Dirichlet-to-Neumann and elliptic operators on exterior Lipschitz domains

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Abstract

Via form methods we investigate the Dirichlet-to-Neumann operator $\mathcal{N}$ associated with a uniformly elliptic pure second-order operator on an exterior domain Ω with Lipschitz boundary Γ. We consider two versions of the Dirichlet-to-Neumann operator and a variational problem on Ω associated with each case. We prove that for bounded data, solutions of the variational problem are continuous on $\overline{\Omega}$ and decay at infinity. We then characterise the Dirichlet-to-Neumann operator $\mathcal{N}$ in terms of a j-elliptic sesquilinear form and establish that $-\mathcal{N}$ generates an asymptotically stable submarkovian holomorphic C_0-semigroup on $L_2(\Gamma)$ that leaves $C(\Gamma)$ invariant. Finally we prove that the associated heat kernel is jointly continuous on $\Sigma_\theta \times \Gamma \times \Gamma$, satisfies uniform bounds in complex time and converges uniformly on $\Gamma \times \Gamma$ to an equilibrium, where $\Sigma_\theta \subset \mathbb{C}$ is an open sector of angle $\theta \in (0, \frac{\pi}{2})$.

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1 Introduction

Consider a smooth function φ on the boundary Γ of a bounded smooth domain. Then the Dirichlet problem admits a solution u and the Dirichlet-to-Neumann operator $\mathcal{N}$ maps φ to the normal derivative of u. Using form methods we recast the problem weakly and investigate $\mathcal{N}$ in a generalised setting. Our departure from the classical situation is twofold. First, we assume an exterior domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary Γ. Next, we replace the Laplacian with a uniformly elliptic operator $A = -\sum \partial_l(a_{kl}\partial_k)$ with real measurable coefficients. In order to study the resolvent of the Dirichlet-to-Neumann operator, we consider the Robin-type problem

$$\begin{aligned} Au &= 0 \quad \text{on } \Omega \\ \lambda \operatorname{Tr} u + \partial_\nu u &= \psi \quad \text{on } \Gamma \end{aligned} \tag{1}$$

with boundary data $\psi \in L_2(\Gamma)$ and $\lambda > 0$. Then it remains to specify a boundary condition at infinity in order to ensure that (1) is well-posed. We consider two possibilities, namely Dirichlet and Neumann boundary conditions at infinity. Hence we obtain two versions of the boundary value problem (1) and consequently, two realisations of the Dirichlet-to-Neumann operator. Throughout this thesis we simultaneously examine the two cases, investigating such matters as elliptic regularity, resolvent convergence, semigroup asymptotics and heat kernel bounds.

The classical Dirichlet-to-Neumann operator resides within the pseudo-differential framework, where it is equal to the difference between the square root of the Laplace–Beltrami operator and a pseudo-differential operator of order zero, defined on the boundary of a compact Riemannian manifold [Tay96] Section 12C. Recent decades have seen the development of numerous generalisations, with the Dirichlet-to-Neumann operator emerging as an object of both theoretical and practical interest (see [SU90], [Hug95], [SU98], [LU01], [GK04], [Fok05], [MS10], [CT10], [War18], [SY22] for a selection of examples). One widely-known situation in which the Dirichlet-to-Neumann operator appears is that of the Calderón problem [Cal80], an inverse problem wherein one seeks to determine the coefficients of an elliptic operator on the interior of a region using knowledge of the Dirichlet-to-Neumann operator at the boundary [SU87], [Nac88], [AP06], [Uhl09], [BR12]. Another significant line of enquiry is the relationship between the Dirichlet-to-Neumann operator and the spectral properties of the associated elliptic operator. For a singular Sturm–Liouville operator L, it is known that the spectral data of self-adjoint realisations of L in $L_2([0, \infty))$ are captured by the limiting behaviour of the associated Titchmarsh–Weyl m-function [Tit62]. This theory has been extended to the case of self-adjoint elliptic operators in $L_2(\mathbb{R}^d)$, where the Dirichlet-to-Neumann map happens to fulfil the role of the m-function [AP04], [AM07], [AM12], [BR16], [BR15], [BGH+16]. In recent times connections have also been drawn between the Dirichlet-to-Neumann operator $\mathcal{N}$ and the theory of stochastic processes, where the C_0-semigroup generated by $-\mathcal{N}$ turns out to coincide with the transition function for a Markov process whose state space is the boundary Γ of the bounded Lipschitz domain on which the Neumann or Robin problem is considered [BV17].

In [DL90] the Dirichlet-to-Neumann operator $\mathcal{N}$ was studied on $C(\Gamma)$ for a bounded C^1-domain. Positivity and analyticity of the semigroup generated by $-\mathcal{N}$ on $C(\Gamma)$ were subsequently investigated for the case where Γ is smooth [Esc94], [Eng03]. In [AE12] the classical form methods of Kato [Kat80] and Lions [Lio57] were extended so as to enable the association of an m-sectorial operator in a Hilbert space H with a sectorial form whose domain need not be contained in H. In particular, the form method presented in [AE12] yields a convenient framework for the study of $\mathcal{N}$ as an m-sectorial operator in $L_2(\Gamma)$, which has facilitated the treatment of the Dirichlet-to-Neumann operator in a variety of contexts (see [AE11], [EO14], [BE17], [AE17], [ARP19], [AE20], [EW20], [BE21], [EO22] and the references therein). In [AE15] the Dirichlet-to-Neumann operator was considered on an exterior Lipschitz domain for $A = -\Delta$. Several results extend to the case $A = -\sum \partial_l (a_{kl} \partial_k)$ with minimal modification of the arguments, demonstrating the robustness of the form method. Nevertheless, arguments relying on symmetry or smoothness of the coefficients break down in the general case, necessitating a different approach.

In our setting, the regularity theorem of Nash [Nas58] and De Giorgi [De 57] provides immediately that solutions of (1) are locally Hölder continuous on Ω. Regularity at the boundary Γ is less apparent. While the utility of Morrey and Campanato estimates in the study of elliptic operators with non-smooth coefficients is well evidenced [Cam63], [Gia83], [Aus96], [Lie03], [GM05], the application of this theory in the case of an unbounded domain Ω becomes problematic because the Morrey and Campanato spaces cease to be subspaces of $L_2(\Omega)$. Alternatively, if the boundary is too rough to satisfy the so-called inner volume condition, then the classical Morrey–Campanato theory is again rendered inapplicable. In [ER15] pointwise Morrey and Campanato seminorms were introduced in order to derive global Hölder estimates for solutions on domains with outward cusps (which fail to satisfy the inner volume condition). This technique enabled the separate treatment of boundary and interior regularity and was subsequently applied in [EW20] in order to obtain Hölder Gaussian heat kernel bounds for elliptic operators on bounded Lipschitz domains. Using a similar approach, we apply elliptic regularity and bootstrap along a scale of pointwise Morrey–Campanato seminorms in order to prove that solutions of (1) extend continuously to Γ.

Given data ψ and λ, we denote by $B_\lambda^D \psi$ and $B_\lambda \psi$ the unique solutions of (1) satisfying Dirichlet and Neumann boundary conditions at infinity, respectively.

Theorem 1.1. (a) *Let $\psi \in L_\infty(\Gamma)$ and $\lambda \geq 0$. Then $B_\lambda^D \psi \in C(\overline{\Omega})$.*

(b) *Let $\psi \in L_\infty(\Gamma)$ and $\lambda > 0$. Then $B_\lambda \psi \in C(\overline{\Omega})$.*

A typical consideration in the study of boundary value problems on unbounded domains is the behaviour of solutions at infinity [Mes92], [HK14], [Elt20]. If $A = -\Delta$, the unique solvability of the Dirichlet problem yields that solutions of (1) decay radially on Ω [AE15]. For non-symmetric variable coefficients, this property becomes delicate. Using the elliptic regularity of very weak solutions [AEG20] we prove that if the coefficients (a_{kl}) are Lipschitz continuous, then solutions of (1) decay in the following manner.

Fix $d \geq 3$. We denote by $B_R \subset \mathbb{R}^d$ the open ball of radius $R > 0$ centred at the origin.

Theorem 1.2. *Suppose that the coefficients (a_{kl}) are Lipschitz continuous and fix $R > 0$ sufficiently large. Then there exists a $c > 0$ such that the following are valid.*

(a) *Let $\psi \in L_\infty(\Gamma)$ and $\lambda > 0$. Then*

$$|(B_\lambda^D \psi)(x)| \leq \frac{c \, \|\psi\|_{L_\infty(\Gamma)}}{\lambda} \cdot \frac{1}{|x|^{d-2}}$$

for all $x \in \Omega \setminus B_R$.

(b) *Let* $\psi \in L_\infty(\Gamma)$ *and* $\lambda > 0$. *Then*

$$|(B_\lambda \psi)(x) - \langle B_\lambda \psi \rangle| \le c \Big(\frac{\|\psi\|_{L_\infty(\Gamma)}}{\lambda} + |\langle B_\lambda \psi \rangle| \Big) \frac{1}{|x|^{d-2}}$$

for all $x \in \Omega \setminus B_R$, *where* $\langle B_\lambda \psi \rangle$ *is the average of* $B_\lambda \psi$ *over* Ω.

In Section 5 we use the form method from [AE12] to characterise the Dirichlet-to-Neumann operator $\mathcal{N}$ associated with the elliptic operator $A = -\sum \partial_l (a_{kl} \partial_k)$ on the exterior domain Ω. Each version of (1) gives rise to a distinct realisation of $\mathcal{N}$ and in each case $-\mathcal{N}$ generates an ultracontractive holomorphic C_0-semigroup on $L_2(\Gamma)$. For second-order elliptic operators, Gaussian heat kernel bounds serve as a valuable tool in the investigation of spectral and regularity properties. Existence of such bounds carries a variety of consequences, including L_p-analyticity of the semigroup, p-independence of the spectrum and existence of H^∞-functional calculi, and the associated corpus is commensurately extensive (see [Aro67], [Dav89], [SC92], [VSCC92], [AE97], [ER97], [ER98], [AT01], [Ouh05], [AMP06], [EO19b] and the references therein). If Γ is smooth, then the Laplace–Beltrami heat kernel satisfies Gaussian bounds and the kernel of the semigroup generated by $-\mathcal{N}$ on $C^\infty(\Gamma)$ satisfies Poisson bounds [EO14]. In [EO14] Poisson bounds were also obtained in the L_2-setting and these results were extended in [EO19a] and [EO19b] to bounded $C^{1+\kappa}$-domains and operators with symmetric Hölder continuous coefficients. More recently in [AE20], bounded Lipschitz domains and symmetric Lipschitz continuous coefficients were considered. In that paper it was proved that the C_0-semigroup generated by $-\mathcal{N}$ on $L_2(\Gamma)$ leaves $C(\Gamma)$ invariant and that its kernel is continuous on $\Gamma \times \Gamma$. In the case of non-symmetric measurable coefficients and an exterior Lipschitz domain, we establish joint continuity of the heat kernel on $\Sigma_\theta \times \Gamma \times \Gamma$, a result that seems yet to appear in the literature, even for the case $A = -\Delta$. Moreover, we prove that the semigroup generated by $-\mathcal{N}$ again leaves $C(\Gamma)$ invariant and that its kernel satisfies uniform bounds on a sector in $\mathbb{C}$.

Let S^D and S denote the holomorphic C_0-semigroups on $L_2(\Gamma)$ generated by $-\mathcal{N}$, corresponding to Dirichlet and Neumann boundary conditions at infinity respectively. We denote by $\theta^D, \theta^N \in (0, \frac{\pi}{2}]$ their respective angles of analyticity and by $\Sigma_{\theta^D}, \Sigma_{\theta^N} \subset \mathbb{C}$ the corresponding open sectors.

Theorem 1.3. (a) *There exists a continuous function* $K^D \colon \Sigma_{\theta^D} \times \Gamma \times \Gamma \to \mathbb{C}$ *such that*

$$(S^D_z \varphi)(w_1) = \int_\Gamma K^D_z(w_1, w_2) \varphi(w_2) \, \mathrm{d}w_2$$

for all $w_1 \in \Gamma$, $\varphi \in L_1(\Gamma)$ *and* $z \in \Sigma_{\theta^D}$.

(b) *The map* $z \mapsto K^D_z(w_1, w_2)$ *is analytic on* Σ_{θ^D} *for all* $w_1, w_2 \in \Gamma$.

(c) *For all* $\theta' \in (0, \theta^D)$ *there exist* $c, \delta > 0$ *such that*

$$\|K^D_z\|_{L_\infty(\Gamma \times \Gamma)} \le c \, (\operatorname{Re} z)^{-(d-1)} e^{-\delta \operatorname{Re} z}$$

for all $z \in \Sigma_{\theta'}$.

Theorem 1.4. (a) *There exists a continuous function $K\colon \Sigma_{\theta^N} \times \Gamma \times \Gamma \to \mathbb{C}$ such that*

$$(S_z\varphi)(w_1) = \int_\Gamma K_z(w_1, w_2)\varphi(w_2)\, \mathrm{d}w_2$$

for all $w_1 \in \Gamma$, $\varphi \in L_1(\Gamma)$ and $z \in \Sigma_{\theta^N}$.

(b) *The map $z \mapsto K_z(w_1, w_2)$ is analytic on Σ_{θ^N} for all $w_1, w_2 \in \Gamma$.*

(c) *For all $\theta' \in (0, \theta^N)$ there exists a $c > 0$ such that*

$$\|K_z\|_{L_\infty(\Gamma\times\Gamma)} \leq c\,(1 \wedge \operatorname{Re} z)^{-(d-1)}$$

for all $z \in \Sigma_{\theta'}$.

This thesis is organised as follows. In **Section 2** we introduce the form domain and other preliminary constructions, collecting various properties for later use. In **Section 3** we formulate (1) in terms of an abstract variational problem. One readily obtains well-posedness from the Lax–Milgram theorem and consequently, the existence of a continuous solution operator. We show that the solution operator is compact and submarkovian, before concluding Section 3 with the proof Theorem 1.1. In **Section 4** we consider (1) on the truncated domain $\Omega \cap B_R$ and establish convergence of the associated solution operator in the limit $R \to \infty$. We then prove Theorem 1.2 and obtain a variant of Theorem 1.1 that permits less regular data ψ, at the cost of requiring Lipschitz continuity of the coefficients (a_{kl}). In **Section 5** we introduce two versions of the Dirichlet-to-Neumann operator and in each case we obtain resolvent convergence with respect to the truncated problem under minimal regularity. We then show that if the boundary and coefficients are sufficiently smooth, our two realisations of the Dirichlet-to-Neumann operator differ only by a rank-one operator. In **Section 6** we prove that the holomorphic C_0-semigroup generated by each version of the Dirichlet-to-Neumann operator is submarkovian and uniformly mean ergodic. In [AE15] irreducibility of the semigroups was obtained using the self-adjointness of the Laplacian and Dirichlet-to-Neumann operator. Since we do not assume symmetry of the matrix (a_{kl}), the operator $\mathcal{N}$ is no longer self-adjoint in general. Hence we instead proceed via ergodicity in order to obtain that the semigroup generated by $-\mathcal{N}$ is irreducible when Neumann boundary conditions are imposed at infinity. We then establish irreducibility in the Dirichlet case, assuming that (a_{kl}) is symmetric. Finally in **Section 7** we consider the heat kernel associated with the Dirichlet-to-Neumann operator. Existence follows from ultracontractivity of the semigroup and the elliptic regularity afforded by Theorem 1.1 provides that the kernel is jointly continuous on $\Sigma_\theta \times \Gamma \times \Gamma$. We prove Theorems 1.3 and 1.4 and subsequently deduce that the semigroup leaves $C(\Gamma)$ invariant and that the heat kernel converges uniformly on $\Gamma \times \Gamma$ to an equilibrium.

2 The form domain

In this section we introduce the function spaces that underlie our study. We use the localised Sobolev spaces presented in [LO05], which are suited to the investigation of exterior variational problems. These spaces were used in [AE15] and [ARP19] to study the Dirichlet-to-Neumann operator, and in [KL20] to study the Robin Laplacian.

Throughout this thesis we fix $d \geq 3$. Define

$$W(\mathbb{R}^d) = \big\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |\nabla u|^2 < \infty \big\}.$$

We denote by $\mathfrak{p} > 2$ the Sobolev conjugate of 2, that is, $\frac{1}{\mathfrak{p}} = \frac{1}{2} - \frac{1}{d}$. Then [Bré11] Theorem 9.9 provides that $H^1(\mathbb{R}^d) \subset L_{\mathfrak{p}}(\mathbb{R}^d)$ and there exists a $c_s > 0$ such that

$$\|u\|_{L_{\mathfrak{p}}(\mathbb{R}^d)} \leq c_s \|\nabla u\|_{L_2(\mathbb{R}^d)}$$

for all $u \in H^1(\mathbb{R}^d)$.

For all $R > 0$ we write $B_R = \{x \in \mathbb{R}^d : |x| < R\}$. If $u \in L_1(B_R)$, we denote by

$$\langle u \rangle_R = \frac{1}{|B_R|} \int_{B_R} u$$

the average of u over the ball B_R. We write $\langle u \rangle_R = \langle u|_{B_R} \rangle_R$ for all $u \in W(\mathbb{R}^d)$.

Lemma 2.1. *There exists a $c > 0$ such that*

$$\|u - \langle u \rangle_R\|^2_{L_{\mathfrak{p}}(B_R)} \leq c \int_{\mathbb{R}^d} |\nabla u|^2$$

for all $u \in W(\mathbb{R}^d)$ and $R > 0$.

Proof. Let $v \in H^1(B_1)$. Since B_1 has the extension property, there exists a $c_0 > 0$ (independent of v) and a $\widetilde{v} \in H^1(\mathbb{R}^d)$ such that $\widetilde{v}|_{B_1} = v$ and $\|\widetilde{v}\|_{H^1(\mathbb{R}^d)} \leq c_0 \|v\|_{H^1(B_1)}$. Then

$$\|v\|_{L_{\mathfrak{p}}(B_1)} \leq \|\widetilde{v}\|_{L_{\mathfrak{p}}(\mathbb{R}^d)} \leq c_s \|\widetilde{v}\|_{H^1(\mathbb{R}^d)} \leq c_s c_0 \|v\|_{H^1(B_1)}.$$

Hence by Proposition A.1 there exists a $c > 0$ such that

$$\|v - \langle v \rangle_1\|^2_{L_{\mathfrak{p}}(B_1)} \leq (c_s c_0)^2 \Big(\int_{B_1} |\nabla v|^2 + \int_{B_1} |v - \langle v \rangle_1|^2 \Big) \leq c \int_{B_1} |\nabla v|^2$$

for all $v \in H^1(B_1)$.

Let $u \in W(\mathbb{R}^d)$ and $R > 0$. Then $u|_{B_R} \in H^1(B_R)$. Define $u_R \colon B_1 \to \mathbb{C}$ by $u_R(x) = u(Rx)$. Then $u_R \in H^1(B_1)$ and a change of variable yields that

$$\langle u_R \rangle_1 = \frac{1}{\omega_d} \int_{B_1} u(Rx)\, \mathrm{d}x = \frac{1}{R^d \omega_d} \int_{B_R} u(x)\, \mathrm{d}x = \langle u \rangle_R.$$

Similarly,

$$\int_{B_1} |u_R|^{\mathfrak{p}} = R^{-d} \int_{B_R} |u|^{\mathfrak{p}}$$

and

$$\int_{B_1} |\nabla u_R|^2 = R^{2-d} \int_{B_R} |\nabla u|^2.$$

Therefore

$$\begin{aligned}
\|u - \langle u \rangle_R\|^2_{L_\mathfrak{p}(B_R)} &= R^{\frac{2d}{\mathfrak{p}}} \|u_R - \langle u_R \rangle_1\|^2_{L_\mathfrak{p}(B_1)} \\
&\leq R^{\frac{2d}{\mathfrak{p}}} c \int_{B_1} |\nabla u_R|^2 = c R^{\frac{2d}{\mathfrak{p}}} R^{2-d} \int_{B_R} |\nabla u|^2 = c \int_{B_R} |\nabla u|^2 \leq c \int_{\mathbb{R}^d} |\nabla u|^2
\end{aligned}$$

as required. □

Proposition 2.2. *Let $u \in W(\mathbb{R}^d)$. Then the limit*

$$\langle u \rangle = \lim_{R \to \infty} \langle u \rangle_R \tag{2}$$

exists and $u - \langle u \rangle \in L_\mathfrak{p}(\mathbb{R}^d)$. Moreover, there exists a $c > 0$ such that

$$\|u - \langle u \rangle\|^2_{L_\mathfrak{p}(\mathbb{R}^d)} \leq c \int_{\mathbb{R}^d} |\nabla u|^2$$

for all $u \in W(\mathbb{R}^d)$.

Proof. Let $c > 0$ be as in Lemma 2.1 and let $n \in \mathbb{N}$. Then

$$\begin{aligned}
|\langle u \rangle_{2^{n+1}} - \langle u \rangle_{2^n}| &= \Big| \frac{1}{|B_{2^n}|} \int_{B_{2^n}} u - \langle u \rangle_{2^{n+1}} \Big| \leq \frac{1}{|B_{2^n}|} \int_{B_{2^{n+1}}} |u - \langle u \rangle_{2^{n+1}}| \\
&= \frac{2^d}{|B_{2^{n+1}}|} \int_{B_{2^{n+1}}} |u - \langle u \rangle_{2^{n+1}}| \leq 2^d \Big(\frac{1}{|B_{2^{n+1}}|} \int_{B_{2^{n+1}}} |u - \langle u \rangle_{2^{n+1}}|^\mathfrak{p} \Big)^{1/\mathfrak{p}} \\
&= \frac{2^{d(1 - \frac{n+1}{\mathfrak{p}})}}{\omega_d^{1/\mathfrak{p}}} \Big(\int_{B_{2^{n+1}}} |u - \langle u \rangle_{2^{n+1}}|^\mathfrak{p} \Big)^{1/\mathfrak{p}} \leq \frac{2^{d(1 - \frac{n+1}{\mathfrak{p}})}}{\omega_d^{1/\mathfrak{p}}} c^{1/2} \Big(\int_{\mathbb{R}^d} |\nabla u|^2 \Big)^{1/2}.
\end{aligned}$$

So $\sum |\langle u \rangle_{2^{n+1}} - \langle u \rangle_{2^n}| < \infty$ and it follows that the limit $\lim \langle u \rangle_{2^n}$ exists.

Write $\alpha = \lim \langle u \rangle_{2^n}$. Then by Fatou's lemma

$$\int_{\mathbb{R}^d} |u - \alpha|^\mathfrak{p} = \int_{\mathbb{R}^d} \liminf_{n \to \infty} |u - \langle u \rangle_{2^n}|^\mathfrak{p} \mathbb{1}_{B_{2^n}} \leq \liminf_{n \to \infty} \int_{B_{2^n}} |u - \langle u \rangle_{2^n}|^\mathfrak{p} \leq c^{\mathfrak{p}/2} \Big(\int_{\mathbb{R}^d} |\nabla u|^2 \Big)^{\mathfrak{p}/2},$$

so $u - \alpha \in L_\mathfrak{p}(\mathbb{R}^d)$. Since

$$|\langle u - \alpha \rangle_R| \leq \Big(\frac{1}{|B_R|} \int_{B_R} |u - \alpha|^\mathfrak{p} \Big)^{1/\mathfrak{p}} \leq (R^d \omega_d)^{-1/\mathfrak{p}} \|u - \alpha\|_{L_\mathfrak{p}(\mathbb{R}^d)}$$

for all $R > 0$, one deduces that $\lim \langle u - \alpha \rangle_R = 0$. Hence the limit $\lim \langle u \rangle_R = \alpha$ exists and the claim follows. □

For all $u \in W(\mathbb{R}^d)$ we define the average $\langle u \rangle$ of u over $\mathbb{R}^d$ by (2). We define the norm

$$\|u\|_{W(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |\nabla u|^2 + |\langle u \rangle|^2 \Big)^{1/2}$$

on $W(\mathbb{R}^d)$.

Proposition 2.3. *The space $W(\mathbb{R}^d)$ is a Hilbert space.*

Proof. It is easy to verify that $W(\mathbb{R}^d)$ is a pre-Hilbert space with respect to the inner product associated with the norm $\|\cdot\|_{W(\mathbb{R}^d)}$. Hence it remains only to show that the space $\big(W(\mathbb{R}^d), \|\cdot\|_{W(\mathbb{R}^d)}\big)$ is complete.

Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W(\mathbb{R}^d)$ and let $c > 0$ be as in Proposition 2.2. For each $n \in \mathbb{N}$ write $v_n = u_n - \langle u_n\rangle \in W(\mathbb{R}^d)$. Let $n, m \in \mathbb{N}$. Then

$$\|v_n - v_m\|^2_{L_{\mathfrak{p}}(\mathbb{R}^d)} = \|(u_n - u_m) - \langle u_n - u_m\rangle\|^2_{L_{\mathfrak{p}}(\mathbb{R}^d)} \le c\int_{\mathbb{R}^d} |\nabla(u_n - u_m)|^2$$

and

$$\int_{\mathbb{R}^d} |\nabla(v_n - v_m)|^2 = \int_{\mathbb{R}^d} |\nabla(u_n - u_m)|^2.$$

So $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_{\mathfrak{p}}(\mathbb{R}^d)$ and $(\nabla v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_2(\mathbb{R}^d)^d$. Then by completeness there exist $v \in L_{\mathfrak{p}}(\mathbb{R}^d)$ and $w \in L_2(\mathbb{R}^d)^d$ such that $\lim v_n = v$ in $L_{\mathfrak{p}}(\mathbb{R}^d)$ and $\lim \nabla v_n = w$ in $L_2(\mathbb{R}^d)^d$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ and let $R > 0$ be such that $\operatorname{supp}\chi \subset B_R$. Since $\lim v_n|_{B_R} = v|_{B_R}$ in $L_2(B_R)$, it follows that

$$\int_{\mathbb{R}^d} v\,\overline{\partial_k\chi} = \lim_{n\to\infty}\int_{B_R} v_n\,\overline{\partial_k\chi} = -\lim_{n\to\infty}\int_{B_R} (\partial_k v_n)\overline{\chi} = -\int_{\mathbb{R}^d} w_k\,\overline{\chi}$$

for all $k \in \{1, \ldots, d\}$. So $\nabla v = w$ and $v \in W(\mathbb{R}^d)$, since $L_{\mathfrak{p}}(\mathbb{R}^d) \subset L_{2,\mathrm{loc}}(\mathbb{R}^d)$.

Note that

$$|\langle v\rangle_R| \le \Big(\frac{1}{|B_R|}\int_{B_R} |v|^{\mathfrak{p}}\Big)^{1/\mathfrak{p}} \le (R^d\omega_d)^{-1/\mathfrak{p}}\|v\|_{L_{\mathfrak{p}}(\mathbb{R}^d)}$$

for all $R > 0$. Then $\langle v\rangle = \lim \langle v\rangle_R = 0$. Moreover, since $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W(\mathbb{R}^d)$ it follows that $(\langle u_n\rangle)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$, so $\lim \langle u_n\rangle$ exists. Write $u = v + \lim \langle u_n\rangle$. Then $u \in W(\mathbb{R}^d)$ and

$$\lim_{n\to\infty} \|u - u_n\|^2_{W(\mathbb{R}^d)} = \lim_{n\to\infty}\int_{\mathbb{R}^d} |\nabla v - \nabla v_n|^2 + \lim_{n\to\infty}\Big|\lim_{k\to\infty}\langle u_k\rangle - \langle u_n\rangle\Big|^2 = 0$$

as required. $\square$

We call a connected open set $U \subset \mathbb{R}^d$ a **domain**. We equip the boundary ∂U with the $(d-1)$-dimensional Hausdorff measure σ.

