Note from the third to fourth lines we interchange A_d and A_k, which multiplies the result by -1 unless $k = d$, so we collect another of the Kronecker delta factors.

However

$$(-1)^{1-\delta_{k,d}} \cdot (-1)^{1-\delta_{k,d}} = \left((-1)^{1-\delta_{k,d}}\right)^2 = 1,$$

so those terms do not appear in the following lines.

Finally, it remains to sum the above terms over k:

$$\begin{aligned}
&\left(\sum_{k=1}^{d} N_k^{A_0} \nabla_i N^k_{[A_d}\right) N_{A_1 A_2 \cdots A_{d-1}] A_0} \\
&\quad = (d-1)! \cdot \sum_{k=1}^{d} N^1_{[A_1} N^2_{A_2} \cdots N^{k-1}_{A_{k-1}} \left(\nabla_i N^k_{A_k}\right) N^{k+1}_{A_{k+1}} \cdots N^{d-1}_{A_{d-1}} N^d_{A_d]} \\
&\quad = (d-1)! \cdot \nabla_i \left(N^1_{[A_1} N^2_{A_2} \cdots N^{d-1}_{A_{d-1}} N^d_{A_d]}\right)
\end{aligned}$$

$$= \frac{(d-1)!}{d!} \cdot \nabla_i \left(d! \cdot N^1_{[A_1} N^2_{A_2} \cdots N^{d-1}_{A_{d-1}} N^d_{A_d]} \right)$$
$$= \frac{1}{d} \cdot \nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d}.$$

Now that we have computed both of the sums from equation (4.4.12), we have

$$\begin{aligned} -\mathbb{L}_{i[A_d}{}^{A_0} N_{A_1 A_2 \cdots A_{d-1}]A_0} &= \left(\sum_{k=1}^{d} N^{A_0}_k \nabla_i N^k_{[A_d} \right) N_{A_1 A_2 \cdots A_{d-1}]A_0} \\ &\quad + \left(\sum_{k \neq \ell} \left(N^B_k \nabla_i N^\ell_B \right) N^{A_0}_k N^\ell_{[A_d} \right) N_{A_1 A_2 \cdots A_{d-1}]A_0} \\ &= \frac{1}{d} \cdot \nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d} + 0, \end{aligned}$$

whence

$$\nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d} = -d \cdot \mathbb{L}_{i[A_d}{}^{A_0} N_{A_1 A_2 \cdots A_{d-1}]A_0}.$$

□

Using Theorem 4.4.5, we see that the tractor second fundamental form may also be written purely in terms of the tractor normal form.

Proposition 4.4.6. *The tractor second fundamental form is related to the tractor normal form via*

$$N^{A_{d+1} A_2 \cdots A_d} \nabla_i N_{A_1 A_2 \cdots A_d} = -(d-1)! \cdot \mathbb{L}_{iA_1}{}^{A_{d+1}}. \tag{4.4.14}$$

Proof. Theorem 4.4.5 gives that

$$\nabla_i N_{A_1 A_2 \cdots A_{d-1} A_d} = -d \cdot \mathbb{L}_{i[A_d}{}^{A_0} N_{A_1 A_2 \cdots A_{d-1}]A_0}.$$

We will contract both sides of this display with $N^{A_{d+1} A_2 \cdots A_d}$. First, we see that

$$\begin{aligned} &N^{A_{d+1} A_2 \cdots A_d} \mathbb{L}_{i[A_d}{}^{A_0} N_{A_1 A_2 \cdots A_{d-1}]A_0} \\ &\quad = -N^{A_{d+1} A_2 \cdots A_d} \mathbb{L}_{i[A_1}{}^{A_0} N_{A_d A_2 \cdots A_{d-1}]A_0} \\ &\quad = -\frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn}(\sigma) \cdot \mathbb{L}_{iA_{\sigma(1)}}{}^{A_0} N_{A_{\sigma(d)} A_{\sigma(2)} \cdots A_{\sigma(d-1)} A_0} N^{A_{d+1} A_2 \cdots A_d}. \end{aligned}$$

Now, the contraction of the tractor second fundamental form and the first tractor normal form will be zero unless $\sigma(1) = 1$, since otherwise the normal form will be contracted into the (tangential) lower index of the tractor second fundamental form. Any such permutation σ will mean the two normal forms are contracted on all indices except one on each copy. Hence

$$
\begin{aligned}
&-\frac{1}{d!}\sum_{\sigma\in\mathfrak{S}_d}\operatorname{sgn}(\sigma)\cdot\mathbb{L}_{iA_{\sigma(1)}}{}^{A_0}N_{A_{\sigma(d)}A_{\sigma(2)}\cdots A_{\sigma(d-1)}A_0}N^{A_{d+1}A_2\cdots A_d}\\
&=\frac{1}{d!}\sum_{\sigma\in\mathfrak{S}_d}\operatorname{sgn}(\sigma)\mathbb{L}_{iA_{\sigma(1)}}{}^{A_0}N_{A_0A_{\sigma(2)}\cdots A_{\sigma(d-1)}A_{\sigma(d)}}N^{A_{d+1}A_2\cdots A_d}\\
&=\frac{1}{d!}\sum_{\sigma\in\mathrm{Sym}\{2,\ldots,d-1,d\}}\operatorname{sgn}(\sigma)\cdot\mathbb{L}_{iA_1}{}^{A_0}N_{A_0A_{\sigma(2)}\cdots A_{\sigma(d-1)}A_{\sigma(d)}}N^{A_{d+1}A_2\cdots A_d}\\
&=\frac{1}{d!}\sum_{\sigma\in\mathrm{Sym}\{2,\ldots,d-1,d\}}\operatorname{sgn}(\sigma)\cdot\operatorname{sgn}(\sigma)\cdot\mathbb{L}_{iA_1}{}^{A_0}N_{A_0A_{\sigma(2)}\cdots A_{\sigma(d-1)}A_{\sigma(d)}}N^{A_{d+1}A_{\sigma(2)}\cdots A_{\sigma(d)}}\\
&=\frac{1}{d!}\sum_{\sigma\in\mathrm{Sym}\{2,\ldots,d-1,d\}}\mathbb{L}_{iA_1}{}^{A_0}\cdot(d-1)!\cdot N_{A_0}^{A_{d+1}}\\
&=\frac{(d-1)!}{d!}\sum_{\sigma\in\mathrm{Sym}\{2,\ldots,d-1,d\}}\mathbb{L}_{iA_1}{}^{A_{d+1}}\\
&=\frac{1}{d}\cdot(d-1)!\cdot\mathbb{L}_{iA_1}{}^{A_{d+1}}\\
&=\frac{(d-1)!}{d}\cdot\mathbb{L}_{iA_1}{}^{A_{d+1}}.
\end{aligned}
$$

Thus, equation (4.4.10) implies that

$$
\begin{aligned}
N^{A_{d+1}A_2\cdots A_{d-1}A_d}\nabla_i N_{A_1A_2\cdots A_{d-1}A_d}&=-d\cdot N^{A_{d+1}A_2\cdots A_{d-1}A_d}\mathbb{L}_{[A_d}{}^{A_0}N_{A_1A_2\cdots A_{d-1}]A_0}\\
&=-d\cdot\frac{(d-1)!}{d}\cdot\mathbb{L}_{iA_1}{}^{A_{d+1}}\\
&=-(d-1)!\cdot\mathbb{L}_{iA_1}{}^{A_{d+1}}.
\end{aligned}
$$

□

We have already seen in Theorem 4.2.7 that the vanishing of the tractor second fundamental form is equivalent to the tractor normal projector being parallel. Using the results of this section, we can now prove a similar equivalence involving the tractor normal form.

Theorem 4.4.7. *The tractor normal form is parallel if, and only if, the tractor second fundamental form vanishes.*

Proof. First, suppose that $\mathbb{L}_{iA_d}{}^{A_{d+1}}=0$. Then the result of Theorem 4.4.5 shows that $\nabla_i N_{A_1A_2\cdots A_{d-1}A_d}=0$.

For the other direction, from Proposition 4.4.6, we have

$$N^{A_{d+1}A_2\cdots A_{d-1}A_d}\nabla_i N_{A_1A_2\cdots A_{d-1}A_d} = -(d-1)!\cdot \mathbb{L}_{iA_1}{}^{A_{d+1}},$$

and it follows that if $\nabla_i N_{A_1A_2\cdots A_{d-1}A_d} = 0$, then $\mathbb{L}_{iA_1}{}^{A_{d+1}} = 0$. □

Remark 4.4.8. Of note, many of the results of this Chapter used nothing more than a local orthonormal basis for the normal bundle and the Gauß formula. Since the normal tractor bundle is isomorphic to the usual normal bundle, and we have a Gauß formula in both cases, the proofs of these results may be repeated mutatis mutandis for the Riemannian objects to yield analogous statements and formulae.

4.5 The dual formulation

We have already noted in Section 4.4.1 that the tractor volume form gives an isomorphism $\Lambda^d\mathcal{T}^* \to \Lambda^{n+2-d}\mathcal{T}$. Let $N_{A_1A_2\cdots A_d}$ be the tractor normal form of a submanifold Γ. Define

$$\Sigma^{A_1A_2\cdots A_{m+2}} := \epsilon^{A_1A_2\cdots A_{m+2}A_{m+3}\cdots A_{n+2}} N_{A_{m+3}\cdots A_{n+2}}. \tag{4.5.1}$$

Then, since the tractor volume form is parallel and the map induced by contraction with it is an isomorphism, one observes

Theorem 4.5.1. *Let $\Gamma \hookrightarrow M$ be a submanifold of dimension m. Then the $m+2$-tractor $\Sigma^{A_1A_2\cdots A_{m+2}}$ defined in (4.5.1) is parallel if, and only if, the tractor normal form of Γ is parallel.*

Throughout the rest of this work, Σ will always denote the dual of a tractor normal form via the tractor volume form.

In light of this observation, one may equally well phrase results concerning the normal d-form dually, in terms of $m+2$-forms. Indeed, the results of [39] are presented from this perspective, with conformal distinguished curves being characterized via $1+2=3$-tractor fields. The point of this section is to note that these approaches are equivalent, and also to introduce this "dual" picture so that we can refer to it later.

In this section we calculate an explicit formula in terms of tractor projectors for the $m+2$-form Σ defined above. The calculation is split across two lemmata which combine to give the final formula.

Lemma 4.5.2. *The contraction of the ambient tractor volume form with the $\mathbb{Z}$ slot of the tractor normal form is*

$$\begin{aligned}&\left(\mathrm{vol}_M^{a_3\cdots a_{m+2}a_{m+3}\cdots a_{n+2}}\mathbb{W}^{A_1A_2A_3\cdots A_{m+2}A_{m+3}\cdots A_{n+2}}_{\quad a_3\cdots a_{m+2}a_{m+3}\cdots a_{n+2}}\right)\left(N_{b_{m+3}\cdots b_{n+2}}\mathbb{Z}^{b_{m+3}\cdots b_{n+2}}_{A_{m+3}\cdots A_{n+2}}\right)\\ &= \frac{(m+2)(m+1)}{(n+2)(n+1)}\cdot d!\cdot \mathrm{vol}_\Gamma^{a_3\cdots a_{m+2}}\cdot \mathbb{W}^{A_1A_2A_3\cdots A_{m+2}}_{\quad a_3\cdots a_{m+2}}.\end{aligned} \tag{4.5.2}$$

Proof. The idea is to partition the individual tractor projectors that make up the $\mathbb{W}$ projector into d contracted indices and the $n+2-d = m+2$ uncontracted indices. Do this by fixing the order of the tractor indices and assigning a tractor projector, X, Y or one of the Z_{a_i}, to those indices. Note that if either the X or Y are in the contracted partition, the result will be zero. Therefore we only consider the ${}^{(n+2)-2}C_d$ partitions where the only contracted projectors are Z projectors:

$$\begin{aligned}
&\mathrm{vol}_M^{a_3\cdots a_{m+2}a_{m+3}\cdots a_{n+2}} N_{b_{m+3}\cdots b_{n+2}} X^{[A_1} Y^{A_2} Z^{A_3}_{a_3} \cdots Z^{A_{m+2}}_{a_{m+2}} Z^{A_{m+3}}_{a_{m+3}} \cdots Z^{A_{n+2}]}_{a_{n+2}} Z^{b_{m+3}}_{[A_{m+3}} \cdots Z^{a_{n+2}}_{A_{n+2}]} \\
&= \mathrm{vol}_M^{a_3\cdots a_{m+2}a_{m+3}\cdots a_{n+2}} N_{b_{m+3}\cdots b_{n+2}} \cdot \frac{d! \cdot (n+2-d)!}{(n+2)!} \\
&\quad \cdot \Big(X^{[A_1} Y^{A_2} Z^{A_3} \cdots Z^{A_{m+2}]}_{a_{m+2}} Z^{[A_{m+3}}_{a_{m+3}} \cdots Z^{A_{n+2}]}_{a_{n+2}} \\
&\qquad - X^{[A_1} Y^{A_2} Z^{A_3} \cdots Z^{A_{m+2}]}_{a_{m+3}} Z^{[A_{m+3}}_{a_{m+2}} \cdots Z^{A_{n+2}]}_{a_{n+2}} - \cdots \Big) Z^{b_{m+3}}_{A_{m+3}} \cdots Z^{b_{n+2}}_{A_{n+2}} \\
&= \mathrm{vol}_M^{a_3\cdots a_{m+2}a_{m+3}\cdots a_{n+2}} N_{b_{m+3}\cdots b_{n+2}} \cdot \frac{d! \cdot (m+2)!}{(n+2)!} \\
&\quad \cdot \left(\frac{n!}{d!(n-d)!} \cdot X^{[A_1} Y^{A_2} Z^{A_3} \cdots Z^{A_{m+2}]}_{a_{m+2}} \delta^{b_{m+3}}_{a_{m+3}} \cdots \delta^{b_{n+2}}_{a_{n+2}} \right) \\
&= \mathrm{vol}_M^{a_3\cdots a_{m+2}a_{m+3}\cdots a_{n+2}} N_{a_{m+3}\cdots a_{n+2}} \cdot \frac{(m+2)(m+1)}{(n+2)(n+1)} \cdot X^{[A_1} Y^{A_2} Z^{A_3} \cdots Z^{A_{m+2}]}_{a_{m+2}} \\
&= d! \cdot \mathrm{vol}_\Gamma^{a_3\cdots a_{n+2-d}} \cdot \frac{(m+2)(m+1)}{(n+2)(n+1)} \cdot \mathbb{W}^{A_1 A_2 A_3 \cdots A_{n+2-d}}_{ a_3 \cdots a_{m+2}}.
\end{aligned}$$

Note that any permutation of the tensorial indices during the partitioning will can be reversed by applying the same permutation to the volume form and then relabelling indices. Therefore all non-zero terms remaining after the contraction will be equal and added together, and as already explained there are ${}^nC_d$ of these. These two observations combined explain the factor in the third-from-last line. We have also used Proposition 4.1.12 to simplify the contraction of the ambient volume form and the normal form in the final line of the above display. □

Lemma 4.5.3. *The contraction of the ambient tractor volume form with the* $\mathbb{X}$ *slot of the tractor normal form is*

$$\begin{aligned}
&\left(\mathrm{vol}_M^{a_3\cdots a_{m+2}a_{m+3}a_{m+4}\cdots a_{n+2}} \mathbb{W}^{A_1 A_2 A_3 \cdots A_{m+2} A_{m+3} A_{m+4} \cdots A_{n+2}}_{ a_3 \cdots a_{m+2} a_{m+3} a_{m+4} \cdots a_{n+2}}\right) \\
&\qquad \left(d \cdot N_{c b_{m+4}\cdots b_{n+2}} H^c \cdot \mathbb{X}_{A_{m+3}A_{m+4}\cdots A_{n+2}}^{\phantom{A_{m+3}} b_{m+4}\cdots b_{n+2}}\right) \\
&= (-1)^{m+1} d! \cdot \frac{(m+2)(m+1)}{(n+2)(n+1)} \mathrm{vol}_\Gamma^{[a_2\cdots a_{m+1}} H^{a_{m+2}]} \mathbb{X}^{A_1 A_2 \cdots A_{m+2}}_{ a_2 \cdots a_{m+2}}
\end{aligned} \tag{4.5.3}$$

Proof. We follow the same partitioning strategy as in the previous lemma. Note that the X projector in $\mathbb{X}$ *must* be contracted into the Y projector of $\mathbb{W}$; any other contraction yields zero.

