Locally bi-2-transitive graphs and cycle-regular graphs, and the answer to a 2001 problem posed by Fouquet and Hahn

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Abstract

A vertex-transitive but not edge-transitive graph Γ is called *locally bi-2-transitive* if the stabiliser S in the full automorphism group of Γ of every vertex v of Γ has two orbits of equal size on the neighbourhood of v, and S acts 2-transitively on each of these two orbits. Also a graph is called *cycle-regular* if the number of cycles of a given length passing through a given edge in the graph is a constant, and a graph with girth g is called *edge-girth-regular* if the number of cycles of length g passing through any edge in the graph is a constant.

In this paper, we prove that a graph of girth 3 is edge-girth-regular and locally bi-2-transitive if and only if Γ is the line graph of a semi-symmetric locally 3-transitive graph. Then as an application, we prove that every tetravalent edge-girth-regular locally bi-2-transitive graph of girth 3 is cycle-regular. This shows that vertex-transitive cycle-regular graphs need not to be edge-transitive, and hence resolves the problem posed by Fouquet and Hahn at the end of their paper 'Cycle regular graphs need not be transitive', in *Discrete Appl. Math.* 113 (2001) 261–264.

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1 Introduction

The main purpose of this paper is to resolve the problem posed by Fouquet and Hahn at the end of their 2001 paper on cycle regular graphs [12].

All graphs we consider are finite, connected, simple and undirected. For any graph Γ, we let $V(\Gamma)$, $E(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ be its vertex set, edge set, and full automorphism group, respectively. Also we say that Γ is *vertex-transitive* if for any two vertices of Γ, there exists an automorphism of Γ that sends one to the other. Note that every vertex-transitive graph is regular, in the sense of having constant valency, but the converse does not hold. Similarly, we say that Γ is *edge-transitive* if for any two edges of Γ, there exists an automorphism of Γ that sends one to the other. If Γ is regular and edge-transitive but not vertex-transitive, then we say that Γ is *semi-symmetric.*

Now let Γ be a graph with girth g. For each $v \in V(\Gamma)$ and $e \in E(\Gamma)$, and for each integer k with $g \le k \le |V(\Gamma)|$, let $c_k(v)$ and $c_k(e)$ be the number of simple cycles of length k in Γ that pass through v and e, respectively. We say that Γ is *cycle-regular* if c_k is constant on edges for every k (so that the value of $c_k(e)$ depends only on k), and say that Γ is *vertex-cycle-regular* if c_k is constant on vertices for every k (so that the value of $c_k(v)$ depends only on k). It is easy to see that every cycle-regular graph having constant valency is also vertex-cycle-regular, and that every edge-transitive graph is cycle-regular, while every vertex-transitive graph is vertex-cycle-regular.

As pointed out by Fouquet and Hahn in [12], vertex-cycle-regular graphs that are regular need not be vertex-transitive. Indeed every semi-symmetric graph is cycle-regular and hence vertex-cycle-regular, but is not vertex-transitive. Moreover, there are infinitely many finite counter-examples, because there are infinitely many finite semi-symmetric graphs; see [4, 6, 7, 11] for example. Also Fouquet and Hahn constructed in [12] an infinite tetravalent vertex-cycle-regular graph that is not vertex-transitive, but they could not determine whether vertex-transitive cycle-regular graphs are edge-transitive, so they posed the following problem:

Problem 1 [12, Problem] *Is there a cycle-regular, vertex transitive but not edge transitive graph, finite or infinite?*

We will give a positive answer to this problem by investigating what we will call 'locally bi-2-transitive' graphs. Recall that an *arc* in a graph is an ordered edge, or equivalently, an ordered pair of adjacent vertices. Similarly, a 2-*arc* in a graph Γ is an ordered triple (u, v, w) of three distinct vertices of Γ such that v is adjacent to both u and w.

A vertex-transitive graph Γ is said to be *bi-arc-transitive* if $\mathrm{Aut}(\Gamma)$ has two orbits of equal size on the set of all arcs (ordered edges) of Γ, and similarly, *bi-edge-transitive* if $\mathrm{Aut}(\Gamma)$ has two orbits of equal size on the edge set of Γ. Bi-arc-transitive graphs that are edge-transitive are also called *half-arc-transitive.* Such graphs have been extensively studied in the literature; see [8, 21, 25, 27, 33], for example. Analogously, a graph is called *half-edge-transitive* if it is bi-arc-transitive and bi-edge-transitive. The latter kind of graphs were introduced in [31], where the authors proved that tetravalent half-edge-transitive graphs can have arbitrarily large vertex-stabilisers. We will show that every bi-arc-transitive graph has even valency ≥ 4; see Lemma 2.1.

Now let Γ be a half-edge-transitive graph with valency $2k \ge 4$, and let E_1 and E_2 be the two orbits of $\mathrm{Aut}(\Gamma)$ on E. Then $\Gamma_1 = (V(\Gamma), E_1)$ and $\Gamma_2 = (V(\Gamma), E_2)$ are two subgraphs of Γ, admitting $\mathrm{Aut}(\Gamma)$ as an arc-transitive automorphism group. We say that Γ is *locally bi-2-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on the 2-arcs of both Γ_1 and Γ_2.

In this paper, we will characterise the locally bi-2-transitive graphs with girth 3 having the property that the number of 3-cycles passing through any edge is a constant. (Actually, a graph with girth g having the property that c_g is constant on edges is called *edge-girth-regular.* Such graphs were introduced in [15], where several of their basic properties were given, and the trivalent and tetravalent cases were investigated systematically.)

Before stating our main results, we introduce some more definitions and notation.

For a permutation group G on a set Ω, we use G_α to denote the stabiliser in G of a point $\alpha \in \Omega$, and we say that G is *t-transitive* on Ω if for any two ordered t-tuples of pairwise distinct elements of Ω, there exists $g \in G$ sending one to the other. Also we denote by $\mathbb{Z}_n$ the cyclic group of order n, and by K_n the complete graph with n vertices.

Next, let Γ be a graph. For $u, v \in V(\Gamma)$, denote by $\{u, v\}$ the edge incident to u and v in Γ, and by $\Gamma(u)$ the set of vertices adjacent to u in Γ, and for a subset S of $V(\Gamma)$, denote by $\Gamma[S]$ the subgraph of Γ induced by S. The *line graph* $L(\Gamma)$ of Γ is the graph with vertex set $E(\Gamma)$ where two edges of Γ are adjacent in $L(\Gamma)$ if and only if they share a vertex in Γ. It is easy to see that if Γ has at least one 2-arc, then $\mathrm{Aut}(\Gamma)$ acts transitively on the 2-arcs of Γ if and only if Γ is vertex-transitive and $\mathrm{Aut}(\Gamma)_u$ acts 2-transitively on $\Gamma(u)$ for some $u \in V(\Gamma(u))$. Finally, a semi-symmetric graph Γ is called *locally 3-transitive* if $\mathrm{Aut}(\Gamma)_u$ acts 3-transitively on $\Gamma(u)$, for every $u \in V(\Gamma)$.

Our first main theorem gives a characterisation of edge-girth-regular locally bi-2-transitive graphs of girth 3.

Theorem 1.1 *A graph Γ of girth 3 is locally bi-2-transitive and edge-girth-regular if and only if Γ is the line graph of a semi-symmetric locally 3-transitive graph.*

Applying this gives the following two theorems, and a positive answer to Problem 1.

Theorem 1.2 *Every connected tetravalent edge-girth-regular locally bi-2-transitive graph of girth 3 is cycle-regular.*

Theorem 1.3 *For every integer $n \geq 3$, there exist infinitely many connected semi-symmetric locally 3-transitive graphs of valency n.*

By Theorems 1.1 and 1.3, there exist infinitely many connected tetravalent edge-girth-regular locally bi-2-transitive graphs of girth 3, and by Theorem 1.2, there are infinitely many connected vertex-transitive cycle-regular graphs that are not edge-transitive.

Finally, before proceeding, we point out that we have been unable to decide if there exists a trivalent cycle-regular graph that is vertex-transitive but not edge-transitive. We leave the existence or non-existence of such a graph as an open problem for future consideration. (We know that there exists no such graph with girth less than 6, by using the classification of cubic vertex-transitive graphs of girth at most 5 given in [10, Theorems 6.1–6.3], and we believe there is also no such graph with girth equal to 6. Also we have verified that every trivalent vertex-transitive cycle-regular graph of order at most 300 is edge-transitive, with the help of Magma [3] and the census of trivalent vertex-transitive graphs of order up to 1280 (see [23, 24]).)

2 Locally bi-2-transitive graphs of girth 3

In this section, we prove Theorem 1.1, using the following lemma that establishes some basic properties of bi-arc-transitive graphs.

Lemma 2.1 *A graph* Γ *is bi-arc-transitive if and only if* $\mathrm{Aut}(\Gamma)$ *is transitive on* $V(\Gamma)$ *and* $\mathrm{Aut}(\Gamma)_u$ *has two orbits of equal size on* $\Gamma(u)$*, for some* $u \in V(\Gamma(u))$*. In particular, every bi-arc-transitive graph has even valency at least* 4*.*

Proof For necessity in the first part, assume that Γ is bi-arc-transitive. Then $\mathrm{Aut}(\Gamma)$ is transitive on $V(\Gamma)$ and has two orbits of equal size on the arc set of Γ. Now take any vertex u of Γ. Then $\mathrm{Aut}(\Gamma)_u$ is intransitive on $\Gamma(u)$, so take $x, y \in \Gamma(u)$ such that

$$U_1 := \{(u, x^g) \mid g \in \mathrm{Aut}(\Gamma)_u\} \neq \{(u, y^g) \mid g \in \mathrm{Aut}(\Gamma)_u\} =: U_2.$$

Then $O_1 = \{(u, x)^a \mid a \in \mathrm{Aut}(\Gamma)\}$, so $O_2 = \{(u, y)^a \mid a \in \mathrm{Aut}(\Gamma)\}$ are the two orbits of $\mathrm{Aut}(\Gamma)$ on the arcs of Γ, and it follows that $|O_1| = |O_2|$, and that $U_1 \cup U_2 = \Gamma(u)$. Since Γ is vertex-transitive, also $|O_i| = |V(\Gamma)||U_i|$ for $i = 1, 2$, and hence $|U_1| = |U_2|$. Thus $\mathrm{Aut}(\Gamma)_u$ has two orbits U_1 and U_2 of equal size on $\Gamma(u)$, as required.