Lemma 2.4. *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain.*

(a) *Let $A \subset U$ be a measurable set with $|A| > 0$. Then the norm*

$$u \mapsto \Big(\int_U |\nabla u|^2 + \int_A |u|^2\Big)^{1/2}$$

is equivalent to the norm $\|\cdot\|_{H^1(U)}$ on $H^1(U)$.

(b) *Let $Z \subset \overline{U}$ be a measurable set with $0 < \sigma(Z) < \infty$. Suppose that the restriction $u \mapsto u|_Z$ from $H^1(U) \cap C(\overline{U})$ into $L_2(Z)$ admits a compact extension $T\colon H^1(U) \to L_2(Z)$. Then the norm*

$$u \mapsto \Big(\int_U |\nabla u|^2 + \int_Z |Tu|^2\Big)^{1/2}$$

is equivalent to the norm $\|\cdot\|_{H^1(U)}$ on $H^1(U)$.

Proof. We first prove (a). Clearly

$$\int_U |\nabla u|^2 + \int_A |u|^2 \leq \|u\|^2_{H^1(U)}$$

for all $u \in H^1(U)$, since $A \subset U$. It remains to show that there exists a $c > 0$ such that

$$\|u\|^2_{H^1(U)} \leq c\Big(\int_U |\nabla u|^2 + \int_A |u|^2\Big)$$

for all $u \in H^1(U)$. Note that it suffices to prove that

$$\int_U |u|^2 \leq c\Big(\int_U |\nabla u|^2 + \int_A |u|^2\Big)$$

for all $u \in H^1(U)$. Suppose to the contrary that for each $n \in \mathbb{N}$ there exists a $u_n \in H^1(U)$ such that

$$\int_U |\nabla u_n|^2 + \int_A |u_n|^2 < \frac{1}{n}\int_U |u_n|^2.$$

Without loss of generality we may assume that $\int_U |u_n|^2 = 1$ for all $n \in \mathbb{N}$. Then

$$\|u_n\|^2_{H^1(U)} = \int_U |\nabla u_n|^2 + 1 < 2$$

for all $n \in \mathbb{N}$, so the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $H^1(U)$. Passing to a subsequence if necessary, we may assume that there exists a $u \in H^1(U)$ such that $\lim u_n = u$ weakly in $H^1(U)$. By [EE87] Theorem V.4.17 the embedding $H^1(U) \hookrightarrow L_2(U)$ is compact, so $\lim u_n = u$ in $L_2(U)$. Then $\|u\|_{L_2(U)} = 1$ and

$$\begin{aligned}\int_U |\nabla u|^2 + 1 = \|u\|^2_{H^1(U)} &\leq \liminf_{n\to\infty} \|u_n\|^2_{H^1(U)} \\ &= \liminf_{n\to\infty} \int_U |\nabla u_n|^2 + 1 \leq \liminf_{n\to\infty} \frac{1}{n} + 1 = 1.\end{aligned}$$

Hence $\int_U |\nabla u|^2 = 0$ and it follows that u is constant, as U is connected. Moreover, since the embedding $H^1(U) \hookrightarrow L_2(U)$ is compact and the restriction $u \mapsto u|_A$ from $L_2(U)$ into $L_2(A)$ is continuous, the map $u \mapsto u|_A$ from $H^1(U)$ into $L_2(A)$ is compact. Therefore $\lim u_n|_A = u|_A$ in $L_2(A)$ and

$$\int_A |u|^2 = \lim_{n\to\infty} \int_A |u_n|^2 \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Then $u|_A = 0$ and it follows that $u = 0$. So $0 = \|u\|_{L_2(U)} = 1$, a contradiction.

We now prove (b). By hypothesis the map $T \colon H^1(U) \to L_2(Z)$ is continuous, so there exists a $c_1 > 0$ such that $\|Tu\|_{L_2(Z)} \leq c_1 \|u\|_{H^1(U)}$ for all $u \in H^1(U)$. Hence

$$\int_U |\nabla u|^2 + \int_Z |Tu|^2 \leq (c_1^2 + 1)\, \|u\|^2_{H^1(U)}$$

for all $u \in H^1(U)$. The converse estimate follows from a contradictory argument similar to the above, together with the assumed compactness of the map T. $\square$

We require the following equivalent norm on $W(\mathbb{R}^d)$.

Lemma 2.5. *Let $A \subset \mathbb{R}^d$ be a bounded measurable set with $|A| > 0$. Then the norm*

$$u \mapsto \Big(\int_{\mathbb{R}^d} |\nabla u|^2 + \int_A |u|^2 \Big)^{1/2}$$

is equivalent to the norm $\| \cdot \|_{W(\mathbb{R}^d)}$ on $W(\mathbb{R}^d)$.

Proof. Define $|||\cdot|||\colon W(\mathbb{R}^d) \to [0, \infty)$ by

$$|||u||| = \Big(\int_{\mathbb{R}^d} |\nabla u|^2 + \int_A |u|^2 \Big)^{1/2}.$$

We prove the claim using the closed graph theorem. Hence we first verify that the space $\big(W(\mathbb{R}^d), |||\cdot|||\big)$ is complete.

Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\big(W(\mathbb{R}^d), |||\cdot|||\big)$ and let $R > 0$ be such that $A \subset B_R$. Then by Lemma 2.4(a) there exists a $c > 0$ such that

$$\|u_n|_{B_R}\|^2_{H^1(B_R)} \le c\Big(\int_{B_R} |\nabla u_n|^2 + \int_A |u_n|^2 \Big) \le c|||u_n|||^2$$

for all $n \in \mathbb{N}$, so the sequence $(u_n|_{B_R})_{n\in\mathbb{N}}$ is Cauchy in $H^1(B_R)$. Since $H^1(B_R)$ is complete, the sequence $(u_n|_{B_R})_{n\in\mathbb{N}}$ is convergent. By a diagonal argument one deduces that there exists a function $u\colon \mathbb{R}^d \to \mathbb{C}$ such that $\lim u_n|_{B_R} = u|_{B_R}$ in $H^1(B_R)$ for all $R > 0$ with $A \subset B_R$. Hence $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$. Let $n \in \mathbb{N}$. Then

$$\int_{B_R} |\nabla(u - u_n)|^2 = \lim_{m\to\infty} \int_{B_R} |\nabla(u_m - u_n)|^2 \le \liminf_{m\to\infty} |||u_m - u_n|||^2$$

for all $R > 0$ with $A \subset B_R$. So

$$\int_{\mathbb{R}^d} |\nabla u - \nabla u_n|^2 = \lim_{R\to\infty} \int_{B_R} |\nabla(u - u_n)|^2 \le \liminf_{m\to\infty} |||u_m - u_n|||^2 < \infty$$

by the monotone convergence theorem. Hence $\nabla u - \nabla u_n \in L_2(\mathbb{R}^d)^d$ for all $n \in \mathbb{N}$, so $u \in W(\mathbb{R}^d)$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |\nabla u - \nabla u_n|^2 \le \lim_{n\to\infty} \liminf_{m\to\infty} |||u_m - u_n|||^2 = 0.$$

Moreover, $\lim u_n|_A = u|_A$ in $L_2(A)$ since $L_2(A)$ is complete. Therefore $\lim |||u - u_n||| = 0$ and $\big(W(\mathbb{R}^d), |||\cdot|||\big)$ is complete.

Next we show that there exists a $c > 0$ such that $|||u||| \le c\|u\|_{W(\mathbb{R}^d)}$ for all $u \in W(\mathbb{R}^d)$. Suppose to the contrary that for each $n \in \mathbb{N}$ there exists a $w_n \in W(\mathbb{R}^d)$ such that $|||w_n||| > n\|w_n\|_{W(\mathbb{R}^d)}$. Without loss of generality we may assume that $\|w_n\|_{W(\mathbb{R}^d)} = 1$ for all $n \in \mathbb{N}$. Write $u_n = n^{-1/2} w_n \in W(\mathbb{R}^d)$. Then $\|u_n\|_{W(\mathbb{R}^d)} = n^{-1/2}$ and $|||u_n||| > n^{1/2}$ for all $n \in \mathbb{N}$. So $\lim \|u_n\|_{W(\mathbb{R}^d)} = 0$ and $\lim |||u_n||| = \infty$. Write $v_n = u_n - \langle u_n \rangle \in W(\mathbb{R}^d)$. Then Proposition 2.2 provides that $v_n \in L_{\mathfrak{p}}(\mathbb{R}^d)$ for all $n \in \mathbb{N}$ and $\lim \|v_n\|_{L_{\mathfrak{p}}(\mathbb{R}^d)} = 0$. So $\lim \|v_n|_A\|_{L_2(A)} = 0$, since $|A| < \infty$ and $\mathfrak{p} > 2$. Moreover, $\lim \langle u_n \rangle = 0$, so $\lim \|\langle u_n \rangle \mathbb{1}_A\|_{L_2(A)} = 0$ and it follows that $\lim \|u_n|_A\|_{L_2(A)} = 0$. Then since $\lim \|\nabla u_n\|_{L_2(\mathbb{R}^d)} = 0$ it follows that $\lim |||u_n||| = 0$, a contradiction.

Since the inclusion $\big(W(\mathbb{R}^d), \|\cdot\|_{W(\mathbb{R}^d)}\big) \hookrightarrow \big(W(\mathbb{R}^d), |||\cdot|||\big)$ is continuous, it follows from Proposition 2.3 together with the closed graph theorem that the norms are equivalent. $\square$

In Proposition 2.7 we show that the test functions are dense in $W(\mathbb{R}^d) \cap L_{\mathfrak{p}}(\mathbb{R}^d)$. We need a lemma.

Lemma 2.6. *Let $u \in W(\mathbb{R}^d) \cap L_{\mathfrak{p}}(\mathbb{R}^d)$ and let $\varepsilon > 0$. Then there exists a $\chi \in C_c^\infty(\mathbb{R}^d)$ such that the estimate*

$$\|u - \chi\|_{L_{\mathfrak{p}}(\mathbb{R}^d)} + \|\nabla(u - \chi)\|_{L_2(\mathbb{R}^d)} < \varepsilon$$

is valid.

Proof. Fix $\tau \in C_c^\infty(\mathbb{R}^d)$ such that $\tau|_{B_1} = \mathbb{1}$, $\operatorname{supp} \tau \subset B_2$, $|\tau| \leq \mathbb{1}$ and $|\nabla \tau| \leq 2$. For each $R > 0$ define $\tau_R \in C_c^\infty(\mathbb{R}^d)$ by $\tau_R(x) = \tau\big(\frac{x}{R}\big)$. Since $|u|^2 \in L_{\mathfrak{p}/2}(\mathbb{R}^d)$, it follows from Hölder's inequality that

$$\begin{aligned}\int_{\mathbb{R}^d} |u \, \nabla \tau_R|^2 &\leq \frac{4}{R^2} \int_{B_{2R} \setminus B_R} |u|^2 \leq \frac{4}{R^2} \Big(\int_{B_{2R} \setminus B_R} |u|^{\mathfrak{p}} \Big)^{2/\mathfrak{p}} |B_{2R} \setminus B_R|^{1 - \frac{2}{\mathfrak{p}}} \\ &\leq \frac{4}{R^2} \, {\omega_d}^{1 - \frac{2}{\mathfrak{p}}} (2R)^{d(1 - \frac{2}{\mathfrak{p}})} \Big(\int_{B_{2R} \setminus B_R} |u|^{\mathfrak{p}} \Big)^{2/\mathfrak{p}} = 16 \, {\omega_d}^{2/d} \Big(\int_{B_{2R} \setminus B_R} |u|^{\mathfrak{p}} \Big)^{2/\mathfrak{p}}\end{aligned}$$

for all $R > 0$, since $\frac{2}{d} = 1 - \frac{2}{\mathfrak{p}}$. Hence

$$\begin{aligned}\|\nabla(u - u\tau_R)\|_{L_2(\mathbb{R}^d)}^2 &\leq 2\|(1 - \tau_R)\nabla u\|_{L_2(\mathbb{R}^d)}^2 + 2\|u \, \nabla \tau_R\|_{L_2(\mathbb{R}^d)}^2 \\ &\leq 2 \int_{\mathbb{R}^d \setminus B_R} |\nabla u|^2 + 32 \, {\omega_d}^{2/d} \Big(\int_{B_{2R} \setminus B_R} |u|^{\mathfrak{p}} \Big)^{2/\mathfrak{p}}.\end{aligned}$$

for all $R > 0$, so

$$\lim_{R \to \infty} \|u - u\tau_R\|_{L_{\mathfrak{p}}(\mathbb{R}^d)} + \|\nabla(u - u\tau_R)\|_{L_2(\mathbb{R}^d)} = 0.$$

Choose $R > 0$ such that

$$\|u - u\tau_R\|_{L_{\mathfrak{p}}(\mathbb{R}^d)} + \|\nabla(u - u\tau_R)\|_{L_2(\mathbb{R}^d)} < \frac{\varepsilon}{2}.$$

Since $\operatorname{supp} u\tau_R$ is bounded, by mollification we may assume that there exists a $\chi \in C_c^\infty(\mathbb{R}^d)$ such that

$$\|u\tau_R - \chi\|_{L_{\mathfrak{p}}(\mathbb{R}^d)} + \|\nabla(u\tau_R - \chi)\|_{L_2(\mathbb{R}^d)} < \frac{\varepsilon}{2}.$$

The claim then follows from the triangle inequality. □

We define

$$W^D(\mathbb{R}^d) = \overline{C_c^\infty(\mathbb{R}^d)}^{W(\mathbb{R}^d)}$$

and equip $W^D(\mathbb{R}^d)$ with the norm $\| \cdot \|_{W^D(\mathbb{R}^d)}$ induced by the norm on $W(\mathbb{R}^d)$. Then $W^D(\mathbb{R}^d)$ is a Hilbert space and

$$\|u\|_{W^D(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |\nabla u|^2 \Big)^{1/2}$$

by the following assertion.

Proposition 2.7. *The space $W(\mathbb{R}^d)$ admits the orthogonal decomposition*

$$W(\mathbb{R}^d) = W^D(\mathbb{R}^d) \oplus \mathbb{C}\mathbb{1}.$$

Moreover, $W^D(\mathbb{R}^d) = \{u \in W(\mathbb{R}^d) : \langle u \rangle = 0\} = W(\mathbb{R}^d) \cap L_{\mathfrak{p}}(\mathbb{R}^d)$.

Proof. Let $u \in W^D(\mathbb{R}^d)$. Then there exists a sequence $(\chi_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ such that $\lim \chi_n = u$ in $W(\mathbb{R}^d)$. Since $\langle \chi_n \rangle = 0$ for all $n \in \mathbb{N}$, it follows that $\langle u \rangle = 0$. So $W^D(\mathbb{R}^d) \subset \{u \in W(\mathbb{R}^d) : \langle u \rangle = 0\}$. Moreover, $\{u \in W(\mathbb{R}^d) : \langle u \rangle = 0\} \subset W(\mathbb{R}^d) \cap L_{\mathfrak{p}}(\mathbb{R}^d)$ by Proposition 2.2.

Now let $u \in W(\mathbb{R}^d) \cap L_{\mathfrak{p}}(\mathbb{R}^d)$. By Lemma 2.6 there exists a sequence $(\chi_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ such that $\lim \chi_n = u$ in $L_{\mathfrak{p}}(\mathbb{R}^d)$ and $\lim \nabla \chi_n = \nabla u$ in $L_2(\mathbb{R}^d)^d$. Moreover,

$$\Big|\frac{1}{|B_R|}\int_{B_R} u - \chi_n\Big| \le \Big(\frac{1}{|B_R|}\int_{B_R} |u - \chi_n|^{\mathfrak{p}}\Big)^{1/\mathfrak{p}} \le (R^d\omega_d)^{-1/\mathfrak{p}} \|u - \chi_n\|_{L_{\mathfrak{p}}(\mathbb{R}^d)}$$

for all $R > 0$ and $n \in \mathbb{N}$. So $\langle u - \chi_n \rangle = 0$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|u - \chi_n\|^2_{W(\mathbb{R}^d)} = \lim_{n\to\infty} \int_{\mathbb{R}^d} |\nabla(u - \chi_n)|^2 + |\langle u - \chi_n \rangle|^2 = 0.$$

Hence $u \in W^D(\mathbb{R}^d)$ and $W^D(\mathbb{R}^d) = \{u \in W(\mathbb{R}^d) : \langle u \rangle = 0\} = W(\mathbb{R}^d) \cap L_{\mathfrak{p}}(\mathbb{R}^d)$.

The above implies that $u - \langle u \rangle \in W^D(\mathbb{R}^d)$ for all $u \in W(\mathbb{R}^d)$. Therefore $W(\mathbb{R}^d) \subset W^D(\mathbb{R}^d) + \mathbb{C}\mathbb{1}$. Since $\mathbb{1} \perp W^D(\mathbb{R}^d)$, the claim follows. □

Corollary 2.8. $H^1(\mathbb{R}^d) \subset W^D(\mathbb{R}^d)$.

Proof. Since $H^1(\mathbb{R}^d) \subset L_{\mathfrak{p}}(\mathbb{R}^d)$ by the Sobolev embedding theorem and $H^1(\mathbb{R}^d) \subset W(\mathbb{R}^d)$, the corollary follows. □

Corollary 2.9. *Let $u \in W(\mathbb{R}^d)$. Then $\langle |u| \rangle = |\langle u \rangle|$.*

Proof. Write $u = v + \lambda\mathbb{1}$, where $v \in W^D(\mathbb{R}^d)$ and $\lambda \in \mathbb{C}$. Then $|v| \in W(\mathbb{R}^d) \cap L_{\mathfrak{p}}(\mathbb{R}^d) = W^D(\mathbb{R}^d)$, so $\langle |v| \rangle = 0$. Moreover, for all $R > 0$

$$\begin{aligned} \big|\langle |u| \rangle_R - |\lambda|\big| &\le \frac{1}{|B_R|}\int_{B_R} \big||v + \lambda\mathbb{1}| - |\lambda|\mathbb{1}\big| \\ &\le \frac{1}{|B_R|}\int_{B_R} \big||v| + |\lambda|\mathbb{1} - |\lambda|\mathbb{1}\big| = \langle |v| \rangle_R. \end{aligned}$$

Hence $\langle |u| \rangle = \lim \langle |u| \rangle_R = |\lambda| = |\langle v + \lambda\mathbb{1} \rangle| = |\langle u \rangle|$. □

We now introduce the exterior domain Ω. Throughout this thesis we fix a bounded open set $\Omega_0 \subset \mathbb{R}^d$ with Lipschitz boundary and consider the exterior domain

$$\Omega = \mathbb{R}^d \setminus \overline{\Omega_0}.$$

We assume that Ω is connected. We write $\Gamma = \partial\Omega = \partial\Omega_0$ and equip Γ with the $(d-1)$-dimensional Hausdorff measure σ. Moreover, for all $R > 0$ we write $\Omega_R = \Omega \cap B_R$.

Define

$$W(\Omega) = \big\{u \in H^1_{\mathrm{loc}}(\Omega) : \int_\Omega |\nabla u|^2 < \infty\big\}.$$

We fix $R_0 > 3$ such that $\overline{\Omega_0} \subset B_{R_0 - 3}$. Since Ω_R has Lipschitz boundary, it follows from [Maz85] Lemma 1.1.11 that $u|_{\Omega_R} \in H^1(\Omega_R)$ for all $u \in W(\Omega)$ and $R \geq R_0$. Hence

$$W(\Omega) = \big\{u \in \mathbb{C}^\Omega : u \text{ is measurable}, u|_{\Omega_R} \in H^1(\Omega_R) \text{ for all } R \geq R_0 \text{ and } \int_\Omega |\nabla u|^2 < \infty\big\}.$$

We regularly invoke this characterisation of $W(\Omega)$.

Since Ω_{R_0} has Lipschitz boundary, there exists a bounded operator $E_0 \colon H^1(\Omega_{R_0}) \to H^1(B_{R_0})$ such that $(E_0 u)|_{\Omega_{R_0}} = u$. Define $E \colon W(\Omega) \to W(\mathbb{R}^d)$ by

$$(Eu)(x) = \begin{cases} u(x) & \text{if } x \in \mathbb{R}^d \setminus B_{R_0}, \\ \big(E_0(u|_{\Omega_{R_0}})\big)(x) & \text{if } x \in B_{R_0}. \end{cases}$$

Then $(Eu)|_\Omega = u$. For all $u \in W(\Omega)$ we define $\langle u \rangle = \langle Eu \rangle$. Then it follows that

$$\langle u \rangle = \lim_{R \to \infty} \frac{1}{|\Omega_R|} \int_{\Omega_R} u$$

since $|\Omega_0| < \infty$. We equip $W(\Omega)$ with the norm

$$\|u\|_{W(\Omega)} = \Big(\int_\Omega |\nabla u|^2 + |\langle u \rangle|^2 \Big)^{1/2}.$$

In Proposition 2.11 we shall prove that $W(\Omega)$ is a Hilbert space. First we establish that the following Sobolev–Poincaré-type inequality is valid on $W(\Omega)$.

Proposition 2.10. *There exists a $c > 0$ such that*

$$\|u - \langle u \rangle\|_{L_\mathfrak{p}(\Omega)}^2 \leq c \int_\Omega |\nabla u|^2$$

for all $u \in W(\Omega)$.

Proof. Let $u \in W(\Omega)$ and let $c > 0$ be as in Proposition 2.2. Write

$$\alpha = \frac{1}{|\Omega_{R_0}|} \int_{\Omega_{R_0}} u.$$

Then by the Proposition A.1 there exists a $c_1 > 0$ such that

$$\|u|_{\Omega_{R_0}} - \alpha\|_{H^1(\Omega_{R_0})}^2 \leq c_1 \int_{\Omega_{R_0}} |\nabla u|^2.$$

Moreover, by Proposition A.2 we may assume that $E_0 \mathbb{1}_{\Omega_{R_0}} = \mathbb{1}_{B_{R_0}}$. Then

$$\begin{aligned} \|u - \langle u \rangle\|_{L_\mathfrak{p}(\Omega)}^2 &\leq \|Eu - \langle Eu \rangle\|_{L_\mathfrak{p}(\mathbb{R}^d)}^2 \leq c \int_{\mathbb{R}^d} |\nabla Eu|^2 = c \int_{\mathbb{R}^d} |\nabla (Eu - \alpha)|^2 \\ &\leq c \Big(\int_{B_{R_0}} |\nabla (Eu - \alpha)|^2 + \int_\Omega |\nabla (Eu - \alpha)|^2 \Big) \\ &= c \Big(\int_{B_{R_0}} |\nabla E_0 (u|_{\Omega_{R_0}} - \alpha)|^2 + \int_\Omega |\nabla u|^2 \Big) \end{aligned}$$

$$\leq c\Big(\|E_0(u|_{\Omega_{R_0}} - \alpha)\|^2_{H^1(B_{R_0})} + \int_\Omega |\nabla u|^2\Big)$$

$$\leq c\Big(\|E_0\|^2 \|u|_{\Omega_{R_0}} - \alpha\|^2_{H^1(\Omega_{R_0})} + \int_\Omega |\nabla u|^2\Big)$$

$$\leq c\Big(\|E_0\|^2 c_1 \int_{\Omega_{R_0}} |\nabla u|^2 + \int_\Omega |\nabla u|^2\Big) \leq c_2 \int_\Omega |\nabla u|^2$$

where $c_2 = c(\|E_0\|^2 c_1 + 1)$. □

Next we define the trace map on $W(\Omega)$. Let $R \geq R_0$. Then Ω_R is a bounded Lipschitz domain. Denote by $\mathrm{Tr}_{\Omega_R} \colon H^1(\Omega_R) \to L_2(\partial\Omega_R)$ the trace map on $H^1(\Omega_R)$ and define $\mathrm{Tr}_R \colon H^1(\Omega_R) \to L_2(\Gamma)$ by $\mathrm{Tr}_R u = (\mathrm{Tr}_{\Omega_R} u)|_\Gamma$. We then define $\mathrm{Tr} : W(\Omega) \to L_2(\Gamma)$ by

$$\mathrm{Tr}\, u = \mathrm{Tr}_R(u|_{\Omega_R}).$$

Note that the map Tr does not depend on R and is therefore well-defined.

Throughout this thesis, we make frequent use of the following facts.

Proposition 2.11. (a) *The space $W(\Omega)$ is a Hilbert space.*

(b) *The trace map $\mathrm{Tr} : W(\Omega) \to L_2(\Gamma)$ is compact.*

(c) *The extension operator $E \colon W(\Omega) \to W(\mathbb{R}^d)$ is bounded.*

(d) *Let $A \subset \Omega$ be a bounded measurable set with $|A| > 0$. Then the norm*

$$u \mapsto \Big(\int_\Omega |\nabla u|^2 + \int_A |u|^2\Big)^{1/2}$$

is equivalent to the norm $\|\cdot\|_{W(\Omega)}$ on $W(\Omega)$.

(e) *The norm*

$$u \mapsto \Big(\int_\Omega |\nabla u|^2 + \int_\Gamma |\mathrm{Tr}\, u|^2\Big)^{1/2} \tag{3}$$

is equivalent to the norm $\|\cdot\|_{W(\Omega)}$ on $W(\Omega)$.

Proof. Let $A \subset \Omega$ be a bounded measurable set with $|A| > 0$. Write

$$|||u|||_A = \Big(\int_\Omega |\nabla u|^2 + \int_A |u|^2\Big)^{1/2}$$

for all $u \in W(\Omega)$. Arguing as in the second paragraph of the proof of Lemma 2.5 with Ω_R in place of B_R, one obtains that the space $\big(W(\Omega), |||\cdot|||_A\big)$ is complete.

We now show that the norm $|||\cdot|||_A$ is equivalent on $W(\Omega)$ to the norm defined by (3). Let $R \geq R_0$ be such that $A \subset B_R$. By Lemma 2.4(a) there exists a $c_1 > 0$ such that

$$\|u|_{\Omega_R}\|^2_{H^1(\Omega_R)} \leq c_1\Big(\int_{\Omega_R} |\nabla u|^2 + \int_A |u|^2\Big) \leq c_1 |||u|||^2_A$$

for all $u \in W(\Omega)$. On the other hand, since Ω_R has Lipschitz boundary it follows that the map $\mathrm{Tr}_R \colon H^1(\Omega_R) \to L_2(\Gamma)$ is compact, so by Lemma 2.4(b) there exists a $c_2 > 0$ such that

$$\int_{\Omega_R} |\nabla u|^2 + \int_\Gamma |\mathrm{Tr}\, u|^2 \leq c_2 \|u|_{\Omega_R}\|^2_{H^1(\Omega_R)}$$

for all $u \in W(\Omega)$. Then

$$\int_\Omega |\nabla u|^2 + \int_\Gamma |\mathrm{Tr}\, u|^2 \le (c_1 c_2 + 1) |||u|||_A^2$$

for all $u \in W(\Omega)$. One similarly deduces from Lemma 2.4 that there exists a $c_3 > 0$ such that

$$|||u|||_A^2 \le c_3 \int_\Omega |\nabla u|^2 + \int_\Gamma |\mathrm{Tr}\, u|^2$$

for all $u \in W(\Omega)$. Therefore the norm $|||\cdot|||_A$ is equivalent on $W(\Omega)$ to the norm defined by (3). Moreover, it follows that $|||\cdot|||_A$ does not depend, up to equivalence, on the set A.