$$\mathbb{W}^{A_1A_2A_3\cdots A_{m+2}A_{m+3}A_{m+4}\cdots A_{n+2}}{}_{a_3\cdots a_{m+2}a_{m+3}a_{m+4}\cdots a_{n+2}}\mathbb{X}_{A_{m+3}A_{m+4}\cdots A_{n+2}}{}^{b_{m+4}\cdots b_{n+2}}$$
$$X^{[A_1}Y^{A_2}Z^{A_3}_{a_3}\cdots Z^{A_{m+2}}_{a_{m+2}}Z^{A_{m+3}}_{a_{m+3}}Z^{A_{m+4}}_{a_{m+4}}\cdots Z^{A_{n+2}]}_{a_{n+2}}X_{A_{m+3}}Z^{b_{m+4}}_{A_{m+4}}\cdots Z^{b_{n+2}}_{A_{n+2}}$$
$$-X^{[A_1}Z^{A_2}_{a_{m+3}}Z^{A_3}_{a_3}\cdots Z^{A_{m+2}}_{a_{m+2}}Y^{A_{m+3}}Z^{A_{m+4}}_{a_{m+4}}\cdots Z^{A_{n+2}]}_{a_{n+2}}X_{A_{m+3}}Z^{b_{m+4}}_{A_{m+4}}\cdots Z^{b_{n+2}}_{A_{n+2}}$$
$$=-\frac{(n+1)!\cdot 1!}{(n+2)!}X^{[A_1}Z^{A_2}_{a_{m+3}}Z^{A_3}_{a_3}\cdots Z^{A_{m+2}}_{a_{m+2}}Z^{A_{m+4}}_{a_{m+4}}\cdots Z^{A_{n+2}]}_{a_{n+2}}\left(Y^{A_{m+3}}X_{A_{m+3}}\right)Z^{b_{m+4}}_{A_{m+4}}\cdots Z^{b_{n+2}}_{A_{n+2}}$$
$$=-\frac{1}{n+2}X^{[A_1}Z^{A_2}_{a_{m+3}}Z^{A_3}_{a_3}\cdots Z^{A_{m+2}}_{a_{m+2}}Z^{A_{m+4}}_{a_{m+4}}\cdots Z^{A_{n+2}]}_{a_{n+2}}Z^{b_{m+4}}_{A_{m+4}}\cdots Z^{b_{n+2}}_{A_{n+2}}$$
$$=-\frac{1}{n+2}\cdot(-1)^m\cdot X^{[A_1}Z^{A_2}_{a_3}Z^{A_3}_{a_4}\cdots Z^{A_{m+2}}_{a_{m+3}}Z^{A_{m+4}}_{a_{m+4}}\cdots Z^{A_{n+2}]}_{a_{n+2}}Z^{b_{m+4}}_{A_{m+4}}\cdots Z^{b_{n+2}}_{A_{n+2}}$$
$$=(-1)^{m+1}\cdot\frac{1}{n+2}\cdot X^{[A_1}Z^{A_2}_{a_3}Z^{A_3}_{a_4}\cdots Z^{A_{m+2}}_{a_{m+3}}Z^{A_{m+4}}_{a_{m+4}}\cdots Z^{A_{n+2}]}_{a_{n+2}}Z^{b_{m+4}}_{A_{m+4}}\cdots Z^{b_{n+2}}_{A_{n+2}}$$
$$=(-1)^{m+1}\cdot\frac{1}{n+2}\cdot\mathbb{X}^{A_1A_2A_3\cdots A_{m+2}A_{m+4}\cdots A_{n+2}}{}_{a_3a_4\cdots a_{m+3}a_{m+4}\cdots a_{n+2}}\cdot\mathbb{Z}^{b_{m+4}\cdots b_{n+2}}_{A_{m+4}\cdots A_{n+2}}.$$

For convenience, we relabel indices here before proceeding with the computation. We will compute just the contraction of the tractor projectors, and only restore the constant factors at the end. We again use our standard partition strategy, noting that for this particular contraction, the only non-zero contractions arise from partitions of the projectors making up the $\mathbb{X}$ projector where the X is in the uncontracted partition. There are ${}^{n+1}C_{m+2} ={}^{n+1} C_{d-1}$ possible partitions, of which ${}^{n}C_{m+1} ={}^{n} C_{d-1}$ have the X in the uncontracted partition.

Thus

$$\mathbb{X}^{A_1A_2A_3\cdots A_{m+2}A_{m+3}\cdots A_{n+1}}{}_{a_2a_3\cdots a_{m+2}a_{m+3}\cdots a_{n+1}}\cdot\mathbb{Z}^{b_{m+3}\cdots b_{n+1}}_{A_{m+3}\cdots A_{n+1}}$$
$$=\frac{{}^{n}C_{m+1}}{{}^{n+1}C_{m+2}}\mathbb{X}^{A_1A_2A_3\cdots A_{m+2}}{}_{[a_2a_3\cdots a_{m+2}}\cdot\delta^{b_{m+3}}_{a_{m+3}}\cdots\delta^{b_{n+1}}_{a_{n+1}]}$$
$$=\frac{n!}{(m+1)!\cdot(d-1)!}\cdot\frac{(m+2)!\cdot(d-1)!}{(n+1)!}\cdot\mathbb{X}^{A_1A_2A_3\cdots A_{m+2}}{}_{[a_2a_3\cdots a_{m+2}}\cdot\delta^{b_{m+3}}_{a_{m+3}}\cdots\delta^{b_{n+1}}_{a_{n+1}]}$$
$$=\frac{m+2}{n+1}\cdot\mathbb{X}^{A_1A_2A_3\cdots A_{m+2}}{}_{[a_2a_3\cdots a_{m+2}}\cdot\delta^{b_{m+3}}_{a_{m+3}}\cdots\delta^{b_{n+1}}_{a_{n+1}]}.$$

Returning the coefficients of the tractor terms yields

$$d\cdot\mathrm{vol}_M^{a_3\cdots a_{m+2}a_{m+3}a_{m+4}\cdots a_{n+2}}N_{cb_{m+4}\cdots b_{n+2}}H^c.$$

$$
\begin{aligned}
&\mathbb{W}^{A_1A_2A_3\cdots A_{m+2}A_{m+3}A_{m+4}\cdots A_{n+2}}{}_{a_3\cdots a_{m+2}a_{m+3}a_{m+4}\cdots a_{n+2}}\mathbb{X}_{A_{m+3}A_{m+4}\cdots A_{n+2}}{}^{b_{m+4}\cdots b_{n+2}}\\
&= (-1)^{m+1}\cdot d\cdot \frac{m+2}{(n+2)(n+1)}\mathrm{vol}_M^{a_2\cdots a_{m+1}a_{m+2}a_{m+3}\cdots a_{n+1}}N_{cb_{m+3}\cdots b_{n+1}}H^c\\
&\qquad \mathbb{X}^{A_1A_2A_3\cdots A_{m+2}}{}_{[a_2a_3\cdots a_{m+2}}\cdot\delta^{b_{m+3}}_{a_{m+3}}\cdots\delta^{b_{n+1}}_{a_{n+1}]}\\
&= (-1)^{m+1}\cdot d\cdot \frac{m+2}{(n+2)(n+1)}\mathrm{vol}_M^{a_2\cdots a_{m+3}\cdots a_{n+1}}N_{ca_{m+3}\cdots a_{n+1}}H^c\cdot\mathbb{X}^{A_1A_2A_3\cdots A_{m+2}}{}_{a_2a_3\cdots a_{m+2}}.
\end{aligned}
$$

Note that in the second line we have relabelled the tensor indices.

To complete the proof, we simplify the term involving the volume form, normal form and mean curvature. Again for simplicity, we relabel the indices. We pick local orthonormal frames for the intrinsic tangent bundle $T\Gamma$ and the normal bundle $N\Gamma$, $\{u_1,\dots u_m\}$ and $\{n_1,\dots,n_d\}$ respectively.

$$
\begin{aligned}
&\mathrm{vol}_M^{a_1\cdots a_m a_{m+1}a_{m+2}\cdots a_n}N_{ca_{m+2}\cdots a_n}H^c\\
&= n!\cdot u_1^{[a_1}\cdots u_m^{a_m}n_1^{a_{m+1}}n_2^{a_{m+2}}\cdots n_d^{a_n]}\cdot d!\cdot n^1_{[c}n^2_{a_{m+2}}\cdots n^d_{a_n]}H^c\\
&= n!\cdot\frac{(m+1)!\cdot(d-1)!}{n!}\Big(u_1^{[a_1}\cdots u_m^{a_m}n_1^{a_{m+1}]}n_2^{[a_{m+2}}\cdots n_d^{a_n]}\\
&\qquad -u_1^{[a_1}\cdots u_m^{a_m}n_2^{a_{m+1}]}n_1^{[a_{m+2}}\cdots n_d^{a_n]}-\cdots\Big)\\
&\qquad\quad \cdot d!\cdot\frac{(d-1)!\cdot 1!}{d!}\Big(n^1_c n^2_{[a_{m+2}}\cdots n^d_{a_n]}-n^2_c n^1_{[a_{m+2}}\cdots n^d_{a_n]}-\cdots\Big)H^c\\
&= (m+1)!\cdot((d-1)!)^2\Big(u_1^{[a_1}\cdots u_m^{a_m}n_1^{a_{m+1}]}(n^1_cH^c)n_2^{[a_{m+2}}\cdots n_d^{a_n]}n^2_{[a_{m+2}}\cdots n^d_{a_n]}\\
&+u_1^{[a_1}\cdots u_m^{a_m}n_2^{a_{m+1}]}(n^2_cH^c)n_1^{[a_{m+2}}\cdots n_d^{a_n]}n^1_{[a_{m+2}}\cdots n^d_{a_n]}+\cdots\\
&+u_1^{[a_1}\cdots u_m^{a_m}n_d^{a_{m+1}]}(n^d_cH^d)n_1^{[a_{m+2}}\cdots n_{d-1}^{a_n]}n^1_{[a_{m+2}}\cdots n^{d-1}_{a_n]}\Big)\\
&= (m+1)!\cdot((d-1)!)^2\Bigg(\frac{1}{(d-1)!}\cdot u_1^{[a_1}\cdots u_m^{a_m}n_1^{a_{m+1}]}(n^1_cH^c)+\\
&+\frac{1}{(d-1)!}\cdot u_1^{[a_1}\cdots u_m^{a_m}n_2^{a_{m+1}]}(n^2_cH^c)+\cdots\\
&+\frac{1}{(d-1)!}\cdot u_1^{[a_1}\cdots u_m^{a_m}n_d^{a_{m+1}]}(n^d_cH^d)\Bigg)\\
&= (m+1)!\cdot(d-1)!\cdot u_1^{[a_1}\cdots u_m^{a_m}H^{a_{m+1}]}\\
&= (m+1)\cdot(d-1)!\cdot m!\cdot u_1^{[[a_1}\cdots u_m^{a_m]}H^{a_{m+1}]}\\
&= (m+1)\cdot(d-1)!\cdot\mathrm{vol}_\Gamma^{[a_1\cdots a_m}H^{a_{m+1}]}.
\end{aligned}
$$

Note that the contractions of the form $n_2^{[a_{m+2}}\cdots n_d^{a_n]}n^2_{[a_{m+2}}\cdots n^d_{a_n]}$ give the factors of $\frac{1}{(d-1)!}$.

Substituting this back into our earlier equation and relabelling the indices appropriately gives

$$\begin{aligned}
&d \cdot \mathrm{vol}_M^{a_3\cdots a_{m+2}a_{m+3}a_{m+4}\cdots a_{n+2}} N_{cb_{m+4}\cdots b_{n+2}} H^c \cdot \\
&\qquad \mathbb{W}^{A_1A_2A_3\cdots A_{m+2}A_{m+3}A_{m+4}\cdots A_{n+2}}{}_{a_3\cdots a_{m+2}a_{m+3}a_{m+4}\cdots a_{n+2}} \mathbb{X}_{A_{m+3}A_{m+4}\cdots A_{n+2}}{}^{b_{m+4}\cdots b_{n+2}} \\
&= (-1)^{m+1} \cdot \frac{m+2}{(n+2)(n+1)} \mathrm{vol}_M^{a_2\cdots a_{m+3}\cdots a_{n+1}} N_{ca_{m+3}\cdots a_{n+1}} H^c \cdot \mathbb{X}^{A_1A_2A_3\cdots A_{m+2}}{}_{a_2a_3\cdots a_{m+2}} \\
&= (-1)^{m+1} \cdot d! \cdot \frac{(m+2)(m+1)}{(n+2)(n+1)} \cdot \mathrm{vol}_\Gamma^{a_2\cdots a_{m+1}} H^{a_{m+2}} \cdot \mathbb{X}^{A_1A_2A_3\cdots A_{m+2}}{}_{a_2a_3\cdots a_{m+2}}.
\end{aligned}$$

□

Combining Lemmata 4.5.2 and 4.5.3 gives an explicit formula for Σ.

Theorem 4.5.4. *Let $\Gamma \hookrightarrow$ be a m-dimensional submanifold of a conformal manifold $(M, \boldsymbol{c})$. Fix $d := \operatorname{codim}\Gamma$, and let $N_{A_1\cdots A_d}$ denote the tractor normal form of Γ. Define*

$$\Sigma^{A_1A_2A_3\cdots A_{m+2}} := \epsilon^{A_1A_2A_3\cdots A_{m+2}A_{m+3}\cdots A_{n+2}} N_{A_{m+3}\cdots A_{n+2}}.$$

Then

$$\begin{aligned}
\Sigma^{A_1A_2A_3\cdots A_{m+2}} = \frac{(m+2)(m+1)\cdot d!}{(n+2)(n+1)} \Big[&\mathrm{vol}_\Gamma^{a_3\cdots a_{m+2}} \mathbb{W}^{A_1A_2A_3\cdots A_{m+2}}{}_{a_3\cdots a_{m+3}} \\
&+ (-1)^{m+1} \cdot \mathrm{vol}_\Gamma^{a_2\cdots a_{m+1}} H^{a_{m+2}} \cdot \mathbb{X}^{A_1A_2A_3\cdots A_{m+2}}{}_{a_2a_3\cdots a_{m+2}} \Big].
\end{aligned} \tag{4.5.4}$$

Proof. From the formulae for the tractor volume form (4.4.5) and the normal form (4.4.6), it follows that Σ is simply the sum of equations (4.5.2) and (4.5.3). □

As the notation suggests, the Σ defined in (4.5.1) is a generalization to arbitrary codimension of the Σ used to characterize distinguished conformal curves from [39]. We conclude this chapter with some remarks on this subject.

With the conventions and definitions of Chapter 3, it was shown that

$$\Sigma^{ABC} = \pm 6\mathbf{u}^c X^{[A} Y^B Z_c^{C]} + 6\mathbf{u}^b \mathbf{a}^c X^{[A} Z_b^B Z_c^{C]}, \tag{4.5.5}$$

where $\mathbf{u}^b$ and $\mathbf{a}^c$ are the weighted velocity and acceleration vectors of the curve.

For a curve γ viewed as a 1-dimensional submanifold, one has $\mathrm{vol}_\gamma^{a_1} = \mathbf{u}^{a_1}$, and we will shortly see (equation (5.1.5)) that $H^{a_2} = \mathbf{a}^{a_2}$. In [39] and Chapter 3, we allowed both space- and time-like curves; this is the reason for the $\pm$ in (4.5.5). Since in this thesis we mainly assume Riemannian signature (to avoid the complications of null submanifolds), that sign will always be positive. (In fact, if we allow split signature but still assume that the submanifold Γ is non-null, the $\mathbb{W}$ slot of Σ will have a prefactor of $(-1)^q$, where (p, q) is the signature of the induced metric on Γ.) Thus in the case of a curve, the tractor form Σ of equation (4.5.4) becomes the Σ^{ABC} of equation (3.1.26), up to a constant factor. The exact value of this factor is not particularly important, and rescaling Σ by a constant factor does not change the statement of Theorem 4.5.1.

In $\Sigma^{A_1A_2\cdots A_{m+2}}\Sigma_{A_1A_2\cdots A_{m+2}}$, the only non-zero contraction of tractor projectors is that of the two $\mathbb{W}$ projectors. Therefore

$$\Sigma^{A_1A_2\cdots A_{m+2}}\Sigma_{A_1A_2\cdots A_{m+2}} = -\left(\frac{(m+2)(m+1)\cdot d!}{(n+2)(n+1)}\right)^2 \cdot \mathrm{vol}_\Gamma^{a_3\cdots a_{n+2}}\mathrm{vol}^\Gamma_{a_3\cdots a_{n+2}} \cdot \frac{1}{(m+2)!},$$

which is constant and depends only on the dimensions of M and Γ.

The particular factor here is a consequence of initially working with normal forms, and defining those in such a way that they are volume forms for the normal bundle. Equally, one may start with the right-hand side of (4.5.4) *without* any prefactors involving the dimensions of M or Γ, and then define $\Sigma^{A_1\cdots A_{m+2}}$ to be this expression scaled such that $\Sigma^{A_1\cdots A_{m+2}}\Sigma_{A_1\cdots A_{m+2}}$ has a prescribed value. This was done in [39], where we chose the factor such that $\Sigma^{A_1A_2A_3}\Sigma_{A_1A_2A_3} = \mp 1$, with the sign according to whether the submanifold (curve) was spacelike or timelike respectively. From that viewpoint, now it is $\epsilon_{A_1A_2\cdots A_{m+2}A_{m+3}\cdots A_{n+2}}\Sigma^{A_1A_2\cdots A_{m+2}}$ which then recovers the tractor normal form of the submanifold *up to some factor* that will again depend only on $\dim M$ and $\dim \Gamma$.

Chapter 5

Distinguished conformal submanifolds

In this chapter, we define a notion of a distinguished submanifold of a conformal manifold. Our proposed definition is motivated by results from Chapter 4, and it generalizes some existing special classes of conformal submanifolds.

Recall that, for 1-manifolds, there is already a notion of distinguished conformal submanifold: that of a conformal circle, which we have already discussed in Chapter 3. We begin by recasting a characterization of conformal circles from Chapter 3 in terms of the submanifold tractor theory developed in Chapter 4. Specifically, we show that a curve (viewed as a 1-dimensional submanifold) satisfies the conformal circle equation if, and only if, its tractor second fundamental form vanishes.