For sufficiency (in the first part), assume that B_1 and B_2 are two orbits of $\mathrm{Aut}(\Gamma)_u$ of equal size on $\Gamma(u)$ for some vertex u of Γ, and take $b_1 \in B_1$ and $b_2 \in B_2$. As $\mathrm{Aut}(\Gamma)$ is transitive on $V(\Gamma)$, by hypothesis, $A_1 := \{(u, b_1)^g \mid g \in \mathrm{Aut}(\Gamma)\}$ and $A_2 := \{(u, b_2)^g \mid g \in \mathrm{Aut}(\Gamma)\}$ are the two orbits of $\mathrm{Aut}(\Gamma)$ on the arcs of Γ. Then by an easy computation, $|A_1| = |V(\Gamma)||B_1| = |V(\Gamma)||B_2| = |A_2|$, and so Γ is bi-arc-transitive.

The second part follows easily. □

Proof of Theorem 1.1

First, we establish sufficiency in the statement of Theorem 1.1. Let Γ be the line graph of a semi-symmetric locally 3-transitive graph Π with valency d. Then Γ is edge-girth-regular, and has girth 3. A well known theorem about the line graphs (for example, see [1, p.1455]) states that if a connected graph X has at least 5 vertices then $\mathrm{Aut}(X) \cong \mathrm{Aut}(L(X))$, where $L(X)$ is the line graph of X. Since Π is semi-symmetric, Π has more than 5 vertices. Accordingly, if we view $\mathrm{Aut}(\Pi)$ as a permutation group on $E(\Pi)$, then we see that $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Pi)$, and hence that Γ is vertex-transitive. Now take an edge $e = \{x, y\}$ of Π, and let B_x be the set of edges of Π incident with x, and B_y be the set of edges of Π incident with y. Then $\Gamma[\{e\} \cup B_x] \cong \Gamma[\{e\} \cup B_y] \cong K_d$. Also because Π is semi-symmetric, $\mathrm{Aut}(\Pi)_e = \mathrm{Aut}(\Pi)_{xy}$, and because Π is locally 3-transitive, $\mathrm{Aut}(\Pi)_{xy}$ acts 2-transitively on both B_x and B_y. Hence by Lemma 2.1, we find Γ is bi-arc-transitive. To show that Γ is locally bi-2-transitive, we need only show that Γ is bi-edge-transitive. If that is not the case, then Γ is edge-transitive. So now take any $x_a = \{a, x\} \in B_x$ and any $y_b = \{y, b\} \in B_y$. Then $\{x_a, e\}$ and $\{e, y_b\}$ are two edges of the line graph Γ, so there exists some $\alpha \in \mathrm{Aut}(\Pi)$ taking $\{x_a, e\}$ to $\{e, y_b\}$. But these give 2-paths axy and xyb in the graph Π, and it follows that α sends x to y. This, however, is impossible, because Π is semi-symmetric and so no automorphism of Π can take a vertex of Π to one of its neighbours. This establishes sufficiency.

For necessity, suppose first that Γ is a locally bi-2-transitive edge-girth-regular graph of girth 3 with valency $2k$ for some $k > 1$. Now every locally bi-2-transitive graph is also bi-edge-transitive, and so $\mathrm{Aut}(\Gamma)$ has two orbits of equal size on $E(\Gamma)$, say E' and E''.

Let $\Gamma' = (V(\Gamma), E')$ and $\Gamma'' = (V(\Gamma), E'')$. Then $\mathrm{Aut}(\Gamma)$ acts 2-arc-transitively on both Γ' and Γ''. Take a vertex u in $V(\Gamma)$. Then $\Gamma(u) = \Gamma'(u) \cup \Gamma''(u)$ and $|\Gamma'(u)| = |\Gamma''(u)|$, and $\mathrm{Aut}(\Gamma)_u$ acts 2-transitively on both $\Gamma'(u)$ and $\Gamma''(u)$. Also because Γ is an edge-girth-regular graph of girth 3, there exists at least one triangle passing through any given edge of Γ, and hence that in any 3-cycle in Γ, there exist two incident edges that lie in the same orbit of $\mathrm{Aut}(\Gamma)$ on $E(\Gamma)$. Moreover, by vertex-transitivity of Γ, we may assume that $\Gamma'(u)$ contains two vertices that are adjacent in Γ, and then because $\mathrm{Aut}(\Gamma)_u$ acts 2-transitively on $\Gamma'(u)$, it follows that $\Gamma[\{u\} \cup \Gamma'(u)] \cong K_{k+1}$.

Next we show that $\Gamma[\{u\} \cup \Gamma''(u)] \cong K_{k+1}$. Since $\mathrm{Aut}(\Gamma)_u$ acts 2-transitively on $\Gamma''(u)$, it suffices to show that $\Gamma''(u)$ contains two vertices that are adjacent in Γ. By way of contradiction, suppose that no two vertices of $\Gamma''(u)$ are adjacent in Γ. As Γ is edge-girth-regular (by hypothesis), we know that $c_3(\{u, v\}) \geq k - 1 > 0$ for any $v \in \Gamma'(u)$. Hence in particular, $c_3(\{u, w\}) \geq k - 1 > 0$ for all $w \in \Gamma''(u)$. Then since no two vertices of $\Gamma''(u)$ are adjacent in Γ, we have $k - 1 \leq c_3(\{u, w\}) \leq k$, and so u and w share at least $k - 1$ common neighbours in $\Gamma'(u)$. Without loss of generality, we may assume that v is a common neighbour of u and w, and then because $w \in \Gamma''(u)$, it follows that $c_3(\{u, v\}) \geq k$. But $c_3(\{u, w\}) \leq k$, so the edge-girth-regularity of Γ implies that $c_3(\{u, w\}) = k$, and therefore w is adjacent to all vertices in $\Gamma'(u)$. Then since w was an arbitrary vertex in $\Gamma''(u)$, this shows that every vertex in $\Gamma'(u)$ is adjacent to every vertex in $\Gamma''(u)$. That, however, implies that $c_3(\{u, v\}) = 2k - 1 > k = c_3(\{u, w\})$, which is a contradiction, allowing us conclude that $\Gamma[\{u\} \cup \Gamma''(u)] \cong K_{k+1}$.

Now we shall show that both Γ' and Γ'' are unions of cliques isomorphic to K_{k+1}.

If $k = 2$, we can do this by showing that $c_3(\{u, v\}) = 1$ for every $v \in \Gamma'(u)$ and $c_3(\{u, w\}) = 1$ for every $w \in \Gamma''(u)$. First, if $c_3(\{u, v\}) = 3$ then $\Gamma \cong K_5$, which is arc-transitive, a contradiction. Second, if $c_3(\{u, v\}) = 2$, then there are two exactly parallel edges in Γ between $\Gamma'(u)$ and $\Gamma''(u)$, and v has a unique neighbour, say w, which is not adjacent to u. Moreover, again since $c_3(\{v, w\}) = 2$, we find that w shares at least three common neighbours with u. If there exists a vertex $x \in \Gamma(u)$ which is not adjacent to w, then x has a neighbour, say y, which is not adjacent to u, but then y would share at least three common neighbours with u, which is impossible because Γ has valency 4. Thus $\Gamma(w) = \Gamma(u)$, and Γ is the octahedron, which again is arc-transitive, a contradiction. Thus, $c_3(\{u, v\}) = 1$. Next let z be the unique common neighbour of u and v. By vertex-transitivity, some $g \in \mathrm{Aut}(\Gamma)$ takes u to v, and then $\{v, u\}$ and $\{v, z\}$ lie in the same orbit of $\mathrm{Aut}(\Gamma)$ on $E(\Gamma)$. This implies that Γ' is a union of triangles. Similarly, we have $c_3(\{u, w\}) = 1$ for any $w \in \Gamma''(u)$, and hence also Γ'' is a union of triangles.

On the other hand, if $k > 2$, then $\mathrm{Aut}(\Gamma)_u$ acts 2-transitively on $\Gamma'(u)$ and on $\Gamma''(u)$, and also $E(\Gamma[\Gamma'(u)])$ is contained in one orbit of $\mathrm{Aut}(\Gamma)$ on $E(\Gamma)$, while $E(\Gamma[\Gamma''(u)])$ is contained in the other. As $k > 2$, there exists at least one triangle whose edges are contained in the same orbits of $\mathrm{Aut}(\Gamma)$ on $E(\Gamma)$. By vertex-transitivity of Γ, there exists a triangle Δ passing through u whose edges are contained in the same orbit of $\mathrm{Aut}(\Gamma)$ on

$E(\Gamma)$. Without loss of generality, we may assume that Δ is contained in $\{u\} \cup \Gamma'(u)$, and then all edges of $\Gamma[\{u\} \cup \Gamma'(u)]$ are contained in E'. It now follows from Lemma 2.1 that $\Gamma' = (V(\Gamma), E')$ has valency k, and so Γ' is a union of cliques isomorphic to K_{k+1}.