Next we show that the extension operator E is continuous from $\big(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\big)$ into $W(\mathbb{R}^d)$, where $|||\cdot|||_{\Omega_{R_0}} = |||\cdot|||_A$ with the choice $A = \Omega_{R_0}$. Let $u \in W(\Omega)$. Then

$$\begin{aligned}
\int_{\mathbb{R}^d} |\nabla E u|^2 + \int_{\Omega_{R_0}} |Eu|^2 &= \int_{\mathbb{R}^d \setminus B_{R_0}} |\nabla E u|^2 + \int_{B_{R_0}} |\nabla E u|^2 + \int_{\Omega_{R_0}} |Eu|^2 \\
&\le \int_{\mathbb{R}^d \setminus B_{R_0}} |\nabla u|^2 + \|(Eu)|_{B_{R_0}}\|_{H^1(B_{R_0})}^2 \\
&= \int_{\mathbb{R}^d \setminus B_{R_0}} |\nabla u|^2 + \|E_0(u|_{\Omega_{R_0}})\|_{H^1(B_{R_0})}^2 \\
&\le \int_\Omega |\nabla u|^2 + \|E_0\|^2 \, \|u|_{\Omega_{R_0}}\|_{H^1(\Omega_{R_0})}^2 \le (1 + \|E_0\|^2) |||u|||_{\Omega_{R_0}}^2 .
\end{aligned}$$

Hence by Lemma 2.5 the operator E maps $\big(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\big)$ continuously into $W(\mathbb{R}^d)$. Moreover,

$$\|u\|_{W(\Omega)}^2 = \int_\Omega |\nabla E u|^2 + |\langle E u\rangle|^2 \le \|Eu\|_{W(\mathbb{R}^d)}^2 \le \|E\|^2_{(W(\Omega), |||\cdot|||_{\Omega_{R_0}}) \to W(\mathbb{R}^d)} |||u|||_{\Omega_{R_0}}^2$$

for all $u \in W(\Omega)$, so the inclusion $\big(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\big) \hookrightarrow W(\Omega)$ is continuous.

We now prove (a). Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W(\Omega)$ and write $v_n = u_n - \langle u_n \rangle$. By Proposition 2.10 the sequence $(v_n)_{n\in\mathbb{N}}$ is Cauchy in $L_{\mathfrak{p}}(\Omega)$. Since $\mathfrak{p} > 2$ and $|\Omega_{R_0}| < \infty$, one deduces that the sequence $(v_n|_{\Omega_{R_0}})_{n\in\mathbb{N}}$ is Cauchy in $L_2(\Omega_{R_0})$. Moreover, by assumption the sequence $(\langle u_n \rangle)_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$, so $(u_n|_{\Omega_{R_0}})_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_2(\Omega_{R_0})$. Hence $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\big(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\big)$. Since $\big(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\big)$ is complete, there exists a $u \in W(\Omega)$ such that $\lim |||u - u_n|||_{\Omega_{R_0}} = 0$. In particular, $\lim \|\nabla(u - u_n)\|_{L_2(\Omega)} = 0$. Moreover, the map $u \mapsto \langle u \rangle$ is continuous from $\big(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\big)$ into $\mathbb{C}$, since the inclusion $\big(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\big) \hookrightarrow W(\Omega)$ is continuous. So $\lim \langle u - u_n \rangle = 0$ and it follows that $\lim u_n = u$ in $W(\Omega)$.

Since $\big(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\big)$ is complete, it follows from the closed graph theorem that the norms $|||\cdot|||_{\Omega_{R_0}}$ and $\|\cdot\|_{W(\Omega)}$ are equivalent on $W(\Omega)$. Hence Statement (c) is valid. Moreover, since the equivalence of $|||\cdot|||_A$ with the norm (3) does not depend on the set A, Statements (d) and (e) follow from the conclusion of the second paragraph.

Lastly we prove (b). Since Ω_{R_0} has Lipschitz boundary, the map $\mathrm{Tr}_{R_0} \colon H^1(\Omega_{R_0}) \to L_2(\Gamma)$ is compact. Moreover, the restriction $u \mapsto u|_{\Omega_{R_0}}$ is continuous from $\big(W(\Omega), |||\cdot|||_{\Omega_{R_0}}\big)$ into $H^1(\Omega_{R_0})$. Hence the equivalence of the norms $|||\cdot|||_{\Omega_{R_0}}$ and $\|\cdot\|_{W(\Omega)}$ yields that the composition Tr is compact on $W(\Omega)$. $\square$

We define

$$W^D(\Omega) = \overline{\{\chi|_\Omega : \chi \in C_c^\infty(\mathbb{R}^d)\}}^{W(\Omega)}$$

and equip $W^D(\Omega)$ with the norm $\|\cdot\|_{W^D(\Omega)}$ induced by the norm on $W(\Omega)$. Then $W^D(\Omega)$ is a Hilbert space and

$$\|u\|_{W^D(\Omega)} = \Big(\int_\Omega |\nabla u|^2\Big)^{1/2}$$

by the following assertion.

Proposition 2.12. *The space $W(\Omega)$ admits the orthogonal decomposition*

$$W(\Omega) = W^D(\Omega) \oplus \mathbb{C}\mathbb{1}_\Omega.$$

Moreover, $W^D(\Omega) = \{u \in W(\Omega) : \langle u \rangle = 0\} = W(\Omega) \cap L_\mathfrak{p}(\Omega)$.

Proof. By an argument similar to the proof of Proposition 2.7, with Proposition 2.10 in place Proposition 2.2, one deduces that $W^D(\Omega) \subset \{u \in W(\Omega) : \langle u \rangle = 0\} \subset W(\Omega) \cap L_\mathfrak{p}(\Omega)$.

Let $u \in W(\Omega) \cap L_\mathfrak{p}(\Omega)$. Then $Eu \in W(\mathbb{R}^d)$. Since

$$\Big|\frac{1}{|\Omega_R|} \int_{\Omega_R} u\Big| \le |\Omega_R|^{-1/\mathfrak{p}} \|u\|_{L_\mathfrak{p}(\Omega)}$$

for all $R > 0$, it follows that $\langle u \rangle = 0$. So $\langle Eu \rangle = \langle u \rangle = 0$ and by Proposition 2.7 one has that $Eu \in W^D(\mathbb{R}^d)$. Then there exists a sequence $(\chi_n)_{n\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ such that $\lim \chi_n = Eu$ in $W(\mathbb{R}^d)$. Hence $\lim \chi_n|_\Omega = (Eu)|_\Omega = u$ in $W(\Omega)$. Therefore $u \in W^D(\Omega)$ and $W(\Omega) \cap L_\mathfrak{p}(\Omega) \subset W^D(\Omega)$.

Since $u - \langle u \rangle \in W^D(\Omega)$ for all $u \in W(\Omega)$, one deduces that $W(\Omega) \subset W^D(\Omega) + \mathbb{C}\mathbb{1}_\Omega$. The claim then follows from the fact that $\mathbb{1}_\Omega \perp W^D(\Omega)$. □

Corollary 2.13. *$H^1(\Omega) \subset W^D(\Omega)$.*

Proof. The claim follows from an argument similar to the proof of Corollary 2.8. □

Corollary 2.14. *Let $u \in W(\Omega)$. Then $\langle |u| \rangle = |\langle u \rangle|$.*

Proof. The claim follows from an argument similar to the proof of Corollary 2.9. □

Our final consideration for this section is the lattice structure of the subspace

$$W(\Omega, \mathbb{R}) = \{u \in W(\Omega) : u \text{ is real-valued}\}$$

of $W(\Omega)$. The space $W^D(\Omega, \mathbb{R})$ is defined similarly. We note the following basic properties.

Proposition 2.15. (a) *Let $u \in W(\Omega, \mathbb{R})$. Then $u^+, u^-, |u| \in W(\Omega)$ and $\| |u| \|_{W(\Omega)} = \|u\|_{W(\Omega)}$.*

(b) *Let $u \in W^D(\Omega, \mathbb{R})$. Then $u^+, u^-, |u| \in W^D(\Omega)$.*

(c) *The maps $u \mapsto u^+$, $u \mapsto u^-$ and $u \mapsto |u|$ are continuous from $W(\Omega, \mathbb{R})$ into $W(\Omega)$.*

(d) *Let $u \in W(\Omega, \mathbb{R})$. Then $\operatorname{Tr}(u^+) = (\operatorname{Tr} u)^+$, $\operatorname{Tr}(u^-) = (\operatorname{Tr} u)^-$ and $\operatorname{Tr}|u| = |\operatorname{Tr} u|$.*

Proof. Note that by the identities $u^+ = \frac{1}{2}(|u|+u)$ and $u^- = \frac{1}{2}(|u|-u)$, it suffices to prove each statement for the case $|u|$.

We first prove (a). Let $k \in \{1, \ldots, d\}$. Then by [GT83] Lemmas 7.6 and 7.7

$$\partial_k|u| = \partial_k(u^+) + \partial_k(u^-) = \mathbb{1}_{[u>0]}\partial_k u + \mathbb{1}_{[u<0]}\partial_k u = \partial_k u - \mathbb{1}_{[u=0]}\partial_k u = \partial_k u.$$

So $u^+, u^-, |u| \in W(\Omega)$ and it follows from Corollary 2.14 that $\| \, |u| \, \|_{W(\Omega)} = \|u\|_{W(\Omega)}$. Statement (b) then follows from the fact that $W^D(\Omega) = W(\Omega) \cap L_{\mathfrak{p}}(\Omega)$, together with the lattice structure of $L_{\mathfrak{p}}(\Omega)$.

We now prove (c). Let $u, u_1, u_2, \ldots \in W(\Omega, \mathbb{R})$ and suppose that $\lim u_n = u$ in $W(\Omega)$. Then $\lim |\nabla u_n| = |\nabla u|$ in $L_2(\Omega)$. Moreover, Proposition 2.11(d) provides that $\lim |u_n| = |u|$ in $L_{2,\mathrm{loc}}(\Omega)$. Since $\| \, |u_n| \, \|_{W(\Omega)} = \|u_n\|_{W(\Omega)}$ for all $n \in \mathbb{N}$, by passing to a subsequence if necessary we may assume that there exists a $w \in W(\Omega)$ such that $\lim |u_n| = w$ weakly in $W(\Omega)$. Then $|u| = \lim |u_n| = w$ weakly in $L_{2,\mathrm{loc}}(\Omega)$, so $\lim |u_n| = |u|$ weakly in $W(\Omega)$. Moreover,

$$\lim_{n\to\infty} \| \, |u_n| \, \|_{W(\Omega)} = \lim_{n\to\infty} \|u_n\|_{W(\Omega)} = \|u\|_{W(\Omega)} = \| \, |u| \, \|_{W(\Omega)}.$$

Hence $\lim |u_n| = |u|$ in $W(\Omega)$ and Statement (c) follows.

Finally we prove (d). By density there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\Omega_{R_0}) \cap C(\overline{\Omega_{R_0}})$ such that $\lim u_n = u|_{\Omega_{R_0}}$ in $H^1(\Omega_{R_0})$. Then $\lim |u_n| = |u|_{\Omega_{R_0}}|$ in $H^1(\Omega_{R_0})$. Moreover, since $\mathrm{Tr}_{\Omega_{R_0}}$ is continuous it follows that $\lim \mathrm{Tr}_{R_0} u_n = \mathrm{Tr}\, u$ in $L_2(\Gamma)$ and

$$|\mathrm{Tr}\, u| = \lim_{n\to\infty} |\mathrm{Tr}_{R_0} u_n| = \lim_{n\to\infty} |u_n|_\Gamma| = \lim_{n\to\infty} \mathrm{Tr}_{R_0}|u_n| = \mathrm{Tr}_{R_0}|u|_{\Omega_{R_0}}| = \mathrm{Tr}\,|u|$$

as required. $\square$

3 A Robin problem on an exterior domain

In this section we formulate the Robin-type problem (1) in a variational sense, using the spaces $W(\Omega)$ and $W^D(\Omega)$ to distinguish boundary conditions at infinity. In each case well-posedness follows from the Lax–Milgram theorem and we subsequently obtain that the associated solution operator is compact and submarkovian. We conclude this section with the proof of Theorem 1.1, which states that solutions of (1) are continuous on $\overline{\Omega}$. While basic properties of the solution operator follow from minimal modifications of the arguments used in [AE15] for the Laplacian, Theorem 1.1 requires a new approach.

Throughout this thesis we assume bounded measurable real-valued coefficients $a_{kl} \in L_\infty(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$. Where additional regularity of the coefficients is required, we state this explicitly. We further assume that there exists a $\mu > 0$ such that

$$\operatorname{Re} \sum_{k,l=1}^{d} a_{kl}(x)\xi_k\overline{\xi_l} \geq \mu|\xi|^2 \tag{4}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{C}^d$.

Define the sesquilinear form $\mathfrak{a}\colon W(\Omega) \times W(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \sum_{k,l=1}^{d} \int_\Omega a_{kl}(\partial_k u)\overline{\partial_l v}.$$

Then $\mathfrak{a}$ is continuous and elliptic in the sense of [AE12] (2.1) and (2.2) respectively, where $V = H = W(\Omega)$ and $j = \mathrm{id}_{W(\Omega)}$. Moreover, $\mathfrak{a}$ induces an equivalent norm on $W(\Omega)$.

Lemma 3.1. *The norm*

$$u \mapsto \Big(\operatorname{Re}\mathfrak{a}(u) + \int_\Gamma |\operatorname{Tr} u|^2\Big)^{1/2} \tag{5}$$

is equivalent to the norm $\|\cdot\|_{W(\Omega)}$ *on* $W(\Omega)$.

Proof. By the continuity and ellipticity of the form $\mathfrak{a}$, the norm defined by (5) is equivalent to the norm $u \mapsto \big(\int_\Omega |\nabla u|^2 + \int_\Gamma |\operatorname{Tr} u|^2\big)^{1/2}$ on $W(\Omega)$. Then the assertion follows from Proposition 2.11(e). □

We realise the elliptic operator $-\sum \partial_l(a_{kl}\partial_k)$ on Ω in a distributional sense as follows. Define the map $\mathcal{A}\colon H^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}(\Omega)'$ by

$$\langle \mathcal{A}u, v\rangle = \sum_{k,l=1}^{d} \int_\Omega a_{kl}(\partial_k u)\overline{\partial_l v}, \qquad v \in \mathcal{D}(\Omega).$$

For all $u \in W(\Omega)$ and $f \in L_2(\Omega)$, we write $\mathcal{A}u = f$ if $\mathfrak{a}(u, v) = (f, v)_{L_2(\Omega)}$ for all $v \in C_c^\infty(\Omega)$. We write $\mathcal{A}u \in L_2(\Omega)$ if there exists an $f \in L_2(\Omega)$ such that $\mathcal{A}u = f$. Similarly, for all $R \geq R_0$ we write $(\mathcal{A}u)|_{C_c^\infty(\Omega_R)} \in L_2(\Omega_R)$ if there exists an $f \in L_2(\Omega_R)$ such that $\mathfrak{a}(u, v) = (f, v)_{L_2(\Omega)}$ for all $v \in C_c^\infty(\Omega)$ with $\operatorname{supp} v \subset \Omega_R$.

Note that $\{\chi|_\Gamma : \chi \in C_c^\infty(\mathbb{R}^d)\} \subset \operatorname{Tr} W(\Omega)$. Since by the Stone–Weierstraß theorem the set $\{\chi|_\Gamma : \chi \in C_c^\infty(\mathbb{R}^d)\}$ is dense in $(C(\Gamma), \|\cdot\|_\infty)$ and by [EG92] Theorem 2.1 the measure σ is Borel regular, one deduces that $\operatorname{Tr} W(\Omega)$ is dense in $L_2(\Gamma)$.

On a Lipschitz domain the potential presence of corner points means that the conormal derivative may not exist in the classical sense. Hence we adopt the following variational definition, formulated in terms of the Gauß–Green formula.

We say that $u \in W(\Omega)$ has a **conormal derivative on** Γ if there exist $R \geq R_0$ and $\psi \in L_2(\Gamma)$ such that $(\mathcal{A}u)|_{C_c^\infty(\Omega_R)} \in L_2(\Omega_R)$ and

$$\mathfrak{a}(u, v) - \int_{\Omega_R} (\mathcal{A}u)\overline{v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v} \tag{6}$$

for all $v \in C_c^\infty(B_R)$. Then ψ is unique since $\overline{\operatorname{Tr} W(\Omega)} = L_2(\Gamma)$ and we write $\partial_\nu u = \psi$.

Let $S \geq R_0$ and suppose that $u \in W(\Omega)$ with $(\mathcal{A}u)|_{C_c^\infty(\Omega_S)} \in L_2(\Omega_S)$. Then u has a conormal derivative on Γ if and only if (6) is valid for all $R \in [R_0, S]$, since $C_c^\infty(B_R) \subset C_c^\infty(B_S)$. Moreover, if (6) is valid for all $v \in C_c^\infty(B_R)$, then by density (6) is also valid for all $v \in H^1(B_R)$ with $\mathbb{1}_{\partial B_R} \operatorname{Tr}_{\Omega_R} v = 0$. These properties extend to Ω in the following sense.

Proposition 3.2. *Let $u \in W(\Omega)$. Suppose that $\mathcal{A}u \in L_2(\Omega)$ and that u has a conormal derivative $\psi \in L_2(\Gamma)$. Then*

$$\mathfrak{a}(u, v) - \int_\Omega (\mathcal{A}u)\overline{v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v} \tag{7}$$

for all $v \in W^D(\Omega)$.

Proof. Since $\partial_\nu u = \psi$ there exists an $R \geq R_0$ such that (6) is valid for all $v \in C_c^\infty(B_R)$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$. Then there exist $\chi_1, \chi_2 \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} \chi_1 \subset B_R$, $\operatorname{supp} \chi_2 \subset \mathbb{R}^d \setminus \overline{B_{R_0 - 1}}$ and $\chi = \chi_1 + \chi_2$. So $\chi_1 \in C_c^\infty(B_R)$ and $\chi_2 \in C_c^\infty(\Omega)$. Therefore

$$\begin{aligned}
\mathfrak{a}(u, \chi|_\Omega) - \int_\Omega (\mathcal{A}u)\overline{\chi} &= \mathfrak{a}(u, \chi_1|_\Omega) - \int_{\Omega_R} (\mathcal{A}u)\overline{\chi_1} + \mathfrak{a}(u, \chi_2) - \int_\Omega (\mathcal{A}u)\overline{\chi_2} \\
&= \int_\Gamma \psi \, \overline{\operatorname{Tr}(\chi_1|_\Omega)} + \mathfrak{a}(u, \chi_2) - \mathfrak{a}(u, \chi_2) \\
&= \int_\Gamma \psi \, \overline{\operatorname{Tr}(\chi|_\Omega)}.
\end{aligned}$$

So (7) is valid for all $v \in \{\chi|_\Omega : \chi \in C_c^\infty(\mathbb{R}^d)\}$ and then by density for all $v \in W^D(\Omega)$. □

We use the following assertion to formulate (1) precisely.

Proposition 3.3. (a) *Let $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$. Then there exists a unique $u \in W^D(\Omega)$ such that*

$$\mathfrak{a}(u, v) + \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v} \tag{8}$$

for all $v \in W^D(\Omega)$.

(b) *Let $\psi \in L_2(\Gamma)$ and $\lambda > 0$. Then there exists a unique $u \in W(\Omega)$ such that*

$$\mathfrak{a}(u, v) + \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v} \tag{9}$$

for all $v \in W(\Omega)$.

Proof. We prove (b). Define the sesquilinear form $\mathfrak{a}_\lambda \colon W(\Omega) \times W(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}_\lambda(u, v) = \mathfrak{a}(u, v) + \lambda(\operatorname{Tr} u, \operatorname{Tr} v)_{L_2(\Gamma)}.$$

Since the form $\mathfrak{a}$ and trace map Tr are continuous on $W(\Omega)$ it follows that $\mathfrak{a}_\lambda$ is continuous. Moreover, by Lemma 3.1 the form $\mathfrak{a}_\lambda$ is coercive on $W(\Omega)$. Consider the linear functional $f \colon v \mapsto \int_\Gamma \psi \, \overline{\operatorname{Tr} v}$ on $W(\Omega)$. Then f is continuous since Tr is continuous. The claim then follows from the Lax–Milgram theorem applied to the pair $(\mathfrak{a}_\lambda, f)$.

The proof of (a) is similar. $\square$

Given $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$, we call $u \in W^D(\Omega)$ a **solution of (1) with Dirichlet boundary conditions at infinity** if u satisfies (8) for all $v \in W^D(\Omega)$. Moreover, we define the solution operator $B_\lambda^D \colon L_2(\Gamma) \to W^D(\Omega)$ by $B_\lambda^D \psi = u$. It follows from Lemma 3.1 together with the continuity of Tr that the operator B_λ^D is bounded.

Given $\psi \in L_2(\Gamma)$ and $\lambda > 0$, we call $u \in W(\Omega)$ a **solution of (1) with Neumann boundary conditions at infinity** if u satisfies (9) for all $v \in W(\Omega)$. Moreover, we define the bounded solution operator $B_\lambda \colon L_2(\Gamma) \to W(\Omega)$ by $B_\lambda \psi = u$.

Observe that if $\lambda = 0$ then $u \in W(\Omega)$ satisfies (9) for all $v \in W(\Omega)$ only when $\int_\Gamma \psi = 0$, since $\mathbb{1}_\Omega \in W(\Omega)$. It is for this reason that we exclude the case $\lambda = 0$ when considering Neumann boundary conditions at infinity.

As a consequence of the characterisation $W^D(\Omega) = \{u \in W(\Omega) : \langle u \rangle = 0\}$, we obtain the following relationship between solutions of the two versions of (1).

Proposition 3.4. *Let $\psi \in L_2(\Gamma)$. Then*

$$B_\lambda \psi = B_\lambda^D \psi + \langle B_\lambda \psi \rangle \big(\mathbb{1}_\Omega - \lambda B_\lambda^D \mathbb{1}_\Gamma\big)$$

for all $\lambda > 0$.

Proof. Let $\lambda > 0$ and write $u = B_\lambda \psi$. Then $u - \langle u \rangle \mathbb{1}_\Omega \in W^D(\Omega)$ by Proposition 2.12. Moreover, for all $v \in W^D(\Omega)$

$$\begin{aligned}
\mathfrak{a}(u - \langle u \rangle \mathbb{1}_\Omega, v) + \lambda \int_\Gamma \operatorname{Tr}(u - \langle u \rangle \mathbb{1}_\Omega) \, \overline{\operatorname{Tr} v} \\
&= \mathfrak{a}(u, v) + \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr} v} - \lambda \int_\Gamma \langle u \rangle \mathbb{1}_\Gamma \overline{\operatorname{Tr} v} \\
&= \int_\Gamma \psi \, \overline{\operatorname{Tr} v} - \int_\Gamma \lambda \langle u \rangle \mathbb{1}_\Gamma \, \overline{\operatorname{Tr} v} \\
&= \int_\Gamma (\psi - \lambda \langle u \rangle \mathbb{1}_\Gamma) \, \overline{\operatorname{Tr} v}.
\end{aligned}$$

Then $B_\lambda^D(\psi - \lambda \langle u \rangle \mathbb{1}_\Gamma) = u - \langle u \rangle \mathbb{1}_\Omega$ and

$$u = B_\lambda^D \psi - \lambda \langle u \rangle B_\lambda^D \mathbb{1}_\Gamma + \langle u \rangle \mathbb{1}_\Omega = B_\lambda^D \psi + \langle u \rangle \big(\mathbb{1}_\Omega - \lambda B_\lambda^D \mathbb{1}_\Gamma\big)$$

as required. $\square$

We denote by

$$L_2(\Gamma)^+ = \{\varphi \in L_2(\Gamma) : \varphi \geq 0\}$$

the positive cone in $L_2(\Gamma)$.

The operators B_λ and B_λ^D are positivity preserving, in that positive data correspond to positive solutions.

Proposition 3.5. (a) *Let $\psi \in L_2(\Gamma)^+$ and $\lambda \geq 0$. Then $B_\lambda^D \psi \geq 0$.*

(b) *Let $\psi \in L_2(\Gamma)^+$ and $\lambda > 0$. Then $B_\lambda \psi \geq 0$.*

Proof. We first prove (b). Write $\xi = -\psi$ and $u = B_\lambda \xi$. Then $u \in W(\Omega, \mathbb{R})$, so $u^+ \in W(\Omega)$ by Proposition 2.15(a). Moreover, $\mathfrak{a}(u, u^+) = \mathfrak{a}(u^+)$ by [GT83] Lemma 7.6. Then the choice $v = u^+$ in (9) yields that

$$\mathfrak{a}(u^+) + \lambda \int_\Gamma |\mathrm{Tr}\,(u^+)|^2 = \int_\Gamma \xi\,(\mathrm{Tr}\,u)^+ \leq 0,$$

since $\mathrm{Tr}\,(u^+) = (\mathrm{Tr}\,u)^+$ by Proposition 2.15(d). Hence $\|u^+\|_{W(\Omega)} = 0$ by Lemma 3.1 and it follows that $-B_\lambda \psi = B_\lambda \xi = u \leq 0$.

We now prove (a). Again write $\xi = -\psi$ and $u = B_\lambda^D \xi$. Then $u^+ \in W^D(\Omega)$. Using the ellipticity condition (4), one deduces in a manner similar to the above that

$$\mu \|u^+\|^2_{W^D(\Omega)} = \mu \int_\Omega |\nabla(u^+)|^2 \leq \mathfrak{a}(u^+) + \lambda \int_\Gamma |\mathrm{Tr}\,(u^+)|^2 = \int_\Gamma \xi\,(\mathrm{Tr}\,u)^+ \leq 0.$$

Hence $u^+ = 0$ as claimed. □

Solutions satisfying Dirichlet boundary conditions at infinity are dominated by their Neumann counterpart.

Proposition 3.6. *Let $\psi \in L_2(\Gamma)^+$ and $\lambda > 0$. Then $B_\lambda^D \psi \leq B_\lambda \psi$.*

Proof. Write $u = B_\lambda^D \psi$ and $w = B_\lambda \psi$. Then by definition

$$\mathfrak{a}(u, v) + \lambda \int_\Gamma \mathrm{Tr}\,u\,\overline{\mathrm{Tr}\,v} = \int_\Gamma \psi\,\overline{\mathrm{Tr}\,v}$$

for all $v \in W^D(\Omega)$ and

$$\mathfrak{a}(w, v) + \lambda \int_\Gamma \mathrm{Tr}\,w\,\overline{\mathrm{Tr}\,v} = \int_\Gamma \psi\,\overline{\mathrm{Tr}\,v}$$

for all $v \in W(\Omega)$. So

$$\mathfrak{a}(u - w, v) + \lambda \int_\Gamma \mathrm{Tr}\,(u - w)\,\overline{\mathrm{Tr}\,v} = 0 \tag{10}$$

for all $v \in W^D(\Omega)$.