This result is even more significant when taken together with a similar characterization of totally umbilic hypersurfaces in conformal manifolds. Total umbilicity is another conformally invariant property, and Corollary 4.2.12 shows that, for hypersurfaces, this is also equivalent to the tractor second fundamental form being zero. These observations naturally motivate a notion of distinguished conformal submanifold which interpolates between these extremal cases, giving a notion that is valid in all codimensions.

We should also consider alternative ways that such a notion could be defined. Particularly influential to the content of this chapter was the work of Belgun [6]. Specifically, the notions of weak and total conformal circularity (Definitions 5.3.2 and 5.3.3 respectively), as well as the characterization of these conditions in terms of conformally invariant tensors (essentially Theorems 5.3.8 and 5.3.9 if one considers the individual slots of the tractor fields concerned, instead of viewing the tractors as atomic objects, as we do) all appear in [6]. Belgun also observed that the notions of total conformal circularity, weak conformal circularity, and umbilicity form a nested chain, where any of these conditions holding implies that the subsequent ones do also. This is Theorem 5.3.4, but it is clear from the characterizations of these notions in terms of conformal tractors. Belgun also observed that all of these notions coincide for conformally flat manifolds (Theorem 5.3.11). The machinery developed in Chapter 4 allows us to recover and restate these results using tractor calculus. In this way, the statements are manifestly invariant and are more natural in a native conformal geometry setting.

We also summarize which of the results presented in this chapter are new. Theorem 5.2.1 may be regarded as the main result of this thesis. It motivates the notion of *distinguished* for submanifolds which we propose in Definition 5.2.2. Section 5.4 extends many of the ideas of [38] from curves to submanifolds of any codimension.

5.1 Conformal circles as distinguished 1-submanifolds

We will maintain our convention of using γ to denote a curve, even when we are viewing it as a 1-dimensional submanifold. In this case, the projector $\Pi_i^a = u^a u_i$, where u^a is the unit velocity of the curve viewed as a section of $TM|_\gamma$, and u_i is the unit (co)velocity viewed as a section of $T^*\gamma$. Therefore also

$$\mathrm{N}_b^a = \delta_b^a - u^a u_b. \tag{5.1.1}$$

So far, the only conformally invariant condition on submanifolds we have seen is that of total umbilicity. Any general notion of a "distinguished conformal submanifold" should recover the conformal circle condition when the submanifold is 1-dimensional. The following proposition shows that total umbilicity alone does not give this.

Proposition 5.1.1. *Let $\gamma \hookrightarrow M$ be* any *smooth curve in M, viewed as a 1-dimensional submanifold. Then γ is totally umbilic.*

Proof. Since γ is 1-dimensional, the second fundamental form of γ must have the form

$$I\!I_{ij}{}^c = u_i u_j H^c, \tag{$*$}$$

where $u_i \in \mathcal{E}_i[1]$ is a unit length tangent covector and $H^c \in \mathcal{E}^c[-2]$ is the mean curvature vector of γ. (Equally, one may also take unweighted $u_i \in \mathcal{E}_i$ and $H^c \in \mathcal{E}^c$; either convention gives a second fundamental form of conformal weight zero.) Our convention is to use weighted velocity for two reasons: firstly, it was shown in Chapter 3 a weighted conformal circle equation is naturally parametrization-independent so a weighted velocity is more natural when giving the equations for conformal circles, and secondly, with u_i weighted, we have $\boldsymbol{g}_{ij} = u_i u_j$. Hence equation ($*$) is really

$$I\!I_{ij}{}^c = \boldsymbol{g}_{ij} H^c,$$

i.e. the second fundamental form is pure trace. Thus $\mathring{I\!I}_{(ij)_0}{}^c = 0$ and γ is totally umbilic. □

Following Remark 4.2.5, we have that

$$II_{ij}{}^{c} = \Pi^a_j \mathrm{N}^c_d \nabla_i \Pi^d_a. \tag{5.1.2}$$

Specializing to the case of a curve,

$$\begin{aligned} II_{ij}{}^{c} &= \Pi^a_j \mathrm{N}^c_d \nabla_i \Pi^d_a \\ &= u^a u_j \left(\delta^c_d - u^c u_d\right) \nabla_i \left(u_a u^d\right) \\ &= u^a u_j \left(\delta^c_d - u^c u_d\right) \left(u_a \nabla_i u^d + u^d \nabla_i u_a\right) \\ &= u^a u_j \left(u_a \nabla_i u^c + u^c \nabla_i u^a - u^c u_a u_d \nabla_i u^d - u^c \nabla_i u_a\right) \\ &= u_j \nabla_i u^c, \end{aligned}$$

where we have used that u^d has unit length, and this therefore also implies the vanishing of $u_d \nabla_i u^d$.

Finally, writing $\widetilde{u}^c$ for an extension of u^c to the whole of M and using the definition of the pullback connection given in equation (4.1.11),

$$\nabla_i u^c = \Pi^a_i \nabla_a \widetilde{u}^c = u_i u^a \nabla_a \widetilde{u}^c \overset{\text{along } \gamma}{=} u_i a^c, \tag{5.1.3}$$

where $a^c := u^a \nabla_a u^c$ is the (weighted) acceleration of the curve.

Thus

$$II_{ij}{}^{c} = u_i u_j a^c. \tag{5.1.4}$$

Comparing with $(*)$ above,

$$H^c = \boldsymbol{g}^{ij} II_{ij}{}^{c} = u^i u^j u_i u_j a^c = a^c, \tag{5.1.5}$$

i.e. the weighted mean curvature of a curve is exactly the weighted acceleration.

While writing [38], Mike Eastwood sent us an extract from his then-upcoming article with Lenka Zalabová [30] in which the following result was proved. Our result that the mean curvature of a curve is equal to its acceleration, combined with Theorem 4.1.11 on the existence of minimal scales now gives an alternate proof.

Proposition 5.1.2. *Let γ be a smooth curve in a conformal manifold. Then there exists a scale in the conformal class for which γ is an affine geodesic.*

Proposition 5.1.1 showed that total umbilicity does not give a notion of distinguished submanifolds which in the 1-dimensional case exactly recovers the curve being a conformal circle. We now prove a theorem which provides the link between the theory of conformal circles and our work on conformal submanifolds from Chapter 4.

Theorem 5.1.3. *A curve γ is a conformal circle if, and only if, its tractor second fundamental form vanishes.*

Proof. The characterization of conformal circles we will use is that of Proposition 3.1.3, namely that a curve is an unparametrized conformal circle if, and only if, there is a scale in the conformal class for which that curve is an affine geodesic, and $P_a{}^b u^a \propto u^b$, where u^a is the velocity of the curve, and P_{ab} is the Schouten tensor in the geodesic scale. Since we have seen that a minimal scale is exactly a scale in which γ is an affine geodesic, and moreover that minimal scales for submanifolds always exist, we may work in a minimal scale without loss of generality. By equation (4.2.26) and Proposition 5.1.1, in such a scale, one has

$$\mathbb{L}_{iK}{}^B = P_i{}^b \mathrm{N}_b^c X_K Z_c^B,$$

and so $\mathbb{L}$ will vanish if, and only if, $P_i{}^b \mathrm{N}_b^c = 0$.

In the case of a curve, one has $\Pi_i^a P_a{}^b = u_i u^a P_a{}^b$, and so $\Pi_i^a P_a{}^b \mathrm{N}_b^c = 0$ if, and only if, $u^a P_a{}^b$ is in the kernel of N_b^c, i.e. $u^a P_a{}^b \propto u^b$.

Thus we have that the tractor second fundamental form of a curve is zero if, and only if, there exists a minimal scale for γ such that $u^a P_a{}^b \propto u^b$ where P is the Schouten tensor of that minimal scale. But by Proposition 3.1.3, this is exactly equivalent to γ being a conformal circle, since the mean curvature of a curve is its acceleration. □

The following theorem gives a link between umbilic hypersurfaces and conformal circles.

Theorem 5.1.4. *Suppose that $\Gamma_1, \ldots, \Gamma_{n-1}$ are a collection of $(n-1)$ totally umbilic hypersurfaces in M, and let $\gamma := \bigcap \Gamma_i$. Suppose moreover that the hypersurfaces Γ_i are mutually transverse, i.e., $N\gamma = N\Gamma_1 \oplus \cdots \oplus N\Gamma_{n-1}$, where $N\gamma$ is the usual normal bundle of the submanifold γ. (This transversality condition is a generic one.) Then γ is a conformal circle.*

Proof. The transversality condition ensures that $\dim \bigcap \Gamma_i = 1$, i.e. γ is a curve. By the characterization of conformal circles given in Theorem 3.1.8, it suffices to construct a totally-skew 3-tractor which is parallel along γ and satisfies the incidence relation involving the canonical tractor.

Let N_A^i denote the (unit) conormal tractor of the hypersurface Γ_i. Then

$$N_{A_1 A_2 \cdots A_{n-1}} := d! \cdot N^1_{[A_1} N^2_{A_2} \cdots N^{n-1}_{A_{n-1}]}$$

is the tractor normal form of the submanifold γ: the $N^i_{A_i}$ form an orthonormal basis for the tractor normal bundle $\mathcal{N}^*\gamma$ at every point and equation (4.4.7) gives an expression for the tractor normal form in such a case. Since each N_A^i is parallel along Γ_i by total umbilicity, $N_{A_1 A_2 \cdots A_{n-1}}$ is parallel along the intersection γ.

Thus the 3-tractor Σ defined by

$$\Sigma^{A_1A_2A_3} := \epsilon^{A_1A_2A_3A_4\cdots A_{n+2}} N_{A_4\cdots A_{n+2}} \tag{5.1.6}$$

following equation (4.5.1) is also parallel via Theorem 4.5.1. It remains to show that the incidence relation is satisfied, i.e. $X^{[A_0}\Sigma^{A_1A_2A_3]}$ is zero. Theorem 4.5.4 gives the explicit formula for a Σ defined as the tractor volume dual of a normal form. In particular, such a Σ only involves the $\mathbb{W}$ and $\mathbb{X}$ tractor form projectors, both of which already contain the canonical tractor X. Hence it follows that the $\Sigma^{A_1A_2A_3}$ defined here for the curve γ will satisfy $X^{[A_0}\Sigma^{A_1A_2A_3]} = 0$. Theorem 3.1.8 now gives that γ is a conformal circle. □

In fact, one easily see that the above Theorem is not limited to curves; the intersection of a collection of d mutually transverse totally umbilic hypersurfaces will give a codimension d submanifold whose normal form is parallel and satisfies the incidence relation with the canonical tractor. Note that when phrased in terms of the normal form, the "incidence relation" $X \wedge \Sigma = 0$ becomes $X^{A_1}N_{A_1A_2\cdots A_d} = 0$, which a tractor normal form is readily seen to satisfy (c.f. equation (4.4.6)).

5.2 A definition

We have now seen that in both extremal cases of a submanifold (namely curves and hypersurfaces), an existing notion of distinguished submanifold is characterized by the vanishing of the tractor second fundamental form. Before formally stating our proposed definition for distinguished submanifold in conformal geometry, we begin with a theorem which summarizes many of the main results from the previous chapter.

Theorem 5.2.1. *Let $(M, \boldsymbol{c})$ be a conformal manifold and $\Gamma \hookrightarrow M$ a conformal submanifold of codimension d. Then the following are equivalent:*

1. $\mathbb{L}_{iA_d}{}^{A_{d+1}} = 0$;
2. $\nabla_i \mathrm{N}^{A_1}_{A_2} = 0$;
3. $\nabla_i N_{A_1A_2\cdots A_{d-1}A_d} = 0$.

Proof. Theorem 4.2.7 shows that (1) and (2) are equivalent, and the equivalence of (1) and (3) is Theorem 4.4.7. □

In light of these equivalences, we make the following definition.

Definition 5.2.2 (Distinguished submanifold)**.** Let $\Gamma \hookrightarrow M$ be a submanifold in a conformal manifold $(M, \boldsymbol{c})$. We will say that Γ is a *distinguished submanifold* if Γ satisfies one (equivalently any) of the conditions of Theorem 5.2.1.

Recall that Corollary 4.2.12 shows that a hypersurface is distinguished exactly when it is totally umbilic, and we have just seen in Theorem 5.1.3 that a curve is distinguished in the sense of this definition if, and only if, it is a conformal circle. So this general notion of distinguished encompasses two existing classes of distinguished conformal submanifolds and extends the notion to arbitrary codimension.

Additionally, a notable feature of the theory of distinguished curves in Chapter 3 is a concrete way to produce conserved quantities along these curves. The proliferation of such quantities is made possible by the characterization of the curve via a parallel (along the curve) tractor field. Theorem 5.2.1 characterizes submanifolds of arbitrary codimension in terms of a parallel tractor field, namely the tractor normal form. Section 4.5 describes the relationship between the normal form and its dual via the volume form, the latter object being the direct generalization of the 3-tractor Σ from Chapter 3 which characterizes distinguished curves. We view both of these points as strong justification for the notion of distinguished that we propose.

5.3 Alternative notions of distinguished conformal submanifolds

Defining a notion of distinguished object in a class is a significant matter. In this section, we present several different ways that one could define distinguished for conformal submanifolds. These have their origins as conformal generalizations of the various equivalent conditions to total geodesicity for Riemannian submanifolds. We first recall the theorem from Riemannian submanifold geometry where these originate. A detailed proof may be found in many places, e.g. [56], but it is mostly just an exercise in using the Gauß formula.

Theorem 5.3.1. *Let (M, g) be a* Riemannian *manifold, $\Gamma \hookrightarrow M$ a submanifold. The following are equivalent:*

1. *Γ is totally geodesic in M, i.e. $II_{ij}{}^{c} = 0$;*
2. *If $v \in T_pM$ is tangent to Γ, then the M-geodesic with initial velocity v lies in Γ;*
3. *Every geodesic of Γ is also a geodesic of M.*

While these properties are not conformally invariant, they admit slight modifications which have the desired invariance property. However, in the conformal context, these generalizations are no longer equivalent. The conformal analog of Property (1) is that of totally umbilicity, see Definition 4.1.9. We introduce new definitions for the conformal analogs of Properties (2) and (3). Belgun [6] calls these *weakly geodesic* and *strongly geodesic* respectively; we use terminology that we feel makes clearer their role as conformal analogs of the notions in Theorem 5.3.1. The notion of weakly conformally circular was also studied in [4, Proposition 2.13] for hypersurfaces, although it was not given an explicit name.

Definition 5.3.2 (Weakly conformally circular)**.** A submanifold Γ is *weakly conformally circular* if any M-conformal circle whose 2-jet at a point lies in Γ remains in Γ. That is, for any M-conformal circle γ whose 2-jet at some point p lies in Γ, with $\gamma(0) = p$, $\gamma(t) \in \Gamma$ for all t.

Definition 5.3.3 (Totally conformally circular)**.** Let Γ be a submanifold in a conformal manifold M. Then Γ is *totally conformally circular* if any Γ-conformal circle is also an M-conformal circle.

We will see shortly that there is a minor subtlety around whether we consider parametrized or unparametrized curves in the notion of total conformal circularity. We will say more on this when when we come to characterize this property in terms of tractors.

As the choice of terminology suggests, total conformal circularity implies weak conformal circularity. In fact, more can be said.

Theorem 5.3.4. *The classes of umbilic, weakly and totally conformally circular submanifolds form a nested chain:*

$$\text{totally conformally circular} \;\Rightarrow\; \text{weakly conformally circular} \;\Rightarrow\; \text{umbilic.}$$

We defer the proof of this until after we have characterized weakly and totally conformally circular submanifolds in terms of tractors and/or invariant tensors, since when cast in this language the result is clear. We comment on the extremal cases. For curves, the first two notions are equivalent, while the third is always satisfied. For hypersurfaces, we have already seen that umbilicity is equivalent to the vanishing of the tractor second fundamental form, and we will shortly see that this in turn is equivalent to weak conformal circularity. Thus in the hypersurface case, the second and third notions in the above display are equivalent.

To facilitate these characterizations, we introduce a new conformally invariant tensor field due to Belgun. Define $\mu \in \Gamma(T^*\Gamma \otimes N\Gamma)$ by

$$\mu_i{}^c := \mathrm{N}^c_b \left(P_i^b - \nabla_i H^b - \frac{1}{m-1} D^j \mathring{I\!I}_{ij}{}^b \right), \tag{5.3.1}$$

where the intrinsic Levi-Civita connection D is coupled to the normal connection $\nabla^\perp$. In the $m = 1$ case (i.e. Γ is a curve) we adopt the convention of omitting the divergence of the trace-free second fundamental form term. Recall that for a curve, one has $\mathring{I\!I}_{ij}{}^c = 0$ trivially, so this is reasonable.

In our derivation of the alternate formula (4.2.30) for the tractor second fundamental form, the intermediate calculation (4.2.31) shows that, for $m \neq 1$,

$$\mu_i{}^c = \frac{1}{m-1} W_{ij}{}^{di} \mathrm{N}^c_d. \tag{5.3.2}$$

The conformal invariance of the Weyl tensor and the normal projector then immediately implies that the same property is true of $\mu_i{}^c$.

The μ tensor is closely related to the tractor second fundamental form. Without the divergence of the trace-free second fundamental form, it is the coefficient of $X_J Z^C_c$ in (4.2.26). Thus one sees that if $\mathbb{L}_{iJ}{}^C = 0$, then it must be that $\mathring{I\!I}_{ij}{}^c = 0$ and then also $\mu_i{}^c = 0$ by this observation. This essentially shows that $\mathring{I\!I}_{ij}{}^c$ is the obstruction to $\mathrm{N}^c_b \left(P_i{}^b - \nabla_i H^b\right)$ being conformally invariant. The converse also holds, as the following lemma shows.