Next recall that $\Gamma[\{u\} \cup \Gamma''(u)] \cong K_{k+1}$, and $E(\Gamma[\Gamma''(u)]) \subseteq E''$ or $E(\Gamma[\Gamma''(u)]) \subseteq E'$. If $E(\Gamma[\Gamma''(u)]) \subseteq E''$, then because Γ'' has valency k, we find that Γ'' is a union of copies of K_{k+1}, as required. So suppose that $E(\Gamma[\Gamma''(u)]) \subseteq E'$. We will show this case is impossible. As Γ' is a union of copies of K_{k+1}, there exists $w \in V(\Gamma)$ for which $\Gamma'[\{w\} \cup \Gamma''(u)] \cong K_{k+1}$, and so $\Gamma''(u) = \Gamma'(w)$. By vertex-transitivity of Γ, for any $x \in V(\Gamma)$ there exists a unique $y \in V(\Gamma)$ such that $\Gamma'(x) = \Gamma''(y)$, and consequently $c_3(e) \geq k$ for every edge e of Γ. Hence for each $x \in \Gamma'(u)$, there exists a unique $y \in \Gamma''(u)$ such that $\Gamma'(x) = \Gamma''(y)$. Moreover, because $\Gamma'(u)$ and $\Gamma''(u)$ are two orbits of $\mathrm{Aut}(\Gamma)_u$, each $x \in \Gamma'(u)$ is adjacent (in Γ'') to $k-1$ vertices in $\Gamma''(u)$. Now take $x \in \Gamma'(u)$. Then there exists a unique $z \in \Gamma(x)$ which is not adjacent to u in Γ. Clearly $x \in \Gamma''(z)$, so $\Gamma''(z) = \Gamma'(u)$. Also $|\Gamma''(x) \cap \Gamma''(u)| = k-1$ and $\Gamma[\Gamma''(x)] \cong K_k$, and it follows that z is adjacent in Γ' to $k-1$ vertices of $\Gamma''(u)$. But now if $a \in \Gamma''(u)$ is adjacent in Γ' to z, then $\{z\} \cup \{w\} \cup (\Gamma''(u) \backslash \{a\}) \subseteq \Gamma'(a)$, and clearly $z \neq w$ because w is not adjacent in Γ' to x, and therefore a has at least $k+1$ neighbours in Γ', which is impossible because Γ' has valency k.

Hence we know that both Γ' and Γ'' are unions of copies of K_{k+1}.

Furthermore, $\{u\} \cup \Gamma'(u)$ and $\{u\} \cup \Gamma''(u)$ are two blocks of imprimitivity of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$. As $E = E' \cup E''$, there exist no edges between $\Gamma'(u)$ and $\Gamma''(u)$ in Γ.

Now by a theorem of Krausz (see [14]), we know that a graph is a line graph if and only if its edge-set can be partitioned into cliques such that every vertex is contained in at most two cliques. In our context, let Π be the graph whose vertex set is the set of all cliques K_{k+1} of Γ, with two such cliques being adjacent if and only if they share a common vertex in Γ. It is easy to see that Γ is isomorphic to the line graph of Π.

If we view $\mathrm{Aut}(\Gamma)$ as a permutation group on $V(\Pi)$, then $\mathrm{Aut}(\Pi) \cong \mathrm{Aut}(\Gamma)$, and as $\mathrm{Aut}(\Gamma)$ acts 2-arc-transitively on both Γ' and Γ'', we see that $\mathrm{Aut}(\Gamma)$ has exactly two orbits on $V(\Pi)$. Also there are exactly two copies of K_{k+1} containing any given vertex of Γ, and it follows that $\mathrm{Aut}(\Gamma)$ is edge-transitive on Π, and hence Π is semi-symmetric.

Moreover, $\mathrm{Aut}(\Gamma)_u$ acts 2-transitively on $\Gamma'(u)$, and so the subgroup H of $\mathrm{Aut}(\Gamma)$ preserving the clique $\Gamma[\{u\} \cup \Gamma'(u)]$ set-wise acts 3-transitively on $\{u\} \cup \Gamma'(u)$, and hence H acts 3-transitively on the neighbourhood of the clique $\Gamma[\{u\} \cup \Gamma'(u)]$ in Π. Similarly, $\mathrm{Aut}(\Gamma)_{\{u\} \cup \Gamma''(u)}$ acts 3-transitively on the neighbourhood of the clique $\Gamma[\{u\} \cup \Gamma''(u)]$ in Π, and this establishes necessity in the statement of Theorem 1.1. $\Box$

3 Proof of Theorem 1.2

Lemma 3.1 *Let Γ be a connected edge-girth-regular locally bi-2-transitive graph of girth 3. Let E_1, E_2 be the two orbits of $\mathrm{Aut}(\Gamma)$ on $E(\Gamma)$, and let $\{u, x\} \in E_1$ and $\{u, y\} \in E_2$. Then for every $k \geq 4$,*

$$|\bigcup_{\{x,a\} \in E_2} C_k(uxa)| = |\bigcup_{\{y,b\} \in E_1} C_k(uyb)|,$$

where $C_k(uxa)$ and $C_k(uyb)$ are the sets of k-cycles of Γ passing through the 2-paths uxa and uyb, respectively.

Proof First we note that by Lemma 2.1, the graph Γ has valency $2d$ for some $d > 1$, and then by Theorem 1.1, we know that Γ is the line graph of a semi-symmetric locally 3-transitive graph Δ of valency $d+1$. Moreover, by the proof of Theorem 1.1, the edge set of Γ can be partitioned into edge-disjoint copies of K_{d+1}, in such a way that every vertex of Γ is contained in exactly two of these cliques.

Now let X and Y be disjoint subsets of $\Gamma(u)$ whose union is $\Gamma(u)$ and having the property that $\Gamma[\{u\} \cup X] \cong \Gamma[\{u\} \cup Y] \cong K_{d+1}$. From the last paragraph in the proof of Theorem 1.1, we know that X and Y are two orbits of $\mathrm{Aut}(\Gamma)_u$. Also let

$$\Gamma' = \bigcup_{g \in \mathrm{Aut}(\Gamma)} \Gamma[\{u^g\} \cup X^g] \quad \text{and} \quad \Gamma'' = \bigcup_{g \in \mathrm{Aut}(\Gamma)} \Gamma[\{u^g\} \cup Y^g].$$

As Γ is locally bi-2-transitive, the edge sets of Γ' and Γ'' are the two equal-length orbits of $\mathrm{Aut}(\Gamma)$ on the $E(\Gamma)$, and $\mathrm{Aut}(\Gamma)_u$ acts 2-transitively on each of X and Y.

Next, let $X(u) = \{u\} \cup (X \setminus \{x\})$ and $Y(u) = \{u\} \cup (Y \setminus \{y\})$. Then $X(u)$ is an orbit of $\mathrm{Aut}(\Gamma)_x$ on $\Gamma(x)$, and $Y(u)$ is an orbit of $\mathrm{Aut}(\Gamma)_y$ on $\Gamma(y)$.

Let A be the orbit of $\mathrm{Aut}(\Gamma)_x$ on $\Gamma(x)$ that is distinct from $X(u)$, and let B be the orbit of $\mathrm{Aut}(\Gamma)_y$ on $\Gamma(y)$ that is distinct from $Y(u)$.

In order to prove our lemma, it suffices to show that

$$|\bigcup_{a \in A} C_k(uxa)| = |\bigcup_{b \in B} C_k(uyb)|.$$

By vertex-transitivity of Γ, there exists an automorphism δ' of Γ taking x to y. This automorphism δ' takes every clique K_{d+1} containing x to a clique K_{d+1} containing y, and so δ' takes the two orbits of $\mathrm{Aut}(\Gamma)_x$ on $\Gamma(x)$ to the two orbits of $\mathrm{Aut}(\Gamma)_y$ on $\Gamma(y)$, in some order; that is, $\{X(u), A\}^{\delta'} = \{Y(u), B\}$. But the edges of $\Gamma[X(u)]$ and $\Gamma[Y(u)]$ are contained in different orbits of $\mathrm{Aut}(\Gamma)$ on the edge set of Γ, and so it follows that δ' takes $X(u)$ to B, and A to $Y(u)$. Hence for every $a \in A$, we find that

$$(uxa)^{\delta'} = u'yb' \ \text{ where } u' \in Y(u) \text{ and } b' \in B.$$

Also we know that $\mathrm{Aut}(\Gamma)_y$ acts 2-transitively on both $Y(u)$ and B, and so there exists an automorphism $\alpha' \in \mathrm{Aut}(\Gamma)_y$ taking u' to u, and then $(uxa)^{\delta'\alpha'} = uy(b')^{\alpha'}$. Hence for every $a \in A$, there exists some $\delta \in \mathrm{Aut}(\Gamma)$ such that

$$(uxa)^{\delta} = uyb \text{ where } b \in B.$$

By a similar argument, for every $b' \in B$ there exists some $\sigma \in \mathrm{Aut}(\Gamma)$ such that

$$(uyb')^{\sigma} = uxa' \text{ where } a' \in A.$$

Now let a_1 and a_2 be two vertices that lie in different orbits of $\mathrm{Aut}(\Gamma)_{ux}$ on A.