It follows from Proposition 3.5 that $u \geq 0$ and $w \geq 0$. Since $W^D(\Omega) = W(\Omega) \cap L_{\mathfrak{p}}(\Omega)$ by Proposition 2.12, one obtains that $(u - w)^+ \leq u \in L_{\mathfrak{p}}(\Omega)$. Moreover, $(u - w)^+ \in W(\Omega)$ by Proposition 2.15(a), so $(u - w)^+ \in W^D(\Omega)$. The choice $v = (u - w)^+$ in (10) then yields that

$$\mathfrak{a}((u - w)^+) + \lambda \int_\Gamma |\mathrm{Tr}\,(u - w)^+|^2 = 0.$$

Hence $\|(u - w)^+\|_{W(\Omega)} = 0$ by Lemma 3.1, so $(u - w)^+ = 0$ and $u \leq w$. □

The operators B_λ and B_λ^D are decreasing in the parameter λ.

Proposition 3.7. (a) *Let* $\psi \in L_2(\Gamma)^+$ *and* $0 \le \lambda_1 < \lambda_2$. *Then* $B^D_{\lambda_2}\psi \le B^D_{\lambda_1}\psi$.

(b) *Let* $\psi \in L_2(\Gamma)^+$ *and* $0 < \lambda_1 < \lambda_2$. *Then* $B_{\lambda_2}\psi \le B_{\lambda_1}\psi$.

Proof. We prove (b). Write $u_1 = B_{\lambda_1}\psi$ and $u_2 = B_{\lambda_2}\psi$. Then

$$\mathfrak{a}(u_1, v) + \lambda_1 \int_\Gamma \operatorname{Tr} u_1 \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}$$

and

$$\mathfrak{a}(u_2, v) + \lambda_2 \int_\Gamma \operatorname{Tr} u_2 \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}$$

for all $v \in W(\Omega)$. Note that $u_2 \ge 0$ by Proposition 3.5. Let $v \in W(\Omega)$ be such that $v \ge 0$. Then $\operatorname{Tr} u_2 \ge 0$ and $\operatorname{Tr} v \ge 0$ by Proposition 2.15(d), so

$$\begin{aligned}
&\mathfrak{a}(u_2 - u_1, v) + \lambda_1 \int_\Gamma \operatorname{Tr}(u_2 - u_1) \operatorname{Tr} v \\
&\qquad \le \mathfrak{a}(u_2 - u_1, v) + \lambda_1 \int_\Gamma \operatorname{Tr}(u_2 - u_1) \operatorname{Tr} v + (\lambda_2 - \lambda_1) \int_\Gamma \operatorname{Tr} u_2 \operatorname{Tr} v \\
&\qquad = \mathfrak{a}(u_2, v) + \lambda_2 \int_\Gamma \operatorname{Tr} u_2 \operatorname{Tr} v - \mathfrak{a}(u_1, v) - \lambda_1 \int_\Gamma \operatorname{Tr} u_1 \operatorname{Tr} v = 0.
\end{aligned}$$

Then the choice $v = (u_2 - u_1)^+$ yields that

$$\mathfrak{a}((u_2 - u_1)^+) + \lambda_1 \int_\Gamma |\operatorname{Tr}(u_2 - u_1)^+|^2 \le 0.$$

Hence $(u_2 - u_1)^+ = 0$ by Lemma 3.1 and the result follows.

The proof of (a) is similar. $\square$

Let X denote a σ-finite measure space and let $p \in [1, \infty]$. An operator $B \in \mathcal{L}\big(L_p(X)\big)$ satisfying $|Bu| \le \mathbb{1}$ for all $u \in L_p(X)$ with $|u| \le \mathbb{1}$ is often called submarkovian. The operators λB_λ and λB^D_λ are submarkovian and extrapolate consistently to $L_p(\Gamma)$ in the following sense.

Corollary 3.8. (a) *Let* $\lambda \ge 0$. *Then* $\lambda B^D_\lambda \mathbb{1}_\Gamma \le \mathbb{1}_\Omega$.

(b) *Let* $\lambda > 0$. *Then* $\lambda B_\lambda \mathbb{1}_\Gamma = \mathbb{1}_\Omega$.

(c) *Let* $R \ge R_0$ *and* $\lambda \ge 0$. *Then the map* $\psi \mapsto \mathbb{1}_{\Omega_R} B^D_\lambda \psi$ *is continuous from* $L_p(\Gamma)$ *into* $L_p(\Omega)$ *for all* $p \in [2, \infty]$.

(d) *Let* $R \ge R_0$ *and* $\lambda > 0$. *Then the map* $\psi \mapsto \mathbb{1}_{\Omega_R} B_\lambda \psi$ *is continuous from* $L_p(\Gamma)$ *into* $L_p(\Omega)$ *for all* $p \in [2, \infty]$.

Proof. We first prove (b). Since

$$\mathfrak{a}\big(\tfrac{1}{\lambda}\mathbb{1}_\Omega, v\big) + \lambda \int_\Gamma \operatorname{Tr}\big(\tfrac{1}{\lambda}\mathbb{1}_\Omega\big) \, \overline{\operatorname{Tr} v} = \int_\Gamma \mathbb{1}_\Gamma \, \overline{\operatorname{Tr} v}$$

for all $v \in W(\Omega)$, it follows that $B_\lambda \mathbb{1}_\Gamma = \frac{1}{\lambda}\mathbb{1}_\Omega$. Then Statement (a) follows from Proposition 3.6.

We now prove (c). Let $\psi \in L_2(\Gamma)$ and write $u = B_\lambda^D \psi$. Recall that $\Omega_R \subset \Omega$ is a bounded Lipschitz domain. Then by Proposition 2.11(d) together with Lemma 3.1, there exists a $c > 0$ such that

$$\begin{aligned}
\|\mathbb{1}_{\Omega_R} B_\lambda^D \psi\|_{L_2(\Omega)}^2 = \int_{\Omega_R} |u|^2 &\leq \int_\Omega |\nabla u|^2 + \int_{\Omega_R} |u|^2 \\
&\leq c\big(\operatorname{Re} \mathfrak{a}(u) + \lambda \int_\Gamma |\operatorname{Tr} u|^2 \big) = c \operatorname{Re} \int_\Gamma \psi \, \overline{\operatorname{Tr} u} \\
&\leq c \|\operatorname{Tr}\|_{\mathcal{L}(W(\Omega), L_2(\Gamma))} \|B_\lambda^D\|_{\mathcal{L}(L_2(\Gamma), W(\Omega))} \|\psi\|_{L_2(\Gamma)}^2.
\end{aligned}$$

Hence the map $\psi \mapsto \mathbb{1}_{\Omega_R} B_\lambda^D \psi$ is continuous from $L_2(\Gamma)$ into $L_2(\Omega)$.

Now let $\psi \in L_\infty(\Gamma)$. Then $|\psi| \leq \|\psi\|_{L_\infty(\Gamma)} \mathbb{1}_\Gamma$ and it follows from Proposition 3.5(a) that $B_\lambda^D |\psi| \leq \|\psi\|_{L_\infty(\Gamma)} B_\lambda^D \mathbb{1}_\Gamma$. Moreover,

$$|B_\lambda^D \psi| = \sup_{\alpha \in [0, 2\pi]} \operatorname{Re}(e^{i\alpha} B_\lambda^D \psi) = \sup_{\alpha \in [0, 2\pi]} B_\lambda^D (\operatorname{Re} e^{i\alpha} \psi) \leq \sup_{\alpha \in [0, 2\pi]} B_\lambda^D |e^{i\alpha} \psi| = B_\lambda^D |\psi|$$

and therefore $|B_\lambda^D \psi| \leq B_\lambda^D |\psi| \leq \|\psi\|_{L_\infty(\Gamma)} \, B_\lambda^D \mathbb{1}_\Gamma \leq \|\psi\|_{L_\infty(\Gamma)} \, \frac{1}{\lambda} \mathbb{1}_\Omega$ by (a). Then

$$\|\mathbb{1}_{\Omega_R} B_\lambda^D \psi\|_{L_\infty(\Omega)} \leq \frac{1}{\lambda} \|\psi\|_{L_\infty(\Gamma)},$$

so the map $\psi \mapsto \mathbb{1}_{\Omega_R} B_\lambda^D \psi$ is continuous from $L_\infty(\Gamma)$ into $L_\infty(\Omega)$. Now (c) follows from an interpolation argument.

The proof of (d) is similar. □

Next we show that the operators B_λ and B_λ^D are compact. We use this fact in the proof of Proposition 4.5 to obtain convergence of the solution operator associated with a truncated version of the boundary value problem (1), and in the proof of Proposition 5.5 to establish that the Dirichlet-to-Neumann operator has compact resolvent.

Proposition 3.9. (a) *Let $\lambda \geq 0$. Then the operator B_λ^D is compact.*

(b) *Let $\lambda > 0$. Then the operator B_λ is compact.*

Proof. We prove (b). Define the norm $|||\cdot|||$ on $W(\Omega)$ by

$$|||u||| = \big(\operatorname{Re} \mathfrak{a}(u) + \lambda \int_\Gamma |\operatorname{Tr} u|^2 \big)^{1/2}.$$

Let $\psi, \psi_1, \psi_2, \ldots \in L_2(\Gamma)$ and suppose that $\lim \psi_n = \psi$ weakly in $L_2(\Gamma)$. Write $u = B_\lambda \psi$ and $u_n = B_\lambda \psi_n$. Then $\lim u_n = u$ weakly in $W(\Omega)$, since B_λ is bounded. Hence $\lim u_n = u$ weakly in $(W(\Omega), |||\cdot|||)$ by Lemma 3.1. Moreover, by Proposition 2.11(b) the trace map Tr is compact, so $\lim \operatorname{Tr} u_n = \operatorname{Tr} u$ in $L_2(\Gamma)$ and $\lim (\psi_n, \operatorname{Tr} u_n)_{L_2(\Gamma)} = (\psi, \operatorname{Tr} u)_{L_2(\Gamma)}$. Since

$$\mathfrak{a}(u_n) + \lambda \int_\Gamma |\operatorname{Tr} u_n|^2 = \int_\Gamma \psi_n \, \overline{\operatorname{Tr} u_n}$$

for all $n \in \mathbb{N}$, it follows that

$$\lim_{n \to \infty} \operatorname{Re} \mathfrak{a}(u_n) + \lambda \int_\Gamma |\operatorname{Tr} u_n|^2 = \operatorname{Re} \int_\Gamma \psi \, \overline{\operatorname{Tr} u} = \operatorname{Re} \mathfrak{a}(u) + \lambda \int_\Gamma |\operatorname{Tr} u|^2.$$

Hence $\lim |||u_n||| = |||u|||$ and one therefore deduces that $\lim u_n = u$ in $(W(\Omega), |||\cdot|||)$. Applying Lemma 3.1 once again then yields that $\lim u_n = u$ in $W(\Omega)$.

The proof of (a) is similar. □

Finally we consider the proof of the following.

Theorem 1.1. (a) *Let $\psi \in L_\infty(\Gamma)$ and $\lambda \geq 0$. Then $B_\lambda^D \psi \in C(\overline{\Omega})$.*

(b) *Let $\psi \in L_\infty(\Gamma)$ and $\lambda > 0$. Then $B_\lambda \psi \in C(\overline{\Omega})$.*

Note that by Nash [Nas58] the solutions $B_\lambda \psi$ and $B_\lambda^D \psi$ each admit a continuous representative on Ω, so it suffices to verify continuity at the boundary Γ. In the proof of the above we use the following pointwise Morrey and Campanato seminorms introduced in [ER15], defined for a reference space $E^- \subset \mathbb{R}^d$.

We denote by

$$E = \{x = (\widetilde{x}, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R} : \|\widetilde{x}\|_{\mathbb{R}^{d-1}} < 1 \text{ and } x_d \in (-1,1)\}$$

the open cylinder in $\mathbb{R}^d$ and write $E^- = \{x \in E : x_d < 0\}$. For all $x \in \mathbb{R}^d$ and $r > 0$ we define

$$E_r^-(x) = E^- \cap B_r(x).$$

If $|E_r^-(x)| > 0$ then we write $\langle u \rangle_{E_r^-(x)} = \frac{1}{|E_r^-(x)|} \int_{E_r^-(x)} u$.

For all $\gamma \in [0,d]$ and $x \in E^-$ we define $\|\cdot\|_{M_{\gamma,x}} \colon L_2(E^-) \to [0,\infty]$ by

$$\|u\|_{M_{\gamma,x}} = \sup_{r \in (0,\frac{1}{2}]} \Big(r^{-\gamma} \int_{E_r^-(x)} |u|^2 \Big)^{1/2}.$$

Moreover, for all $\gamma \in [0,d+2]$ and $x \in E^-$ we define $\|\cdot\|_{\mathcal{M}_{\gamma,x}} \colon L_2(E^-) \to [0,\infty]$ by

$$\|u\|_{\mathcal{M}_{\gamma,x}} = \sup_{r \in (0,\frac{1}{2}]} \Big(r^{-\gamma} \int_{E_r^-(x)} |u - \langle u \rangle_{E_r^-(x)}|^2 \Big)^{1/2}.$$

Then the seminorms $\|\cdot\|_{M_{\gamma,x}}$ and $\|\cdot\|_{\mathcal{M}_{\gamma,x}}$ on $L_2(E^-)$ correspond to those introduced in [ER15] Section 3, with $\Omega = E^-$ and $R_e = \frac{1}{2}$.

The proof of Theorem 1.1 relies on the following extension of [EW20] Proposition 3.1.

Lemma 3.10. *There exists a $\kappa \in (0,1)$ such that for all $\gamma \in [0,d)$ and $\delta \in (0,2]$ with $\gamma + \delta < d - 2 + 2\kappa$, there exists a $c > 0$ such that the following is valid.*

Let $U \subset \mathbb{R}^d$ be an open set and let Φ be a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\Phi(U) = E$ and $\Phi(\Omega \cap U) = E^-$. Let $\psi \in L_\infty(\Gamma)$ and $u \in W(\Omega)$ and suppose that

$$\mathfrak{a}(u,v) + \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}$$

for all $v \in W(\Omega)$. Then

$$\|\nabla(u \circ \Phi^{-1})\|_{M_{\gamma+\delta,x}} \leq c \left(\|\nabla u\|_{L_2(\Omega)} + \|\psi\|_{L_\infty(\Gamma)} + \|\nabla(u \circ \Phi^{-1})\|_{M_{\gamma,x}} + \|u \circ \Phi^{-1}\|_{M_{\gamma+\delta,x}} \right)$$

for all $x \in \frac{1}{2} E^-$.

Proof. The proof is similar to the proof of [EW20] Proposition 3.1. □

We are now able to prove Theorem 1.1.

Proof of Theorem 1.1. We first prove (b). Without loss of generality we may assume that $\lambda = 1$. Let $\kappa \in (0,1)$ be such that Lemma 3.10 is valid. Let $U \subset \mathbb{R}^d$ be an open set and let Φ be a bi-Lipschitz map from an open neighbourhood of $\overline{U}$ onto an open subset of $\mathbb{R}^d$ such that $\Phi(U) = E$ and $\Phi(\Omega \cap U) = E^-$.

Write $u = B_\lambda \psi \in W(\Omega)$. Since E is bounded and Φ^{-1} is uniformly continuous, it follows that $U = \Phi^{-1}(E)$ is bounded. Then $u|_{\Omega \cap U} \in H^1(\Omega \cap U)$, so $u \circ \Phi^{-1} \in H^1(E^-)$ and $\nabla(u \circ \Phi^{-1}) \in L_2(E^-)$, where we write $u \circ \Phi^{-1} = u \circ (\Phi^{-1}|_{E^-})$. Hence there exists a $c_1 > 0$ such that

$$\|\nabla(u \circ \Phi^{-1})\|_{M_0,x} \le c_1 \tag{11}$$

for all $x \in \frac{1}{2}E^-$. By [ER15] Lemma 6.2 there exists a $c > 0$ such that

$$\|u \circ \Phi^{-1}\|_{\mathcal{M}_2,x} \le c$$

for all $x \in \frac{1}{2}E^-$. Then [ER15] Lemma 3.1(a) provides that there exists a $c_2 > 0$ such that

$$\|u \circ \Phi^{-1}\|_{M_2,x} \le c_2 \tag{12}$$

for all $x \in \frac{1}{2}E^-$, since $u \circ \Phi^{-1} \in L_2(E^-)$.

Suppose first that $d = 3$. By Lemma 3.10 there exists a $c > 0$ such that

$$\begin{aligned}\|\nabla(u \circ \Phi^{-1})\|_{M_{1+\kappa},x} &\le c\big(\|\nabla u\|_{L_2(\Omega)} + \|\psi\|_{L_\infty(\Gamma)} + \|\nabla(u \circ \Phi^{-1})\|_{M_0,x} + \|u \circ \Phi^{-1}\|_{M_{1+\kappa},x}\big) \\ &\le c\big(\|\nabla u\|_{L_2(\Omega)} + \|\psi\|_{L_\infty(\Gamma)} + c_1 + c_2\big)\end{aligned}$$

for all $x \in \frac{1}{2}E^-$. Then by [ER15] Lemma 6.2 there exists a $c > 0$ such that

$$\|u \circ \Phi^{-1}\|_{\mathcal{M}_{3+\kappa},x} \le c$$

for all $x \in \frac{1}{2}E^-$. Since by [Nas58] the function u is continuous on Ω, it follows from [ER15] Lemma 3.1(c) that there exists a $c > 0$ such that

$$|(u \circ \Phi^{-1})(x) - (u \circ \Phi^{-1})(y)| \le c|x-y|^{\frac{\kappa}{2}}$$

for all $x, y \in \frac{1}{2}E^-$ with $|x - y| \le \frac{1}{4}$. Hence $u \circ \Phi^{-1}$ is uniformly continuous on $\frac{1}{2}E^-$ and extends to a continuous function on $\overline{\frac{1}{2}E^-}$. Consequently one deduces that $u|_{\Phi^{-1}(\frac{1}{2}E^-)}$ extends to a continuous function on $\overline{\Phi^{-1}\big(\frac{1}{2}E^-\big)} = \Phi^{-1}\big(\overline{\frac{1}{2}E^-}\big)$.

Next suppose that $d > 3$ is odd. Then there exists a $k \ge 2$ such that $d = 2k+1$. Note that $2j + 2 < d - 2 + 2\kappa$ for all $j \in \{0, \ldots, k-2\}$. Then it follows from the estimates (11) and (12), together with an iterative argument using Lemma 3.10 with $\gamma = 2j$ and $\delta = 2$, that there exists a $c_3 > 0$ such that

$$\|\nabla(u \circ \Phi^{-1})\|_{M_{2k-2},x} \le c_3$$

for all $x \in \frac{1}{2}E^-$. So by [ER15] Lemmas 6.2 and 3.1(a) there exists a $c_4 > 0$ such that

$$\|u \circ \Phi^{-1}\|_{M_{2k},x} \le c_4$$

for all $x \in \frac{1}{2}E^-$ and therefore

$$\|u \circ \Phi^{-1}\|_{M_{2k-1+\kappa},x} \le c_4$$

for all $x \in \frac{1}{2}E^-$. Note that

$$2k - 1 + \kappa = d - 2 + \kappa < d - 2 + 2\kappa.$$

Hence by Lemma 3.10 with $\gamma = 2k - 2$ and $\delta = 1 + \kappa$, there exists a $c > 0$ such that

$$\begin{aligned}
&\|\nabla(u \circ \Phi^{-1})\|_{M_{2k-1+\kappa},x} \\
&\quad \leq c\big(\|\nabla u\|_{L_2(\Omega)} + \|\psi\|_{L_\infty(\Gamma)} + \|\nabla(u \circ \Phi^{-1})\|_{M_{2k-2},x} + \|u \circ \Phi^{-1}\|_{M_{2k-1+\kappa},x}\big) \\
&\quad \leq c\big(\|\nabla u\|_{L_2(\Omega)} + \|\psi\|_{L_\infty(\Gamma)} + c_3 + c_4\big)
\end{aligned}$$

for all $x \in \frac{1}{2}E^-$. Therefore [ER15] Lemma 6.2 provides that there exists a $c > 0$ such that

$$\|u \circ \Phi^{-1}\|_{\mathcal{M}_{d+\kappa},x} \leq c$$

for all $x \in \frac{1}{2}E^-$. As in the case $d = 3$, one then concludes that $u|_{\Phi^{-1}(\frac{1}{2}E^-)}$ extends to a continuous function on $\overline{\Phi^{-1}\big(\frac{1}{2}E^-\big)}$.

Finally suppose that d is even. Then there exists a $k \geq 2$ such that $d = 2k$. Since again $2j + 2 < d - 2 + 2\kappa$ for all $j \in \{0, \ldots, k-2\}$, the estimates (11) and (12) together with Lemma 3.10 and an iterative argument yield that there exists a $c_5 > 0$ such that

$$\|\nabla(u \circ \Phi^{-1})\|_{M_{2k-2},x} \leq c_5$$

for all $x \in \frac{1}{2}E^-$. Then by [ER15] Lemmas 6.2 and 3.1(a) there exists a $c_6 > 0$ such that

$$\|u \circ \Phi^{-1}\|_{M_{2k},x} \leq c_6$$

for all $x \in \frac{1}{2}E^-$, so

$$\|u \circ \Phi^{-1}\|_{M_{2k-2+\kappa},x} \leq c_6$$

for all $x \in \frac{1}{2}E^-$. Note that $2k - 2 + \kappa < d - 2 + 2\kappa$. Hence by Lemma 3.10 with $\gamma = 2k - 2$ and $\delta = \kappa$, there exists a $c > 0$ such that

$$\begin{aligned}
&\|\nabla(u \circ \Phi^{-1})\|_{M_{2k-2+\kappa},x} \\
&\quad \leq c\big(\|\nabla u\|_{L_2(\Omega)} + \|\psi\|_{L_\infty(\Gamma)} + \|\nabla(u \circ \Phi^{-1})\|_{M_{2k-2},x} + \|u \circ \Phi^{-1}\|_{M_{2k-2+\kappa},x}\big) \\
&\quad \leq c\big(\|\nabla u\|_{L_2(\Omega)} + \|\psi\|_{L_\infty(\Gamma)} + c_5 + c_6\big)
\end{aligned}$$

for all $x \in \frac{1}{2}E^-$. So [ER15] Lemma 6.2 provides that there exists a $c > 0$ such that

$$\|u \circ \Phi^{-1}\|_{\mathcal{M}_{d+\kappa},x} \leq c$$

for all $x \in \frac{1}{2}E^-$. As in the case $d = 3$, one then concludes that $u|_{\Phi^{-1}(\frac{1}{2}E^-)}$ extends to a continuous function on $\overline{\Phi^{-1}\big(\frac{1}{2}E^-\big)}$.

In each of the above three cases, one deduces by a compactness argument that $u \in C(\overline{\Omega})$.

Since Lemma 3.10 remains valid with the space $W^D(\Omega)$ in place of $W(\Omega)$, Statement (a) follows from an argument similar to the proof of (b). □

4 Convergence of the truncated problem

In this section we consider a version of the Robin-type problem (1) for the truncated domain Ω_R and study convergence in the limit $R \to \infty$. First we establish that the associated solution operator $B_\lambda^D(R)$ converges to B_λ^D in $\mathcal{L}\big(L_2(\Gamma), W^D(\Omega)\big)$. We then show that if the coefficients (a_{kl}) are Lipschitz continuous, solutions of the truncated problem converge locally uniformly on Ω to solutions of (1) that satisfy Dirichlet boundary conditions at infinity. Using this result we prove the following.

Theorem 1.2. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$. Then there exists a $c > 0$ such that the following are valid.*

(a) *Let $\psi \in L_\infty(\Gamma)$ and $\lambda > 0$. Then*

$$|(B_\lambda^D \psi)(x)| \le \frac{c\,\|\psi\|_{L_\infty(\Gamma)}}{\lambda} \cdot \frac{1}{|x|^{d-2}}$$

for all $x \in \Omega \setminus \Omega_{R_0}$.

(b) *Let $\psi \in L_\infty(\Gamma)$ and $\lambda > 0$. Then*

$$|(B_\lambda \psi)(x) - \langle B_\lambda \psi \rangle| \le c \left(\frac{\|\psi\|_{L_\infty(\Gamma)}}{\lambda} + |\langle B_\lambda \psi \rangle| \right) \frac{1}{|x|^{d-2}}$$

for all $x \in \Omega \setminus \Omega_{R_0}$.

We conclude this section with a variant of Theorem 1.1 that permits less regular boundary data ψ, at the cost of requiring Lipschitz continuity of the coefficients. Subsequently we obtain that $B_\lambda^D(R)$ converges to B_λ^D in $\mathcal{L}\big(L_\infty(\Gamma), L_\infty(\Omega)\big)$ under the same hypotheses.

Let $R \ge R_0$. Then Ω_R is a Lipschitz domain. We define

$$W_R^D(\Omega) = \big\{u \in W^D(\Omega) : u|_{\Omega \setminus \Omega_R} = 0\big\}.$$

Then $W_R^D(\Omega)$ is a closed subspace of $W^D(\Omega)$ with the induced norm.

Lemma 4.1. *Let $R \ge R_0$. Then $W_R^D(\Omega)$ is a Hilbert space. Moreover, the map $u \mapsto u|_{\Omega_R}$ defines a homeomorphism from $W_R^D(\Omega)$ onto $\{v \in H^1(\Omega_R) : \mathbb{1}_{\partial B_R} \mathrm{Tr}_{\Omega_R} v = 0\}$.*

Proof. Clearly the closed subspace $W_R^D(\Omega)$ of the Hilbert space $W^D(\Omega)$ is a Hilbert space. By identifying the sets $\{v|_{B_R} : v \in H^1(\mathbb{R}^d)$ and $v|_{\mathbb{R}^d \setminus B_R} = 0\}$ and $\{v \in H^1(B_R) : \mathrm{Tr}_{B_R} v = 0\}$ in the natural way, one deduces that the map $u \mapsto u|_{\Omega_R}$ places $W_R^D(\Omega)$ and $\{v \in H^1(\Omega_R) : \mathbb{1}_{\partial B_R} \mathrm{Tr}_{\Omega_R} v = 0\}$ in bijective correspondence. Moreover, by Proposition 2.11(d) there exists a $c > 0$ such that

$$\|u|_{\Omega_R}\|_{H^1(\Omega_R)}^2 \le \int_\Omega |\nabla u|^2 + \int_{\Omega_R} |u|^2 \le c\|u\|_{W(\Omega)}^2 = c\|u\|_{W^D(\Omega)}^2$$

for all $u \in W_R^D(\Omega)$. On the other hand,

$$\int_{\Omega_R} |\nabla u|^2 \le \|u\|_{H^1(\Omega_R)}^2$$

for all $u \in H^1(\Omega_R)$. □

The alluded-to truncated version of (1) is as follows. Let $R \geq R_0$. Given $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$, we consider the problem

$$\begin{aligned} Au &= 0 \quad \text{on } \Omega_R \\ \lambda \operatorname{Tr} u + \partial_\nu u &= \psi \quad \text{on } \Gamma \end{aligned} \tag{13}$$

with Dirichlet boundary conditions at ∂B_R.

Proposition 4.2. *Let $R \geq R_0$. Let $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$. Then there exists a unique $u \in W_R^D(\Omega)$ such that*

$$\mathfrak{a}(u,v) + \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v} \tag{14}$$

for all $v \in W_R^D(\Omega)$.