Lemma 5.3.5. *Let $\Gamma \hookrightarrow M$ be a submanifold in a conformal manifold with $\mathbb{L}_{iJ}{}^C$ its tractor second fundamental. Then $\mathbb{L}_{iJ}{}^C = 0$ if, and only if, $\mathring{I\!I}_{ij}{}^c = 0$ and $\mu_i{}^c = 0$.*

Proof. First, suppose that $\mathbb{L}_{iJ}{}^C = 0$. Then from (4.2.26) we immediately have $\mathring{I\!I}_{ij}{}^c = 0$ and hence (5.3.1) now reads

$$\mu_i{}^c = \mathrm{N}^c_b \left(P_i{}^b - \nabla_i H^b\right).$$

Equation (4.2.26) then becomes

$$\mathbb{L}_{iJ}{}^C = \mu_i{}^c Z^C_c X_J + H_c \mu_i{}^c X^C X_J, \tag{5.3.3}$$

and thus $\mathbb{L}_{iJ}{}^C = 0$ implies that $\mu_i{}^c = 0$.

Conversely, suppose $\mathring{I\!I}_{ij}{}^c = 0$ and $\mu_i{}^c = 0$. Then (5.3.3) holds once more, and thus the vanishing of $\mu_i{}^c$ gives that $\mathbb{L}_{iJ}{}^C = 0$. □

In fact, the data $(\mathring{I\!I}_{ij}{}^c, \mu_i{}^c)$ is equivalent to the tractor second fundamental form. To see this, for one direction, note that $\mathbb{L}_{iJ}{}^C$ may be constructed from $(\mathring{I\!I}_{ij}{}^c, \mu_i{}^c)$ according to

$$\begin{aligned}(\mathring{I\!I}_{ij}{}^c, \mu_i{}^c) \mapsto \mathring{I\!I}_{ij}{}^c Z^j_J Z^C_c &+ \left(\mu_i{}^c + \frac{1}{m-1} D^j \mathring{I\!I}_{ij}{}^c\right) X_J Z^C_c \\ &+ H_c \mathring{I\!I}_{ij}{}^c Z^j_J X^C + H_c \left(\mu_i{}^c + \frac{1}{m-1} D^j \mathring{I\!I}_{ij}{}^c\right) X_J X^C.\end{aligned} \tag{5.3.4}$$

The converse requires some more work. It is clear that there is a map $\mathbb{L}_{iJ}{}^C \mapsto \mathring{I\!I}_{ij}{}^c$, since the trace-free second fundamental form is in the projecting part (i.e. the top slot) of the tractor second fundamental form, and thus one can simply extract this component using the tractor projectors. We show now that there is also an invariant operator which maps $\mathbb{L}_{iJ}{}^C \mapsto \mu_i{}^c$. To this end, we first prove a lemma.

Lemma 5.3.6. *There is an invariant map* $\mathbb{M} : S^2_0 T^*\Gamma \otimes N\Gamma \to T^*\Gamma \otimes \mathcal{T}^*\Gamma \otimes \mathcal{N}$. *Written in tractor projectors, this takes the form*

$$\begin{aligned}\omega_{ij}{}^c \mapsto \mathbb{M}_{cJ}{}^{jC}\omega_{ij}{}^c :=& \omega_{ij}{}^c Z^j_J Z^C_c - \frac{1}{m-1} D^j \omega_{ij}{}^c X_J Z^C_c \\ &+ H_c \omega_{ij}{}^c Z^j_J X^C - \frac{1}{m-1} H_c D^j \omega_{ij}{}^c X_J X^C,\end{aligned} \tag{5.3.5}$$

where again the intrinsic Levi-Civita connection D *is coupled to the normal Levi-Civita connection* $\nabla^\perp$ *when acting on* ω.

Proof. Using (2.2.3) and (4.1.14), one computes that

$$\widehat{D}^j \omega_{ij}{}^c = D^j \omega_{ij}{}^c + (m-1)\Upsilon^j \omega_{ij}{}^c - \Upsilon_i \omega_{kl}{}^c \boldsymbol{g}^{kl} = D^j \omega_{ij}{}^c + (m-1)\Upsilon^j \omega_{ij}{}^c \tag{5.3.6}$$

since $\omega_{ij}{}^c$ is trace-free over the pair of indices (i, j).

Therefore, using the above together with equations (4.1.22) and (2.3.11), and that the X tractor is conformally invariant,

$$\begin{aligned}&\omega_{ij}{}^c \widehat{Z}^j_J \widehat{Z}^C_c - \frac{1}{m-1}\widehat{D}^j \omega_{ij}{}^c \widehat{X}_J \widehat{Z}^C_c + \widehat{H}_c \omega_{ij}{}^c \widehat{Z}^j_J \widehat{X}^C - \frac{1}{m-1}\widehat{H}_c \widehat{D}^j \omega_{ij}{}^c \widehat{X}_J \widehat{X}^C \\ &= \omega_{ij}{}^c \left(Z^j_J + \Upsilon^j X_J\right)\left(Z^C_c + \Upsilon_c X^C\right) \\ &\quad - \frac{1}{m-1}\left(D^j \omega_{ij}{}^c + (m-1)\,\Upsilon^j \omega_{ij}{}^c\right) X_J \left(Z^C_c + \Upsilon_c X^C\right) \\ &\quad + \left(H_c - \mathrm{N}^d_c \Upsilon_d\right)\omega_{ij}{}^c \left(Z^j_J + \Upsilon^j X_J\right) X_C \\ &\quad - \frac{1}{m-1}\left(H_c - \mathrm{N}^d_c \Upsilon_d\right)\left(D^j \omega_{ij}{}^c + (m-1)\Upsilon^j \omega_{ij}{}^c\right) X_J X^C \\ &= \omega_{ij}{}^c Z^j_J Z^C_c - \frac{1}{m-1} D^j \omega_{ij}{}^c X_J Z^C_c + \left(H_c \omega_{ij}{}^c - \omega_{ij}{}^c \Upsilon_c + \omega_{ij}{}^c \Upsilon_c\right) Z^j_J X^C \\ &\quad + \Bigg(-\frac{1}{m-1} H_c D^j \omega_{ij}{}^c - H_c \omega_{ij}{}^c \Upsilon^j + \frac{1}{m-1}\Upsilon_c D^j \omega_{ij}{}^c + \omega_{ij}{}^c \Upsilon^j \Upsilon_c - \omega_{ij}{}^c \Upsilon^j \Upsilon_c \\ &\quad - \frac{1}{m-1}\Upsilon_c D^j \omega_{ij}{}^c + H_c \omega_{ij}{}^c \Upsilon^j - \omega_{ij}{}^c \Upsilon^j \Upsilon_c + \omega_{ij}{}^c \Upsilon^j \Upsilon_c\Bigg) X_J X^C \\ &= \omega_{ij}{}^c Z^j_J Z^C_c - \frac{1}{m-1} D^j \omega_{ij}{}^c X_J Z^C_c + H_c \omega_{ij}{}^c Z^j_J X^C - \frac{1}{m-1} H_c D^j \omega_{ij}{}^c X_J X^C,\end{aligned}$$

which verifies the claimed conformal invariance of the operator $\mathbb{M}_{cJ}{}^{jC}$. □

We will use this operator $\mathbb{M}$ to recover Belgun's invariant $\mu_i{}^c$ tensor from the tractor second fundamental form.

Theorem 5.3.7. *The tensor $\mu_i{}^c$ is equal to the projecting part of the tractor*

$$\left(\delta_J^K \delta_D^C - \mathbb{M}_{Jc}{}^{Cj} Z_j^K Z_D^c\right) \mathbb{L}_{iK}{}^D = \mathbb{L}_{iJ}{}^C - \mathbb{M}_{Jc}{}^{Cj} \mathring{I\!I}_{ij}{}^c. \tag{5.3.7}$$

Proof. By inspection, one sees that (5.3.7) has zero in the $Z_J^j Z_c^C$ slot (since $Z_j^K Z_D^c \mathbb{L}_{iK}{}^D = \mathring{I\!I}_{ij}{}^c$) and hence projecting out the $X_J Z_c^C$ slot must yield a conformally invariant object. Such projection is accomplished by contraction with $Y^J Z_C^c$, and from equations (4.2.26) and (5.3.5) one sees that this projection is equal to

$$Y^J Z_C^c \left(\mathbb{L}_{iJ}{}^C - \mathbb{M}_{cJ}{}^{jC} \mathring{I\!I}_{ij}{}^c\right) = \mathrm{N}_b^c \left(P_i^b - \nabla_i H^b\right) - \frac{1}{m-1} D^j \mathring{I\!I}_{ij}{}^c, \tag{5.3.8}$$

which is exactly $\mu_i{}^c$ as defined in (5.3.1) (since the c index of $D^j \mathring{I\!I}_{ij}{}^c$ is already normal). In particular, this is another way to establish the conformal invariance of $\mu_i{}^c$. □

Thus the previously-mentioned map $\mathbb{L}_{iJ}{}^C \mapsto \mu_i{}^c$ is realized by mapping

$$\mathbb{L}_{iJ}{}^C \mapsto Y^J Z_C^c \left(\mathbb{L}_{iJ}{}^C - \mathbb{M}_{Jc}{}^{Cj} Z_j^K Z_D^c \mathbb{L}_{iK}{}^D\right).$$

We have already noted (and used) that there is also a map $\mathbb{L}_{iJ}{}^C \mapsto \mathring{I\!I}_{ij}{}^c$, since this is in the projecting part of the tractor $\mathbb{L}_{iJ}{}^C$. Thus both $\mathring{I\!I}_{ij}{}^c$ and $\mu_i{}^c$ may be recovered from $\mathbb{L}_{iJ}{}^C$. Taken together with equation (5.3.4), this shows that the package $(\mathring{I\!I}_{ij}{}^c, \mu_i{}^c)$ is indeed equivalent to the tractor second fundamental form $\mathbb{L}_{iJ}{}^C$.

We now have the required objects and machinery to give tractor characterizations of Definitions 5.3.2 and 5.3.3.

Theorem 5.3.8. *The submanifold $\Gamma \hookrightarrow M$ is weakly conformally circular if, and only if, $\mathbb{L}_{iJ}{}^C = 0$. Thus weakly conformally circular and our notion of distinguished coincide.*

Proof. First, suppose that Γ is weakly conformally circular. Let γ be an M-conformal circle whose 2-jet at $p \in \Gamma$ lies in Γ. Then by assumption γ remains in Γ. We need to introduce some notation. Let

- $N^{\Gamma \hookrightarrow M}_{A_1 A_2 \cdots A_d}$ be the normal form of $\Gamma \hookrightarrow M$,
- $N^{\gamma \hookrightarrow M}_{A_1 A_2 \cdots A_{n-1}}$ be the normal form of $\gamma \hookrightarrow M$, and
- $N^{\gamma \hookrightarrow \Gamma}_{A_1 A_2 \cdots A_{m-1}}$ be the normal form of $\gamma \hookrightarrow N$.

We note a couple of important relations between these various normal forms. First, since the curve γ lies in the submanifold Γ, we have

$$N_{\gamma \hookrightarrow \Gamma}^{A_1 A_2 \cdots A_{m-1}} N^{\Gamma \hookrightarrow M}_{A_1 A_2 \cdots A_d} = 0. \tag{5.3.9}$$

Second, adapting the argument from the proof of Corollary 4.1.29, we see that

$$N^{\gamma\hookrightarrow\Gamma}_{A_1A_2\cdots A_{m-1}} \wedge N^{\Gamma\hookrightarrow M}_{A_m\cdots A_{n-1}} = N^{\gamma\hookrightarrow M}_{A_1A_2\cdots A_{n-1}}. \tag{5.3.10}$$

Finally, since γ is an M-conformal circle, it follows from Theorem 5.1.3 that

$$u^i\nabla_i N^{\gamma\hookrightarrow M}_{A_1A_2\cdots A_{n-1}} = 0.$$

Therefore, using the above,

$$\left(u^i\nabla_i N^{\gamma\hookrightarrow\Gamma}_{[A_1A_2\cdots A_{m-1}}\right) N^{\Gamma\hookrightarrow M}_{A_m\cdots A_{n-1}]} + N^{\gamma\hookrightarrow\Gamma}_{[A_1A_2\cdots A_{m-1}} \left(u^i\nabla_i N^{\Gamma\hookrightarrow M}_{A_m\cdots A_{n-1}]}\right) = 0,$$

and hence

$$\left(u^i\nabla_i N^{\gamma\hookrightarrow\Gamma}_{[A_1A_2\cdots A_{m-1}}\right) N^{\Gamma\hookrightarrow M}_{A_m\cdots A_{n-1}]} + N^{\gamma\hookrightarrow\Gamma}_{[A_1A_2\cdots A_{m-1}} \left(-d\cdot u^i\mathbb{L}_{iA_{n-1}}{}^{A_0} N^{\Gamma\hookrightarrow M}_{A_m\cdots A_{n-2}]A_0}\right) = 0$$

by Theorem 4.4.5. Now, we contract $N_{\gamma\hookrightarrow\Gamma}^{A_1A_2\cdots A_{m-1}}$ into both sides of the above display. Since $N^{\gamma\hookrightarrow\Gamma}_{A_1A_2\cdots A_{m-1}} N_{\gamma\hookrightarrow\Gamma}^{A_1A_2\cdots A_{m-1}}$ is constant, and $N^{\Gamma\hookrightarrow M}_{A_1A_2\cdots A_d} N_{\gamma\hookrightarrow\Gamma}^{A_1B_2\cdots B_{m-1}} = 0$, this completely annihilates the first term. Hence only the contraction with the second term remains. This yields two terms: one term where $N^{\gamma\hookrightarrow\Gamma}$ is completely contracted with itself, and another term where one index is contracted into the tractor second fundamental form, and the remaining indices are all contracted into $N^{\gamma\hookrightarrow\Gamma}_{A_1A_2\cdots A_{m-1}}$. If any indices of $N^{\gamma\hookrightarrow\Gamma}$ are contracted into $N^{\Gamma\hookrightarrow M}$ the result will be zero. After some calculation and simplifying, we find that

$$\begin{aligned} d\cdot\Big((m-1)!\cdot u^i\mathbb{L}_{i[A_m}{}^{A_0} N^{\Gamma\hookrightarrow M}_{A_{n-1}A_{m+1}\cdots A_{n-2}]A_0} \\ -(m-1)(m-2)!\cdot u^i\mathbb{L}_{i[A_1}{}^{A_0} N^{\Gamma\hookrightarrow M}_{A_{n-1}A_{m+1}\cdots A_{n-2}]A_0} \mathrm{N}^{\gamma\hookrightarrow\Gamma A_1}{}_{A_m}\Big) = 0, \end{aligned}$$

where $\mathrm{N}^{\gamma\hookrightarrow\Gamma A_1}{}_{A_m}$ is the normal projector of γ viewed as a submanifold of Γ. We may contract the normal form $N^{\Gamma\hookrightarrow M}$ to isolate the tractor second fundamental forms, c.f. Proposition 4.4.6. After resolving the A_m index of the first term into γ-tangential and γ-normal components and simplifying, we see that

$$u^i\mathbb{L}_{iA_m}\Pi^{\gamma\hookrightarrow\Gamma A_1}{}_{A_m} = 0,$$

Therefore contracting the velocity tractor of the curve into the above annihilates the second term, and we see that

$$u^i\mathbb{L}_{iA_{n-1}}{}^{A_0}U^{A_{n-1}} = 0,$$

which implies in particular that $\mathring{I\!I}_{ij}{}^c u^i u^j = 0$. But the above must hold for any M-conformal circle γ, and hence $\mathring{I\!I}_{ij}{}^c u^i u^j = 0$ for all $u^i \in \Gamma(\mathcal{E}^i)$, whence Γ is totally umbilic by polarization.

We must also establish that $\mu_i{}^c$ vanishes. Since we have already seen that $\mathring{I\!I}_{ij}{}^c = 0$, it suffices to show that $\mathrm{N}^c_b \left(P_i{}^b - \nabla_i H^b\right) = 0$. We may assume without loss of generality that γ is projectively parametrized, since such parametrizations are available for any curve. Then γ satisfies the (projectively parametrized) ambient conformal circle equation (3.1.6). Moreover, since $\mathring{I\!I}_{ij}{}^c = 0$, the Gauß formula gives

$$a^b = \frac{d^\nabla u^b}{dt} = \Pi^b_j u^i D_i u^j + u^i u_i H^b$$

and

$$\frac{d^\nabla a^b}{dt} = \Pi^b_j u^i D_i a^j + 3(a^k u_k) H^b + (u^k u_k) u^i \nabla_i H^b,$$

In particular, one sees that

$$\mathrm{N}^c_b a^b = u^k u_k H^b$$

and

$$\mathrm{N}^c_b \frac{d^\nabla a^b}{dt} = 3(a^k u_k) H^b + u^k u_k \mathrm{N}^c_b u^i \nabla_i H^b.$$

Applying the normal projector to both sides of (3.1.6) yields

$$3(u \cdot a) H^c + (u \cdot u) \mathrm{N}^c_b u^i \nabla_i H^b = (u \cdot u) \cdot u^d P_d{}^b \mathrm{N}^c_b + 3u^{-2} (u \cdot a)(u \cdot u) H^c, \qquad (5.3.11)$$

where we use that $a_c u^c = a_k u^k$ since u is tangential, and hence we may write simply $a \cdot u$ for either term. After simplifying,

$$\mathrm{N}^c_b \left(P_i{}^b - \nabla_i H^b\right) u^i = 0. \qquad (5.3.12)$$

Note that the scale in which the above holds does not depend on the particular curve γ, and so (5.3.12) must hold for all $u^i \in \mathcal{E}^i$, and it follows that $\mathrm{N}^c_b \left(P_i{}^b - \nabla_i H^b\right) = 0$. Thus we have shown that $\mathbb{L}_{iJ}{}^C = 0$.