The above argument shows that there exist $\delta_1, \delta_2 \in \mathrm{Aut}(\Gamma)$ and $b_1, b_2 \in B$ such that

$$(uxa_1)^{\delta_1} = uyb_1 \quad \text{and} \quad (uxa_2)^{\delta_2} = uyb_2.$$

If there exists $\sigma \in \mathrm{Aut}(\Gamma)_{uy}$ taking b_1 to b_2, then $(uxa_1)^{\delta_1\sigma} = (uxa_2)^{\delta_2}$ and it follows that $(uxa_1)^{\delta_1\sigma\delta_2^{-1}} = uxa_2$, and so $\delta_1\sigma\delta_2^{-1}$ fixes x, but then since $\mathrm{Aut}(\Gamma)_x$ preserves both $X(u)$ and A set-wise, we find that $\delta_1\sigma\delta_2^{-1}$ fixes u and takes a_1 to a_2. This, however, contradicts the assumption that a_1 and a_2 lie in different orbits of $\mathrm{Aut}(\Gamma)_{ux}$ on A. Thus b_1 and b_2 must lie in different orbits of $\mathrm{Aut}(\Gamma)_{uy}$ on B.

Similarly, if b'_1 and b'_2 are two vertices that lie in different orbits of $\mathrm{Aut}(\Gamma)_{uy}$ on B, then there exist $\sigma_1, \sigma_2 \in \mathrm{Aut}(\Gamma)$ and $a'_1, a'_2 \in A$ such that

$$(uyb'_1)^{\sigma_1} = uxa'_1 \quad \text{and} \quad (uyb'_2)^{\sigma_2} = uxa'_2,$$

and it follows that a'_1 and a'_2 lie in different orbits of $\mathrm{Aut}(\Gamma)_{ux}$ on A.

We summarise the observations just made, as follows:

Fact 1. Let $A_1, A_2, ..., A_t$ be the distinct orbits of $\mathrm{Aut}(\Gamma)_{ux}$ on A, with representatives $a_i \in A_i$ for $1 \le i \le t$. Then for $1 \le i \le t$, there exists $\delta_i \in \mathrm{Aut}(\Gamma)_{ux}$ such that $(uxa_i)^{\delta_i} = uyb_i$, and moreover, the vertices $b_1, b_2, \ldots, b_t$ are representatives of the t distinct orbits $B_1, B_2, \ldots, B_t$ of $\mathrm{Aut}(\Gamma)_{uy}$ on B, with $b_i \in B_i$ for $1 \le i \le t$.

Before proceeding, note that for any two paths of the same length in Γ, if there exists an automorphism of Γ sending one to the other, then the numbers of k-cycles passing through these two paths are equal.

By Fact 1, we have

$$|\bigcup_{a\in A} C_k(uxa)| = \sum_{i=1}^{t} |A_i||C_k(uxa_i)|$$

and

$$|\bigcup_{b\in B} C_k(uyb)| = \sum_{i=1}^{t} |B_i||C_k(uyb_i)|.$$

If $\mathrm{Aut}(\Gamma)_{ux}$ is transitive on A, then $t = 1$ and we have

$$|\bigcup_{a\in A} C_k(uxa)| = |A||C_k(uxa)| = |B||C_k(uyb)| = |\bigcup_{b\in B} C_k(uyb)|,$$

as required, and similarly, if $\mathrm{Aut}(\Gamma)_{uy}$ is transitive on B, then we have

$$|\bigcup_{a\in A} C_k(uxa)| = |\bigcup_{b\in B} C_k(uyb)|,$$

as required.

In what follows, assume that $t > 1$, so that $\mathrm{Aut}(\Gamma)_{ux}$ is intransitive on A, and $\mathrm{Aut}(\Gamma)_{uy}$ is intransitive on B. For any subset D of $V(\Gamma)$, let $\mathrm{Aut}(\Gamma)_{(D)}$ be the subgroup of $\mathrm{Aut}(\Gamma)$ fixing D point-wise. Then $\mathrm{Aut}(\Gamma)_{(X(u))} \trianglelefteq \mathrm{Aut}(\Gamma)_x$ and $\mathrm{Aut}(\Gamma)_{(X(u))} \le \mathrm{Aut}(\Gamma)_{ux}$. Also because $\mathrm{Aut}(\Gamma)_x$ acts 2-transitively on A, we find that $\mathrm{Aut}(\Gamma)_{(X(u))}$ acts trivially on A and so $\mathrm{Aut}(\Gamma)_{(X(u))} \leq \mathrm{Aut}(\Gamma)_{(A)}$.

By Fact 1, for every $b \in B$ there exists $\sigma \in \mathrm{Aut}(\Gamma)$ such that $(uyb)^\sigma = uxa$ for some $a \in A$, and then $(u,y,b)^\sigma = (a,x,u)$, and so $(\mathrm{Aut}(\Gamma)_{uy})^\sigma = \mathrm{Aut}(\Gamma)_{ax}$ is intransitive on $B^\sigma = X(u)$. Now 2-transitivity of $\mathrm{Aut}(\Gamma)_x$ on $X(u)$ implies that $\mathrm{Aut}(\Gamma)_{(A)} \leq \mathrm{Aut}(\Gamma)_{(X(u))}$, and so $\mathrm{Aut}(\Gamma)_{(A)} = \mathrm{Aut}(\Gamma)_{(X(u))}$.

Next, let G be the set-wise stabiliser in $\mathrm{Aut}(\Gamma)$ of $\{u\} \cup X$ set-wise, and H set-wise stabiliser in $\mathrm{Aut}(\Gamma)$ of $\{x\} \cup A$. Then $\mathrm{Aut}(\Gamma)_{(X(u))}$ is normal in each of G and H. Note also that G and H are stabilisers of the two end-vertices of an edge of the graph Δ (of which Γ is the line graph), and so $\mathrm{Aut}(\Gamma) = \langle G, H\rangle$, and since Δ is edge-transitive, $\mathrm{Aut}(\Gamma)_{(X(u))} = 1$.

Thus, $\mathrm{Aut}(\Gamma)_x$ acts faithfully on both $X(u)$ and A. Similarly, $\mathrm{Aut}(\Gamma)_y$ acts faithfully on both $Y(u)$ and B. We now make a second key observation, as follows:

Fact 2. Assume that $\mathrm{Aut}(\Gamma)_{ux}$ has two orbits on A, say A_1 and A_2. If there exists a vertex $a_1 \in A_1$ such that $\mathrm{Aut}(\Gamma)_{a_1x}$ has also two orbits on $X(u)$, say X_1 and X_2, with $u \in X_1$ and $|X_1| = |A_1|$, then $|\bigcup_{a\in A} C_k(uxa)| = |\bigcup_{b\in B} C_k(uyb)|$.

To see this, note that by Fact 1 there exists $\delta \in \mathrm{Aut}(\Gamma)$ taking (u,x,a_1) to (b_1,y,u) for some $b_1 \in B$, with $b_1 = u^\delta \in X_1^\delta$, and then X_1^δ and X_2^δ are two orbits of $\mathrm{Aut}(\Gamma)_{uy} = (\mathrm{Aut}(\Gamma)_{a_1x})^\delta$ on B. Clearly $|X_1^\delta| = |X_1| = |A_1|$, and $|X_2^\delta| = |X_2| = |A_2| = d - |A_1|$. Next let $a_2 \in A_2$. Then by Fact 1 there exists $\delta' \in \mathrm{Aut}(\Gamma)$ such that $(u,x,a_2)^{\delta'} = (b_2,y,u)$ for some $b_2 \in X_2^\delta$, and it follows that

$$\begin{aligned}
|\textstyle\bigcup_{a\in A} C_k(uxa)| &= |A_1||C_k(uxa_1)| + |A_2||C_k(uxa_2)| \\
&= |A_1||C_k(uxa_1)| + (d - |A_1|)|C_k(uxa_2)| \\
&= |X_1^\delta||C_k(uyb_1)| + |X_2^\delta||C_k(uyb_2)| \\
&= |\textstyle\bigcup_{b\in B} C_k(uyb)|,
\end{aligned}$$

as required.

Now we are ready to finish the proof of our lemma.

Suppose the actions of $\mathrm{Aut}(\Gamma)_x$ on $X(u)$ and A are equivalent. Then there exists $a \in A$ such that $\mathrm{Aut}(\Gamma)_{ux} = \mathrm{Aut}(\Gamma)_{xa}$. Clearly, $\{a\}$ and $A\backslash\{a\}$ are two orbits of $\mathrm{Aut}(\Gamma)_{ux}$ on A, while $\{u\}$ and $X(u)\backslash\{u\}$ are two orbits of $\mathrm{Aut}(\Gamma)_{xa}$ on $X(u)$. Hence by Fact 2, we have $|\bigcup_{a\in A} C_k(uxa)| = |\bigcup_{b\in B} C_k(uyb)|$, as required.

Suppose (on the other hand) that the actions of $\mathrm{Aut}(\Gamma)_x$ on $X(u)$ and A are inequivalent. Then letting $G = \mathrm{Aut}(\Gamma)_{\{u\}\cup X}$, which acts faithfully and 3-transitively on $\{u\} \cup X$, we may deduce from [5, Theorem 5.3] and [16, Appendix 1] that G is isomorphic to one of the following permutation groups of degree $d+1$:

(a) S_{d+1} when $d \geq 3$;

(b) A_{d+1} when $d \geq 4$;

(c) $\mathrm{AGL}(n,2)$ when $d = 2^n - 1 \geq 3$;

(d) $\mathbb{Z}_2^4 : A_7$ when $d = 15$;

(e) one of the five Mathieu simple groups M_{d+1} when $d = 10, 11, 21, 22$ or 23, or M_{11} when $d = 11$, or $\mathrm{Aut}(M_{22}) \cong M_{22}.\mathbb{Z}_2$ when $d = 21$;

(f) a 3-transitive group G satisfying $\mathrm{PGL}(2,d) \leq G \leq \mathrm{P\Gamma L}(2,d)$ for some prime-power $d \geq 3$; noting that $\mathrm{PGL}(2,4) \cong \mathrm{PGL}(2,5) \cong A_5$.