Proof. The claim follows from an argument similar to the proof of Proposition 3.3, using Lemma 4.1 together with the Lax–Milgram theorem. □

Let $R \geq R_0$. Given $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$, we call $u \in W_R^D(\Omega)$ a **solution of (13) with Dirichlet boundary conditions at** ∂B_R if u satisfies (14) for all $v \in W_R^D(\Omega)$. Moreover, we define the bounded solution operator $B_\lambda^D(R) \colon L_2(\Gamma) \to W_R^D(\Omega)$ by $B_\lambda^D(R)\psi = u$.

We first verify that solutions of the truncated problem are continuous on Ω.

Proposition 4.3. *Let $R \geq R_0$. Let $\psi \in L_2(\Gamma)$ and $\lambda \geq 0$. Then $B_\lambda^D(R)\psi \in C(\Omega)$.*

Proof. Without loss of generality we may assume that ψ is real-valued. Write $u = B_\lambda^D(R)\psi$. We may assume that $u|_{\partial B_R} = 0$. Clearly $u|_{\Omega \setminus \Omega_R}$ is continuous and Nash [Nas58] provides that $u|_{\Omega_R}$ is continuous. Then it remains to show that u is continuous in a neighbourhood of ∂B_R. Consider the annulus

$$Z = \{x \in \mathbb{R}^d : R_0 - 1 < |x| < R\}.$$

We shall prove that $u|_{\overline{Z}}$ is continuous.

Fix $\tau \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ such that $\tau = 0$ on $B_{R_0-2} \cup \mathbb{R}^d \setminus \overline{B_{R_0-\frac{1}{2}}}$ and $\tau = \mathbb{1}$ in a neighbourhood of ∂B_{R_0-1}. Define $\eta \colon \overline{Z} \to \mathbb{R}$ by $\eta(x) = (\tau u)(x)$. Then $\eta|_Z \in H^1(Z)$ and $\operatorname{Tr}_Z(\eta|_Z) = \operatorname{Tr}_Z(u|_Z)$, where $\operatorname{Tr}_Z \colon H^1(Z) \to L_2(\partial Z)$ is the trace map on Z. Moreover, by the Lax–Milgram theorem there exists a unique $v \in H_0^1(Z)$ such that

$$\sum_{k,l=1}^d \int_Z a_{kl} (\partial_k v) \overline{\partial_l \chi} = \sum_{k,l=1}^d \int_Z a_{kl} (\partial_k \eta) \overline{\partial_l \chi}$$

for all $\chi \in H_0^1(Z)$. Write $w = \eta|_Z - v \in H^1(Z)$. Then w is harmonic on Z. Since $\eta \in C(\overline{Z})$ and $\eta|_Z \in H^1(Z)$, it follows from [AE19] Proposition 2.14 that v extends to a continuous function on $\overline{Z}$. Hence w extends to a continuous function on $\overline{Z}$ and $w(x) = \eta(x)$ for all $x \in \partial Z$, where we continue to denote by w the extension to $\overline{Z}$.

Since $\operatorname{Tr}_Z(w|_Z) = \operatorname{Tr}_Z(\eta|_Z) = \operatorname{Tr}_Z(u|_Z)$, it follows that $\operatorname{Tr}_Z((w-u)|_Z) = 0$. Therefore $(w-u)|_Z \in H_0^1(Z)$. Moreover, since $u = B_\lambda^D(R)\psi$ one has that

$$\sum_{k,l=1}^d \int_Z a_{kl} (\partial_k u) \overline{\partial_l \chi} = \mathfrak{a}(u, \chi) = \int_\Gamma \psi \, \overline{\operatorname{Tr} \chi} - \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr} \chi} = 0$$

for all $\chi \in C_c^\infty(Z)$. So

$$\sum_{k,l=1}^{d} \int_Z a_{kl} (\partial_k (w-u)) \overline{\partial_l \chi} = 0$$

first for all $\chi \in C_c^\infty(Z)$, and then for all $\chi \in H_0^1(Z)$ by density. The choice $\chi = (w-u)|_Z$ then yields that

$$\mu \int_Z |\nabla (w-u)|^2 \leq \sum_{k,l=1}^{d} \int_Z a_{kl} (\partial_k (w-u)) \overline{\partial_l (w-u)} = 0,$$

so $(w-u)|_Z$ is constant and it follows that $(w-u)|_Z = 0$. Hence $w(x) = u(x)$ for all $x \in Z$. Now let $x \in \partial B_R$ and let $(x_n)_{n \in \mathbb{N}}$ be a sequence in Z such that $\lim x_n = x$. Then

$$\lim_{n \to \infty} u(x_n) = \lim_{n \to \infty} w(x_n) = w(x) = \eta(x) = 0 = u(x)$$

and the claim follows. □

Corollary 4.4. *Let $R \geq R_0$. Let $\psi \in L_\infty(\Gamma)$ and $\lambda \geq 0$. Then $B_\lambda^D(R)\psi \in C(\overline{\Omega})$.*

Proof. Since Lemma 3.10 remains valid with the space $W_R^D(\Omega)$ in place of $W(\Omega)$, the claim follows from an argument similar to the proof of Theorem 1.1. □

Next we prove that the solution operator $B_\lambda^D(R)$ converges to B_λ^D in a uniform manner. We apply this result in the proof of Proposition 4.9 to obtain locally uniform convergence of solutions of (13), and in the proof of Proposition 5.7 to deduce resolvent convergence for the Dirichlet-to-Neumann operator.

Proposition 4.5. *Let $\lambda \geq 0$. Then*

$$\lim_{R \to \infty} B_\lambda^D(R) = B_\lambda^D$$

in $\mathcal{L}\big(L_2(\Gamma), W^D(\Omega)\big)$.

Proof. We argue as in the proof of [AE15] Theorem 4.3. Let $(R_n)_{n \in \mathbb{N}}$ be a sequence in $[R_0, \infty)$ such that $\lim R_n = \infty$. We shall prove that $\lim B_\lambda^D(R_n) = B_\lambda^D$ in $\mathcal{L}\big(L_2(\Gamma), W^D(\Omega)\big)$. Let $\psi, \psi_1, \psi_2, \ldots \in L_2(\Gamma)$ and suppose that $\lim \psi_n = \psi$ weakly in $L_2(\Gamma)$. Since by Proposition 3.9(a) the operator B_λ^D is compact, by Proposition A.7 we need only show that $\lim B_\lambda^D(R_n)\psi_n = B_\lambda^D \psi$ in $W^D(\Omega)$.

Write $u = B_\lambda^D \psi$ and for each $n \in \mathbb{N}$ write $u_n = B_\lambda^D(R_n)\psi_n$. Let $n \in \mathbb{N}$. Then

$$\mathfrak{a}(u_n, v) + \lambda \int_\Gamma \operatorname{Tr} u_n \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi_n \, \overline{\operatorname{Tr} v} \tag{15}$$

for all $v \in W_{R_n}^D(\Omega)$. The choice $v = u_n$ together with the ellipticity of $\mathfrak{a}$ then yields that

$$\mu \|u_n\|_{W^D(\Omega)}^2 \leq \operatorname{Re} \mathfrak{a}(u_n) + \lambda \int_\Gamma |\operatorname{Tr} u_n|^2 = \operatorname{Re} \int_\Gamma \psi_n \, \overline{\operatorname{Tr} u_n}.$$

Since $\operatorname{Tr} : W(\Omega) \to L_2(\Gamma)$ is continuous, it follows that

$$\mu \|u_n\|_{W^D(\Omega)}^2 \leq \operatorname{Re} \int_\Gamma \psi_n \, \overline{\operatorname{Tr} u_n} \leq \|\operatorname{Tr}\|_{\mathcal{L}(W(\Omega), L_2(\Gamma))} \, \|\psi_n\|_{L_2(\Gamma)} \, \|u_n\|_{W^D(\Omega)}$$

for all $n \in \mathbb{N}$. Then the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^D(\Omega)$, since $(\psi_n)_{n\in\mathbb{N}}$ is bounded in $L_2(\Gamma)$. Passing to a subsequence if necessary, we may assume that there exists a $w \in W^D(\Omega)$ such that $\lim u_n = w$ weakly in $W^D(\Omega)$. Then $\lim \operatorname{Tr} u_n = \operatorname{Tr} w$ in $L_2(\Gamma)$, since by Proposition 2.11(b) the map Tr is compact. Let $R \geq R_0$ and let $v \in W^D_R(\Omega)$. Then $v \in W^D_{R_n}(\Omega)$ for all $n \in \mathbb{N}$ sufficiently large. Hence (15) yields that

$$\begin{aligned}\mathfrak{a}(w,v) + \lambda \int_\Gamma \operatorname{Tr} w \, \overline{\operatorname{Tr} v} &= \lim_{n\to\infty} \mathfrak{a}(u_n,v) + \lambda \int_\Gamma \operatorname{Tr} u_n \, \overline{\operatorname{Tr} v} \\ &= \lim_{n\to\infty} \int_\Gamma \psi_n \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}.\end{aligned}$$

Note that since $W^D(\Omega) = \overline{\{\chi|_\Omega : \chi \in C_c^\infty(\mathbb{R}^d)\}}^{W(\Omega)}$, it follows that $\bigcup_{R\geq R_0} W^D_R(\Omega)$ is dense in $W^D(\Omega)$. So

$$\mathfrak{a}(w,v) + \lambda \int_\Gamma \operatorname{Tr} w \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}$$

for all $v \in W^D(\Omega)$. Therefore $w = B^D_\lambda \psi = u$ and $\lim \operatorname{Tr} u_n = \operatorname{Tr} u$ in $L_2(\Gamma)$. Then

$$\begin{aligned}\lim_{n\to\infty} \operatorname{Re} \mathfrak{a}(u_n) + \lambda \int_\Gamma |\operatorname{Tr} u_n|^2 &= \lim_{n\to\infty} \operatorname{Re} \int_\Gamma \psi_n \, \overline{\operatorname{Tr} u_n} \\ &= \operatorname{Re} \int_\Gamma \psi \, \overline{\operatorname{Tr} u} = \operatorname{Re} \mathfrak{a}(u) + \lambda \int_\Gamma |\operatorname{Tr} u|^2.\end{aligned}$$

Moreover, $\lim u_n = u$ weakly in $W^D(\Omega)$. Hence by using Lemma 3.1 and arguing as in the proof of Proposition 3.9, one deduces that $\lim u_n = u$ in $W^D(\Omega)$. □

The operator $B^D_\lambda(R)$ is positivity preserving and is dominated by B^D_λ. Moreover, $B^D_\lambda(R)$ is increasing in the parameter R.

Proposition 4.6. *Let $\psi \in L_2(\Gamma)^+$ and $\lambda \geq 0$.*

(a) *Let $R \geq R_0$. Then $B^D_\lambda(R)\psi \geq 0$.*

(b) *Let $R \geq R_0$. Then $B^D_\lambda(R)\psi \leq B^D_\lambda \psi$.*

(c) *Let $R_2 \geq R_1 \geq R_0$. Then $B^D_\lambda(R_1)\psi \leq B^D_\lambda(R_2)\psi$.*

Proof. The proofs of Statements (a) and (b) are similar to the proofs of the Propositions 3.5 and 3.6, respectively. We prove (c).

Write $u_1 = B^D_\lambda(R_1)\psi$ and $u_2 = B^D_\lambda(R_2)\psi$. Then

$$\mathfrak{a}(u_1,v) + \lambda \int_\Gamma \operatorname{Tr} u_1 \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}$$

for all $v \in W^D_{R_1}(\Omega)$ and

$$\mathfrak{a}(u_2,v) + \lambda \int_\Gamma \operatorname{Tr} u_2 \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}$$

for all $v \in W^D_{R_2}(\Omega)$. So

$$\mathfrak{a}(u_1 - u_2, v) + \lambda \int_\Gamma \operatorname{Tr}(u_1 - u_2) \, \overline{\operatorname{Tr} v} = 0 \tag{16}$$

for all $v \in W^D_{R_1}(\Omega)$. Moreover, $u_1 \geq 0$ and $u_2 \geq 0$ by Statement (a). Since $u_1|_{\Omega \setminus \Omega_{R_1}} = 0$, it follows that $(u_1 - u_2)^+|_{\Omega \setminus \Omega_{R_1}} = 0$. Then by Proposition 2.15(b) one deduces that $(u_1 - u_2)^+ \in W^D_{R_1}(\Omega)$ and the choice $v = (u_1 - u_2)^+$ in (16) yields that

$$\mu \|(u_1 - u_2)^+\|^2_{W^D(\Omega)} \leq \mathfrak{a}((u_1 - u_2)^+) + \lambda \int_\Gamma |\mathrm{Tr}\,(u_1 - u_2)^+|^2 = 0.$$

So $(u_1 - u_2)^+ = 0$ and $u_1 \leq u_2$. □

Corollary 4.7. *Let $R \geq R_0$ and $\lambda \geq 0$. Then the map $\psi \mapsto B^D_\lambda(R)\psi$ is continuous from $L_p(\Gamma)$ into $L_p(\Omega)$ for all $p \in [2, \infty]$.*

Proof. The proof is similar to that of Corollary 3.8(c). Note that since $\operatorname{supp} B^D_\lambda(R)\psi \subset \Omega_R$ for all $\psi \in L_2(\Gamma)$, the cut-off function $\mathbb{1}_{\Omega_R}$ is no longer required. □

For the proof of the next lemma we introduce the following definitions. Let X denote a σ-finite measure space and let $p, q \in [1, \infty]$. Two operators $B_p \in \mathcal{L}(L_p(X))$ and $B_q \in \mathcal{L}(L_q(X))$ are called **consistent** if $B_p|_{L_p \cap L_q} = B_q|_{L_p \cap L_q}$. Two semigroups $T^{(p)} = (T^{(p)}_t)_{t>0}$ on $L_p(X)$ and $T^{(q)} = (T^{(q)}_t)_{t>0}$ on $L_q(X)$ are called **consistent** if the operators $T^{(p)}_t$ and $T^{(q)}_t$ are consistent for all $t > 0$. A semigroup $T^{(p)}$ on $L_p(X)$ is said to **extend consistently to a semigroup on** $L_q(X)$ if there exists a semigroup $T^{(q)}$ on $L_q(X)$ such that $T^{(p)}$ and $T^{(q)}$ are consistent. We revisit these notions in Section 6 when considering the semigroup generated by the Dirichlet-to-Neumann operator.

For the remainder of this section we assume that the coefficients (a_{kl}) are Lipschitz continuous on Ω. Note that by [Ste70] Theorem VI.5 we may assume that the coefficients extend to $\mathbb{R}^d$ such that $a_{kl} \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$ and

$$\mathrm{Re} \sum_{k,l=1}^d a_{kl}(x)\xi_k\overline{\xi_l} \geq \frac{\mu}{2}\,|\xi|^2$$

for a.e. $x \in \mathbb{R}^d$ and all $\xi \in \mathbb{C}^d$, where μ is as in (4) and we continue to denote by a_{kl} the extension to $\mathbb{R}^d$. We define the continuous elliptic form $\widetilde{\mathfrak{a}}\colon H^1(\mathbb{R}^d) \times H^1(\mathbb{R}^d) \to \mathbb{C}$ by

$$\widetilde{\mathfrak{a}}(u, v) = \sum_{k,l=1}^d \int_{\mathbb{R}^d} a_{kl}(\partial_k u)\overline{\partial_l v}$$

and denote by A the associated m-sectorial operator in $L_2(\mathbb{R}^d)$. Then $-A$ generates a C_0-semigroup $(e^{-tA})_{t>0}$ on $L_2(\mathbb{R}^d)$. Moreover, for all $p \in [1, \infty)$ the semigroup $(e^{-tA})_{t>0}$ extends consistently to a C_0-semigroup $(e^{-tA_p})_{t>0}$ on $L_p(\mathbb{R}^d)$ with generator $-A_p$.

The following proof uses the resolvent consistency and optimal regularity associated with the elliptic operator A in $L_2(\mathbb{R}^d)$.

Lemma 4.8. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$. Let $u \in W(\Omega)$ and suppose that $\mathcal{A}u = 0$. Then $u \in W^{2,2d}_{\mathrm{loc}}(\Omega)$.*

Proof. We first show that $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. Let $\chi \in C^\infty_c(\Omega)$. Then $\int_\Omega |\nabla(\chi u)|^2 < \infty$ and it follows that $\chi u \in W^{1,2}(\mathbb{R}^d)$, where we continue to denote by χu the zero extension to $\mathbb{R}^d$. Then for all $\tau \in C^\infty_c(\mathbb{R}^d)$,

$$\widetilde{\mathfrak{a}}(\chi u, \tau) = \sum_{k,l=1}^d \int_{\mathbb{R}^d} a_{kl}\big(\partial_k(\chi u)\big)\overline{\partial_l \tau}$$

$$
\begin{aligned}
&= \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} a_{kl} (u \partial_k \chi) \overline{\partial_l \tau} + \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} a_{kl} (\chi \partial_k u) \overline{\partial_l \tau} \\
&= \Big(\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} a_{kl} (\partial_k \chi) \overline{\partial_l (\overline{u} \tau)} - \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} a_{kl} (\partial_k \chi) (\partial_l u) \overline{\tau} \Big) \\
&\quad + \Big(\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} a_{kl} (\partial_k u) \overline{\partial_l (\overline{\chi} \tau)} - \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} a_{kl} (\partial_k u) (\partial_l \chi) \overline{\tau} \Big) \\
&= - \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} u \big(\partial_l (a_{kl} \partial_k \chi) \big) \overline{\tau} - \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} a_{kl} (\partial_k \chi) (\partial_l u) \overline{\tau} \\
&\quad + \langle \mathcal{A} u, \overline{\chi} \tau \rangle - \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} a_{kl} (\partial_k u) (\partial_l \chi) \overline{\tau} \\
&= \big(u A \chi, \tau \big)_{L_2(\mathbb{R}^d)} - \sum_{k,l=1}^{d} \big(a_{kl} (\partial_k \chi) \partial_l u, \tau \big)_{L_2(\mathbb{R}^d)} - \sum_{k,l=1}^{d} \big(a_{kl} (\partial_k u) \partial_l \chi, \tau \big)_{L_2(\mathbb{R}^d)}.
\end{aligned}
$$

So $\chi u \in D(A)$ and

$$
A(\chi u) = u A \chi - \sum_{k,l=1}^{d} a_{kl} (\partial_k \chi) \partial_l u - \sum_{k,l=1}^{d} a_{kl} (\partial_k u) \partial_l \chi \tag{17}
$$

with $A(\chi u) \in L_2(\mathbb{R}^d)$. Hence by [GT83] Theorem 8.8 together with the fact that $\operatorname{supp} \chi \subset \Omega$, one obtains that $\chi u \in W^{2,2}(\Omega)$.

For each $n \in \{1, \ldots, d\}$ define $p_n \in [2, \infty]$ by $\frac{1}{p_n} = \frac{n}{2d}$. Then $p_1 = 2d$, $p_d = 2$ and $\frac{1}{p_n} - \frac{1}{p_{n-1}} = \frac{1}{2d}$ for all $n \in \{1, \ldots, d\}$. By downwards induction we shall prove that for each $n \in \{1, \ldots, d\}$, one has that $\chi u \in W^{2,p_n}(\Omega)$ for all $\chi \in C_c^\infty(\Omega)$. Since the conclusion of the preceding paragraph implies that $\chi u \in W^{2,2}(\Omega) = W^{2,p_d}(\Omega)$ for all $\chi \in C_c^\infty(\Omega)$, it follows that the base case $n = d$ is valid.

Let $j \in \{1, \ldots, d\}$ and suppose that $\chi u \in W^{2,p_j}(\Omega)$ for all $\chi \in C_c^\infty(\Omega)$. Then $u \in W^{2,p_j}_{\mathrm{loc}}(\Omega)$. Let $\chi \in C_c^\infty(\Omega)$. By a zero extension $\chi u \in W^{2,p_j}(\mathbb{R}^d)$ and the Sobolev embedding theorem provides that $\chi u \in L_{p_{j-1}}(\mathbb{R}^d)$. Note that since $a_{kl} \partial_k \chi \in W^{1,\infty}_c(\Omega)$ for all $k, l \in \{1, \ldots, d\}$, it follows that $A\chi \in L_{\infty,c}(\Omega)$. By Nash [Nas58] the function u is continuous on the compact set $\operatorname{supp} \chi \subset \Omega$, so $uA\chi \in L_2(\Omega) \cap L_\infty(\Omega)$. Then $uA\chi \in L_2(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ by a zero extension and consequently $uA\chi \in L_{p_{j-1}}(\mathbb{R}^d)$. Moreover, since $u \in W^{2,p_j}_{\mathrm{loc}}(\Omega)$ by the inductive hypothesis and $\operatorname{supp} \chi \subset \Omega$ is compact, it follows that $(\partial_k \chi) \partial_l u \in W^{1,p_j}(\Omega)$ for all $k, l \in \{1, \ldots, d\}$. Then by a zero extension and the Sobolev embedding theorem, one obtains that

$$
\sum_{k,l=1}^{d} a_{kl} (\partial_k \chi) \partial_l u \in L_{p_{j-1}}(\mathbb{R}^d)
$$

since a_{kl} is bounded for all $k, l \in \{1, \ldots, d\}$. One similarly deduces that $\sum a_{kl} (\partial_k u) \partial_l \chi \in L_{p_{j-1}}(\mathbb{R}^d)$ and it then follows from (17) that $A(\chi u) \in L_{p_{j-1}}(\mathbb{R}^d)$. Write $f = (I + A)(\chi u) \in L_2(\mathbb{R}^d) \cap L_{p_{j-1}}(\mathbb{R}^d)$. Since the semigroup $(e^{-tA})_{t>0}$ on $L_2(\mathbb{R}^d)$ extends consistently to

$(e^{-tA_{p_{j-1}}})_{t>0}$ on $L_{p_{j-1}}(\mathbb{R}^d)$, the resolvents $(I+A)^{-1}$ and $(I+A_{p_{j-1}})^{-1}$ of the respective generators are consistent. So

$$\chi u = (I+A)^{-1} f = (I+A_{p_{j-1}})^{-1} f \in D(A_{p_{j-1}}) = W^{2,p_{j-1}}(\mathbb{R}^d)$$

by [ER97] Theorem 1.5.II. Therefore $\chi u \in W^{2,p_{j-1}}(\Omega)$ for all $\chi \in C_c^\infty(\Omega)$. Then by induction it follows that for all $n \in \{1,\ldots,d\}$, one has $\chi u \in W^{2,p_n}(\Omega)$ for all $\chi \in C_c^\infty(\Omega)$. The claim then follows from the case $n=1$. □

Proposition 4.9. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega,\mathbb{R})$ for all $k,l \in \{1,\ldots,d\}$. Let $K \subset \Omega$ be a compact set and let $\lambda \geq 0$. Then*

$$\lim_{R\to\infty} \sup_{\|\psi\|_{L_2(\Gamma)}\leq 1} \|\big(B_\lambda^D \psi\big)|_K - \big(B_\lambda^D(R)\psi\big)|_K\|_{C(K)} = 0.$$

In particular, $\lim_{R\to\infty} B_\lambda^D(R)\psi = B_\lambda^D\psi$ locally uniformly on Ω for all $\psi \in L_2(\Gamma)$.

Proof. For each $n \in \{0,\ldots,d\}$ define $p_n \in [2,\infty]$ by $\frac{1}{p_n} = \frac{n}{2d}$. Then $p_0=\infty$, $p_1 = 2d$, $p_d = 2$ and $\frac{1}{p_n} - \frac{1}{p_{n-1}} = \frac{1}{2d}$ for all $n \in \{1,\ldots,d\}$. Moreover, there exists a collection $\{U_n\}_{n=0}^d$ of Lipschitz bounded open subsets of Ω such that $K \subset U_{n-1} \subset \overline{U}_{n-1} \subset U_n \subset \overline{U}_n \subset \Omega$ for all $n \in \{1,\ldots,d\}$. Since $|U_d| < \infty$ and $p_n \leq p_1$ for all $n \in \{1,\ldots,d\}$, it follows from Hölder's inequality that there exists a $c>0$ such that for all $n \in \{1,\ldots,d\}$ one has $\|u\|_{L_{p_n}(U_d)} \leq c\|u\|_{L_{p_1}(U_d)}$ for all $u \in L_{p_1}(U_d)$.

Let $n \in \{1,\ldots,d\}$. By the Sobolev embedding theorem there exists a $c_n > 0$ such that

$$\|u\|_{L_{p_{n-1}}(U_{n-1})} \leq c_n \|u\|_{W^{1,p_n}(U_{n-1})}$$

for all $u \in W^{1,p_n}(U_{n-1})$. Moreover, by [GT83] Theorem 9.11 there exists a $\widetilde{c}_n > 0$ such that

$$\|u\|_{W^{2,p_n}(U_{n-1})} \leq \widetilde{c}_n \big(\big\|\textstyle\sum \partial_l(a_{kl}\partial_k u)\big\|_{L_{p_n}(U_n)} + \|u\|_{L_{p_n}(U_n)}\big)$$

for all $u \in W^{2,p_n}(U_n)$. Let $u \in W^{2,p_1}(U_d)$. Then for all $n \in \{1,\ldots,d\}$, one has that $u \in W^{2,p_n}(U_n) \cap W^{1,p_n}(U_{n-1})$ and

$$\|u\|_{L_{p_{n-1}}(U_{n-1})} \leq c_n \|u\|_{W^{2,p_n}(U_{n-1})} \leq c_n\widetilde{c}_n \big(\big\|\textstyle\sum \partial_l(a_{kl}\partial_k u)\big\|_{L_{p_n}(U_n)} + \|u\|_{L_{p_n}(U_n)}\big).$$

So

$$\begin{aligned}
\|u\|_{L_{p_{n-1}}(U_{n-1})} &\leq c_n\widetilde{c}_n \big(\big\|\textstyle\sum \partial_l(a_{kl}\partial_k u)\big\|_{L_{p_n}(U_n)} + \|u\|_{L_{p_n}(U_n)}\big) \\
&\leq c_n\widetilde{c}_n \big(c\big\|\textstyle\sum \partial_l(a_{kl}\partial_k u)\big\|_{L_{p_1}(U_d)} + \|u\|_{L_{p_n}(U_n)}\big) \\
&\leq \widehat{c}_n \big(\big\|\textstyle\sum \partial_l(a_{kl}\partial_k u)\big\|_{L_{p_1}(U_d)} + \|u\|_{L_{p_n}(U_n)}\big)
\end{aligned}$$

for all $n \in \{1,\ldots,d\}$, where $\widehat{c}_n = c_n\widetilde{c}_n(c+1)$. Hence there exists a $\widehat{c} > 0$ such that

$$\begin{aligned}
\|u\|_{C(K)} \leq \|u\|_{L_\infty(U_0)} &\leq \widehat{c}_1 \big\|\textstyle\sum \partial_l(a_{kl}\partial_k u)\big\|_{L_{p_1}(U_d)} + \widehat{c}_1\|u\|_{L_{p_1}(U_1)} \\
&\leq \widehat{c}_1 \big\|\textstyle\sum \partial_l(a_{kl}\partial_k u)\big\|_{L_{p_1}(U_d)} + \widehat{c}_1\widehat{c}_2 \big\|\textstyle\sum \partial_l(a_{kl}\partial_k u)\big\|_{L_{p_1}(U_d)} + \widehat{c}_1\widehat{c}_2\|u\|_{L_{p_2}(U_2)}
\end{aligned}$$

$$\le \Big(\sum_{n=1}^{d}\prod_{k=1}^{n}\widehat{c}_k\Big)\big\|\sum\partial_l(a_{kl}\partial_k u)\big\|_{L_{p_1}(U_d)} + \Big(\prod_{n=1}^{d}\widehat{c}_n\Big)\|u\|_{L_{p_d}(U_d)}$$
$$\le \widehat{c}\,\big(\big\|\sum\partial_l(a_{kl}\partial_k u)\big\|_{L_{p_1}(U_d)} + \|u\|_{L_{p_d}(U_d)}\big)$$
$$= \widehat{c}\,\big(\big\|\sum\partial_l(a_{kl}\partial_k u)\big\|_{L_{2d}(U_d)} + \|u\|_{L_2(U_d)}\big)$$

for all $u \in W^{2,2d}(U_d)$.