Conversely, suppose that $\mathbb{L}_{iJ}{}^C = 0$. For convenience, we work in a minimal scale. Our approach is as follows: first, we define a family of curves which solve a third order ODE **on Γ** which is *similar* to the conformal circle equation but whose solution curves are not in general conformal circles of Γ. We then show that the vanishing of the second fundamental form is sufficient for such curves to be ambient conformal circles. By uniqueness of solution to an ODE with initial conditions (now the *ambient* conformal circle equation), one concludes that a conformal circle of the ambient manifold which satisfies the given (tangential) initial conditions will remain in the submanifold for some positive time.

Let I be an interval centered on 0 and suppose $\gamma : I \to \Gamma$ is a projectively parametrized smooth curve in Γ with velocity u and acceleration a. Suppose moreover that the 2-jet of γ at 0 is tangential to Γ. We make the following temporary definition, which we will not use outside of this proof. Define the *adapted conformal geodesic equation* to be the following ODE on Γ:

$$\frac{d^D a^j}{dt} = u^2 \cdot u^i P_i{}^j + 3u^{-2}\left(u_k a^k\right) a^j - \frac{3}{2}u^{-2}\left(a_k a^k\right) u^j - 2u^k u^l P_{kl} u^j, \tag{5.3.13}$$

where as usual $P_i{}^j$ and P_{kl} denote the restriction of the ambient Schouten tensor to the intrinsic tangent and cotangent bundles, and $\frac{d^D}{dt}$ denotes $u^i D_i$ where D is the intrinsic Levi-Civita connection for the pullback of the ambient scale. We say that γ is an *adapted conformal geodesic* if it satisfies this equation. Note that equation (5.3.13) is a third order ODE on Γ, and therefore the initial value problem with given initial data has a unique solution on Γ for some interval centered at 0. This solution may also be viewed as a curve in M, and one may ask whether it solves a related ODE there. Since the 2-jet of γ is initially tangential and $\mathbb{L}_{iJ}{}^C = 0$ implies in particular that $\mathring{I\!I}_{ij}{}^c = 0$, it follows that in our minimal scale $\Pi^b_j u^j = u^b$, $\Pi^b_j a^j = a^b$ and $\Pi^b_j \frac{d^D a^j}{dt} = \frac{d^\nabla a^b}{dt}$, and therefore $u_k a^k = u_c a^c$, $a_k a^k = a_c a^c$ and $u^k u^l P_{kl} = u^c u^d P_{cd}$. Thus applying Π^b_j to both sides of (5.3.13) shows that, viewed as a curve on M, γ satisfies

$$\frac{d^\nabla a^b}{dt} = u^2\left(\Pi^b_j u^i P_i{}^j\right) + 3u^{-2}\left(u_c a^c\right) a^b - \frac{3}{2}u^{-2}\left(a_c a^c\right) u^b - 2u^c u^d P_{cd} u^b.$$

But $\mathbb{L}_{iJ}{}^C = 0$ also implies (again for the minimal scale) that $\mathrm{N}^c_b P_i{}^b = 0$, i.e. $\Pi^c_b P_i{}^b = \delta^c_b \Pi_i{}^b = P_i{}^c$. Hence

$$\Pi^b_j P_i{}^j = \Pi^b_j \Pi^j_d P_i{}^d = \Pi^b_d P_i{}^d = P_i{}^b,$$

and so

$$\Pi^b_j u^i P_i{}^j = u^d \Pi^i_d P_i{}^b = u^d \Pi^i_d \Pi^d_i P_d{}^b = u^d \Pi^c_d P_c{}^b = u^c P_c{}^b.$$

Hence viewed as a curve in M, γ satisfies the ODE

$$\frac{d^\nabla a^b}{dt} = u^2 \cdot u^c P_c{}^b + 3u^{-2}\left(u_c a^c\right) a^b - \frac{3}{2}u^{-2}\left(a_c a^c\right) u^b - 2u^c u^d P_{cd} u^b,$$

which is exactly the (projectively parametrized) M-conformal circle equation. So if $\mathbb{L}_{iJ}{}^C = 0$, and γ is an adapted conformal circle, then it is an M-conformal circle, and by the uniqueness of solution to an initial value problem, the curve γ, which lies in Γ, *is* the unique M-conformal circle with the given initial conditions. Hence any M-conformal circle whose 2-jet at a point $p \in \Gamma$ is tangential will remain in Γ, i.e. Γ is weakly conformally circular.

□

We now turn to the characterization of total conformal circularity. Recall from equation (4.2.10) that the intrinsic tractor connection is not simply the projection of the ambient pullback connection. Therefore, a projectively parametrized curve γ in Γ satisfying

$d^\Gamma A^J/dt = 0$ (where we are calculating entirely with intrinsic objects) is not sufficient to conclude that the curve (viewed now as a curve in M) satisfies $dA^B/dt = 0$ (identifying A^J with an ambient tractor and calculating with the ambient tractor connection). Thus a Γ-conformal circle is not necessarily an M-conformal circle, even if $\mathbb{L}_{iJ}{}^C = 0$. The precise statement characterizing total conformal circularity depends on whether one considers parametrized or unparametrized conformal circles. We state both results separately.

Theorem 5.3.9. *Any projectively parametrized Γ-conformal circle is also an M-conformal circle if, and only if $\mathbb{L}_{iJ}{}^C = 0$ and $\mathsf{S}_{iJK} = 0$ (i.e. $\mathcal{F}_{ij} = 0$).*

Theorem 5.3.10. *Any unparametrized Γ-conformal circle is also an unparametrized M-conformal circle if, and only if $\mathbb{L}_{iJ}{}^C = 0$ and $\mathsf{S}_{iJK} \propto \boldsymbol{g}_{ij} Z^j_{[J} X_{K]}$ (i.e. $\mathcal{F}_{ij} \propto \boldsymbol{g}_{ij}$).*

Note that following our conventions for the Fialkow tensor in Section 4.3.1, these results still hold when Γ is a submanifold of dimension 2. The Γ-conformal circle equation is then the usual conformal circle equation (either the projectively parametrized equation (3.1.6) or the parametrization-independent weighted equation (3.1.23)) with Schouten tensor induced by the Fialkow tensor playing the role of the usual Schouten tensor. For a 1-dimensional submanifold, weakly conformally circular and totally conformally circular are equivalent, and thus the Fialkow tensor vanishing identically does not result in any loss of information.

Proof of Theorem 5.3.9. First suppose that Γ is totally conformally circular. Suppose that γ is a Γ-conformal circle satisfying some initial conditions at $p \in \Gamma$. Then by assumption γ is also an M-conformal circle with those same initial conditions, identified with sections of TM. By uniqueness of the solution to an initial value problem, one sees that a totally conformally circular submanifold is necessarily weakly conformally circular. In particular, by Theorem 5.3.8, Γ is totally umbilic, and so $\mathring{I\!I}_{ij}{}^c = 0$.

The curve γ must satisfy the intrinsic and ambient versions of the conformal circle equation, namely

$$\frac{d^D a^j}{dt} = u^2 \cdot u^i p_i{}^j + 3u^{-2}\left(u_k a^k\right) a^j - \frac{3}{2} u^{-2}\left(a_k a^k\right) u^j - 2u^k u^l p_{kl} u^j \tag{5.3.14}$$

and

$$\frac{d^\nabla a^b}{dt} = u^2 \cdot u^c P_c{}^b + 3u^{-2}\left(u_c a^c\right) a^b - \frac{3}{2} u^{-2}\left(a_c a^c\right) u^b - 2u^c u^d P_{cd} u^b \tag{5.3.15}$$

respectively.

From the Gauß formula (4.1.15), we have that

$$\Pi^b_j a^b = a^j$$

and

$$\Pi^j_b \frac{d^\nabla a^b}{dt} = \frac{d^D a^j}{dt} + u^2 (u^i \nabla_i H^b) \Pi^j_b,$$

where $a^j = \frac{d^D u^j}{dt}$, i.e. the acceleration of the curve calculated intrinsically. Therefore, applying Π^j_b to (5.3.15) and comparing the result to (5.3.14), we see that

$$u^2 \cdot u^k P_k{}^b \Pi^j_b - \frac{3}{2} u^2 H_b H^b u^j - 2u^k u^l P_{kl} u^j + u^2 u^j H_b H^b = u^2 \cdot u^k p_k{}^j - 2u^k u^l p_{kl} u^j,$$

after using that $\Pi^j_b u^b = u^j$ and $\Pi^j_b a^b = a^j$, $u_c a^c = u_k a^k$ since u is already tangent to the submanifold Γ, $a^c a_c = a^k a_k + (u^2)^2 H_b H^b$, and that $\Pi^j_b \nabla_i H^b = -\delta^j_i H_b H^b$. (Note that the last two relations here use that Γ is umbilic.)

Contracting the above display with u_j yields

$$u^2 \cdot u_j u^k P_k{}^j - \frac{1}{2} (u^2)^2 H_b H^b - 2u^k u^l P_{kl} u_j u^j = u^2 \cdot u_j u^k p_k{}^j - 2u^k u^l p_{kl} u_j u^j,$$

and hence

$$\left(P_{ij} - p_{ij} + \frac{1}{2} \boldsymbol{g}_{ij} H_b H^b \right) u^i u^j = 0, \tag{5.3.16}$$

and when $\mathring{I\!I}_{ij}{}^c = 0$, the term in parentheses is exactly the Fialkow tensor $\mathcal{F}_{ij}$ from (4.2.11). (Note that in fact one can contrive to have the full equation (4.2.11) for the Fialkow here by carrying $\mathring{I\!I}_{ij}{}^c$ from the earlier $\nabla_i H^b$ term through. Alternatively, since it is zero, one can even insert $H_b \mathring{I\!I}_{ij}{}^b$ into the above parentheses.)

Now, any $u^i \in \mathcal{E}^i$ can be the initial velocity for a conformal circle, and hence (5.3.16) must hold for all $u^i \in \mathcal{E}^i$. Hence $\mathcal{F}_{ij} = 0$. Together with our earlier observation that total conformal circularity implies weak conformal circularity, this establishes that $\mathbb{L}_{iJ}{}^C = 0$ and $\mathsf{S}_{iJK} = 0$.

Conversely, suppose that $\mathbb{L}_{iJ}{}^C = 0$ and $\mathsf{S}_{iJK} = 0$. Let γ be a projectively parametrized Γ-conformal circle. Then the intrinsic acceleration tractor of γ satisfies $\frac{d^D A^J}{dt} = 0$. We must also show that γ satisfies the ambient (projectively parametrized) conformal circle equation. Writing U^J for the intrinsic velocity tractor of γ, we see from equation (3.1.4) (the explicit form of the velocity tractor) and equation (4.2.5) (the formula for the isomorphism Π^B_J) that $U^B = \Pi^B_J U^J$. Then applying the tractor Gauß formula shows that

$$A^B = \frac{d^\nabla A^B}{dt} = \Pi^B_J (u^i D_i U^J + \mathsf{S}_i{}^J{}_K U^K) + u^i \mathbb{L}_{iJ}{}^B U^J = \Pi^B_J \frac{d^D U^J}{dt} = \Pi^B_J A^J$$

and

$$\frac{d^\nabla A^B}{dt} = \Pi^B_J \frac{d^D A^J}{dt}.$$

So if γ satisfies $d^D A^J/dt = 0$, then also $d^\nabla A^B/dt = 0$. Finally, recall that the isomorphism Π^B_J is also metric-preserving, and so

$$A^B A_B = \Pi^B_J A^J \Pi^K_B A^K = \delta^K_J A^J A_K = A^K A_K = 0.$$

Thus by Proposition 3.1.4 γ is an M-conformal circle, and therefore Γ is totally conformally circular. □

We next prove the parametrization-independent version of Theorem 5.3.9, namely Theorem 5.3.10. As this Theorem is a statement about unparametrized conformal circles, we use the 3-tractor Σ introduced in Chapter 3.

Proof of Theorem 5.3.10. First, suppose that Γ is totally conformally circular, i.e. every Γ-conformal circle is an M-conformal circle. Let γ be a Γ-conformal circle. In the previous proof, we observed that total conformal circularity implies weak conformal circularity. Parametrization was not used at all in this part of the proof and hence we may employ the same argument here. Thus $\mathbb{L}_{iJ}{}^C = 0$ by Theorem 5.3.8.

Per Chapter 3, γ determines an intrinsic 3-tractor $\Sigma^{IJK} \in \mathcal{E}^{[IJK]}$ which satisfies $X^{[I}\Sigma^{JKL]} = 0$ and $u^i D_i \Sigma^{IJK} = 0$, where X^I is the intrinsic canonical tractor and D_i is the intrinsic tractor connection.

Explicitly,

$$\Sigma^{IJK} = 6\sigma^{-1} X^{[I} U^J A^{K]},$$

with U^J and A^K defined as in equations (3.1.2) and (3.1.3) respectively, using the intrinsic position tractor and tractor connection.

On the other hand, viewing γ as an ambient curve also defines a 3-tractor; denote this by Ξ^{ABC}. It is defined by

$$\Xi^{ABC} = 6\sigma^{-1} X^{[A} V^B B^{C]}, \tag{5.3.17}$$

where X^A is the ambient position tractor and V^B and B^C are the velocity and acceleration tractors of the curve $\iota(\gamma)$, defined using the *ambient* tractor connection. Since Γ is totally conformally circular, $\iota(\gamma)$ is an M-conformal circle and hence Ξ^{ABC} must satisfy the ambient incidence relation $X^{[A}\Xi^{BCD]} = 0$ and be parallel along the curve, i.e. $u^a \nabla_a \Xi^{ABC} = 0$.

Using the calculation from the proof of Proposition 3.1.4, the derivatives of the intrinsic and ambient 3-tractors are

$$\mathbf{u}^i D_i \Sigma^{IJK} = 6\left(\mathbf{u}^i D_i \mathbf{a}^k \mp \mathbf{u}^l p_l{}^k\right) \mathbf{u}^j X^{[I} Z_j^J Z_k^{K]},$$

and

$$\mathbf{u}^i \nabla_i \Xi^{ABC} = 6\left(\mathbf{u}^i \nabla_i \mathbf{a}^c \mp \mathbf{u}^d P_d{}^c\right) \mathbf{u}^b X^{[A} Z_b^B Z_c^{C]},$$

respectively. We may freely replace the ambient tractor connection from the proof of Proposition (3.1.4) with the pullback connection since these agree along Γ.

Both of the above displays are zero since γ is a Γ- and M-conformal circle. Hence

$$\begin{aligned}
0 &= \Pi^I_A \Pi^J_B \Pi^K_C \left(\mathbf{u}^i \nabla_i \Xi^{ABC}\right) - \mathbf{u}^i D_i \Sigma^{IJK} \\
&= 6\left[\left(\mathbf{u}^i D_i \mathbf{a}^k \mp \mathbf{u}^l P_l{}^k\right) - \left(\mathbf{u}^i D_i \mathbf{a}^k \mp \mathbf{u}^l p_l{}^k\right)\right] \mathbf{u}^j X^{[I} Z^J_j Z^{K]}_k \\
&= \mp 6 \mathbf{u}^l \left(P_l{}^k - p_l{}^k\right) \mathbf{u}^j X^{[I} Z^J_j Z^{K]}_k,
\end{aligned}$$

so it must be that

$$\mathbf{u}^l \left(P_l{}^k - p_l{}^k\right) \propto \mathbf{u}^k$$

and in fact this means that

$$\mathbf{u}^l \mathcal{F}_l{}^k = \mathbf{u}^l \left(P_l{}^k - p_l{}^k + H_c \mathring{I\!I}_l{}^{kc} + \frac{1}{2} H_c H^c \delta^k_l\right) \propto \mathbf{u}^k,$$

since we have already seen that $\mathring{I\!I}_{ij}{}^c = 0$, and moreover adding any amount of the metric (here appearing as the Kronecker delta since we have raised an index) will not affect this proportionality. Thus it must be that $\mathbf{u}^l \mathcal{F}_l{}^k \propto \mathbf{u}^k$, and so $\mathcal{F}_{ij} \propto \boldsymbol{g}_{ij}$, since the (weighted) velocity can be any element of $T\Gamma[-1]$ and the result follows.

Conversely, suppose that $\mathbb{L}_{iJ}{}^C = 0$ and $\mathcal{F}_{ij} \propto \boldsymbol{g}_{ij}$, and let γ be a Γ-conformal circle. The Γ-conformal circle γ determines an intrinsic 3-tractor $\Sigma^{IJK} \in \mathcal{E}^{[IJK]}$ which is parallel along γ for the intrinsic tractor connection and satisfies $X^{[I}\Sigma^{JKL]} = 0$ where X^I is the position tractor of Γ. To show that γ is an M-conformal circle, we need to show that the ambient 3-tractor $\Xi^{ABC} = 6u^{-1} X^{[A} V^B B^{C]}$ satisfies these same properties, this time using the *ambient* tractor connection and position tractor. We show this by using the conditions on the tractor second fundamental form and the difference tractor to relate the ambient X, U and A tractors to their intrinsic counterparts.