In cases (a) and (b), we have $d = 6$ because the vertex stabiliser G_x has two inequivalent 2-transitive representations, but then G_{ux} is transitive on A, a contradiction which rules out these two cases.

In case (c), we have $G_x = \mathrm{SL}(n,2)$. Here we may assume that $X(u)$ and A are the set of points and the set of hyperplanes of the projective space $\mathrm{PG}(n-1,2)$, respectively. Then the hyperplanes containing u form an orbit A_1 of G_{xu} on A, while the hyperplanes not containing u form another orbit A_2 of G_{xu} on A. It is easy to see that $|A_1| = 2^{n-1} - 1$, and if $a_1 \in A_1$, then the set X_1 of points contained in hyperplane a_1 is an orbit of G_{xa_1} on $X(u)$, and the set X_2 of points not contained in a_1 is another orbit of G_{xa_1} on $X(u)$. Moreover, by a direct computation we have $|X_1| = 2^{n-1} - 1$, and hence by Fact 2, it follows immediately that $|\bigcup_{a\in A} C_k(uxa)| = |\bigcup_{b\in B} C_k(uyb)|$, as required.

In case (d), we have $G_x = A_7$. Also a computation using Magma [3] shows that G_{ux} has two orbits on A, say A_1 and A_2, with $|A_1| = 7$ and $|A_2| = 8$, and furthermore, there exists $a \in A_1$ such that G_{xa} has two orbits on $X(u)$, say X_1 and X_2, with $|X_1| = 7$ and $|X_2| = 8$, and $u \in X_1$. Again by Fact 2 it follows that $|\bigcup_{a\in A} C_k(uxa)| = |\bigcup_{b\in B} C_k(uyb)|$.

In case (e), we find that $d = 21$ because the vertex stabiliser G_x has two inequivalent 2-transitive representations, and then either $G = M_{22}$ and $G_x = \mathrm{PSL}(3,4)$, or $G = M_{22}.\mathbb{Z}_2$ and $G_x = \mathrm{PSL}(3,4).\mathbb{Z}_2$. A computation using Magma [3] shows that G_{ux} has two orbits on A, say A_1 and A_2, with $|A_1| = 16$ and $|A_2| = 5$, and there exists $a \in A_1$ such that G_{xa} has two orbits on $X(u)$, say X_1 and X_2, with $|X_1| = 16$ and $|X_2| = 5$, and $u \in X_1$. Once again by Fact 2 it follows that $|\bigcup_{a\in A} C_k(uxa)| = |\bigcup_{b\in B} C_k(uyb)|$.

Finally, in case (f), we have $\mathrm{AGL}(1,d) \leq G_x \leq \mathrm{A\Gamma L}(1,d)$, but then G_x has only one 2-transitive representation, a contradiction which rules out that case.

This completes the proof of Lemma 3.1. $\square$

Proof of Theorem 1.2 Let Γ be a connected tetravalent edge-girth-regular locally bi-2-transitive graph of girth 3. By Theorem 1.1, we know that Γ is the line graph of a semi-symmetric locally 3-transitive graph Δ of valency 3. Moreover, by the proof of Theorem 1.1, the edge set of Γ can be partitioned into edge-disjoint copies of K_3, such that every vertex of Γ is contained in exactly two of these cliques.

Now take any vertex u in $V(\Gamma)$, and let $\Gamma(u) = \{x, x', y, y'\}$ be such that $\Gamma[\{u, x, x'\}] \cong \Gamma[\{u, y, y'\}] \cong K_3$. Set $X = \{x, x'\}$ and $Y = \{y, y'\}$. From the last paragraph in the proof of Theorem 1.1, we know that X and Y are two orbits of $\mathrm{Aut}(\Gamma)_u$. Also let

$$\Gamma' = \bigcup_{g\in \mathrm{Aut}(\Gamma)} \Gamma[\{u, x, x'\}^g] \quad \text{and} \quad \Gamma'' = \bigcup_{g\in \mathrm{Aut}(\Gamma)} \Gamma[\{u, y, y'\}^g].$$

As Γ is locally bi-2-transitive, the edge sets of Γ' and Γ'' are the two equal-length orbits of $\mathrm{Aut}(\Gamma)$ on the $E(\Gamma)$, and $\mathrm{Aut}(\Gamma)_u$ acts transitively on each of X and Y.

It is easy to see that $c_k(e) = c_k(\{u, x\})$ or $c_k(\{u, y\})$ for any edge e of Γ and for $3 \leq k \leq |V(\Gamma)|$, since $E(\Gamma')$ and $E(\Gamma'')$ are the edge-orbits of $\mathrm{Aut}(\Gamma)$.

To show that Γ is cycle-regular, we will extend the cycle-count notation by letting $C_k(P)$ be the set of k-cycles of Γ containing a given path P (of length 1 or more) or single vertex $P = \{v\}$, and prove the following.

Claim: $c_k(\{u, x\}) = c_k(\{u, y\})$ and $c_k(uxx') = c_k(uyy')$ for $3 \leq k \leq |V(\Gamma)|$.

We shall prove this claim by using induction on k. It is clearly true for $k = 3$, so we may assume that $k > 3$.

Suppose $C_k(uxx') \neq \emptyset$, and let C be any k-cycle in $C_k(uxx')$. We may consider C as $suxx' \cup P$, where P is a $(k-3)$-path $x' \cdots s$ with $s \in Y$, and then $C' = sux' \cup P$ is a $(k-1)$-cycle containing the edge $\{u, x'\}$. Conversely, for any $C'' \in C_{k-1}(\{u, x'\})$, if C'' does not pass through x, then we may assume that $C'' = sux' \cup P$, where $P = x' \cdots s$ is a $(k-3)$-path with $s \in Y$ and $x \notin P$. So $C''' = suxx' \cup P$ is a k-cycle passing through the 2-path uxx'. This gives a bijection between $C_k(uxx')$ and $C_{k-1}(\{u, x'\}) \setminus (C_{k-1}(\{u, x'\}) \cap C_{k-1}(\{x\}))$.

For an arbitrary $\mathcal{C} \in (C_{k-1}(\{u, x'\}) \cap C_{k-1}(\{x\}))$, if $\mathcal{C} \notin C_{k-1}(ux'x) \cup C_{k-1}(xux')$, then we have $\mathcal{C} = C_1 \cup C_2$, where C_1 is an ℓ_1-path $x \cdots u$ and C_2 is an ℓ_2-path $x' \cdots x$ such that $V(C_1) \cap V(C_2) = \{x\}$, $\ell_1, \ell_2 \geq 2$ and $\ell_1 + \ell_2 = k - 2$. Then $C_1 x \in C_{\ell_1+1}(\{x, u\})$ and $C_2 x' \in C_{\ell_2+1}(\{x, x'\})$. On the other hand, if C is an s-cycle passing through $\{u, x\}$ and C' is a t-cycle passing through $\{x', x\}$ with $s + t = k$ and $V(C) \cap V(C') = \{x\}$, then we have $C = Qx$ and $C' = Q'x$, where Q is an $(s-1)$-path from x to u and Q' is $(t-1)$-path from x' to x. Then QQ' is a $(k-1)$-cycle belonging to $((C_{k-1}(\{u, x'\}) \cap C_{k-1}(\{x\})) \setminus (C_{k-1}(ux'x) \cup C_{k-1}(xux'))$.

It follows that

$$|C_k(uxx')| = |C_{k-1}(\{u, x'\})| - |C_{k-1}(ux'x)| - |C_{k-1}(xux')| - |U|,$$

where

$$U = \{\{C, C'\} : C \in C_s(\{u, x\}), C' \in C_t(\{x', x\}), s + t = k, V(C) \cap V(C') = \{x\}\}.$$

Note that $\mathrm{Aut}(\Gamma)_{\{u,x,x'\}}$ acts 3-transitively on $\{u, x, x'\}$. It follows that $|C_{k-1}(\{u, x'\})| = |C_{k-1}(\{u, x\})|$ and $|C_{k-1}(ux'x)| = |C_{k-1}(uxx')| = |C_{k-1}(xux')|$, giving

$$|C_k(uxx')| = |C_{k-1}(\{u, x\})| - 2|C_{k-1}(uxx')| - |U|.$$

By a similar argument to the one above, we also find that

$$|C_k(uyy')| = |C_{k-1}(\{u, y\})| - 2|C_{k-1}(uyy')| - |W|,$$

where

$$W = \{\{D, D'\} : D \in C_{s'}(\{u, y\}), D' \in C_{t'}(\{y', y\}), s' + t' = k, V(D) \cap V(D') = \{y\}\}.$$

Also we can make an inductive hypothesis that $|C_{k-1}(\{u, x\})| = |C_{k-1}(\{u, y\})|$ and $|C_{k-1}(uxx')| = |C_{k-1}(uyy')|$, and then to show that $|C_k(uxx')| = |C_k(uyy')|$, it suffices to prove that $|U| = |W|$. Since Γ is vertex-transitive, there exists $g \in \mathrm{Aut}(\Gamma)$ sending x to y. Take an arbitrary $\{C, C'\} \in U$. We may assume that $C \in C_s(\{u, x\})$ and $C' \in C_t(\{x', x\})$, where $s+t=k$ and $V(C)\cap V(C')=\{x\}$. Let $c\in V(C)\setminus\{u\}, c'\in V(C')\setminus\{x'\}$ be adjacent to x. Then $\Gamma(x)=\{c,c',u,x'\}$. Furthermore, $\{c,x\},\{c',x\}\in E(\Gamma'')$, and so $\{c,x\}^g,\{c',x\}^g\in E(\Gamma'')$. Since $x^g=y$, one has $\Gamma(y)=\Gamma(x^g)=\{c^g,(c')^g,u^g,(x')^g\}$ and $\{y,u^g\},\{y,(x')^g\}\in E(\Gamma')$. Note that $u,y'\in\Gamma(y)$ and $\{y,u\},\{y,y'\}\in E(\Gamma'')$. It follows that $\{u,y'\}=\{c^g,(c')^g\}$, and so either $C^g\in C_s(\{u,y\})$ and $(C')^g\in C_s(\{y',y\})$, or $C^g\in C_s(\{y',y\})$ and $(C')^g\in C_t(\{u,y\})$. Clearly, $V(C^g)\cap V((C')^g)=\{x^g\}=\{y\}$, so $\{C^g,(C')^g\}\in W$. This implies that g induces a map, say ϕ, from U to W. Since $g\in \mathrm{Aut}(\Gamma)$, ϕ is injective.