Let $\psi \in L_2(\Gamma)$ and write $u = B_\lambda^D\psi$. Then

$$\langle \mathcal{A}u, \chi\rangle = \mathfrak{a}(u,\chi) = \int_\Gamma \psi\,\overline{\operatorname{Tr}\chi} - \lambda\int_\Gamma \operatorname{Tr}u\,\overline{\operatorname{Tr}\chi} = 0$$

for all $\chi \in C_c^\infty(\Omega)$. So $\mathcal{A}u = 0$ and Lemma 4.8 provides that $u|_{U_d} \in W^{2,2d}(U_d)$. Let $R \ge R_0$ be such that $\overline{U_d} \subset \Omega_R$ and write $u_R = B_\lambda^D(R)\psi$. Then $\langle \mathcal{A}u_R, \chi\rangle = 0$, first for all $\chi \in C_c^\infty(\Omega_R)$ and then for all $\chi \in C_c^\infty(\Omega)$, since $u_R|_{\Omega\setminus\Omega_R} = 0$. Hence $\mathcal{A}u_R = 0$ and $u_R|_{U_d} \in W^{2,2d}(U_d)$. Then by the conclusion of the preceding paragraph together with Proposition 2.11(d), there exists a $\widetilde{c} > 0$ such that

$$\begin{aligned}\|u-u_R\|_{C(K)}^2 \le \widehat{c}^{\,2}\|u-u_R\|_{L_2(U_d)}^2 &\le \widehat{c}^{\,2}\|u-u_R\|_{L_2(U_d)}^2 + \int_\Omega |\nabla(u-u_R)|^2\\ &\le \widetilde{c}\,\|u-u_R\|_{W(\Omega)}^2 = \widetilde{c}\,\|u-u_R\|_{W^D(\Omega)}^2\\ &\le \widetilde{c}\,\|B_\lambda^D - B_\lambda^D(R)\|_{\mathcal{L}(L_2(\Gamma),W^D(\Omega))}^2\|\psi\|_{L_2(\Gamma)}^2.\end{aligned}$$

The claim then follows from Proposition 4.5. □

We are now able to prove Theorem 1.2.

Proof of Theorem 1.2. We first prove (a). By [ER97] Theorem 1.1, the C_0-semigroup $(e^{-tA})_{t>0}$ on $L_2(\mathbb{R}^d)$ generated by $-A$ has a kernel $K\colon (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d \to (0,\infty)$ and there exist $a,b>0$ such that

$$0 < K_t(x,y) \le a t^{-d/2} e^{-b\frac{|x-y|^2}{t}} \tag{18}$$

for all $t>0$ and $x,y\in\mathbb{R}^d$. Moreover, K_t is Hölder continuous on $\mathbb{R}^d\times\mathbb{R}^d$ for all $t>0$. Define $G\colon \{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d : x\ne y\} \to (0,\infty)$ by

$$G(x,y) = \int_0^\infty K_t(x,y)\,\mathrm{d}t.$$

Then G is continuous. By (18) together with a change of variable, one deduces that

$$0 < G(x,y) \le \frac{c_1}{|x-y|^{d-2}} \tag{19}$$

for all $x,y\in\mathbb{R}^d$ with $x\ne y$, where $c_1 = \int_0^\infty a t^{-d/2} e^{\frac{-b}{t}}\,\mathrm{d}t < \infty$.

Without loss of generality we may assume that $B_1 \subset \mathbb{R}^d\setminus\overline{\Omega}$. Fix $\tau \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp}\tau \subset B_1$ and $\tau > 0$. Define $w\colon \overline{\Omega} \to (0,\infty)$ by

$$w(x) = \int_{\mathbb{R}^d} G(x,y)\tau(y)\,\mathrm{d}y.$$

Then w is continuous. Since $w(x) > 0$ for all $x \in \overline{\Omega}$, we may assume that $w(x) \geq 1$ for all x in the compact set ∂B_{R_0}. Moreover, by Tonelli

$$w(x) = \int_0^\infty (e^{-tA}\tau)(x)\,\mathrm{d}t$$

for all $x \in \overline{\Omega}$.

Let $R > R_0$ and let $\chi \in C_c^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp}\chi \subset \Omega_{R+1}$. Then $\operatorname{supp}\chi \cap \operatorname{supp}\tau = \varnothing$ and it follows from the Gaussian bound (18) that

$$\begin{aligned}\lim_{t\to\infty} |\big(e^{-tA}\tau, \chi\big)_{L_2(\mathbb{R}^d)}| &\leq \lim_{t\to\infty} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K_t(x,y)\,\tau(y)|\chi(x)|\,\mathrm{d}y\,\mathrm{d}x \\ &\leq \|\tau\|_{L_1(\mathbb{R}^d)}\|\chi\|_{L_1(\mathbb{R}^d)} \lim_{t\to\infty} a t^{-d/2} = 0.\end{aligned}$$

Write $A^\#\chi = -\sum_{k,l=1}^d \partial_k(a_{kl}\,\partial_l\chi) \in L_{\infty,\mathrm{c}}(\Omega_{R+1})$. Then

$$\begin{aligned}\int_{\Omega_{R+1}}\int_0^\infty |(e^{-tA}\tau)(x)\,\overline{(A^\#\chi)(x)}|\,\mathrm{d}t\,\mathrm{d}x &= \int_{\Omega_{R+1}} |(A^\#\chi)(x)|\Big(\int_0^\infty (e^{-tA}\tau)(x)\,\mathrm{d}t\Big)\mathrm{d}x \\ &\leq \|A^\#\chi\|_\infty \int_{\Omega_{R+1}} w < \infty.\end{aligned}$$

Hence by Fubini

$$\begin{aligned}\int_{\Omega_{R+1}} w(x)\overline{(A^\#\chi)(x)}\,\mathrm{d}x &= \int_0^\infty\int_{\Omega_{R+1}} (e^{-tA}\tau)(x)\,\overline{(A^\#\chi)(x)}\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^\infty \big(e^{-tA}\tau, A^\#\chi\big)_{L_2(\Omega_{R+1})}\,\mathrm{d}t = \int_0^\infty \big(e^{-tA}\tau, A^*\chi\big)_{L_2(\mathbb{R}^d)}\,\mathrm{d}t \\ &= \int_0^\infty \big(Ae^{-tA}\tau, \chi\big)_{L_2(\mathbb{R}^d)}\,\mathrm{d}t = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\big(e^{-tA}\tau, \chi\big)_{L_2(\mathbb{R}^d)}\,\mathrm{d}t \\ &= \lim_{t\downarrow 0}\big(e^{-tA}\tau, \chi\big)_{L_2(\mathbb{R}^d)} - \lim_{t\to\infty}\big(e^{-tA}\tau, \chi\big)_{L_2(\mathbb{R}^d)} = 0.\end{aligned}$$

Since our choice of χ was arbitrary, it follows that

$$\big(w|_{\Omega_{R+1}}, A^\#\chi\big)_{L_2(\Omega_{R+1})} = 0$$

for all $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp}\chi \subset \Omega_{R+1}$. Then $w|_{\Omega_{R+1}} \in H^1_{\mathrm{loc}}(\Omega_{R+1})$ by [AEG20] Proposition A.1. Consider the annulus

$$Z_R = \{x \in \mathbb{R}^d : R_0 < |x| < R\}.$$

Then $w|_{Z_R} \in H^1(Z_R)$ and for all $\chi \in C_c^\infty(Z_R)$ one has that

$$\begin{aligned}\sum_{k,l=1}^d \int_{Z_R} a_{kl}\,(\partial_k w)\,\overline{\partial_l\chi} &= \sum_{k,l=1}^d \int_{\Omega_{R+1}} (\partial_k w)\,\overline{a_{kl}\,\partial_l\chi} = -\sum_{k,l=1}^d \int_{\Omega_{R+1}} w\,\overline{\partial_k(a_{kl}\,\partial_l\chi)} \\ &= \big(w|_{\Omega_{R+1}}, A^\#\chi\big)_{L_2(\Omega_{R+1})} = 0,\end{aligned}$$

so w is harmonic on Z_R.

It follows as in the proof of Corollary 3.8(c) that $|B^D_\lambda \psi| \leq B^D_\lambda |\psi| \leq \|\psi\|_{L_\infty(\Gamma)} B^D_\lambda \mathbb{1}_\Gamma$. Hence we may assume that $\psi = \mathbb{1}_\Gamma$. Write $u = B^D_\lambda \psi$ and $u_R = B^D_\lambda(R)\psi$. Then

$$0 \leq u_R \leq u \leq \tfrac{1}{\lambda} \mathbb{1}_\Omega$$

by Proposition 4.6(a), Proposition 4.6(b) and Corollary 3.8(a). So $u_R|_{\partial B_{R_0}} \leq \frac{1}{\lambda} \mathbb{1}_{\partial B_{R_0}}$. Recall that $u_R|_{\partial B_R} = 0$ by definition. Then $u_R \leq \frac{1}{\lambda} w$ on ∂Z_R and since u_R is harmonic on Z_R, the maximum principle [GT83] Theorem 8.1 provides that $u_R \leq \frac{1}{\lambda} w$ on Z_R. Hence (19) provides that for all $x \in Z_R$

$$\begin{aligned} 0 \leq u_R(x) \leq \frac{1}{\lambda} w(x) &= \frac{1}{\lambda} \int_{\mathbb{R}^d} G(x,y) \tau(y) \, \mathrm{d}y \\ &\leq \frac{c_1}{\lambda} \int_{B_1} \frac{\tau(y)}{|x-y|^{d-2}} \, \mathrm{d}y \leq \frac{c_1 \|\tau\|_{L_1(\mathbb{R}^d)}}{\lambda} \cdot \frac{1}{(|x|-1)^{d-2}} \\ &= \frac{c_1 \|\tau\|_{L_1(\mathbb{R}^d)}}{\lambda} \cdot \frac{1}{(1-\frac{1}{|x|})^{d-2}} \cdot \frac{1}{|x|^{d-2}} < \frac{c_1 \|\tau\|_{L_1(\mathbb{R}^d)}}{\lambda (1-\frac{1}{R_0})^{d-2}} \cdot \frac{1}{|x|^{d-2}} \end{aligned}$$

Write $c = c_1 \|\tau\|_{L_1(\mathbb{R}^d)} (1 - \frac{1}{R_0})^{-(d-2)}$. Then

$$0 \leq u_R(x) \leq \frac{c}{\lambda} \cdot \frac{1}{|x|^{d-2}}$$

for all $x \in \Omega \setminus \Omega_{R_0}$, since $u_R = 0$ on $\Omega \setminus \Omega_R$. This is for all $R > R_0$. By Proposition 4.9 $\lim u_R = u$ locally uniformly on Ω, so

$$u(x) = \lim_{R \to \infty} u_R(x) \leq \frac{c}{\lambda} \cdot \frac{1}{|x|^{d-2}}$$

for all $x \in \Omega \setminus \Omega_{R_0}$ as claimed.

We now prove (b). Since by Proposition 3.4

$$B_\lambda \psi - \langle B_\lambda \psi \rangle \mathbb{1}_\Omega = B^D_\lambda (\psi - \lambda \langle B_\lambda \psi \rangle \mathbb{1}_\Gamma),$$

it follows from (a) that there exists a $c > 0$ such that

$$\begin{aligned} |(B_\lambda \psi)(x) - \langle B_\lambda \psi \rangle| &\leq \frac{c \, \|\psi - \lambda \langle B_\lambda \psi \rangle \mathbb{1}_\Gamma\|_{L_\infty(\Gamma)}}{\lambda} \cdot \frac{1}{|x|^{d-2}} \\ &\leq c \Big(\frac{\|\psi\|_{L_\infty(\Gamma)}}{\lambda} + |\langle B_\lambda \psi \rangle| \Big) \frac{1}{|x|^{d-2}} \end{aligned}$$

for all $x \in \Omega \setminus \Omega_{R_0}$. □

Harmonic elements of $W(\Omega)$ converge radially uniformly to their average at infinity.

Corollary 4.10. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$. Let $u \in W(\Omega)$ and suppose that $\mathcal{A} u = 0$. Then*

$$\lim_{R \to \infty} \sup_{|x| \geq R} |u(x) - \langle u \rangle| = 0.$$

Proof. Note that by Nash the function u is continuous on Ω. Moreover, it follows from Lemma 4.8 that $u \in W^{2,2d}_{\rm loc}(\Omega)$, so $\partial_k u \in W^{1,2d}_{\rm loc}(\Omega)$ for all $k \in \{1, \ldots, d\}$. Write $Z = B_{R_0+1} \setminus \overline{B_{R_0-1}} \subset \Omega$. Then $\partial_k u \in W^{1,2d}(Z)$ for all $k \in \{1, \ldots, d\}$ and it follows from [Bré11] Corollary 9.14 that $\partial_k u \in L_\infty(Z)$.

Let $c > 0$ be as in Theorem 1.2. Write $\Omega' = \mathbb{R}^d \setminus \overline{B_{R_0}}$ and define $\psi \colon \partial\Omega' \to \mathbb{R}$ by

$$\psi(z) = u(z) + \sum_{k,l=1}^{d} \nu_l(z) a_{kl}(z) (\partial_k u)(z),$$

where ν denotes the unit outer normal on $\partial\Omega'$. Then $\psi \in L_\infty(\partial\Omega')$. Fix $\lambda = 1$ and let B'_λ denote the solution operator corresponding to the boundary value problem (1) for the exterior domain Ω' with Neumann boundary conditions at infinity. With the trace and conormal derivative now defined for $\partial\Omega'$, one has that $\operatorname{Tr} u = u|_{\partial\Omega'}$ and $(\partial_\nu u)(z) = \sum_{k,l=1}^{d} \nu_l(z) a_{kl}(z) (\partial_k u)(z)$ for all $z \in \partial\Omega'$. Then $B'_\lambda \psi = u|_{\Omega'}$. Moreover, $\langle u \rangle = \langle u|_{\Omega'} \rangle$ since $|\Omega \setminus \Omega'| = |\Omega_{R_0}| < \infty$. Hence

$$\begin{aligned} \lim_{R\to\infty} \sup_{|x|\geq R} |u(x) - \langle u \rangle| &= \lim_{R\to\infty} \sup_{|x|\geq R} |(u|_{\Omega'})(x) - \langle u|_{\Omega'} \rangle| \\ &\leq \lim_{R\to\infty} c\big(\|\psi\|_{L_\infty(\Gamma)} + |\langle u|_{\Omega'} \rangle|\big) \frac{1}{R^{d-2}} = 0 \end{aligned}$$

by Theorem 1.2(b). □

Corollary 4.11. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$. Let $u \in W^D(\Omega)$. Suppose that $\mathcal{A} u = 0$ and $\operatorname{Tr} u = 0$. Then $u = 0$.*

Proof. Without loss of generality we may assume that u is real-valued. Let $\varepsilon > 0$. By Proposition 2.12 one has that $\langle u \rangle = 0$, so $K = \operatorname{supp}(u - \varepsilon)^+$ is compact by Corollary 4.10. Then $u|_K \in W^{2,2d}(K)$ by Lemma 4.8, so $u|_K \in L_\infty(K)$ and it follows that $(u-\varepsilon)^+ \in H^1(\Omega)$. Moreover, $\operatorname{Tr}((u-\varepsilon)^+) = 0$ by Proposition 2.15(c), so $(u-\varepsilon)^+ \in H^1_0(\Omega)$. Since $\mathcal{A} u = 0$, by density one deduces that

$$\sum_{k,l=1}^{d} \int_\Omega a_{kl} (\partial_k u) \overline{\partial_l v} = 0$$

for all $v \in H^1_0(\Omega)$. Then the choice $v = (u - \varepsilon)^+$ yields that

$$\mu \int_\Omega |\nabla((u-\varepsilon)^+)|^2 \leq \sum_{k,l=1}^{d} \int_\Omega a_{kl} \big(\partial_k((u-\varepsilon)^+)\big) \partial_l((u-\varepsilon)^+) = \sum_{k,l=1}^{d} \int_\Omega a_{kl} (\partial_k u) \partial_l((u-\varepsilon)^+) = 0$$

by [GT83] Lemma 7.6. Hence $(u-\varepsilon)^+$ is constant. Since $\operatorname{Tr}((u-\varepsilon)^+) = 0$ it follows that $(u - \varepsilon)^+ = 0$, so $u \leq \varepsilon$. By a similar argument one deduces that $-u \leq \varepsilon$ and the result then follows. □

Our final endeavour for this section is to prove the following variant of Theorem 1.1.

Theorem 4.12. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$. Let $p \in (d-1, \infty]$.*

(a) *Let $\psi \in L_p(\Gamma)$ and $\lambda > 0$. Then $B^D_\lambda \psi \in C(\overline{\Omega})$.*

(b) *Let $\psi \in L_p(\Gamma)$ and $\lambda > 0$. Then $B_\lambda \psi \in C(\overline{\Omega})$.*

We require the following extension of the Nash–De Giorgi result, which can be found at [Nit11] Proposition 3.14 (iv).

Lemma 4.13. *Let $U \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $p \in (d-1,\infty]$. Then there exists a $c > 0$ such that for every $\psi \in L_p(\partial U)$ and $u \in H^1(U)$ satisfying*

$$\sum_{k,l=1}^{d} \int_U a_{kl} (\partial_k u) \overline{\partial_l v} + \int_{\partial U} \mathrm{Tr}_U u \, \overline{\mathrm{Tr}_U v} = \int_{\partial U} \psi \, \overline{\mathrm{Tr}_U v}$$

for all $v \in H^1(U)$, it follows that u is Hölder continuous on U and

$$|u(x)| \leq c \, \|\psi\|_{L_p(\partial U)}$$

for all $x \in U$. Here $\mathrm{Tr}_U \colon H^1(U) \to L_2(\partial U)$ is the trace map on U.

Lemma 4.14. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$. Let $S \geq R_0 - 2$ and $p \in (d-1, \infty]$. Let $\lambda > 0$. Then there exists a $c_N > 0$ such that for every $\psi \in L_p(\Gamma)$ and $u \in H^1(\Omega_{S+2})$ satisfying*

$$\sum_{k,l=1}^{d} \int_{\Omega_{S+2}} a_{kl} (\partial_k u) \overline{\partial_l v} + \lambda \int_{\Gamma} \mathrm{Tr}_{S+2} u \, \overline{\mathrm{Tr}_{S+2} v} = \int_{\Gamma} \psi \, \overline{\mathrm{Tr}_{S+2} v} \tag{20}$$

for all $v \in W^D_{S+2}(\Omega)$, it follows that $u|_{\Omega_S}$ extends to a continuous function on $\overline{\Omega_S}$ and

$$|u(x)| \leq c_N \big(\|\psi\|_{L_p(\Gamma)} + \|u\|_{L_\infty(\partial B_S)} + \|u\|_{H^1(\Omega_{S+1})} \big)$$

for all $x \in \Omega_S$.

Proof. Without loss of generality we may assume that $\lambda = 1$. Let $c_1 > 0$ be such that $\|v\|_{L_2(\partial B_1)} \leq c_1 \|v\|_{H^1(B_1)}$ for all $v \in H^1(B_1) \cap C(\overline{B_1})$. Let $c > 0$ be as in Lemma 4.13 with $U = \Omega_S$. Let $\psi \in L_p(\Gamma)$ and $u \in H^1(\Omega_{S+2})$, and suppose that (20) is valid for all $v \in W^D_{S+2}(\Omega)$. Note that u admits a continuous representative on $H^1(\Omega_{S+2})$, which we continue to denote by u.

We first bound $\partial_\nu u$ on ∂B_S. Let $x_0 \in \partial B_S$ and denote by $\gamma \colon H^{1/2}(\partial B_1(x_0)) \to H^1(B_1(x_0))$ the harmonic lifting associated with the operator $-\sum \partial_l (a_{kl} \partial_k)$ on $B_1(x_0)$. By [EO19b] Proposition 5.5 the map γ has a continuous kernel $K_\gamma \colon B_1(x_0) \times \partial B_1(x_0) \to \mathbb{C}$ given by

$$K_\gamma(x,z) = -\sum_{k,l=1}^{d} \nu_k(z) \, a_{kl}(z) \, (\partial_l^{(1)} G)(z,x),$$

where $G \colon \{(x,y) \in \overline{B_1(x_0)} \times \overline{B_1(x_0)} : x \neq y\} \to \mathbb{C}$ is the Green function corresponding to the operator $-\sum \partial_l (a_{kl} \, \partial_k)$ on $B_1(x_0)$ with Dirichlet boundary conditions, and $\partial_j^{(n)} G$ denotes the j^{th}-partial derivative of G in the n^{th} variable for all $j \in \{1, \ldots, d\}$ and $n \in \{1,2\}$. Note that by [EO19b] Theorem 4.1, the functions $\partial_l^{(1)} G$ and $\partial_k^{(2)} \partial_l^{(1)} G$ are continuous on $\overline{B_1(x_0)} \times \overline{B_1(x_0)}$ for all $k, l \in \{1, \ldots, d\}$. Moreover, there exists a $c_2 > 0$ (independent of x_0) such that

$$|(\partial_k^{(2)} \partial_l^{(1)} G)(x,y)| \leq \frac{c_2}{|x-y|^d} \tag{21}$$

for all $x, y \in \overline{B_1(x_0)}$ with $x \neq y$ and all $k, l \in \{1, \ldots, d\}$. Since by (20)

$$\sum_{k,l=1}^{d} \int_{B_1(x_0)} a_{kl}(\partial_k u)\overline{\partial_l \chi} = \sum_{k,l=1}^{d} \int_{\Omega_{S+2}} a_{kl}(\partial_k u)\overline{\partial_l \chi} = 0$$

for all $\chi \in C_c^\infty\big(B_1(x_0)\big)$, it follows that

$$u(x) = \int_{\partial B_1(x_0)} K_\gamma(x, z) u(z) \, \mathrm{d}z$$

first for a.e. $x \in B_1(x_0)$, and then for all $x \in B_1(x_0)$ by the continuity of K_γ and u. So

$$\begin{aligned}
(\partial_k u)(x) &= \int_{\partial B_1(x_0)} (\partial_k^{(1)} K_\gamma)(x, z) \, u(z) \, \mathrm{d}z \\
&= -\sum_{k,l=1}^{d} \int_{\partial B_1(x_0)} \nu_k(z) \, a_{kl}(z) \, (\partial_k^{(2)} \partial_l^{(1)} G)(z, x) \, u(z) \, \mathrm{d}z
\end{aligned}$$

for all $x \in B_1(x_0)$ and $k \in \{1, \ldots, d\}$. Then (21) provides that

$$|(\partial_k u)(x)| \leq M \sum_{k,l=1}^{d} \int_{\partial B_1(x_0)} |(\partial_k^{(2)} \partial_l^{(1)} G)(z, x)||u(z)| \, \mathrm{d}z \leq M c_2 d^2 \int_{\partial B_1(x_0)} \frac{|u(z)|}{|z - x|^d} \, \mathrm{d}z$$

for all $x \in B_1(x_0)$ and $k \in \{1, \ldots, d\}$, where $M = \sup\big\{\|a_{kl}\|_{L_\infty(\Omega)}\big\}_{k,l=1}^{d}$. Hence

$$\begin{aligned}
|(\partial_k u)(x_0)| &\leq M c_2 d^2 \int_{\partial B_1(x_0)} |u(z)| \, \mathrm{d}z \leq M c_2 d^2 (d\omega_d)^{1/2} \|u\|_{L_2(\partial B_1(x_0))} \\
&\leq M c_2 d^2 (d\omega_d)^{1/2} c_1 \|u\|_{H^1(B_1(x_0))} \leq M c_2 d^2 (d\omega_d)^{1/2} c_1 \|u\|_{H^1(\Omega_{S+1})}
\end{aligned}$$

for all $k \in \{1, \ldots, d\}$ and it follows that

$$\|\partial_\nu u\|_{L_\infty(\partial B_S)} \leq c_3 \|u\|_{H^1(\Omega_{S+1})},$$

where $c_3 = M^2 c_2 d^4 (d\omega_d)^{1/2} c_1$. Therefore $\partial_\nu u \in L_p(\partial B_S)$.

Define $\xi \in L_p(\partial \Omega_S)$ by

$$\xi(x) = \begin{cases} \psi(x) & \text{if } x \in \Gamma, \\ u(x) + (\partial_\nu u)(x) & \text{if } x \in \partial B_S. \end{cases}$$

Then by the divergence theorem [Alt16] Theorem A8.8, one deduces that

$$\sum_{k,l=1}^{d} \int_{\Omega_S} a_{kl}(\partial_k u)\overline{\partial_l v} + \int_{\partial \Omega_S} \mathrm{Tr}_{\Omega_S} u \, \overline{\mathrm{Tr}_{\Omega_S} v} = \int_{\partial \Omega_S} \xi \, \overline{\mathrm{Tr}_{\Omega_S} v}$$

for all $v \in H^1(\Omega_S)$. Hence by Lemma 4.13 the restriction $u|_{\Omega_S}$ is uniformly continuous and therefore extends to a continuous function on $\overline{\Omega_S}$. Moreover,

$$\begin{aligned}
|u(x)| \leq c\|\xi\|_{L_p(\partial\Omega_S)} &\leq c\big(\|\psi\|_{L_p(\Gamma)} + \sigma(\partial B_S)^{1/p}\|u\|_{L_\infty(\partial B_S)} + \sigma(\partial B_S)^{1/p}\|\partial_\nu u\|_{L_\infty(\partial B_S)}\big) \\
&\leq c\big(\|\psi\|_{L_p(\Gamma)} + \sigma(\partial B_S)^{1/p}\|u\|_{L_\infty(\partial B_S)} + \sigma(\partial B_S)^{1/p} c_3 \|u\|_{H^1(\Omega_{S+1})}\big)
\end{aligned}$$

for all $x \in \Omega_S$. □

Proof of Theorem 4.12. We first prove (a). Write $u = B_\lambda^D \psi$. Then $u|_{\Omega_{R_0}} \in H^1(\Omega_{R_0})$ and for all $v \in W_{R_0}^D(\Omega)$ one has that

$$\sum_{k,l=1}^d \int_{\Omega_{R_0}} a_{kl} (\partial_k u) \overline{\partial_l v} + \lambda \int_\Gamma \operatorname{Tr}_{R_0}(u|_{\Omega_{R_0}}) \, \overline{\operatorname{Tr}_{R_0} v}$$

$$= \sum_{k,l=1}^d \int_\Omega a_{kl} (\partial_k u) \overline{\partial_l v} + \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr} v}$$

$$= \int_\Gamma \psi \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr}_{R_0} v}.$$

Hence by Lemma 4.14 with $S = R_0 - 2$, it follows that $u|_{\Omega_{R_0-2}}$ extends to a continuous function on $\overline{\Omega_{R_0-2}}$. Since by [Nas58] the function u is continuous on Ω, Statement (a) follows. Then (b) follows from Proposition 3.4. □

Corollary 4.15. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$. Let $R \geq R_0$ and $p \in (d-1, \infty]$. Let $\psi \in L_p(\Gamma)$ and $\lambda > 0$. Then $B_\lambda^D(R)\psi \in C(\overline{\Omega})$.*

Proof. Write $u = B_\lambda^D(R)\psi$. By an argument similar to the proof of Theorem 4.12(a), one deduces that $u|_{\Omega_{R_0-2}}$ extends to a continuous function on $\overline{\Omega_{R_0-2}}$. The claim then follows from Proposition 4.3. □

In the sequel we use the following result to extrapolate resolvent convergence for the Dirichlet-to-Neumann operator from $\mathcal{L}\big(L_2(\Gamma)\big)$ to $\mathcal{L}\big(L_p(\Gamma)\big)$ for all $p \in [1, \infty]$.