From the isomorphism of Theorem 4.2.3, it follows that $X^A = \Pi^A_I X^I$ and $V^B = \Pi^B_J U^J$, where X^I and U^J are the intrinsic submanifold canonical and velocity tractors of γ respectively. Write $\mathcal{F}_{ij} = f \boldsymbol{g}_{ij}$ where f is a density of weight -2 on Γ. (Equally we can write $\mathcal{F}_{ij} = f g_{ij}$ where f is a smooth function and g_{ij} is the intrinsic metric of the scale which we are using, but we choose to work in a way that uses the conformal metric.) Then

$$\begin{aligned}
B^C &= u^i \nabla_i V^C \\
&= u^i \nabla_i (\Pi^C_J U^J) \\
&= \Pi^C_J \left[u^i D_i U^J + u^i f \boldsymbol{g}_{ij} \left(Z^{Jj} X_K - Z^j_K X^J\right) U^K\right] \\
&= \Pi^C_J \left(A^J - u \cdot f X^J\right),
\end{aligned}$$

and

$$u^i \nabla_i B^C = u^i \nabla_i (\Pi^C_J A^J) - u^i \nabla_i (u \cdot f \Pi^C_J X^J)$$

$$
\begin{aligned}
&= \Pi^C_J(u^i D_i A^J + f\cdot u^j(Z^J_j X_K - Z_{Kj}X^J)A^K) \\
&\qquad\qquad - u^i\nabla_i(u\cdot f)X^J - u\cdot f\Pi^C_J u^i Z^J_i \\
&= \Pi^C_J(u^i D_i A^J - 2u\cdot f\cdot u^j Z^J_j + \rho X^J) \qquad (5.3.18)
\end{aligned}
$$

where as in Chapter 3, $u^2 := u_k u^k$ and we have collected all the terms in the bottom slot into ρ. Its exact form will not be important. Now, recall that $V^B = u^i\nabla_i(u^{-1}X^B)$ and $B^C = u^i\nabla_i U^C$. Hence, using the skew-symmetry,

$$
u^i\nabla_i\Xi^{ABC} = u^i\nabla_i\left(6u^{-1}X^{[A}V^B B^{C]}\right) = 6u^{-1}X^{[A}V^B(u^i\nabla_i B^{C]}).
$$

Now, substituting (5.3.18) for the derivative of the acceleration,

$$
\begin{aligned}
u^i\nabla_i\Xi^{ABC} &= 6u^{-1}X^{[A}V^B(u^i\nabla_i B^{C]}) \\
&= 6u^{-1}X^I U^J(u^i D_i A^K)\Pi^{[A}_I\Pi^B_J\Pi^{C]}_K - 12f\cdot u^j X^I U^J Z^K_j\Pi^{[A}_I\Pi^B_J\Pi^{C]}_K \\
&\qquad + 6u^{-1}\rho X^I U^K X^K\Pi^{[A}_I\Pi^B_J\Pi^{C]}_K \\
&= 6u^{-1}X^{[I}U^J(u^i D_i A^{K]})\Pi^A_I\Pi^B_J\Pi^C_K \\
&= u^i\Pi^A_I\Pi^B_J\Pi^C_K D_i\Sigma^{IJK}, \qquad (5.3.19)
\end{aligned}
$$

where the term that is skew in two copies of the canonical tractor clearly vanishes, and moreover

$$
u^k X^{[I}U^J Z^{K]}_k = u^{-1}u^j u^k X^{[I}Z^J_j Z^{K]}_k - u^{-3}(u_l a^l)u^k X^{[I}X^J Z^{K]}_k = 0
$$

since the first term is skew over two copies of the velocity and the second term is once again skew over two copies of the canonical tractor X. Finally, the right-hand side of (5.3.19) is zero, since γ is a Γ-conformal circle.

The final thing to show is that Ξ^{ABC} also satisfies the incidence relation, but this is easily seen since

$$
\begin{aligned}
X^{[A}\Xi^{BCD]} &= X^I\Pi^{[A}_I\Pi^B_J\Pi^C_K\Pi^{D]}_L\Sigma^{JKL} \\
&= \Pi^A_{[I}\Pi^B_J\Pi^C_K\Pi^D_{K]}X^I\Sigma^{JKL} \\
&= \Pi^A_I\Pi^B_J\Pi^C_K\Pi^D_K X^{[I}\Sigma^{JKL]} \\
&= 0.
\end{aligned}
$$

Thus γ is an unparametrized M-conformal circle. □

The nested chain of Theorem 5.3.4 is now easily seen.

Proof of Theorem 5.3.4. Let Γ be a totally conformally circular submanifold. Then either of Theorem 5.3.9 or Theorem 5.3.10 gives that in particular $\mathbb{L}_{iJ}{}^{C} = 0$, and hence Γ is weakly conformally circular by Theorem 5.3.8. If Γ is weakly conformally circular, then one sees from (4.2.26) that $\mathbb{L}_{iJ}{}^{C} = 0$ implies in particular $\mathring{I\!I}_{ij}{}^{c} = 0$, and thus Γ is umbilic. □

We summarize these notions and their characterizations via tractors and invariant tensors in a table:

Property	**Characterization**
Umbilic	$\mathring{I\!I}_{ij}{}^{c} = 0$
Weakly conformally circular/distinguished	$\mathbb{L}_{iJ}{}^{C} = 0$
Strongly conformally circular (parametrized)	$\mathbb{L}_{iJ}{}^{C} = 0$ and $\mathsf{S}_{iJK} = 0$
Strongly conformally circular (unparametrized)	$\mathbb{L}_{iJ}{}^{C} = 0$ and $\mathsf{S}_{iJK} \propto \boldsymbol{g}_{ij} Z^{j}{}_{[J} X_{K]}$

While in general the inclusions of Theorem 5.3.4 are strict, in the flat case this is not true.

Theorem 5.3.11. *Suppose $(M, \boldsymbol{c})$ is a conformally flat manifold, and $\Gamma \hookrightarrow M$ is a submanifold. Then the following are equivalent:*

1. *Γ is umbilic;*
2. *Γ is weakly conformally circular;*
3. *Γ is totally conformally circular.*

Proof. In light of the inclusions of Theorem 5.3.4, it suffices to prove that if Γ is umbilic then it is totally conformally circular. But if $W_{abcd} = 0$ and $\mathring{I\!I}_{ij}{}^{c} = 0$ then one sees that equations (4.2.27) and (4.2.30) both vanish, i.e. the Fialkow tensor and the tractor second fundamental form are both zero. Theorem 5.3.9 now completes the proof. □

5.4 Characterizations and generalizations of minimal scales

Many of the results from [38] also admit generalizations to submanifolds of arbitrary codimension. Throughout this section, we frequently use a scale tractor corresponding to a choice of distinguished scale. Unless otherwise stated, for the remainder of this section I_A will always be equal to $\frac{1}{n} D_A \sigma$, where $\sigma \in \mathcal{E}[1]$ is the particular choice of scale. We begin with a characterization of minimal scales in terms of the tractor normal form. We will soon generalize what we mean by a minimal scale, but for now we mean simply the usual definition: a *minimal scale* is one for which the corresponding mean curvature is zero.

Proposition 5.4.1. *Let (M, g) be a pseudo-Riemannian manifold. A submanifold Γ is minimal if, and only if $I^{A_1}N_{A_1A_2\cdots A_d} = 0$, where $I_A = \frac{1}{n}D_A\sigma$ with $\sigma \in \mathcal{E}_+[1]$ such that $g = \sigma^{-2}\boldsymbol{g}$. In other words, the scale σ is minimal if, and only if, I_A is an intrinsic tractor of the submanifold.*

Proof. Working in the scale σ, one has $I_A = \sigma Y_A - \frac{1}{n}\mathsf{J}\sigma X_A$. Using equation (4.4.6), we have that

$$\begin{aligned}
I^{A_1}N_{A_1A_2\cdots A_{d-1}A_d} &= \left(\sigma Y^{A_1} - \frac{1}{n}\mathsf{J}\sigma X^{A_1}\right)\Big(N_{a_1a_2\cdots a_{d-1}a_d}\mathbb{Z}^{a_1a_2\cdots a_{d-1}a_d}_{A_1A_2\cdots A_{d-1}A_d} \\
&\qquad + d\cdot N_{ba_2\cdots a_{d-1}a_d}H^b\cdot \mathbb{X}_{A_1A_2\cdots A_{d-1}A_d}^{\ \ a_2\cdots a_{d-1}a_d}\Big) \\
&= \sigma\cdot d\cdot N_{ba_2\cdots a_{d-1}a_d}H^b\cdot Y^{A_1}\mathbb{X}_{A_1A_2\cdots A_{d-1}A_{d-1}}^{\ \ a_2\cdots a_{d-1}a_d} \\
&= \sigma\cdot d\cdot N_{ba_2\cdots a_{d-1}a_d}H^b\cdot \\
&\qquad Y^{A_1}\cdot\frac{1}{d!}\cdot\sum_{\tau\in\mathfrak{S}_d}\operatorname{sgn}\tau X_{A_{\tau(1)}}Z^{a_2}_{A_{\tau(2)}}\cdots Z^{a_{d-1}}_{A_{\tau(d-1)}}Z^{a_d}_{A_{\tau(d)}} \\
&= \sigma\cdot d\cdot N_{ba_2\cdots a_{d-1}a_d}H^b\cdot \\
&\quad Y^{A_1}\cdot\frac{1}{d!}\cdot\Bigg(X_{A_1}\sum_{\substack{\tau\in\mathfrak{S}_d,\\ \tau(1)=1}}\operatorname{sgn}\tau Z^{a_2}_{A_{\tau(2)}}\cdots Z^{a_{d-1}}_{A_{\tau(d-1)}}Z^{a_d}_{A_{\tau(d)}} \\
&\quad + \sum_{\substack{\tau\in\mathfrak{S}_d,\\ \tau(1)\neq 1}}\operatorname{sgn}\tau X_{A_{\tau(1)}}Z^{a_2}_{A_{\tau(2)}}\cdots Z^{a_{d-1}}_{A_{\tau(d-1)}}Z^{a_d}_{A_{\tau(d)}}\Bigg) \\
&= \sigma\cdot d\cdot N_{ba_2\cdots a_{d-1}a_d}H^b\cdot\frac{1}{d!}\left((d-1)!\cdot Z^{a_2}_{[A_2}\cdots Z^{a_{d-1}}_{A_{d-1}}Z^{a_d}_{A_d]}\right) \\
&= \sigma N_{ba_2\cdots a_{d-1}a_d}H^b\,\mathbb{Z}^{a_2\cdots a_{d-1}a_d}_{A_2\cdots A_{d-1}A_d}.
\end{aligned}$$

Since $\sigma \neq 0$ and the mean curvature is normal, the above vanishes if, and only if, $H^b = 0$, i.e. the chosen scale is minimal. □

Remark 5.4.2. This result generalizes Theorem 2 from [38]: one may restate the above as "a submanifold Γ in a Riemannian manifold (M, g) is minimal if, and only if $I^{A_1}N_{A_1\cdots A_d} = 0$, where I is the scale tractor corresponding to the metric g". Note that a minimal 1-dimensional submanifold in a Riemannian manifold is exactly a geodesic. (Recall one may still form the tractor normal form on a Riemannian manifold.)

Suppose now that $(M, \boldsymbol{c}, I)$ is an almost Einstein manifold. If Γ is minimal for the scale σ, then by the above, I_A may be identified with a submanifold tractor. Since I_A is parallel for the standard tractor connection, and I_A is a submanifold tractor, I_A is also parallel for the connection $\check{\nabla}$:

$$\check{\nabla}_i I_J = \Pi_J^A \nabla_i \left(\Pi_A^K I_K\right) = \Pi_J^A \nabla_i I_A = 0.$$

Therefore, from the definition of the checked connection (4.2.10), one sees that I_J is parallel for the submanifold tractor connection if, and only if, $\mathsf{S}_i^J{}_K I^K = 0$.

Choosing a background scale to split the tractor bundles, we have that

$$\begin{aligned}\mathsf{S}_i{}^J{}_K I^K &= \mathcal{F}_{ij}\left(Z^{Jj} X_K - Z_K^j X_J\right)\left(\sigma Y^K + \nabla_k \sigma Z^{Kk} - \frac{1}{n}\left(\Delta + \mathsf{J}\sigma\right) X^K\right) \\ &= \mathcal{F}_{ij}\left(\sigma Z^{Jj} - \nabla^j \sigma X^J\right).\end{aligned}$$

Recall that almost-Einstein manifolds are a special case of almost pseudo-Riemannian manifolds (Definition 2.5.1), and hence the 1-jet $j^1\sigma$ only vanishes at isolated points (see the discussion of Section 2.5 for details). Therefore away from these points we must have $\mathcal{F}_{ij} = 0$, and then also at those points by continuity.

Thus we have proven

Theorem 5.4.3. *Let $\Gamma \hookrightarrow M$ be a minimal submanifold of the almost-Einstein manifold $(M, \boldsymbol{c}, I)$, where I is the parallel standard tractor corresponding to the solution to the almost Einstein equation* (2.3.21)*. Then Γ is almost-Einstein if, and only if, $\mathcal{F}_{ij} = 0$.*

Proposition 5.4.1 suggests a way to extend the notion of minimal submanifolds to conformally singular geometries. This is an extension of the program initiated in [38]. In that work, Theorem 2 was motivation for Definition 2 in which the notion of a *generalized geodesic* was introduced. This notion allows one to extend usual geodesics to/across the singularity locus of certain singular geometries. Here, we extend that notion with a definition inspired by Proposition 5.4.1.

Definition 5.4.4. Let $(M, \boldsymbol{c}, \sigma)$ be an almost-Riemannian manifold. We say that a submanifold $\Gamma \subset M$ of codimension d is a *generalized minimal submanifold* if

$$I^{A_1} N_{A_1 \cdots A_d} = 0, \tag{5.4.1}$$

where as usual $I_A := \frac{1}{n} D_A \sigma$ is the scale tractor for σ.

Proposition 5.4.5. *Let $(M, \boldsymbol{c}, \sigma)$ be an almost-Riemannian manifold. Suppose that Γ is a submanifold of codimension d that is minimal for $g := \sigma^{-2}\boldsymbol{g}$ on $M \backslash \mathcal{Z}(\sigma)$. Then Γ is a generalized minimal submanifold of $(M, \boldsymbol{c}, \sigma)$.*

Proof. Minimality on $M \backslash \mathcal{Z}(\sigma)$ implies that $I^{A_1} N_{A_1 A_2 \cdots A_d} = 0$ there. But this is an open dense set, so this must hold on the closure $\overline{M \backslash \mathcal{Z}(\sigma)} = M$ by smoothness. Hence Γ is a generalized minimal submanifold. □

Proposition 5.4.6. *On a conformally compact manifold, any generalized minimal submanifold which extends to the boundary meets the boundary orthogonally.*

Proof. Since $\partial M = \mathcal{Z}(\sigma)$,

$$I^A|_{\partial M} = \nabla^a \sigma Z_a^A - \frac{1}{n}\Delta\sigma X^A, \tag{5.4.2}$$

and recall that $\nabla_a \sigma$ is nowhere-zero along the boundary. Thus

$$I^{A_1} N_{A_1 A_2 \cdots A_d} \stackrel{\sigma}{=} N_{a_1 a_2 \cdots a_d} \nabla^{a_1} \sigma \mathbb{Z}^{a_2 \cdots a_d}_{A_2 \cdots A_d}.$$

Hence $\nabla^{a_1} N_{a_1 a_2 \cdots a_d} = 0$, and it follows that $\nabla_a \sigma$ (the conormal to the boundary ∂M) is orthogonal to the normal form of Γ. Thus if Γ extends to the boundary, it meets it orthogonally. □

Proposition 5.4.7. *Let $(M, \boldsymbol{c})$ be a conformal manifold with Γ a distinguished submanifold of codimension d. Suppose moreover that there exists a scale $\sigma \in \mathcal{E}_+[1]$ such that $I^{A_1} N_{A_1 \cdots A_d} = 0$, i.e. Γ is a generalized minimal submanifold of the almost-Riemannian manifold $(M, \boldsymbol{c}, \sigma)$. Then Γ is a totally geodesic submanifold of the Riemannian manifold (M, g^σ), with g^σ the Riemannian metric determined by the scale σ.*

Proof. Since the scale tractor is orthogonal to the normal form, by Proposition 5.4.1 we have that Γ is a minimal submanifold in (M, g^σ). Moreover, since Γ is distinguished, the normal form is parallel for the pullback tractor connection. Substituting the minimality condition into (4.4.9), together with the fact that Γ is a distinguished submanifold and therefore has parallel normal tractor form, shows that $\nabla_i N_{a_1 a_2 \cdots a_d} = 0$ and $N_{b a_2 \cdots a_d} P_i{}^b = 0$. And $\nabla_i N_{a_1 a_2 \cdots a_d} = 0$ if, and only if, Γ is totally geodesic in the given scale. □

In general, the implication of the proposition only holds in one direction. However, on an (almost-)Einstein manifold, we get a converse to the above.