To see ϕ is also surjective, take an arbitrary $\{D,D'\}\in W$. We may assume that $D\in C_{s'}(\{u,y\}), D'\in C_{t'}(\{y',y\})$, where $s'+t'=k$ and $V(D)\cap V(D')=\{y\}$. Let $d\in V(D)\setminus\{u\}$ and $d'\in V(D)\setminus\{y'\}$ be adjacent to y. Then $\Gamma(y)=\{u,y',d,d'\}$. Since we already have $\Gamma(y)=\Gamma(x^g)=\{c^g,(c')^g,u^g,(x')^g\}$ and $\{u,y'\}=\{c^g,(c')^g\}$, one has $\{d,d'\}^{g^{-1}}=\{u,x'\}$. This implies that either $D^{g^{-1}}\in C_{s'}(\{u,x\})$ and $(D')^{g^{-1}}\in C_{t'}(\{x',x\})$, or $D^{g^{-1}}\in C_{s'}(\{x',x\})$ and $(D')^{g^{-1}}\in C_{t'}(\{u,x\})$. Furthermore, $V(D^{g^{-1}})\cap V((D')^{g^{-1}})=\{y^{g^{-1}}\}=\{x\}$. So $\{D^{g^{-1}},(D')^{g^{-1}}\}\in U$. Clearly, $\{D^{g^{-1}},(D')^{g^{-1}}\}^\phi=\{D,D'\}$. Thus, ϕ is a bijection between U and W, and hence $|U|=|W|$. Thus, we have shown that

$$|C_k(uxx')|=|C_k(uyy')|. \tag{1}$$

Next, let $X(u)=\{u,x'\}$ and $Y(u)=\{u,y'\}$. Then $X(u)$ is an orbit of $\mathrm{Aut}(\Gamma)_x$ on $\Gamma(x)$, and $Y(u)$ is an orbit of $\mathrm{Aut}(\Gamma)_y$ on $\Gamma(y)$.

Let $A=\{a_1,a_2\}$ be the orbit of $\mathrm{Aut}(\Gamma)_x$ on $\Gamma(x)$ that is distinct from $X(u)$, and let $B=\{b_1,b_2\}$ be the orbit of $\mathrm{Aut}(\Gamma)_y$ on $\Gamma(y)$ that is distinct from $Y(u)$.

Because $k>3$, every k-cycle of Γ containing the edge $\{u,x\}$ must contain the 2-path uxa_1, uxa_2 or uxx'. Hence we find that

$$C_k(\{u,x\})\subseteq C_k(uxa_1)\cup C_k(uxa_2)\cup C_k(uxx').$$

Also every k-cycle in $C_k(uxa_1)\cup C_k(uxa_2)\cup C_k(uxx')$ contains the edge $\{u,x\}$, and therefore

$$C_k(\{u,x\})=C_k(uxa_1)\cup C_k(uxa_2)\cup C_k(uxx'), \tag{2}$$

and similarly, we have

$$C_k(\{u,y\})=C_k(uyb_1)\cup C_k(uyb_2)\cup C_k(uyy'). \tag{3}$$

Now clearly $c_k(\{u,x\})=|C_k(\{u,x\})|=|C_k(uxa_1)\cup C_k(uxa_2)|+|C_k(uxx')|$ and $c_k(\{u,y\})=|C_k(\{u,y\})|=|C_k(uyb_1)\cup C_k(uyb_2)|+|C_k(uyy')|$. By Equation (1), we have $|C_k(uxx')|=|C_k(uyy')|$, and by Lemma 3.1, we see that

$$|C_k(uxa_1)\cup C_k(uxa_2)|=|C_k(uyb_1)\cup C_k(uyb_2)|.$$

Now by Equations (2) and (3), we obtain the proof of Theorem 1.2. □

Note. The method for proving $|U|=|W|$ does not always work for the case where Γ has valency $2d>4$. Indeed for any $\{C,C'\}\in W$, let $c\in V(C)\setminus\{u\}$ and $c'\in V(C')\setminus\{x'\}$ be vertices adjacent to x. When $2d>4$, it might happen that $\{x,c\}$ or $\{x,c'\}$ belongs to

$E(\Gamma')$, and then at least three of the four edges in $C \cup C'$ incident with x are in $E(\Gamma')$. But on the other hand, for any $\{D, D'\} \in W$, at least two edges in $D \cup D'$ incident with y are in $E(\Gamma'')$, and this would imply that the automorphism g of Γ sending x to y will not send $\{C, C'\}$ to some element in U.

4 A class of semi-symmetric locally 3-transitive graphs

In this final section, we prove Theorem 1.3 and thereby solve Problem 1, by constructing a family of semi-symmetric locally 3-transitive graphs. Our construction is based on assumptions and notation given in the following definition:

Definition 4.1

(1) *n is an integer greater than 2;*

(2) $\Omega = \{1, 2, \ldots, n, n+1, \ldots, 2n-1, 2n, \ldots, 3n-1\}$;

(3) *a, b, c, x, y and z are six permutations on Ω, defined as follows:*

$$a = (1, 2, 3, \ldots, n-1, n),$$
$$b = (1, 2, 3, \ldots, n-2, n-1),$$
$$c = (1, 2),$$
$$x = (n+1, n+2, \ldots, 2n-2, 2n-1)(2n+1, 2n+2, \ldots, 3n-2, 3n-1),$$
$$y = (n+1, n+2)(2n+1, 2n+2),$$
$$z = (n, n+1, n+2, \ldots, 2n-2, 2n-1)(2n, 2n+1, 2n+2, \ldots, 3n-2, 3n-1);$$

(4) $G = \langle a, b, c, x, y, z \rangle$, $H = \langle a, b, c, x, y \rangle$ *and* $K = \langle b, c, x, y, z \rangle$;

(5) $\Delta = \{1, 2, \ldots, n\}$, $\Pi = \{n, n+1, \ldots, 2n-1\}$ *and* $\Lambda = \{2n, 2n+1 \ldots, 3n-2, 3n-1\}$.

Before giving the construction (in Theorem 4.4 below), we make two key observations.

Observation 4.2 $G \cong \mathrm{Sym}(\Delta \cup \Pi) \times \mathrm{Sym}(\Lambda) \cong S_{2n-1} \times S_n$.

First G has two orbits on Ω, namely $\Delta \cup \Pi$ and Λ, which have lengths $2n-1$ and n, respectively. Also $\langle a, b, c \rangle = \mathrm{Sym}(\Delta)$, because the conjugates of c by elements of $\langle a, b \rangle$ include a set of transpositions that generate S_n.

Now let

$$x_1 = (n+1, n+2, \ldots, 2n-2, 2n-1),$$
$$y_1 = (n+1, n+2),$$
$$z_1 = (n, n+1, n+2, \ldots, 2n-2, 2n-1),$$
$$x_2 = (2n+1, 2n+2, \ldots, 3n-2, 3n-1),$$
$$y_2 = (2n+1, 2n+2),$$
$$z_2 = (2n, 2n+1, 2n+2, \ldots, 3n-2, 3n-1).$$

Then $\langle x_1, y_1, z_1 \rangle = \mathrm{Sym}(\Pi)$, $\langle x_2, y_2, z_2 \rangle = \mathrm{Sym}(\Lambda)$, and $\langle a, b, c, x_1, y_1, z_1 \rangle = \mathrm{Sym}(\Delta \cup \Pi)$, by similar arguments. Also because $x = x_1 x_2$, $y = y_1 y_2$ and $z = z_1 z_2$, it follows that

$G = \langle a,b,c,x,y,z\rangle$ induces a 2-transitive group on $\Delta \cup \Pi$, and moreover, G induces $\mathrm{Sym}(\Delta\cup\Pi)$ on $\Delta\cup\Pi$ because $(1,2) = c \in G$. In particular, some element $g \in G$ induces the permutation $(1,2,3,\dots,n,n+1,\dots,2n-1)$ on $\Delta\cup\Pi$.