Proposition 4.16. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$ and let $\lambda > 0$. Then*

$$\lim_{R \to \infty} B_\lambda^D(R) = B_\lambda^D$$

in $\mathcal{L}\big(L_\infty(\Gamma), L_\infty(\Omega)\big)$.

Proof. Let $c > 0$ be as in Theorem 1.2. Let $\varepsilon > 0$ and fix $S \geq R_0$ such that $\frac{2c}{\lambda S^{d-2}} < \varepsilon$. Let $c_N > 0$ be as in Lemma 4.14 and let $R \geq S + 2$. Let $\psi \in L_\infty(\Gamma)$. Write $u = B_\lambda^D \psi$ and $u_R = B_\lambda^D(R)\psi$. Then $(u - u_R)|_{\Omega_{S+2}} \in H^1(\Omega_{S+2})$ and

$$\sum_{k,l=1}^d \int_{\Omega_{S+2}} a_{kl} \big(\partial_k (u - u_R)\big) \overline{\partial_l v} + \lambda \int_\Gamma \operatorname{Tr}_{S+2}\big((u - u_R)|_{\Omega_{S+2}}\big) \, \overline{\operatorname{Tr}_{S+2} v} = 0$$

for all $v \in W_{S+2}^D(\Omega)$. So

$$\|u - u_R\|_{L_\infty(\Omega_S)} \leq c_N (\|u - u_R\|_{L_\infty(\partial B_S)} + \|u - u_R\|_{H^1(\Omega_{S+1})}).$$

By Proposition 2.11(d), the restriction map from $W^D(\Omega)$ into $H^1(\Omega_{S+1})$ is continuous. Hence there exists an $M > 0$ such that

$$\begin{aligned} \|u - u_R\|_{L_\infty(\Omega_S)} &\leq c_N \big(\|u - u_R\|_{L_\infty(\partial B_S)} + M \|u - u_R\|_{W^D(\Omega)}\big) \\ &\leq c_N \big(\|u - u_R\|_{L_\infty(\partial B_S)} + M \|B_\lambda^D - B_\lambda^D(R)\|_{\mathcal{L}(L_2(\Gamma), W^D(\Omega))} \|\psi\|_{L_2(\Gamma)}\big) \end{aligned}$$

$$\leq c_N \big(\|u - u_R\|_{L_\infty(\partial B_S)} + M\,\sigma(\Gamma)^{1/2} \|B_\lambda^D - B_\lambda^D(R)\|_{\mathcal{L}(L_2(\Gamma), W^D(\Omega))} \|\psi\|_{L_\infty(\Gamma)} \big).$$

Moreover, by Proposition 4.6(b) and Theorem 1.2(a) one obtains that

$$|\big(B_\lambda^D(R)\psi\big)(x)| \leq |(B_\lambda^D \psi)(x)| \leq \frac{c\,\|\psi\|_{L_\infty(\Gamma)}}{\lambda} \cdot \frac{1}{|x|^{d-2}}$$

for all $x \in \Omega \setminus \Omega_{R_0}$. Consequently

$$\|u - u_R\|_{L_\infty(\Omega \setminus \Omega_S)} \leq \sup_{x \in \Omega \setminus \Omega_S} |(B_\lambda^D \psi)(x)| + |\big(B_\lambda^D(R)\psi\big)(x)| \leq \frac{2c\,\|\psi\|_{L_\infty(\Gamma)}}{\lambda S^{d-2}} \leq \varepsilon \|\psi\|_{L_\infty(\Gamma)}.$$

Therefore

$$\|u - u_R\|_{L_\infty(\Omega)}$$
$$\leq c_N \big(\|u - u_R\|_{L_\infty(\partial B_S)} + M\,\sigma(\Gamma)^{1/2} \|B_\lambda^D - B_\lambda^D(R)\|_{\mathcal{L}(L_2(\Gamma), W^D(\Omega))} \|\psi\|_{L_\infty(\Gamma)} \big) \vee \varepsilon \|\psi\|_{L_\infty(\Gamma)}.$$

Since $\partial B_S \subset \Omega$ is compact, the claim follows from Proposition 4.9 together with Proposition 4.5. $\square$

5 The Dirichlet-to-Neumann operator on $L_2(\Gamma)$

In this section we introduce the Dirichlet-to-Neumann operator on Γ associated with the elliptic operator $-\sum \partial_l(a_{kl}\partial_k)$ on Ω. We characterise the Dirichlet-to-Neumann operator via the form $\mathfrak{a}$ and trace map Tr, before establishing resolvent convergence with respect to the truncated problem. We then prove in Theorem 5.14 that if the boundary Γ and coefficients (a_{kl}) are sufficiently smooth, our two realisations of the Dirichlet-to-Neumann operator differ only by a rank-one operator.

We define the **Dirichlet-to-Neumann operator with Dirichlet boundary conditions at infinity** $\mathcal{N}^D$ in $L_2(\Gamma)$ as follows. Let $\varphi, \psi \in L_2(\Gamma)$. We write $\varphi \in D(\mathcal{N}^D)$ and $\mathcal{N}^D\varphi = \psi$ if there exists a $u \in W^D(\Omega)$ such that $\mathcal{A}u = 0$, $\mathrm{Tr}\, u = \varphi$ and $\partial_\nu u = \psi$. It is a consequence of Proposition 5.1(a) below that the operator $\mathcal{N}^D$ is single-valued and hence well-defined.

Recall that in the case $\lambda = 0$, a solution of (1) satisfying Neumann boundary conditions at infinity can only exist if $\int_\Gamma \psi = 0$. We define the **Dirichlet-to-Neumann operator with Neumann boundary conditions at infinity** $\mathcal{N}$ in $L_2(\Gamma)$ as follows. Let $\varphi, \psi \in L_2(\Gamma)$. We write $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$ if there exists a $u \in W(\Omega)$ such that $\mathcal{A}u = 0$, $\mathrm{Tr}\, u = \varphi$, $\partial_\nu u = \psi$ and $\int_\Gamma \psi = 0$. It is a consequence of Proposition 5.1(b) below that the operator $\mathcal{N}$ is single-valued and hence well-defined.

As in Section 3, several results in this section follow from relatively minor modifications of the corresponding arguments used in [AE15] for the Laplacian, demonstrating again the versatility of the form method. In the case of Dirichlet boundary conditions at infinity however, to obtain that $\mathbb{1}_\Gamma$ is in the domain of the Dirichlet-to-Neumann operator requires a different argument to that used for the Laplacian, which is the reason for the additional regularity hypotheses appearing in Theorem 5.14. We begin with a helpful characterisation.

By Lemma 3.1 the continuous sesquilinear form $\mathfrak{a}$ is Tr-elliptic on $W(\Omega)$. Moreover, by the Stone–Weierstraß theorem $\mathrm{Tr}\, W(\Omega)$ and $\mathrm{Tr}\, W^D(\Omega)$ are dense in $L_2(\Gamma)$. Define the form $\mathfrak{a}^D \colon W^D(\Omega) \times W^D(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}^D = \mathfrak{a}|_{W^D(\Omega)\times W^D(\Omega)}.$$

Then $\mathfrak{a}^D$ is continuous and $\mathrm{Tr}\,|_{W^D(\Omega)}$-elliptic. Hence by [AE12] Theorem 2.1 there exist m-sectorial operators in $L_2(\Gamma)$ associated with $(\mathfrak{a}, \mathrm{Tr}\,)$ and $(\mathfrak{a}^D, \mathrm{Tr}\,|_{W^D(\Omega)})$.

Proposition 5.1. (a) *The operator $\mathcal{N}^D$ is equal to the operator in $L_2(\Gamma)$ associated with $(\mathfrak{a}^D, \mathrm{Tr}\,|_{W^D(\Omega)})$.*

(b) *The operator $\mathcal{N}$ is equal to the operator in $L_2(\Gamma)$ associated with $(\mathfrak{a}, \mathrm{Tr}\,)$.*

Proof. We first prove (a). Let A^D denote the operator associated with $(\mathfrak{a}^D, \mathrm{Tr}\,|_{W^D(\Omega)})$ and let $\varphi, \psi \in L_2(\Gamma)$. Suppose that $\varphi \in D(A^D)$ and $A^D\varphi = \psi$. Then there exists a $u \in W^D(\Omega)$ such that $\mathrm{Tr}\, u = \varphi$ and $\mathfrak{a}^D(u, v) = (\psi, \mathrm{Tr}\, v)_{L_2(\Gamma)}$ for all $v \in W^D(\Omega)$. Hence

$$\langle \mathcal{A}u, v\rangle = \mathfrak{a}^D(u, v) = (\psi, \mathrm{Tr}\, v)_{L_2(\Gamma)} = 0$$

for all $v \in C_c^\infty(\Omega)$ and $\mathcal{A}u = 0 \in L_2(\Omega)$. Then

$$\mathfrak{a}(u, v) - \int_{\Omega_{R_0}} (\mathcal{A}u)\overline{v} = \mathfrak{a}^D(u, v) = (\psi, \mathrm{Tr}\, v)_{L_2(\Gamma)}$$

for all $v \in C_c^\infty(B_{R_0})$, so $\partial_\nu u = \psi$. Therefore $\varphi \in D(\mathcal{N}^D)$ and $\mathcal{N}^D\varphi = \psi$.

Conversely, suppose that $\varphi, \psi \in L_2(\Gamma)$ are such that $\varphi \in D(\mathcal{N}^D)$ and $\mathcal{N}^D\varphi = \psi$. Then there exists a $u \in W^D(\Omega)$ such that $\mathcal{A}u = 0$, $\operatorname{Tr} u = \varphi$ and $\partial_\nu u = \psi$. Hence $\mathfrak{a}^D(u,v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^D(\Omega)$ by Proposition 3.2. So $\varphi \in D(A^D)$ and $A^D\varphi = \psi$. Therefore $\mathcal{N}^D = A^D$ is the operator associated with $(\mathfrak{a}^D, \operatorname{Tr}|_{W^D(\Omega)})$.

We now prove (b). Let A denote the operator associated with $(\mathfrak{a}, \operatorname{Tr})$. Then the inclusion $A \subset \mathcal{N}$ follows as above, with the additional observation that if $\mathfrak{a}(u,v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W(\Omega)$, then the choice $v = \mathbb{1}_\Omega$ implies that $\int_\Gamma \psi = 0$.

Now suppose that $\varphi, \psi \in L_2(\Gamma)$ are such that $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \psi$. As above, there exists a $u \in W(\Omega)$ such that $\mathfrak{a}(u,v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^D(\Omega)$. Since $\int_\Gamma \psi = 0$ it follows that

$$\mathfrak{a}(u, \mathbb{1}_\Omega) = 0 = (\psi, \mathbb{1}_\Gamma)_{L_2(\Gamma)} = (\psi, \operatorname{Tr} \mathbb{1}_\Omega)_{L_2(\Gamma)}.$$

Then by linearity together with the orthogonal decomposition in Proposition 2.12, one deduces that $\varphi \in D(A)$ and $A\varphi = \psi$. This proves the claim. □

For all $\theta \in (0, \frac{\pi}{2}]$ we denote by

$$\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$$

the (open) **sector** of angle θ in $\mathbb{C}$. We say that a holomorphic C_0-semigroup $T = (T_t)_{t>0}$ on $L_2(\Gamma)$ is **contractive on a sector** if there exists a $\theta \in (0, \frac{\pi}{2}]$ such that T admits a holomorphic extension $\widetilde{T} = (\widetilde{T}_z)_{z\in\Sigma_\theta}$ to the sector Σ_θ and $\|\widetilde{T}_z\|_{\mathcal{L}(L_2(\Gamma))} \le 1$ for all $z \in \Sigma_\theta$. We make the identification $T = \widetilde{T}$.

Corollary 5.2. (a) *The operator $-\mathcal{N}^D$ generates a holomorphic C_0-semigroup $S^D = (S_t^D)_{t>0}$ on $L_2(\Gamma)$. Moreover, S^D is contractive on a sector.*

(b) *The operator $-\mathcal{N}$ generates a holomorphic C_0-semigroup $S = (S_t)_{t>0}$ on $L_2(\Gamma)$. Moreover, S is contractive on a sector.*

Proof. By [AE12] Theorem 2.1(ii), the operators $\mathcal{N}$ and $\mathcal{N}^D$ are m-sectorial with vertex 0. Then the claim follows from [Kat80] Theorem IX.1.24. □

Note that it is not possible to replace the form domain $W(\Omega)$ with $H^1(\Omega)$, because the form $\mathfrak{b} = \mathfrak{a}|_{H^1(\Omega)\times H^1(\Omega)}$ fails to be Tr-elliptic on $H^1(\Omega)$. We present this fact via the following example, which can be found at [AE15] Remark 5.2.

Example 5.3. We shall show that for each $\mu, \omega > 0$ there exists a $u \in H^1(\Omega)$ such that $\mu\|u\|^2_{L_2(\Omega)} > \operatorname{Re} \mathfrak{b}(u) + \omega\|\operatorname{Tr} u\|^2_{L_2(\Gamma)}$. The invalidity of the Tr-ellipticity of $\mathfrak{b}$ on $H^1(\Omega)$ then follows.

Let $\mu, \omega > 0$. Without loss of generality we may assume that $B_1 \subset \mathbb{R}^d \setminus \overline{\Omega}$. Fix $\tau \in C_c^\infty(\mathbb{R}^d)$ such that $\tau|_{B_1} = \mathbb{1}$ and $\|\tau\|_\infty = 1$. For all $R > R_0$ define $u_R : \Omega \to \mathbb{C}$ by $u_R(x) = |x|^{-(d-1)/2}\tau\big(\frac{x}{R}\big)$. Then $u_R \in H^1(\Omega) \cap C^\infty(\Omega)$. Since the coefficients (a_{kl}) are bounded, there exists an $M > 0$ such that

$$\begin{aligned}\operatorname{Re} \mathfrak{b}(u_R) \le M \int_\Omega |\nabla u_R|^2 &= M \int_\Omega \Big|\frac{d-1}{2} \cdot \frac{x}{|x|^{(d+3)/2}}\,\tau\Big(\frac{x}{R}\Big) + \frac{1}{|x|^{(d-1)/2}} \cdot \frac{1}{R}(\nabla\tau)\Big(\frac{x}{R}\Big)\Big|^2 \,\mathrm{d}x \\ &\le 2M\Big(\frac{d-1}{2}\Big)^2 \|\tau\|^2_\infty \int_\Omega \frac{1}{|x|^{d+1}}\,\mathrm{d}x + 2M \int_\Omega \frac{1}{R^2|x|^{d-1}}\Big|(\nabla\tau)\Big(\frac{x}{R}\Big)\Big|^2 \,\mathrm{d}x\end{aligned}$$

$$= 2M\Big(\frac{d-1}{2}\Big)^2 \omega_d \int_1^\infty \frac{1}{r^2}\,\mathrm{d}r + 2M \int_\Omega \frac{|(\nabla\tau)(y)|^2}{R|y|^{d-1}}\,\mathrm{d}y$$
$$\le 2Md^2\omega_d + 2M \int_\Omega \frac{|(\nabla\tau)(y)|^2}{|y|^{d-1}}\,\mathrm{d}y < \infty$$

for all $R > R_0$. Note that the integral in the final term of the above estimate is finite because $0 \notin \operatorname{supp} \nabla\tau$. Moreover, $\int_\Gamma |\mathrm{Tr}\, u_R|^2 = \int_\Gamma |z|^{-(d-1)}\,\mathrm{d}z < \infty$ for all $R > R_0$.

Write

$$a = 2Md^2\omega_d + 2M \int_\Omega \frac{|(\nabla\tau)(y)|^2}{|y|^{d-1}}\,\mathrm{d}y$$

and $b = \int_\Gamma |z|^{-(d-1)}\,\mathrm{d}z$. Choose $S > \frac{1}{\mu\omega_d}(a + \omega b) + R_0$. Then $u_S \in H^1(\Omega)$ and

$$\int_\Omega |u_S|^2 = \int_\Omega \Big|\frac{1}{|x|^{(d-1)/2}}\,\tau\Big(\frac{x}{S}\Big)\Big|^2\,\mathrm{d}x$$
$$\ge \int_{\Omega_S} \frac{1}{|x|^{d-1}}\,\mathrm{d}x \ge \int_{B_S \setminus B_{R_0}} \frac{1}{|x|^{d-1}}\,\mathrm{d}x = \omega_d \int_{R_0}^S \mathrm{d}r = \omega_d(S - R_0).$$

Hence

$$\mu \|u_S\|^2_{L_2(\Omega)} \ge \mu\,\omega_d(S - R_0) > a + \omega b \ge \mathrm{Re}\,\mathfrak{b}(u_S) + \omega \|\mathrm{Tr}\, u_S\|^2_{L_2(\Gamma)}$$

as required.

Clearly $\mathcal{N} \neq \mathcal{N}^D$, since $\ker \mathcal{N} \neq \ker \mathcal{N}^D$.

Proposition 5.4. (a) $\ker \mathcal{N}^D = \{0\}$.

(b) $\ker \mathcal{N} = \mathbb{C}\mathbb{1}_\Gamma$.

Proof. We first prove (b). Let $\varphi \in \ker \mathcal{N}$. Then by Proposition 5.1(b) there exists a $u \in W(\Omega)$ such that $\mathrm{Tr}\, u = \varphi$ and $\mathfrak{a}(u, v) = 0$ for all $v \in W(\Omega)$. The choice $v = u$ then yields that $\mu \int_\Omega |\nabla u|^2 \le \mathrm{Re}\,\mathfrak{a}(u) = 0$. So u is constant and it follows that $\varphi = \mathrm{Tr}\, u$ is constant. Conversely, note that $\mathbb{1}_\Omega \in W(\Omega)$. Then $\mathrm{Tr}\, \mathbb{1}_\Omega = \mathbb{1}_\Gamma$ and $\mathfrak{a}(\mathbb{1}_\Omega, v) = 0 = (0, \mathrm{Tr}\, v)_{L_2(\Gamma)}$ for all $v \in W(\Omega)$. Hence $\mathbb{1}_\Gamma \in D(\mathcal{N})$ and $\mathcal{N}\mathbb{1}_\Gamma = 0$.

We now prove (a). Let $\varphi \in \ker \mathcal{N}^D$. By an argument similar to the above, one deduces that there exists a constant function $u \in W^D(\Omega)$ with $\mathrm{Tr}\, u = \varphi$. Since $\langle u \rangle = 0$ by Proposition 2.12, it follows that $u = 0$ and $\varphi = \mathrm{Tr}\, u = 0$. □

From the compactness of the solution operators B_λ and B^D_λ, we obtain that $\mathcal{N}$ and $\mathcal{N}^D$ have compact resolvent. The following proof also demonstrates that the resolvent of the Dirichlet-to-Neumann operator maps the Robin data in (1) to the trace of the corresponding solution.

Proposition 5.5. (a) *Let $\lambda \ge 0$. Then the operator $(\lambda I + \mathcal{N}^D)^{-1}$ is compact.*

(b) *Let $\lambda > 0$. Then the operator $(\lambda I + \mathcal{N})^{-1}$ is compact.*

Proof. We prove (a). Let $\varphi \in D(\mathcal{N}^D)$ and write $\psi = (\lambda I + \mathcal{N}^D)\varphi \in L_2(\Gamma)$. Then $\mathcal{N}^D\varphi = \psi - \lambda\varphi$. Hence by Proposition 5.1(a) there exists a $u \in W^D(\Omega)$ such that $\mathrm{Tr}\, u = \varphi$ and

$$\mathfrak{a}(u, v) = \mathfrak{a}^D(u, v) = \int_\Gamma (\psi - \lambda \mathrm{Tr}\, u)\, \overline{\mathrm{Tr}\, v}$$

for all $v \in W^D(\Omega)$. So $u = B_\lambda^D \psi$ and $(\lambda I + \mathcal{N}^D)^{-1}\psi = \varphi = \operatorname{Tr} B_\lambda^D \psi$. Therefore $(\lambda I + \mathcal{N}^D)^{-1} = \operatorname{Tr} \circ B_\lambda^D$. Then the claim follows from the continuity of Tr together with the compactness of B_λ^D from Proposition 3.9(a).

The proof of (b) is similar. □

By elliptic regularity, the resolvent leaves $C(\Gamma)$ invariant. In Corollary 7.5 we show that the same is true of the corresponding semigroup.

Proposition 5.6. (a) *Let $\lambda \geq 0$. Then $(\lambda I + \mathcal{N}^D)^{-1} C(\Gamma) \subset C(\Gamma)$.*

(b) *Let $\lambda > 0$. Then $(\lambda I + \mathcal{N})^{-1} C(\Gamma) \subset C(\Gamma)$.*

Proof. We prove (b). Let $\psi \in C(\Gamma)$ and write $\varphi = (\lambda I + \mathcal{N})^{-1}\psi \in L_2(\Gamma)$. Then $\mathcal{N}\varphi = \psi - \lambda\varphi$. Hence by Proposition 5.1(b) there exists a $u \in W(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and

$$\mathfrak{a}(u, v) + \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}$$

for all $v \in W(\Omega)$. Then $u = B_\lambda \psi$. Since Γ is compact, it follows that $\psi \in L_\infty(\Gamma)$ and Theorem 1.1(b) therefore provides that $u \in C(\overline{\Omega})$. Hence $\varphi = \operatorname{Tr} u \in C(\Gamma)$.

The proof of (a) is similar. □

We now investigate resolvent convergence with respect to the truncated problem (13). Let $R \geq R_0$. We define the **Dirichlet-to-Neumann operator with Dirichlet boundary conditions at ∂B_R**, denoted by $\mathcal{N}_R^D$, in $L_2(\Gamma)$ as follows. Let $\varphi, \psi \in L_2(\Gamma)$. We write $\varphi \in D(\mathcal{N}_R^D)$ and $\mathcal{N}_R^D \varphi = \psi$ if there exists a $u \in W_R^D(\Omega)$ such that $\mathcal{A}u = 0$, $\operatorname{Tr} u = \varphi$ and $\partial_\nu u = \psi$.

For all $R \geq R_0$ we define the sesquilinear form $\mathfrak{a}_R^D \colon W_R^D(\Omega) \times W_R^D(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}_R^D = \mathfrak{a}|_{W_R^D(\Omega) \times W_R^D(\Omega)}.$$

Then $\mathfrak{a}_R^D$ is continuous and $\operatorname{Tr}|_{W_R^D(\Omega)}$-elliptic with $\operatorname{Tr} W_R^D(\Omega)$ dense in $L_2(\Gamma)$. It follows from an argument similar to the proof of Proposition 5.1(a) that $\mathcal{N}_R^D$ is equal to the operator in $L_2(\Gamma)$ associated with $\big(\mathfrak{a}_R^D, \operatorname{Tr}|_{W_R^D(\Omega)}\big)$.

Proposition 5.7. *Let $\lambda \geq 0$. Then*

$$\lim_{R \to \infty} (\lambda I + \mathcal{N}_R^D)^{-1} = (\lambda I + \mathcal{N}^D)^{-1}$$

in $\mathcal{L}\big(L_2(\Gamma)\big)$.

Proof. By an argument similar to the proof of Proposition 5.5(a), one deduces that

$$(\lambda I + \mathcal{N}_R^D)^{-1} = \operatorname{Tr} \circ B_\lambda^D(R) \tag{22}$$

for all $R \geq R_0$. Since $\operatorname{Tr} \in \mathcal{L}\big(W(\Omega), L_2(\Gamma)\big)$, the claim follows from Proposition 4.5. □

In Theorem 5.9 we show that if the coefficients (a_{kl}) are Lipschitz continuous, the preceding result extrapolates to $\mathcal{L}\big(L_p(\Gamma)\big)$ for all $p \in [1, \infty]$. In the proof we use the fact that Tr is continuous from $\big(W^D(\Omega) \cap L_\infty(\Omega), \|\cdot\|_{L_\infty(\Omega)}\big)$ into $L_\infty(\Gamma)$.

Lemma 5.8. *Let $u \in W^D(\Omega) \cap L_\infty(\Omega)$. Then $\operatorname{Tr} u \in L_\infty(\Gamma)$ and $\|\operatorname{Tr} u\|_{L_\infty(\Gamma)} \leq 2\|u\|_{L_\infty(\Omega)}$.*

Proof. Let $(\chi_n)_{n\in\mathbb{N}}$ be a sequence in $C_c^\infty(\mathbb{R}^d)$ such that $\lim \chi_n|_\Omega = u$ in $W(\Omega)$. Then $\lim \operatorname{Re} \chi_n|_\Gamma = \operatorname{Re} \operatorname{Tr} u$ in $L_2(\Gamma)$, since Tr is continuous. Write $M = \|u\|_{L_\infty(\Omega)}$. For each $n \in \mathbb{N}$ define $u_n = (-M) \vee \operatorname{Re} \chi_n|_\Omega \wedge M$. Then $u_n \in W(\Omega)$ by Proposition 2.15(a) and the identities $w \vee v = w + (v - w)^+$ and $w \wedge v = -((-w) \vee (-v))$. Since $\lim u_n = \operatorname{Re} u$ in $W(\Omega)$, it follows from the continuity of Tr once again that $\lim \operatorname{Tr} u_n = \operatorname{Re} \operatorname{Tr} u$ in $L_2(\Gamma)$. Therefore

$$\operatorname{Re} \operatorname{Tr} u = \lim_{n\to\infty} \operatorname{Tr} u_n = (-M) \vee \lim_{n\to\infty} \operatorname{Re} \chi_n|_\Gamma \wedge M = (-M) \vee \operatorname{Re} \operatorname{Tr} u \wedge M.$$

So $\| \operatorname{Re} \operatorname{Tr} u\|_{L_\infty(\Gamma)} \leq \|u\|_{L_\infty(\Omega)}$. By a similar argument one obtains that $\| \operatorname{Im} \operatorname{Tr} u\|_{L_\infty(\Gamma)} \leq \|u\|_{L_\infty(\Omega)}$ and the result follows. □

Theorem 5.9. *Suppose that $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R})$ for all $k, l \in \{1, \ldots, d\}$ and let $\lambda > 0$. Then*

$$\lim_{R\to\infty} (\lambda I + \mathcal{N}_R^D)^{-1} = (\lambda I + \mathcal{N}^D)^{-1}$$

in $\mathcal{L}\big(L_p(\Gamma)\big)$ for all $p \in [1, \infty]$.