Proposition 5.4.8. *Let $(M, \boldsymbol{c}, \sigma)$ be an almost-Einstein manifold. Suppose that $\Gamma \subset M$ is a submanifold of codimension d. If Γ is an umbilic, generalized minimal submanifold, then Γ is a (conformally) distinguished submanifold.*

Proof. We must show any of the equivalent conditions of Theorem 5.2.1. The Einstein condition implies that $N_{b a_2 \cdots a_d} P_i{}^b = 0$, and that Γ is a generalized geodesic submanifold implies that $H^b = 0$ up to $\mathcal{Z}(\sigma)$, and hence also on/across it by continuity. Thus for a minimal submanifold of an almost-Einstein manifold, $\nabla_i N_{A_1 A_2 \cdots A_d} = 0$ if, and only if, $\nabla_i N_{a_1 a_2 \cdots a_d} = 0$, i.e total geodesicity in the chosen scale. Since the scale is minimal, this is equivalent to total umbilicity. □

Thus we see that if our ambient space is Einstein, minimal umbilic submanifolds conserved quantities may be proliferated using our existing conformal theory. Moreover, if the submanifold in question is generalized minimal, such conserved quantities will extend to/across singularity sets of these geometries where they exist.

Chapter 6

Applications

In this last chapter, we present some applications of the machinery developed in the previous chapters. We obtain generalizations to arbitrary codimension of the distinguished curve theory introduced in Chapter 3. This includes first integrals, and characterizations of distinguished conformal submanifolds via an incidence relation and zero loci. We also mention some further areas of research wherein this machinery may be used.

6.1 BGG theory

In order to discuss conserved quantities and the methods for their proliferation which our machinery provides, we must review some elements of the theory of BGG sequences. We only require a small amount of this vast theory. The BGG sequence originated in work of Bernstein, Gel'fand and Gel'fand as a projective resolution of modules over a flag variety [8, 7]. Later the construction was generalized to parabolic geometries [12]. In this latter context the sequences are sometimes called *curved BGG sequences*, but since we only work in this setting we do not make the distinction. We only require some very basic elements of the general theory, but more detailed treatments can be found in [12, 5, 19] for the parabolic geometry setting. Specifically, certain overdetermined PDEs turn out to be coincide exactly with the kernel of the first operator in one of these sequences. These equations are the *(first) BGG equations*. One obtains various important overdetermined geometric PDEs by simply varying the representation used in the construction. The general theory then gives a correspondence between solutions to the equation and sections of a tractor bundle (determined by the representation) which are, in a suitable sense, "almost parallel" for the induced tractor connection. This correspondence will play a pivotal role in the way that we construct conserved quantities on distinguished submanifolds.

Note that from (2.3.15) it immediately follows that there is an invariant bundle map

$T^*M \to \Lambda^2\mathcal{T}$ given by $u_b \mapsto 2u_b X^{[A}Z^{B]b}$. Using the tractor metric, we may identify sections of $\Lambda^2\mathcal{T}$ with skew endomorphisms in $\mathrm{End}(\mathcal{T})$. Since sections of $\mathrm{End}(\mathcal{T})$ act tensorially on any tractor bundle $\mathcal{V} := \mathcal{G} \times_P \mathbb{V}$, where $\mathbb{V}$ is an irreducible G-representation, we therefore have an action of T^*M on $\mathcal{V}$. We thus have a sequence of bundle maps

$$\partial^* : \Lambda^k T^*M \otimes \mathcal{V} \to \Lambda^{k-1} T^*M \otimes \mathcal{V}, \tag{6.1.1}$$

where $k = 0, 1, \ldots, n$ and ∂^* is the Kostant codifferential of Lie algebra cohomology [52].

We define the cohomology bundles $\mathcal{H}_k$ in the usual way as the quotient bundles $\mathcal{H}_k := \ker(\partial^*)/\mathrm{im}(\partial^*)$. We will denote the canonical projection by $\Pi_k : \Lambda^k T^*M \otimes \mathcal{V} \to \mathcal{H}_k$, or more commonly simply Π since the domain and codomain will usually be clear from the context. There is then the *BGG sequence*:

$$\mathcal{H}_0 \overset{\mathcal{D}_0^{\mathcal{V}}}{\to} \mathcal{H}_1 \overset{\mathcal{D}_1^{\mathcal{V}}}{\to} \cdots \tag{6.1.2}$$

We will only be interested in the first operator $\mathcal{D}_0^{\mathcal{V}}$ which we henceforth denote simply by $\mathcal{D}^{\mathcal{V}}$, and is constructed as follows. Recall that the parabolic subgroup $P \subset G$ determines a filtration on $\mathbb{V}$ by P-invariant subspaces. Writing $\mathbb{V}^0$ for the largest non-trivial filtration component, one sees that $\mathcal{H}_0 = \mathcal{V}/\mathcal{V}^0$, where $\mathcal{V}^0 := \mathcal{G} \times_P \mathbb{V}^0$. The construction of the operator $\mathcal{D}^{\mathcal{V}}$ is then given by the following theorem from [13].

Theorem 6.1.1. *Let $\mathbb{V}$ be an irreducible G-representation and let $\mathcal{V} := \mathcal{G} \times_P \mathbb{V}$. There is a unique differential operator $L : \mathcal{H}_0 \to \mathcal{V}$ such that $\Pi \circ L = \mathrm{id}_{\mathcal{H}_0}$ and $\nabla \circ L$ lies in $\ker(\partial^*) \subset T^*M \otimes \mathcal{V}$. For $\sigma \in \Gamma(\mathcal{H}_0)$, the first BGG operator $\mathcal{D}^{\mathcal{V}}$ is then given by $\mathcal{D}^{\mathcal{V}}\sigma = \Pi(\nabla(L(\sigma)))$. Moreover, the map Π induces an injection from the space of parallel sections of $\mathcal{V}$ to a subspace of $\Gamma(\mathcal{H}_0)$ which is contained in the kernel of the first BGG operator*

$$\mathcal{D}^{\mathcal{V}} : \mathcal{H}_0 \to \mathcal{H}_1.$$

The information of the theorem is summarized in the following diagram:

$$\begin{array}{ccc}
\mathcal{V} & \xrightarrow{\nabla} & T^*M \otimes \mathcal{V} \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\
\ker(\partial^*) & & \ker(\partial^*) \\
{\scriptstyle L}\uparrow & & \downarrow{\scriptstyle \Pi} \\
\mathcal{H}_0 & \xrightarrow{\mathcal{D}_0^{\mathcal{V}}} & \mathcal{H}_1
\end{array}$$

where the maps $i : \ker(\partial^*) \to \Lambda^k T^*M \otimes \mathcal{V}$ are the obvious inclusions.

Any equation of the form $\mathcal{D}^{\mathcal{V}}\sigma = 0$ for some tractor bundle $\mathcal{V}$ is called a *first BGG equation.* This class of equations includes many well-known geometric PDEs on conformal manifolds such as the almost-Einstein equation (2.3.21), the conformal Killing-Yano equation and the conformal Killing tensor equation, as well as important equations in other parabolic geometries, for example the metrizability equation in projective geometry [54, 34]. These equations all arise from this same underlying construction. For example, the almost-Einstein equation comes from taking $\mathbb{V} = \mathbb{R}^{n+2}$ the standard representation of G, and the conformal Killing form equation arises from $\mathbb{V} = \Lambda^2\mathbb{R}^{n+2}$, the second exterior power of the standard representation. We note that we have already seen one example of a BGG splitting operator. Recall that the Thomas-D operator (2.3.18) maps $\mathcal{E}[1] \to \mathcal{T}$, and that $\nabla^{\mathcal{T}}(D(\sigma)) = 0$ if, and only if, σ solves the almost-Einstein equation (2.3.21). Hence the Thomas-D operator is a BGG splitting operator. Recall that parallel standard tractors are in bijective correspondence with solutions to the almost-Einstein equation, so in this case it turns out that the subspace of $\Gamma(\mathcal{H}_0)$ mentioned above is the full kernel of $\mathcal{D}^{\mathcal{V}}$.

We call the elements of the subspace of $\Gamma(\mathcal{H}_0)$ described in the theorem the *normal* solutions of the equation $\mathcal{D}^{\mathcal{V}}\sigma = 0$ [53]. The operator $L : \mathcal{H}_0 \to \mathcal{V}$, whose existence is the subject of the theorem, is called a *BGG splitting operator*. Normal solutions to $\mathcal{D}^{\mathcal{V}}\sigma = 0$ are by definition in bijective correspondence with sections of $\mathcal{V}$ which are parallel for the usual tractor connection on $\mathcal{V}$. We note here that on geometries which are flat (according to either the tractor connection or, equivalently, the Cartan connection), all solutions are normal. It turns out that one can say even more about $\nabla(L(\sigma))$. Specifically, in [46] (see also [47] for examples), it is shown that $\sigma \in \Gamma(\mathcal{H}_0)$ solves $\mathcal{D}^{\mathcal{V}}\sigma = 0$ if, and only if, $L(\sigma)$ is parallel for a certain modified connection, called the *prolongation connection*. This takes the form of the usual tractor connection on $\mathcal{V}$ plus a deformation term. This deformation is expressed in terms of the curvature of the tractor connection on $\nabla^{\mathcal{V}}$. Thus one sees that on flat geometries, where the curvature of this connection vanishes, the prolongation connection is just the usual tractor connection $\nabla^{\mathcal{V}}$ and hence all solutions are normal. Using this result, we are often able to calculate $\nabla(L(\sigma))$ explicitly.

6.2 Conserved quantities

A key feature of the tractor characterization of distinguished curves developed in [39] is the procedure to proliferate conserved quantities. As was showed by Theorem 6.3.1, our notion of *distinguished* for conformal submanifolds exactly generalizes the existing tractor characterization of distinguished curves; the theory and techniques for manufacturing quantities that are constant along distinguished curves similarly generalize to submanifolds of all codimensions.

6.2.1 General theory

The following technical result is Theorem 6.1 from [39] extended from conformal circles to the case of more general submanifolds. Let $G := \mathrm{SO}(n+1,1)$ since we work in Riemannian signature conformal geometry. In that article, $\mathbb{W}_0$ denotes the G-representation determined by the class of distinguished *curve*; here it will denote the G-representation determined by the distinguished submanifold Γ: if $\dim\Gamma = m$, then $\mathbb{W}_0 := \Lambda^{m+2}\mathbb{R}^{n+2}$. Note that we are working here with the dual characterization of distinguished submanifolds, just to be consistent with the approach of [39].

Theorem 6.2.1. *Let $\mathbb{V}_1, \dots \mathbb{V}_k$ be irreducible representations of G, $\mathcal{V}_i := \mathcal{G} \times_P \mathbb{V}_i$, and $\mathcal{D}^{\mathcal{V}_i}$ be the corresponding first BGG operator for each $i = 1, \dots k$. Suppose that, for each i, σ_i is a normal solution of the first BGG equation*

$$\mathcal{D}^{\mathcal{V}_i}\sigma_i = 0, \tag{6.2.1}$$

and $m_i \in \mathbb{Z}_{\geq 0}$. Then for each copy of the trivial G-representation $\mathbb{R}$ in

$$(\odot^{m_0}\mathbb{W}_0) \otimes (\odot^{m_1}\mathbb{V}_1) \otimes \cdots \otimes (\odot^{m_k}\mathbb{V}_k) \tag{6.2.2}$$

there is a corresponding distinguished submanifold first integral.

Proof. The proof from [39] may be repeated *mutatis mutandis.* □

The theorem also provides a method for proliferation of conserved quantities: every copy of the trivial G-representation in (6.2.2) has an associated G-epimorphism mapping (6.2.2) to $\mathbb{R}$. Such a map determines a corresponding parallel tractor field which we denote T which takes values in the bundle

$$(\odot^{m_0}\mathcal{W}_0^*) \otimes (\odot^{m_1}\mathcal{V}_1^*) \otimes \cdots \otimes (\odot^{m_k}\mathcal{V}_k^*)\,. \tag{6.2.3}$$

Writing $S_i := L_i(\sigma_i) \in \Gamma(\mathcal{G} \times_P \mathbb{V}_i)$, where L_i denotes the BGG splitting operator corresponding to the BGG equation $\mathcal{D}^{\mathcal{V}_i}\tau = 0$, the quantity

$$T\left(\odot^{m_0}\Sigma, \odot^{m_1}S_1, \cdots, \odot^{m_k}S_k\right) \tag{6.2.4}$$

is constant along the distinguished submanifold Γ characterized by Σ, and this exactly realizes the first integral of Theorem 6.2.1.

6.2.2 An example of a distinguished submanifold conserved quantity

We give an example to show how this machinery yields conserved quantities for distinguished submanifolds. The space $\mathcal{E}_{a_0[a_1a_2\cdots a_d]}[w] = \mathcal{E}_{a_0} \otimes \mathcal{E}_{[a_1a_2\cdots a_d]}[w]$ is completely irreducible, and has the $O(g)$-decomposition

$$\mathcal{E}_{a_0[a_1a_2\cdots a_d]}[w] = \mathcal{E}_{[a_0a_1a_2\cdots a_d]}[w] \oplus \mathcal{E}_{\{a_0[a_1a_2\cdots a_d]\}_0}[d] \oplus \mathcal{E}_{[a_2\cdots a_d]}[w-2], \tag{6.2.5}$$

where $\mathcal{E}_{\{a_0[a_1a_2\cdots a_d]\}_0}[d]$ consists of sections $k_{a_0a_1\cdots a_d}$ which are completely skew on the indices $a_1, a_2, \ldots a_d$, and for which $k_{[a_0a_1a_2\cdots a_d]} = 0$. A d-form $k_{a_1a_2\cdots a_d} \in \Gamma(\mathcal{E}_{[a_1a_2\cdots a_d]}[d+1])$ is said to be a *conformal Killing-Yano form* or simply *conformal Killing form* if it satisfies

$$\nabla_{\{a_0}k_{a_1a_2\cdots a_d\}_0} = 0, \tag{6.2.6}$$

where the braces and subscript zero denote projection onto the middle factor of (6.2.5). This equation can be checked to be conformally invariant, and is moreover a first BGG equation (which in particular implies conformal invariance anyway). Thus solutions to this equation correspond bijectively to sections of a certain tractor bundle. For this equation, the corresponding tractor bundle is $\Lambda^{d+1}\mathcal{T}^*$ [37], and the BGG splitting operator L : $\mathcal{E}_{[a_1a_2\cdots a_d]}[d+1] \to \mathcal{E}_{[A_0A_1\cdots A_d]}$ is calculated there also:

$$\begin{aligned} L(k_{a_1\cdots a_d}) &= k_{a_1\cdots a_d}\mathbb{Y}_{A_0A_1\cdots A_d}{}^{a_1\cdots a_d} + \frac{1}{d+1}\nabla_{a_0}k_{a_1\cdots a_d}\mathbb{Z}_{A_0A_1\cdots A_d}^{a_0a_1\cdots a_d} \\ &\quad + \frac{d}{n-d+1}\nabla^c k_{ca_2\cdots a_d}\mathbb{W}_{A_0A_1A_2\cdots A_d}{}^{a_2\cdots a_d} + \rho_{a_1\cdots a_d}\mathbb{X}_{A_0A_1\cdots A_d}{}^{a_1\cdots a_d}, \end{aligned} \tag{6.2.7}$$

where the exact form of $\rho_{a_1\cdots a_d} \in \mathcal{E}_{[a_1\cdots a_d]}[d-1]$ is unimportant for our purposes.

Proposition 6.2.2. *Let $k_{a_1\cdots a_d} \in \mathcal{E}_{[a_1\cdots a_d]}[d]$ be a normal solution to the conformal Killing-Yano equation and Γ a distinguished submanifold of codimension $d+1$ with corresponding tractor normal form $N_{A_0A_1\cdots A_d}$. Let $\mathbb{K}_{A_0A_1\cdots A_d} := L(k_{a_1\cdots a_d}) \in \mathcal{E}_{[A_0A_1\cdots A_d]}$ be the image of $k_{a_1\cdots a_d}$ under the BGG splitting operator L of (6.2.7). Then the scalar function $\mathbb{K}_{A_0A_1\cdots A_d}N^{A_0A_1\cdots A_d}$ is constant along Γ.*

Proof. Since $k_{a_1\cdots a_d}$ is a normal solution, we have that $\nabla_i\mathbb{K}_{A_0A_1\cdots A_d} = 0$. Moreover, since Γ is a distinguished submanifold, $\nabla_i N^{A_0A_1\cdots A_d} = 0$ by definition. Hence the scalar quantity $\mathbb{K}_{A_0A_1\cdots A_d}N^{A_0A_1\cdots A_d}$ is constant.

We show the non-triviality of this quantity by calculating it directly. This also allows us to see that the normality is necessary in this case.

From the explicit forms of $\mathbb{K}_{A_0A_1\cdots A_d}$ and $N_{A_0A_1\cdots A_d}$, we see that

$$\begin{aligned} \mathbb{K}_{A_0A_1\cdots A_d}N^{A_0A_1\cdots A_d} &= (d+1)\cdot k_{a_1\cdots a_d}N^{cb_1\cdots b_d}H_c \cdot \mathbb{Y}_{A_0A_1\cdots A_d}{}^{a_1\cdots a_d}\mathbb{X}^{A_0A_1\cdots A_d}{}_{a_1\cdots a_d} \\ &\quad + \frac{1}{d+1}\left(\nabla_{a_0}k_{a_1\cdots a_d}\right)N^{b_0b_1\cdots b_d}\cdot \mathbb{Z}_{A_0A_1\cdots A_d}^{a_0a_1\cdots a_d}\mathbb{Z}^{A_0A_1\cdots A_d}_{b_0b_1\cdots b_d} \\ &= k_{a_1\cdots a_d}N^{ca_1\cdots a_d}H_c + \frac{1}{d+1}\left(\nabla_{a_0}k_{a_1\cdots a_d}\right)N^{a_0a_1\cdots a_d}, \end{aligned} \tag{6.2.8}$$

which verifies non-triviality.