Next, G contains $c^{g^{i-1}} = (1,2)^{g^{i-1}} = (i,i+1)$ for $1 \le i \le 2n-2$, and hence G actually contains $\mathrm{Sym}(\Delta\cup\Pi)$. In particular, G contains x_1, y_1, z_1, and so also contains $x_2 = x_1^{-1}x$, $y_2 = y_1^{-1}y$ and $z_2 = z_1^{-1}z$, and therefore G contains $\langle x_2,y_2,z_2\rangle = \mathrm{Sym}(\Lambda)$ as well. Thus $G \cong \mathrm{Sym}(\Delta\cup\Pi)\times\mathrm{Sym}(\Lambda)$, as claimed. □

Observation 4.3 *If $n > 5$, then every automorphism of G that preserves $H\cap K$ is an inner automorphism of G induced by an element of $H\cap K$; that is, if $\alpha\in\mathrm{Aut}(G)$ satisfies $(H\cap K)^\alpha = H\cap K$, then there exists $g \in H\cap K$ such that $\alpha : u \mapsto g^{-1}ug$ for all $u \in G$.*

To justify this, we first note that $G \cong M\times N$ where $M = \mathrm{Sym}(\Delta\cup\Pi)\cong S_{2n-1}$ and $N = \mathrm{Sym}(\Lambda)\cong S_n$, by Observation 4.2, and furthermore,

$M = \langle a,b,c,x_1,y_1,z_1\rangle \cong S_{2n-1}$, in its single-orbit action on $\Delta\cup\Pi = \{1,2,\dots,2n-1\}$,
$N = \langle x_2,y_2,z_2\rangle \cong S_n$, in its single-orbit action on $\Lambda = \{2n, 2n+1\dots,3n-2,3n-1\}$,
$H = \langle a,b,c\rangle\times\langle x,y\rangle \cong S_n\times S_{n-1}$, in its 3-orbit action on $\Omega = \{1,2,\dots,3n-2,3n-1\}$,
$K = \langle b,c\rangle\times\langle x,y,z\rangle \cong S_{n-1}\times S_n$, in its 3-orbit action on $\Omega\backslash\{1\}$, and
$H\cap K = \langle b,c\rangle\times\langle x,y\rangle \cong S_{n-1}\times S_{n-1}$, in its 3-orbit action on $\Omega\backslash\{n,2n\}$,

noting that the effect of each of x, y and z on $\Lambda = \{2n,2n+1\dots,3n-2,3n-1\}$ is analogous to its effect on $\Pi = \{n,n+1,\dots,2n-1\}$, in that if it takes $n+j$ to $n+k$ in Π, then it takes $2n+j$ to $2n+k$ in Λ.

Now let D be the subgroup of $\mathrm{Aut}(G)$ preserving $H\cap K$. Since $M\cong S_{2n-1}$ and $N\cong S_n$ are characteristic subgroups of $G\cong M\times N\cong S_{2n-1}\times S_n$, we know that M and N are invariant under $\mathrm{Aut}(G)$, and it follows that $M\cap(H\cap K)$ is invariant under D. In fact $M\cap(H\cap K) = \langle a,b,c,x_1,y_1,z_1\rangle\cap\langle b,c,x,y\rangle = \langle b,c\rangle\cong S_{n-1}$ because M fixes Λ and $H\cap K$ fixes n (and the effect of each of x, y and z on Λ is analogous to its effect on Π), and so D preserves $\langle b,c\rangle$. Then since $\langle b,c\rangle\cong S_{n-1}$ has trivial centre, it follows that D also preserves $C_{H\cap K}(\langle b,c\rangle) = \langle x,y\rangle$.

Now let α be any element of D. Then since α preserves $M\cong S_{2n-1}$, it induces an inner automorphism of M, and hence its effect on M can be represented by a permutation π of $\Delta\cup\Pi = \{1,2,\dots,2n-1\}$. Also α preserves $\langle b,c\rangle$ and $\langle x,y\rangle$, and so π must preserve their non-trivial orbits $\{1,2,\dots,n-1\}$ and $\{n+1,n+2,\dots,2n-1\}$ on $\Delta\cup\Pi$, and therefore π fixes n. Moreover, as π preserves $\langle x,y\rangle\cong S_{n-1}$, we find that α must induce a permutation π' on $\Lambda = \{2n,2n+1\dots,3n-2,3n-1\}$ analogous to the one it induces on $\Pi = \{n,n+1,\dots,2n-1\}$, in that if π takes $n+j$ to $n+k$ in Π, then π' takes $2n+j$ to $2n+k$ in Λ. Hence in particular, π' must fix the point $2n$, because n and $2n$ are the fixed points of $\langle x,y\rangle$ on Π and Λ.

It follows that the automorphism α is completely determined by the effects of π and π' on the sets $\{1,2,\dots,n-1\}$ and $\{n+1,n+2,\dots,2n-1\}$, and hence by the effects of α on $\langle b,c\rangle$ and $\langle x,y\rangle$. As these are determined by inner automorphisms of $\langle b,c\rangle$ and $\langle x,y\rangle$, we find that α itself is an inner automorphism of $\langle b,c\rangle\times\langle x,y\rangle = H\cap K$, as claimed. □

We can now state and prove the following:

Theorem 4.4 *Under the notation set out in Definition* 4.1*, let* $\Gamma = \mathrm{Cos}(G, H, K)$ *be a graph with vertex set* $\{Hu : u \in G\} \cup \{Kv : v \in G\}$*, and with edges all pairs* $\{Hu, Kv\}$ *of these cosets having non-empty intersection* $Hu \cap Kv$ *in* G*. Then* Γ *is a connected semi-symmetric locally* 3*-transitive graph of valency* n*.*

Proof First, Γ is bipartite, with parts $P = \{Hu : u \in G\}$ and $Q = \{Kv : v \in G\}$, and as G is generated by $H = \langle a, b, c, x, y\rangle$ and $K = \langle b, c, x, y, z\rangle$, we see that Γ is connected. Also G acts naturally as a group of automorphisms of Γ, with P and Q as its orbits, by right multiplication on the (right) cosets of H and K, respectively. Moreover, since H and K are core-free subgroups of G (each being isomorphic to $S_n \times S_{n-1}$), the action of G is faithful on each of P and Q and hence on $V(\Gamma)$.

Next, H is adjacent to K in Γ (because $H \cap K$ contains b and hence is non-empty), and then since $H \cap K \cong S_{n-1} \times S_{n-1}$ has precisely n right cosets in $H \cong S_n \times S_{n-1}$, the neighbours of H in Γ are the n cosets of the form Kx where $x \in H$ (corresponding to the fact that $H \cap Kx = Hx \cap Kx = (H \cap K)x \neq \emptyset$). Similarly, the neighbours of K in Γ are the n cosets of the form Hy where $y \in K$. Thus Γ is regular with valency n, and moreover, each of the subgroups H and K acts transitively on its neighbourhood in Γ, and then since G acts transitively on each of its two parts, Γ is both edge-transitive and locally arc-transitive.

In fact, the stabiliser of the arc (H, K) is $H \cap K \cong S_{n-1} \times S_{n-1}$, so the action of $H \cong S_n \times S_{n-1}$ on its neighbourhood $\Gamma(H)$ is equivalent to the action of $S_n \times S_{n-1}$ on right cosets of $S_{n-1} \times S_{n-1}$, which is 3-transitive. The analogous property holds for the action of K on $\Gamma(K)$, and so Γ is locally 3-transitive.

All that remains for us to do is prove that Γ is not vertex-transitive (and is therefore semi-symmetric). This can be verified easily using Magma [3] for the cases where $n = 3, 4$ or 5, and so we may assume that $n > 5$ and that Γ is vertex-transitive. Indeed under this assumption, Γ will be 2-arc-transitive.

Now let $A = \mathrm{Aut}(\Gamma)$, let $u = H$ and $v = K$ (as vertices of Γ), let A_u^* be the subgroup of A_u fixing all the neighbours of u, and A_v^* be the subgroup of A_v fixing all the neighbours of v, and define G_u^* and G_v^* in the same way. As noted above for the actions of H and K on $\Gamma(H)$ and $\Gamma(K)$, we have $G_u/G_u^* \cong S_n \cong G_v/G_v^*$, with $G_{uv}/G_u^* \cong S_{n-1} \cong G_{uv}/G_v^*$, and in fact $G_u^* = \langle x, y\rangle \cong S_{n-1}$ and $G_v^* = \langle b, c\rangle \cong S_{n-1}$, and $G_u^* \cap G_v^*$ is trivial. It then follows that also $A_u/A_u^* \cong S_n \cong A_v/A_v^*$ and $A_{uv}/A_u^* \cong S_{n-1} \cong A_{uv}/A_v^*$, since Γ has valency n.

Next, as G_v^* fixes $u \in \Gamma(v)$, we find that $G_v^* \cap A_u^* \leq G_u \cap A_u^* \leq G_u^*$, which implies that $G_u^* = G_u^*(G_v^* \cap A_u^*) = G_u^* G_v^* \cap A_u^*$, and so by the Second Group Homomorphism Theorem,

$$G_v^* G_u^*/G_u^* = G_v^* G_u^*/(G_u^* G_v^* \cap A_u^*) \cong (G_v^* G_u^*)A_u^*/A_u^* = G_v^*(G_u^* A_u^*)/A_u^* \leq A_v^* A_u^*/A_u^* \leq A_{uv}/A_u^*.$$

On the left-hand, we have $G_v^* G_u^*/G_u^* \cong G_u^*/(G_u^* \cap G_v^*) \cong G_u^* \cong S_{n-1}$, while at the right-hand, we have $A_{uv}/A_u^* \cong S_{n-1}$, and hence the inequalities are equalities. Thus $A_u^*/A_{uv}^* = A_u^*/(A_u^* \cap A_v^*) \cong A_v^* A_u^*/A_u^* \cong S_{n-1}$, and by the analogous argument, also $A_v^*/A_{uv}^* \cong S_{n-1}$.

Furthermore, by the Thompson-Wielandt theorem described in [13, 26, 30] (for example), we know that $A^*_{uv} = A^*_u \cap A^*_v$ is a p-group for some prime p, and as the quotients A^*_u/A^*_{uv} and A^*_v/A^*_{uv} are isomorphic to S_{n-1} and hence almost-simple, A^*_{uv} is the unique maximal normal p-subgroup of each of A^*_v and A^*_u, and therefore characteristic in both of them, and hence is normal in each of A_u and A_v. But $\langle A_u, A_v\rangle$ contains $\langle G_u, G_v\rangle = \langle H, K\rangle = G$ and so $\langle A_u, A_v\rangle$ is transitive on the edges of Γ, and it follows that the normal subgroup A^*_{uv} of $\langle A_u, A_v\rangle$ is trivial.