To show that the normality is required for this example, we calculate

$$\nabla_i\left(\mathbb{K}_{A_0A_1\cdots A_d}N^{A_0A_1\cdots A_d}\right) = \left(\nabla_i\mathbb{K}_{A_0A_1\cdots A_d}\right)N^{A_0A_1\cdots A_d}$$

for a general conformal Killing-Yano form. The derivative of the splitting operator may be calculated directly via the prolongation connection [46, 47]. Theorem 3.9 of [37] essentially gives the prolongation connection for the conformal Killing-Yano equation. Recall that this connection is equal to the connection induced on $\mathcal{E}_{[A_0A_1\cdots A_d]}$ by the standard tractor connection, plus a modification term.

In the special case of the conformal Killing equation on forms, one has

$$(\nabla_c - \Psi_c)\,\mathbb{K}_{A_0A_1\cdots A_d} = 0,$$

where ∇_c is the standard tractor connection and $\Psi_c : \mathcal{E}_{[A_0A_1\cdots A_d]} \to \mathcal{E}_{c[A_0A_1\cdots A_d]}$ is defined by

$$\begin{aligned}\Psi_c(\mathbb{K}_{A_0A_1A_2\cdots A_d}) := -\frac{1}{2}W_{a_0a_1c}{}^{p}k_{pa_2\cdots a_d}\mathbb{Z}^{a_0a_1a_2\cdots a_d}_{A_0A_1A_2\cdots A_d} + \varphi_{ca_2\cdots a_d}\mathbb{W}_{A_0A_1A_2\cdots A_d}{}^{a_2\cdots a_d}\\ + \xi_{a_1a_2\cdots a_d}\mathbb{X}_{A_0A_1A_2\cdots A_d}{}^{a_1a_2\cdots a_d},\end{aligned} \tag{6.2.9}$$

where only the explicit form of the $\mathbb{Z}$ slot will be important.

Therefore one has

$$\begin{aligned}\nabla_i\left(\mathbb{K}_{A_0A_1A_2\cdots A_d}N^{A_0A_1A_2\cdots A_d}\right) &= \left(\nabla_i\mathbb{K}_{A_0A_1A_2\cdots A_d}\right)N^{A_0A_1A_2\cdots A_d}\\ &= \Psi_i(\mathbb{K}_{A_0A_1A_2\cdots A_d})N^{A_0A_1A_2\cdots A_d}\\ &= -\frac{1}{2}W_{a_0a_1i}{}^{p}k_{pa_2\cdots a_d}N^{b_0b_1b_2\cdots b_d}\cdot\mathbb{Z}^{a_0a_1a_2\cdots a_d}_{A_0A_1A_2\cdots A_d}\mathbb{Z}^{A_0A_1A_2\cdots A_d}_{b_0b_1b_2\cdots b_d}\\ &= -\frac{1}{2}W_{a_0a_1i}{}^{p}k_{pa_2\cdots a_d}N^{a_0a_1a_2\cdots a_d},\end{aligned}$$

which we will not in general expect to vanish.

□

In [39], some scalar quantities were constructed that were shown to be conserved even when the corresponding BGG solution was not necessarily normal. It seems that this phenomenon may be quite limited for higher-codimension submanifolds, since in the curve case the reason for this is frequently simply a symmetry one: namely that any skew terms vanish on repeated contraction with the velocity vector of the curve (compare, for example, the difference between the formula for the Fialkow tensor in the hypersurface case (4.2.29) and the more general case (4.2.27) where a simplification of this type occurs). Even in the case of a 2-dimensional submanifold, the situation is vastly different, as the above example illustrates.

6.3 Distinguished submanifolds via incidence relations

We next give a result which generalizes the main theorem (for non-null conformal circles) of [39].

Theorem 6.3.1. *Let $\Gamma \hookrightarrow M$ be a submanifold of codimension d in the conformal manifold $(M, \boldsymbol{c})$. Then Γ is distinguished if, and only if, there exists $\Psi_{A_1 A_2 \cdots A_d} \in \Lambda^d \mathcal{T}^*$ such that $\Psi_{A_1 A_2 \cdots A_d} X^{A_1} = 0$ and $\nabla_i \Psi_{A_1 A_2 \cdots A_d} = 0$ along Γ.*

Proof. If Γ is distinguished, then by Theorem 5.2.1, the tractor normal form is parallel in tangential directions. Moreover, it is clear from the definition of the tractor normal form (4.4.6) that $N_{A_1 A_2 \cdots A_d} X^{A_1} = 0$. Thus we may take $\Psi_{A_1 \cdots A_d}$ to be the tractor normal form.

Conversely, suppose that such a Ψ exists along Γ. From (4.4.1), we know that Ψ must be of the form

$$\begin{aligned}\Psi_{A_1 A_2 \cdots A_d} = \sigma_{a_2 \cdots a_d} \mathbb{Y}_{A_1 A_2 \cdots A_d}{}^{a_2 \cdots a_d} + \nu_{a_1 a_2 \cdots a_d} \mathbb{Z}_{A_1 A_2 \cdots A_d}^{a_1 a_2 \cdots a_d} \\ + \varphi_{a_3 \cdots a_d} \mathbb{W}_{A_1 A_2 A_3 \cdots A_d}{}^{a_3 \cdots a_d} + \rho_{a_2 \cdots a_d} \mathbb{X}_{A_1 A_2 \cdots A_d}{}^{a_2 \cdots a_d}.\end{aligned}$$

We immediately see that the condition $\Psi_{A_1 \cdots A_d} X^{A_1} = 0$ implies that $\sigma_{a_2 \cdots a_d} = 0$ and $\varphi_{a_3 \cdots a_d} = 0$.

Moreover, if $u^i \in \mathcal{E}^i$, the above incidence relation together with the parallel condition means that

$$0 = u^i \nabla_i \left(X^{A_1} \Psi_{A_1 A_2 \cdots A_d} \right) = u^i Z_i^{A_1} \Psi_{A_1 A_2 \cdots A_d}$$

so $u^i Z_i^{A_1} \Psi_{A_1 A_2 \cdots A_d} = 0$ for all $u^i \in \mathcal{E}^i$.

Expanding this, one sees that $\nu_{a_1 a_2 \cdots a_d} u^{a_1} = 0$ and $\rho_{a_2 \cdots a_d} u^{a_2} = 0$. Since u^i was an arbitrary submanifold tangent vector, we conclude that $\nu \in (\Lambda^d N^* \Gamma)[d]$ and $\rho \in (\Lambda^{d-1} N^* \Gamma)[d-2]$, i.e. ν and ρ are in appropriate exterior powers of the normal bundle. Thus in particular $\nu_{a_1 a_2 \cdots a_d} = f N_{a_1 a_2 \cdots a_d}$, where $N_{a_1 a_2 \cdots a_d}$ is the Riemannian normal form of Γ and f is a function on Γ.

Now, note that

$$\nabla_i \left(\Psi^{A_1 A_2 \cdots A_d} \Psi_{A_1 A_2 \cdots A_d} \right) = 2 \Psi^{A_1 A_2 \cdots A_d} \nabla_i \Psi_{A_1 A_2 \cdots A_d} = 0,$$

so $\Psi^{A_1 A_2 \cdots A_d} \Psi_{A_1 A_2 \cdots A_d}$ is constant along Γ.

On the other hand,

$$\Psi^{A_1 A_2 \cdots A_d} \Psi_{A_1 A_2 \cdots A_d} = \nu^{a_1 a_2 \cdots a_d} \nu_{a_1 a_2 \cdots a_d} = f^2 N^{a_1 a_2 \cdots a_d} N_{a_1 a_2 \cdots a_d} = f^2 \cdot d!,$$

and therefore the function f is constant. Thus $\nu_{a_1 a_2 \cdots a_d}$ is a constant multiple of the Riemannian normal form.

From equation (4.4.3), we calculate

$$\begin{aligned} \nabla_i \Psi_{A_1 A_2 \cdots A_d} = &\left(f \nabla_i N_{a_1 a_2 \cdots a_d} + \rho_{a_2 \cdots a_d} \boldsymbol{g}_{i a_1}\right) \mathbb{Z}_{A_1 A_2 \cdots A_d}^{a_1 a_2 \cdots a_d} \\ &+ \left(\nabla_i \rho_{a_2 \cdots a_d} - f \cdot d \cdot N_{a_1 a_2 \cdots a_d} P_i{}^{a_1}\right) \mathbb{X}_{A_1 A_2 \cdots A_d}{}^{a_2 \cdots a_d}. \end{aligned} \tag{6.3.1}$$

Now, note that the same argument that yielded equation (4.4.10) may be repeated replacing normal *tractors* with normal *vectors* (as per Remark 4.4.8) to give

$$\nabla_i N_{a_1 a_2 \cdots a_d} = -d \cdot I\!I_{i[a_d}{}^{a_0} N_{a_1 a_2 \cdots a_{d-1}]a_0}. \tag{6.3.2}$$

Substituting this into (6.3.1) gives that in particular

$$-f \cdot d \cdot I\!I_{i[a_d}{}^{a_0} N_{a_1 a_2 \cdots a_{d-1}]a_0} + \boldsymbol{g}_{i[a_1} \rho_{a_2 \cdots a_{d-1} a_d]} = 0. \tag{6.3.3}$$

Contracting the above with $\boldsymbol{g}^{i a_1}$ allows us to express $\rho_{a_2 \cdots a_d}$ explicitly.

The contraction with the second fundamental form term is not completely obvious, so we compute it first.

$$\begin{aligned} \boldsymbol{g}^{i a_1} I\!I_{i[a_d}{}^{a_0} N_{a_1 a_2 \cdots a_{d-1}]a_0} &= -\boldsymbol{g}^{i a_1} \cdot I\!I_{i[a_1}{}^{a_0} N_{a_d a_2 \cdots a_{d-1}]a_0} \\ &= -\frac{1}{d!} \cdot \boldsymbol{g}^{i a_1} \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn} \sigma\, I\!I_{i a_{\sigma(1)}}{}^{a_0} N_{a_{\sigma(d)} a_{\sigma(2)} \cdots a_{\sigma(d-1)} a_0} \\ &= -\frac{1}{d!} \cdot \boldsymbol{g}^{i a_1} \sum_{\substack{\sigma \in \mathfrak{S}_d, \\ \sigma(1)=1}} \operatorname{sgn} \sigma\, I\!I_{i a_1}{}^{a_0} N_{a_{\sigma(d)} a_{\sigma(2)} \cdots a_{\sigma(d-1)} a_0} \\ &= -\frac{1}{d!} \sum_{\substack{\sigma \in \mathfrak{S}_d, \\ \sigma(1)=1}} I\!I_{i a_1}{}^{a_0} N_{a_d a_2 \cdots a_{d-1} a_0} \\ &= \frac{(d-1)!}{d!} \cdot m \cdot H^{a_0} N_{a_0 a_2 \cdots a_{d-1} a_d}. \end{aligned}$$

For the other term,

$$\begin{aligned} \boldsymbol{g}^{i a_1} \boldsymbol{g}_{i[a_1} \rho_{a_2 \cdots a_d]} &= \boldsymbol{g}^{i a_1} \cdot \frac{1}{d!} \sum_{\sigma \in \mathfrak{S}_d} \boldsymbol{g}_{i a_{\sigma(1)}} \rho_{a_{\sigma(2)} \cdots a_{\sigma(d)}} \\ &= \boldsymbol{g}^{i a_1} \cdot \frac{1}{d!} \sum_{\substack{\sigma \in \mathfrak{S}_d \\ \sigma(1)=1}} \boldsymbol{g}_{i a_1} \rho_{a_{\sigma(2)} \cdots a_{\sigma(d)}} \end{aligned}$$

$$= \frac{(d-1)!}{d!}\boldsymbol{g}^{ia_1}\boldsymbol{g}_{ia_1}\rho_{a_2\cdots a_d}$$
$$= \frac{m}{d}\rho_{a_2\cdots a_d}.$$

Hence

$$\begin{aligned}
\boldsymbol{g}^{ia_1}\boldsymbol{g}_{i[a_1}\rho_{a_2\cdots a_{d-1}a_d]} &= f\cdot d\cdot \boldsymbol{g}^{ia_1} I\!I_{i[a_d}{}^{a_0} N_{a_1a_2\cdots a_{d-1}]a_0}\\
\frac{m}{d}\cdot \rho_{a_2\cdots a_{d-1}a_d} &= f\cdot d\cdot \frac{1}{d}\cdot m\cdot H^{a_0}N_{a_0a_2\cdots a_{d-1}a_d}\\
\rho_{a_2\cdots a_{d-1}a_d} &= f\cdot d\cdot H^{a_0}N_{a_0a_2\cdots a_{d-1}a_d}.
\end{aligned}$$

Thus

$$\begin{aligned}
\Psi_{A_1A_2\cdots A_d} &= fN_{a_1a_2\cdots a_d}\mathbb{Z}^{a_1a_2\cdots a_d}_{A_1A_2\cdots A_d} + f\left(d\cdot H^bN_{ba_2\cdots a_d}\right)\mathbb{X}_{A_1A_2\cdots A_d}{}^{a_2\cdots a_d}\\
&= fN_{A_1A_2\cdots A_d},
\end{aligned}$$

where $N_{A_1A_2\cdots A_d}$ is the tractor normal form.

Since the function f is constant, $\nabla_i\Psi_{A_1A_2\cdots A_d} = 0$ implies that the tractor normal form is parallel. Thus Γ satisfies ones of the equivalent conditions of Definition 5.2.2, and is therefore a distinguished submanifold. □

6.4 Conformal distinguished submanifolds as zero loci

In [39], there is also a theorem linking conformal circles to zero loci of conformal Killing 2-forms. As one might expect, this also extends to more general submanifolds that are distinguished according to our definition.

Theorem 6.4.1. *Suppose $k_{a_1\cdots a_d}$ is a normal solution of the conformal Killing form equation on $(M, \boldsymbol{c})$. Then the zero locus of*

$$(k_{a_1\cdots a_d}, \nabla^c k_{ca_2\cdots a_d}) \textit{ for any } g\in\boldsymbol{c} \textit{ with Levi-Civita connection } \nabla,$$

is either empty or a distinguished conformal submanifold of codimension $d+1$.

Proof. Suppose $k_{a_1\cdots a_d} \in \mathcal{E}_{[a_1\cdots a_d]}[d+1]$ is a normal solution to (6.2.6). Then $\mathbb{K}_{A_0A_1\cdots A_d} := L(k_{a_1\cdots a_d})$ is parallel for the standard tractor connection.

From equation (6.2.7), one sees that $X^{A_0}_p\mathbb{K}_{A_0A_1\cdots A_d\,p} = 0$ at some point $p\in M$ is exactly the condition $(k_{a_1\cdots a_d}, \nabla^c k_{ca_2\cdots a_d}) = 0$ at the point p.

In the case of the model, if $\mathbb{K}$ is a parallel simple d-cotractor and $X\lrcorner\mathbb{K}$ is zero at some point p, then $X\lrcorner\mathbb{K}$ is zero along a submanifold of dimension m through p, namely the unique plane through p with normal form $N_{a_1\cdots a_d} := \mathbb{X}^{A_0 A_1 \cdots A_d}{}_{a_1\cdots a_d} \mathbb{K}_{A_0 A_1 \cdots A_d}$ (equivalently, the unique plane through p spanned by the vectors orthogonal to the defined normal form). From Theorem 2.6 of [16] it then follows that on $(M, \boldsymbol{c})$ for a d-tractor of the same algebraic type (namely simple and of signature $(+, +, \ldots, +, -)$, where the signature of a simple k-tractor refers to the restriction of the tractor metric to the span of its factors) is either empty or a distinguished conformal submanifold. □

6.5 Further work

We end by mentioning several further directions of related work or potential applications of the machinery developed here.

The most obvious of these is extensions of our ideas on conformal distinguished curves to the distinguished curves of (hypersurface type) CR geometry. CR geometry is closely related to conformal geometry, see for example the so-called *Fefferman space* construction, which originated in [31] and was later treated using CR tractor calculus in [20]. The Fefferman space $\tilde{M}$ of a CR manifold M is the total space of a circle bundle over M. The CR structure on M induces a conformal structure on $\tilde{M}$, and much of the CR structure on M is recoverable from the conformal structure on the Fefferman space. Notably, there are relations between the CR tractor bundle on M and the conformal tractor bundle on $\tilde{M}$. Given known relations between the distinguished curves of a CR manifold and its Fefferman space (for example, all chains of M are projections of null geodesics in its Fefferman space [10]), we expect this approach to yield tractor characterizations of CR distinguished curves such as chains and null-chains [50] similar to those of conformal and projective distinguished curves in [39].

Another question of interest is the extent to which one can relax the normality condition on solutions to first BGG equations when constructing conserved quantities. Certainly this can be done to some extent for curves, but we would hope for a full theory which, given some distinguished curve or submanifold, completely describes the conditions needed on first BGG solutions in order to get a conserved quantity.

Finally, in [42, 40, 43, 41], Gover and Waldron develop boundary calculus methods and apply these to study hypersurfaces in conformal manifolds through the singular Yamabe problem. The tractor conormal to a hypersurface in a conformal manifold plays an important role in this theory. Since the tractor normal form is a generalization of this conormal to arbitrary submanifolds, it seems promising to try to extend the singular Yamabe approach of Gover and Waldron to submanifolds of arbitrary codimension with the tractor normal form somehow playing the role of the tractor conormal.

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