Thus $A^*_u \cong S_{n-1} \cong A^*_v$, from which it follows that $|A_u| = |S_n||S_{n-1}| = |H| = |G_u|$ and similarly $|A_v| = |S_n||S_{n-1}| = |K| = |G_v|$, so $A_u = G_u$ and $A_v = G_v$. Then since G has two orbits on $V(\Gamma)$ while A has just one, we find that $|A : G| = 2$, and in particular, G is normal in A. Moreover, because Γ is arc-transitive, there exists some $t \in A \setminus G$ such that t interchanges u and v, and then conjugation by t gives an automorphism α of G that interchanges $H = G_u$ with $G_v = K$. This automorphism of G preserves $G_{uv} = H \cap K$, and so by Observation 4.3, we find that α induces an inner automorphism of $H \cap K = G_{uv}$, the same as conjugation by some element $g \in G_{uv}$. But then it follows that $K = G_v = G^\alpha_u = G^g_u = G_u = H$ (because $g \in G_{uv} \le G_u$), a contradiction.

Hence Γ cannot be vertex-transitive, and is therefore semi-symmetric, as required. □

Based on the construction of this family of graphs, we can now prove Theorem 1.3, and hence solve the problem posed by Fouquet and Hahn in 2001.

Proof of Theorem 1.3

Let n be any integer ≥ 3, and let Γ be the semi-symmetric locally 3-transitive graph of valency n given in Theorem 4.4, with $\mathrm{Aut}(\Gamma) \cong S_{2n-1} \times S_n$.

Next let p be any prime > 81. Then by [9, Theorem 2.11] (see also [2, 17, 18]), there exists a connected covering graph Σ of Γ such that $\mathrm{Aut}(\Sigma)$ has an edge-transitive subgroup X satisfying the following conditions:

(a) the subgroup X has an elementary abelian normal p-subgroup N which acts semi-regularly on $V(\Sigma)$ and has order $p^{\beta(\Gamma)}$, where $\beta(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$ (the Betti number of Γ);

(b) the graph Γ is isomorphic to the quotient graph Σ_N, the vertices of which are the orbits of N on $V(\Sigma)$, with two such orbits adjacent in Σ_N whenever there exists an edge in Σ between a pair of vertices lying in those two orbits;

(c) $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\Sigma_N) = X/N$.

For notational convenience, write $\overline{X} = X/N$, and $\overline{g} = gN \in \overline{X}$ for any $g \in X$, and also denote by $\overline{v}$ the vertex of Σ_N representing the orbit v^N of any vertex $v \in V(\Sigma)$. Then the neighbourhood $\Sigma_N(\overline{v})$ of $\overline{v}$ consists of all the vertices $\overline{w}$ representing some $w \in \Sigma(v)$, and because N is semi-regular on $V(\Sigma)$, the stabiliser $\overline{X}_{\overline{v}}$ in $\overline{X}$ of $\overline{v}$ is X_vN/N, which is isomorphic to X_v.

Now let ϕ be the isomorphism from X_v to $\overline{X}_{\overline{v}}$, given by $\phi :\ g \mapsto \overline{g}$ for $g \in X_v$. If we label the vertices in $\Sigma(v)$ as $w_1, w_2, \ldots, w_k$, say, then for any $w_i, w_j \in \Sigma(v)$ and $g \in X_v$, we see that $w_i^g = w_j$ if and only if $\overline{w}_i^{\overline{g}} = (w_i^N)^{gN} = w_i^{gg^{-1}NgN} = (w_i^g)^{NN} = w_j^N = \overline{w}_j$, and so the action of $\overline{X}_{\overline{v}}$ on $\Sigma_N(\overline{v})$ is permutationally isomorphic to the action of X_v on $\Sigma(v)$. Thus $X_v \cong X_vN/N \cong S_n \times S_{n-1}$, and it follows that Σ is locally 3-transitive.

Hence to complete the proof, all we have to do is show that Σ is semisymmetric.

So assume to the contrary that Σ is vertex-transitive, and therefore arc-transitive. Also let $A = \mathrm{Aut}(\Sigma)$, let $\{u, v\}$ be any edge of Σ, and for any subgroup L of A and any vertex w of Σ, let L_w^* be the the subgroup of L_w fixing the neighbourhood $\Sigma(w)$ of w point-wise.

By equations (2) and (3) in Section 3 and the argument in the last four paragraphs of our proof of Theorem 4.4 above, we find that

$$A_u/A_u^* \cong A_v/A_v^* \cong S_n \quad \text{and} \quad A_u^*A_v^*/A_u^* \cong A_u^*A_v^*/A_v^* \cong S_{n-1},$$

and also by the Thompson-Wielandt theorem (as mentioned in [13, 26, 30]), we know that $A_u^* \cap A_v^*$ is a q-group for some prime q.

Now if $n > 5$, then $A_u^*/(A_u^* \cap A_v^*) \cong A_u^*A_v^*/A_v^* \cong S_{n-1}$ is almost simple, and so $A_u^* \cap A_v^*$ is a unique maximal normal q-subgroup of A_u^*, and therefore characteristic in A_u^* and so normal in A_u. The same argument shows that $A_u^* \cap A_v^*$ is normal in A_v, and so $A_u^* \cap A_v^*$ is normal in $\langle A_u, A_v\rangle$, and then since $\langle A_u, A_v\rangle$ is transitive on the edges of Σ, we find that $A_u^* \cap A_v^* = 1$. Similarly, because $X/N \cong \mathrm{Aut}(\Gamma) \cong S_{2n-1} \times S_n$, which is insoluble, the p-subgroup N is the unique maximal normal p-subgroup of X, and so N is characteristic in X. Next, because $X_v \cong X_u \cong S_n \times S_{n-1}$, it follows that $A_u = X_u$ and $A_v = X_v$, and therefore $X = \langle X_u, X_v\rangle = \langle A_u, A_v\rangle$, which is a normal subgroup of A with index 2. Hence N is normal in A, but that makes A/N a vertex-transitive subgroup of $\mathrm{Aut}(\Sigma_N)$, and so $\Gamma \cong \Sigma_N$ is vertex-transitive, contradiction.

Thus $3 \le n \le 5$. Here we need some other information before we can proceed along the same lines. First note that $X_v \le A_v$, and that $X_v \cong S_n \times S_{n-1}$ as above. If $n = 3$, then by Tutte's theory of arc-transitive cubic graphs [28, 29], we find that $|A_v|$ divides $2^4 \cdot 3 = 48$ and then $|A_v : X_v|$ divides 4. Similarly, if $n = 4$, then $|A_v|$ divides $3^6 \cdot 2^4$ (by [20, Theorem 4]), and then $|A_v : X_v|$ divides $3^4 = 81$, while if $n = 5$, then $|A_v|$ divides $2^9 \cdot 3^2 \cdot 5$ and then $|A_v : X_v|$ divides $2^3 = 8$ (by [19, Table 2]). But we know that $|\langle A_u, A_v\rangle| = \frac{1}{2}|V(\Gamma)||A_v|$ (since Σ is bipartite), and $|X| = \frac{1}{2}|V(\Gamma)||X_v|$, and it follows that $|\langle A_u, A_v\rangle : X|$ divides either 8 or 81. Moreover, $X/N \cong \mathrm{Aut}(\Gamma) \cong S_{2n-1} \times S_n$ (the order of which is divisible only by primes ≤ 7), and so N is a characteristic p-subgroup of X. Hence the index of the normaliser of N in $\langle A_u, A_v\rangle$ divides $|\langle A_u, A_v\rangle : X|$, and so cannot be greater than 81, but $p > 81$, and therefore by Sylow theory N is a normal Sylow p-subgroup of $\langle A_u, A_v\rangle$. Again it now follows that N is characteristic in $\langle A_u, A_v\rangle$ and hence normal in A, which leads to the same contradiction as in the case $n > 5$. □

Final note: One of the referees of this paper kindly suggested two alternative ways to prove Theorem 1.3, and we summarise these as follows.

For one way, by [4, Corollary 3], we know that there is a semisymmetric locally 3-transitive graph Υ with valency n for every $n \ge 3$. The stabiliser in $\mathrm{Aut}(\Upsilon)$ of a vertex v of Υ acts as the full symmetric group on the neighbourhood $\Upsilon(v)$, and as Υ is 2-arc-transitive, the order of this vertex-stabiliser is bounded by a function of n (see [32]). Using this fact and a similar argument to the one in our proof of Theorem 1.3, we see that for every large enough prime p, there exists a semi-symmetric locally 3-transitive p^b-fold regular cover $\tilde{\Upsilon}$ of Γ, where $b = |E(\Upsilon)| - |V(\Upsilon)| + 1$.

For the second way, take any regular bipartite graph Γ admitting an edge-transitve and locally 3-transitive but not vertex-transitive group G. (For example, let Γ be the complete bipartite graph $K_{n,n}$, and take $G = S_n \times S_n$.) By Theorem 6 of [22], for every prime $p \geq 3$ there exists a q-fold regular cover $\tilde{\Gamma}$ of Γ for some power q of p, such that the maximal lifted group of automorphisms of Γ is G. In this case, the lift $\tilde{G} = P.G$ of G (with P being a group of order q) acts as a non-vertex-transitive locally 3-transitive group on the covering graph $\tilde{\Gamma}$. Again, using the bound on the order of the vertex-stabiliser in 2-arc-transitive graphs (see [32]) and the Sylow theorem (as in the proof of Theorem 1.3), one can see that for every large enough prime p, the covering graph $\tilde{\Gamma}$ will be semi-symmetric and locally 3-transitive.

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