Proof. By (22) together with Lemma 5.8 and Proposition 4.16, one deduces that the claim is valid for the case $p = \infty$. Hence by Proposition 5.7 and an interpolation argument, the claim holds for all $p \in [2, \infty]$. Since the matrix $\big((a_{kl})_{k,l=1}^d\big)^* = (a_{lk})_{k,l=1}^d$ satisfies the same Lipschitz and ellipticity conditions as $(a_{kl})_{k,l=1}^d$, by duality one deduces that the case $p \in [1, 2]$ is also valid. □

Next we examine resolvent convergence in the case of Neumann boundary conditions at infinity. In this situation it happens not to be possible to define the Dirichlet-to-Neumann operator as acting on traces of harmonic functions. We therefore we proceed via the form method directly.

For all $R \geq R_0$ define the sesquilinear form $\mathfrak{a}_R \colon H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C}$ by

$$\mathfrak{a}_R(u, v) = \sum_{k,l=1}^d \int_{\Omega_R} a_{kl} (\partial_k u) \overline{\partial_l v}.$$

Then $\mathfrak{a}_R$ is continuous and Tr_R-elliptic, with $\operatorname{Tr}_R H^1(\Omega_R)$ dense in $L_2(\Gamma)$. We define the **Dirichlet-to-Neumann operator with Neumann boundary conditions at ∂B_R**, denoted by $\mathcal{N}_R$, to be the operator in $L_2(\Gamma)$ associated with $\big(\mathfrak{a}_R, \operatorname{Tr}_R\big)$.

Theorem 5.10. *Let $\lambda > 0$. Then*

$$\lim_{R\to\infty} (\lambda I + \mathcal{N}_R)^{-1} = (\lambda I + \mathcal{N})^{-1}$$

in $\mathcal{L}\big(L_2(\Gamma)\big)$.

Proof. Let $(R_n)_{n\in\mathbb{N}}$ be a sequence in $[R_0, \infty)$ with $\lim R_n = \infty$. Let $\psi, \psi_1, \psi_2, \ldots \in L_2(\Gamma)$ and suppose that $\lim \psi_n = \psi$ weakly in $L_2(\Gamma)$. By Propositions 5.5(b) and A.7 it suffices to prove that $\lim (\lambda I + \mathcal{N}_{R_n})^{-1}\psi_n = (\lambda I + \mathcal{N})^{-1}\psi$ in $L_2(\Gamma)$.

Let $n \in \mathbb{N}$ and write $\varphi_n = (\lambda I + \mathcal{N}_{R_n})^{-1}\psi_n$. Then $\mathcal{N}_{R_n}\varphi_n = \psi_n - \lambda\varphi_n$, so there exists a $u_n \in H^1(\Omega_{R_n})$ such that $\operatorname{Tr}_{R_n} u_n = \varphi_n$ and

$$\mathfrak{a}_{R_n}(u_n, v) + \lambda \int_\Gamma \operatorname{Tr}_{R_n} u_n \, \overline{\operatorname{Tr}_{R_n} v} = \int_\Gamma \psi_n \, \overline{\operatorname{Tr}_{R_n} v} \tag{23}$$

for all $v \in H^1(\Omega_{R_n})$. The choice $v = u_n$ then yields that

$$\operatorname{Re} \mathfrak{a}_{R_n}(u_n) + \lambda \int_\Gamma |\operatorname{Tr}_{R_n} u_n|^2 = \operatorname{Re} \int_\Gamma \psi_n \, \overline{\operatorname{Tr}_{R_n} u_n} \leq \|\psi_n\|_{L_2(\Gamma)} \|\varphi_n\|_{L_2(\Gamma)}.$$

Hence

$$\lambda \|\varphi_n\|_{L_2(\Gamma)}^2 = \lambda \int_\Gamma |\operatorname{Tr}_{R_n} u_n|^2 \leq \|\psi_n\|_{L_2(\Gamma)} \|\varphi_n\|_{L_2(\Gamma)}$$

for all $n \in \mathbb{N}$ and since the sequence $(\psi_n)_{n\in\mathbb{N}}$ is bounded in $L_2(\Gamma)$, it follows that $(\varphi_n)_{n\in\mathbb{N}}$ is also bounded in $L_2(\Gamma)$. Then $\sup \|\psi_n\|_{L_2(\Gamma)} \|\varphi_n\|_{L_2(\Gamma)} < \infty$ and it follows that

$$M = \sup_{n\in\mathbb{N}} \operatorname{Re} \mathfrak{a}_{R_n}(u_n) + \int_\Gamma |\operatorname{Tr}_{R_n} u_n|^2 < \infty. \tag{24}$$

Note that since the ellipticity condition (4) is valid a.e. on Ω, it follows that

$$\mu \int_{\Omega_R} |\nabla v|^2 \leq \operatorname{Re} \mathfrak{a}_R(v)$$

for all $R \geq R_0$ and $v \in H^1(\Omega_R)$. Let $R \in [R_0, \infty) \cap \mathbb{N}$. Then for all $n \in \mathbb{N}$ with $R_n \geq R$,

$$\begin{aligned}
\int_{\Omega_R} |\nabla u_n|^2 + \int_\Gamma |\operatorname{Tr}_R(u_n|_{\Omega_R})|^2 &\leq \int_{\Omega_{R_n}} |\nabla u_n|^2 + \int_\Gamma |\operatorname{Tr}_{R_n} u_n|^2 \\
&\leq \frac{1}{\mu} \operatorname{Re} \mathfrak{a}_{R_n}(u_n) + \int_\Gamma |\operatorname{Tr}_{R_n} u_n|^2 \leq M\Big(1 + \frac{1}{\mu}\Big).
\end{aligned}$$

Since Ω_R has Lipschitz boundary, the map Tr_R is compact and Lemma 2.4(b) consequently provides that the sequence $(u_n|_{\Omega_R})_{n\in\mathbb{N},\, R_n \geq R}$ is bounded in $H^1(\Omega_R)$. By a diagonal argument and passing to a subsequence if necessary, we may assume that for all $R \geq R_0$ the sequence $(u_n|_{\Omega_R})_{n\in\mathbb{N}}$ is weakly convergent in $H^1(\Omega_R)$. Hence there exists a $u \in H^1_{\mathrm{loc}}(\Omega)$ such that for all $R \geq R_0$, one has that $u|_{\Omega_R} \in H^1(\Omega_R)$ and $\lim u_n|_{\Omega_R} = u|_{\Omega_R}$ weakly in $H^1(\Omega_R)$. Then $\lim \nabla(u_n|_{\Omega_R}) = \nabla(u|_{\Omega_R})$ weakly in $L_2(\Omega_R)$, so

$$\int_{\Omega_R} |\nabla u|^2 \leq \liminf_{n\to\infty} \int_{\Omega_R} |\nabla u_n|^2 \leq M\Big(1 + \frac{1}{\mu}\Big)$$

for all $R \geq R_0$. Therefore $\int_\Omega |\nabla u|^2 < \infty$ and $u \in W(\Omega)$.

Recall that $\operatorname{Tr}_R(v|_{\Omega_R}) = \operatorname{Tr} v$ for all $R \geq R_0$ and $v \in W(\Omega)$. So

$$\lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} \operatorname{Tr}_{R_0}(u_n|_{\Omega_{R_0}}) = \operatorname{Tr}_{R_0}(u|_{\Omega_{R_0}}) = \operatorname{Tr} u$$

in $L_2(\Gamma)$, since Tr_{R_0} is compact. Let $v \in C_c^\infty(\mathbb{R}^d)$. Then there exists an $m \in \mathbb{N}$ such that $\operatorname{supp} v \subset B_{R_m}$, so for all $n \geq m$ one has that $\mathfrak{a}_{R_n}(u_n, v|_{\Omega_{R_n}}) = \mathfrak{a}_{R_m}(u_n|_{\Omega_{R_m}}, v|_{\Omega_{R_m}})$. Hence

$$\lim_{n\to\infty} \mathfrak{a}_{R_n}(u_n, v|_{\Omega_{R_n}}) = \lim_{n\to\infty} \mathfrak{a}_{R_m}(u_n|_{\Omega_{R_m}}, v|_{\Omega_{R_m}}) = \mathfrak{a}_{R_m}(u|_{\Omega_{R_m}}, v|_{\Omega_{R_m}}) = \mathfrak{a}(u, v|_\Omega)$$

since $\lim u_n|_{\Omega_{R_m}} = u|_{\Omega_{R_m}}$ weakly in $H^1(\Omega_{R_m})$. Then it follows from (23) that

$$\mathfrak{a}(u, v|_\Omega) + \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr}(v|_\Omega)} = \lim_{n\to\infty} \mathfrak{a}_{R_n}(u_n, v|_{\Omega_{R_n}}) + \lambda \int_\Gamma \varphi_n \, \overline{\operatorname{Tr}_{R_n}(v|_{\Omega_{R_n}})}$$

$$= \lim_{n\to\infty} \int_\Gamma \psi_n \, \overline{\mathrm{Tr}_{R_n}(v|_{\Omega_{R_n}})} = \int_\Gamma \psi \, \overline{\mathrm{Tr}\,(v|_\Omega)}.$$

So

$$\mathfrak{a}(u,v) + \lambda \int_\Gamma \mathrm{Tr}\, u \, \overline{\mathrm{Tr}\, v} = \int_\Gamma \psi \, \overline{\mathrm{Tr}\, v} \tag{25}$$

for all $v \in W^D(\Omega)$ by density. Moreover, the choice $v = \mathbb{1}_{\Omega_{R_n}}$ in (23) provides that $\lambda \int_\Gamma \varphi_n = \int_\Gamma \psi_n$ for all $n \in \mathbb{N}$, so

$$\lambda \int_\Gamma \mathrm{Tr}\, u = \lim_{n\to\infty} \lambda \int_\Gamma \varphi_n = \lim_{n\to\infty} \int_\Gamma \psi_n = \int_\Gamma \psi$$

and (25) is therefore valid for $v = \mathbb{1}_\Omega$. Then by linearity together with the orthogonal decomposition $W(\Omega) = W^D(\Omega) \oplus \mathbb{C}\mathbb{1}_\Omega$ from Proposition 2.12, it follows that (25) is valid for all $v \in W(\Omega)$. Hence $B_\lambda \psi = u$, so $\lim \varphi_n = \mathrm{Tr}\, B_\lambda \psi = (\lambda I + \mathcal{N})^{-1}\psi$ in $L_2(\Gamma)$. □

The next result concerns the C_0-semigroups S and S^D on $L_2(\Gamma)$, generated by $-\mathcal{N}$ and $-\mathcal{N}^D$ respectively. We establish positivity of the semigroups using [AE12] Proposition 2.9, which extends Ouhabaz' generalisation of the Beurling–Deny criteria to the setting of the form method in [AE12]. From the domination of B_λ^D by B_λ we obtain an analogous relationship between the semigroups generated by the corresponding Dirichlet-to-Neumann operators. In Proposition 7.6 we show that this relationship propagates forward to the associated heat kernels.

A semigroup $T = (T_t)_{t>0}$ on $L_2(\Gamma)$ is called **positive** if $T_t L_2(\Gamma)^+ \subset L_2(\Gamma)^+$ for all $t > 0$.

Proposition 5.11. *The semigroups S and S^D are positive. Moreover,*

$$0 \le S_t^D \varphi \le S_t \varphi$$

for all $\varphi \in L_2(\Gamma)^+$ and $t > 0$.

Proof. We first verify that S^D is positive. Define $P\colon L_2(\Gamma) \to L_2(\Gamma)^+$ by $P\varphi = (\mathrm{Re}\,\varphi)^+$. Let $u \in W^D(\Omega)$. Then $\mathrm{Re}\, u \in W^D(\Omega)$ and Proposition 2.15(b) provides that $(\mathrm{Re}\, u)^+ \in W^D(\Omega)$. Moreover, $\mathrm{Tr}\,((\mathrm{Re}\, u)^+) = (\mathrm{Re}\ \mathrm{Tr}\, u)^+$ by Proposition 2.15(d). Then by [GT83] Lemma 7.6 one deduces that

$$\begin{aligned} \mathrm{Re}\, \mathfrak{a}^D\big((\mathrm{Re}\, u)^+, u - (\mathrm{Re}\, u)^+\big) &= \mathrm{Re}\, \mathfrak{a}^D\big((\mathrm{Re}\, u)^+, -(\mathrm{Re}\, u)^- + i\,\mathrm{Im}\, u\big) \\ &= -\mathfrak{a}^D\big((\mathrm{Re}\, u)^+, (\mathrm{Re}\, u)^-\big) = 0. \end{aligned}$$

Hence by [AE12] Proposition 2.9 '(ii)⇒(i)', the semigroup S^D is positive.

Let $\varphi \in L_2(\Gamma)^+$. Then by Proposition 3.6

$$(\lambda I + \mathcal{N}^D)^{-1}\varphi = \mathrm{Tr}\, B_\lambda^D \varphi \le \mathrm{Tr}\, B_\lambda \varphi = (\lambda I + \mathcal{N})^{-1}\varphi$$

for all $\lambda > 0$. Therefore

$$0 \le S_t^D \varphi = \lim_{n\to\infty} (I + \frac{t}{n}\mathcal{N}^D)^{-n}\varphi \le \lim_{n\to\infty} (I + \frac{t}{n}\mathcal{N})^{-n}\varphi = S_t\varphi$$

for all $t > 0$ and it follows that S is positive. □

An obvious consequence of Proposition 5.4 is that $\mathbb{1}_\Gamma \in D(\mathcal{N})$ and $\mathcal{N}\mathbb{1}_\Gamma = 0$. The following assertion establishes that if the coefficients (a_{kl}) and boundary Γ are sufficiently smooth, then $\mathbb{1}_\Gamma \in D(\mathcal{N}^D)$ also. The need for additional regularity stems from the fact that $\mathbb{1}_\Omega \notin W^D(\Omega)$, as this means that we must construct a harmonic element of $W^D(\Omega)$ with trace equal to $\mathbb{1}_\Gamma$ and normal derivative in $L_2(\Gamma)$. It is clear that if $\mathbb{1}_\Gamma \in D(\mathcal{N}^D)$, then $\mathcal{N}^D\mathbb{1}_\Gamma \neq 0$ since $\ker \mathcal{N}^D$ is trivial. Indeed, Corollary 5.13 shows that $\mathcal{N}^D\mathbb{1}_\Gamma > 0$.

Proposition 5.12. *Suppose that Ω is a C^2-domain and $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R}) \cap C^1(\Omega_{R_0+1})$ for all $k, l \in \{1, \ldots, d\}$. Then $\mathbb{1}_\Gamma \in D(\mathcal{N}^D)$.*

Proof. Let $\tau \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ be such that $\tau|_{\Omega_{R_0}} = \mathbb{1}$. It follows from the Lax–Milgram theorem that for each $n \in \mathbb{N}$ with $n \geq R_0$, there exists a $u_n \in H_0^1(\Omega_n)$ such that

$$\sum_{k,l=1}^{d} \int_{\Omega_n} a_{kl}(\partial_k u_n)\overline{\partial_l v} = \mathfrak{a}^D(\tau, v)$$

for all $v \in H_0^1(\Omega_n)$. By a zero extension we may assume that $u_n \in W^D(\Omega)$. Since $\mathfrak{a}^D$ is continuous, there exists a $c > 0$ such that

$$\begin{aligned}\mu\|u_n\|_{W^D(\Omega)}^2 = \mu \int_\Omega |\nabla u_n|^2 \leq \operatorname{Re} \mathfrak{a}^D(u_n) &= \operatorname{Re} \sum_{k,l=1}^{d} \int_{\Omega_n} a_{kl}(\partial_k u_n)\overline{\partial_l u_n} \\ &= \operatorname{Re} \mathfrak{a}^D(\tau, u_n) \leq c\,\|\tau\|_{W^D(\Omega)}\,\|u_n\|_{W^D(\Omega)}\end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq R_0$. Then $\|u_n\|_{W^D(\Omega)} \leq c\mu^{-1}\|\tau\|_{W^D(\Omega)}$ for all $n \in \mathbb{N}$ with $n \geq R_0$, so the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^D(\Omega)$. Passing to a subsequence if necessary we may assume that there exists a $u \in W^D(\Omega)$ such that $\lim u_n = u$ weakly in $W^D(\Omega)$. Then $\operatorname{Tr} u = \lim \operatorname{Tr} u_n = 0$ in $L_2(\Gamma)$, since by Proposition 2.11(b) the map Tr is compact.

Let $v \in C_c^\infty(\Omega)$. Then $v \in H_0^1(\Omega_n)$ for all n sufficiently large, so

$$\langle \mathcal{A}u, v\rangle = \mathfrak{a}^D(u, v) = \lim_{n\to\infty} \mathfrak{a}^D(u_n, v) = \mathfrak{a}^D(\tau, v) = \Big(-\sum_{k,l=1}^{d} \partial_l(a_{kl}\,\partial_k\tau), v\Big)_{L_2(\Omega)}.$$

Hence $\langle \mathcal{A}u, v\rangle = \langle \mathcal{A}\tau, v\rangle$ for all $v \in C_c^\infty(\Omega)$ and $\mathcal{A}u = -\sum_{k,l=1}^d \partial_l(a_{kl}\,\partial_k\tau) \in L_2(\Omega)$. Note that $u|_{\Omega_{R_0+1}} \in H^1(\Omega_{R_0+1})$. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be such that $\chi|_{B_{R_0}} = \mathbb{1}$ and $\operatorname{supp}\chi \subset B_{R_0+1}$. Then $\operatorname{supp}\chi u \subset \Omega_{R_0+1}$, so $\chi u \in H_0^1(\Omega_{R_0+1})$. Moreover,

$$\mathcal{A}(\chi u) = u\mathcal{A}\chi + \chi\mathcal{A}u - \sum_{k,l=1}^{d} a_{kl}(\partial_k\chi)\partial_l u - \sum_{k,l=1}^{d} a_{kl}(\partial_k u)\partial_l\chi \in L_2(\Omega_{R_0+1}).$$

Then $\chi u \in H^2(\Omega_{R_0+1})$ by [Sim72] Theorem 9.12 and it follows that $a_{kl}\partial_k(\chi u) \in H^1(\Omega_{R_0+1})$ for all $k, l \in \{1, \ldots, d\}$. So $\partial_\nu(\chi u) \in L_2(\partial\Omega_{R_0+1})$ and

$$\mathbb{1}_\Gamma\partial_\nu(\chi u) = \sum_{k,l=1}^{d} \nu_l \operatorname{Tr}_{R_0+1}(a_{kl}\partial_k(\chi u)) \in L_2(\Gamma).$$

Note that $\partial_\nu\chi = \partial_\nu(\chi|_{\Omega_{R_0+1}}) = 0$. Hence

$$\partial_\nu u = \mathbb{1}_\Gamma\chi\partial_\nu u = \mathbb{1}_\Gamma\partial_\nu(\chi u) - \mathbb{1}_\Gamma(\partial_\nu\chi)\operatorname{Tr} u = \mathbb{1}_\Gamma\partial_\nu(\chi u) \in L_2(\Gamma).$$

Write $w = \tau|_\Omega - u$. Then $w \in W^D(\Omega)$ and

$$\langle \mathcal{A}w, v\rangle = \langle \mathcal{A}\tau, v\rangle - \langle \mathcal{A}u, v\rangle = 0$$

for all $v \in C_c^\infty(\Omega)$, so $\mathcal{A}w = 0$. Moreover, $\operatorname{Tr} w = \operatorname{Tr}_{R_0} \mathbb{1}_{\Omega_{R_0}} = \mathbb{1}_\Gamma$ and $\partial_\nu w = \partial_\nu \tau - \partial_\nu u = -\partial_\nu u \in L_2(\Gamma)$. Therefore $\mathbb{1}_\Gamma \in D(\mathcal{N}^D)$ and $\mathcal{N}^D \mathbb{1}_\Gamma = \partial_\nu w$. $\square$

Corollary 5.13. *Suppose that Ω is a C^2-domain and $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R}) \cap C^1(\Omega_{R_0+1})$ for all $k, l \in \{1, \ldots, d\}$. Then $\mathcal{N}^D \mathbb{1}_\Gamma > 0$.*

Proof. Since $\mathcal{N}\mathbb{1}_\Gamma = 0$, it follows from Proposition 5.11 that

$$\mathcal{N}^D \mathbb{1}_\Gamma = \lim_{t\downarrow 0} \frac{1}{t}(I - S_t^D)\mathbb{1}_\Gamma \ge \lim_{t\downarrow 0} \frac{1}{t}(I - S_t)\mathbb{1}_\Gamma = 0.$$

To see that $\mathcal{N}^D \mathbb{1}_\Gamma$ is non-zero on a set of positive measure, suppose to the contrary that $\int_\Gamma \mathcal{N}^D \mathbb{1}_\Gamma = 0$. Let $u \in W^D(\Omega)$ be such that $\operatorname{Tr} u = \mathbb{1}_\Gamma$ and $\mathfrak{a}^D(u, v) = (\mathcal{N}^D \mathbb{1}_\Gamma, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^D(\Omega)$. Then the choice $v = u$ yields that

$$\mu \int_\Omega |\nabla u|^2 \le \operatorname{Re} \mathfrak{a}^D(u) = \operatorname{Re} \big(\mathcal{N}^D \mathbb{1}_\Gamma, \mathbb{1}_\Gamma\big)_{L_2(\Gamma)} = 0.$$

So $\nabla u = 0$ and u is constant. Hence $u = 0$, since $\langle u\rangle = 0$ by Proposition 2.12. Then $0 = \operatorname{Tr} u = \mathbb{1}_\Gamma$, a contradiction. $\square$

If the coefficients and boundary are smooth enough to allow that $\mathbb{1}_\Gamma \in D(\mathcal{N}^D)$, then $\mathcal{N}^D$ is merely a rank-one perturbation of $\mathcal{N}$.

Theorem 5.14. *Suppose that Ω is a C^2-domain and $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R}) \cap C^1(\Omega_{R_0+1})$ for all $k, l \in \{1, \ldots, d\}$. Then $D(\mathcal{N}^D) = D(\mathcal{N})$ and*

$$\mathcal{N}^D \varphi = \mathcal{N}\varphi + \frac{1}{\beta}\big(\varphi, \mathcal{N}^{D^*}\mathbb{1}_\Gamma\big)_{L_2(\Gamma)} \mathcal{N}^D \mathbb{1}_\Gamma$$

for all $\varphi \in D(\mathcal{N})$, where $\beta = \int_\Gamma \mathcal{N}^D \mathbb{1}_\Gamma$.

Proof. We first show that $D(\mathcal{N}) \subset D(\mathcal{N}^D)$. Let $\varphi \in D(\mathcal{N})$ and write $\psi = \mathcal{N}\varphi$. Then $\int_\Gamma \psi = 0$. By Proposition 5.1(b) there exists a $u \in W(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and $\mathfrak{a}(u, v) = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W(\Omega)$. Moreover, $u - \langle u\rangle \mathbb{1}_\Omega \in W^D(\Omega)$ by Proposition 2.12 and it follows that

$$\mathfrak{a}^D(u - \langle u\rangle \mathbb{1}_\Omega, v) = \mathfrak{a}(u - \langle u\rangle \mathbb{1}_\Omega, v) = \mathfrak{a}(u, v) = \int_\Gamma \psi\, \overline{\operatorname{Tr} v}$$

for all $v \in W^D(\Omega)$. Hence Proposition 5.1(a) provides that $\varphi - \langle u\rangle \mathbb{1}_\Gamma = \operatorname{Tr}(u - \langle u\rangle \mathbb{1}_\Omega) \in D(\mathcal{N}^D)$ and $\mathcal{N}^D(\varphi - \langle u\rangle \mathbb{1}_\Gamma) = \psi$. Since $\mathbb{1}_\Gamma \in D(\mathcal{N}^D)$, it follows that $\varphi \in D(\mathcal{N}^D)$. Therefore $D(\mathcal{N}) \subset D(\mathcal{N}^D)$.

Let $\varphi \in D(\mathcal{N})$ and write $\psi = \mathcal{N}\varphi$. It follows from that above that

$$\mathcal{N}\varphi = \mathcal{N}^D \varphi - \langle u\rangle \mathcal{N}^D \mathbb{1}_\Gamma.$$

Since $\big((a_{kl})_{k,l=1}^d\big)^* = (a_{lk})_{k,l=1}^d$ and $a_{lk} \in W^{1,\infty}(\Omega, \mathbb{R}) \cap C^1(\Omega_{R_0+1})$ for all $k, l \in \{1, \ldots, d\}$, it follows from Proposition 5.12 that $\mathbb{1}_\Gamma \in D(\mathcal{N}^{D^*})$. Hence

$$\big(\varphi, \mathcal{N}^{D^*}\mathbb{1}_\Gamma\big)_{L_2(\Gamma)} - \beta\langle u\rangle = \big(\mathcal{N}^D \varphi, \mathbb{1}_\Gamma\big)_{L_2(\Gamma)} - \beta\langle u\rangle$$

$$= \int_\Gamma \mathcal{N}^D \varphi - \langle u \rangle \int_\Gamma \mathcal{N}^D \mathbb{1}_\Gamma = \int_\Gamma \mathcal{N} \varphi = 0.$$

So $\langle u \rangle = \frac{1}{\beta}\big(\varphi, \mathcal{N}^{D*} \mathbb{1}_\Gamma\big)_{L_2(\Gamma)}$. Therefore

$$\mathcal{N}^D \varphi = \mathcal{N}\varphi + \frac{1}{\beta}\big(\varphi, \mathcal{N}^{D*} \mathbb{1}_\Gamma\big)_{L_2(\Gamma)} \mathcal{N}^D \mathbb{1}_\Gamma$$

for all $\varphi \in D(\mathcal{N})$.

Define $B \colon L_2(\Gamma) \to L_2(\Gamma)$ by $B\varphi = \frac{1}{\beta}\big(\varphi, \mathcal{N}^{D*} \mathbb{1}_\Gamma\big)_{L_2(\Gamma)} \mathcal{N}^D \mathbb{1}_\Gamma$. Then B is bounded and it follows from the above that $\mathcal{N} \subset \mathcal{N}^D - B$. Moreover, since $-\mathcal{N}$ generates a C_0-semigroup and by [BKR17] Theorem 11.5 the perturbed operator $-(\mathcal{N}^D - B)$ also generates a C_0-semigroup, it follows from Proposition A.15 that $D(\mathcal{N}^D) = D(\mathcal{N}^D - B) = D(\mathcal{N})$. □

Corollary 5.15. *Suppose that* Ω *is a* C^2*-domain and* $a_{kl} \in W^{1,\infty}(\Omega, \mathbb{R}) \cap C^1(\Omega_{R_0+1})$ *for all* $k, l \in \{1, \ldots, d\}$*. Then* $\mathcal{N} - \mathcal{N}^D$ *is a rank-one operator and* $R(\mathcal{N} - \mathcal{N}^D) = \mathbb{C}\mathcal{N}^D \mathbb{1}_\Gamma$.

6 The semigroup on $L_2(\Gamma)$

In this section we investigate the C_0-semigroups S and S^D on $L_2(\Gamma)$ generated by our two versions of the Dirichlet-to-Neumann operator. We prove that both S and S^D are submarkovian, uniformly mean ergodic and converge in norm to an equilibrium at an exponential rate. Moreover, we establish that the semigroups are irreducible and strictly positive, assuming symmetry of the coefficients (a_{kl}) in the case of S^D. In the sequel we use these results to study the Dirichlet-to-Neumann heat kernel.

A semigroup $T = (T_t)_{t>0}$ on $L_2(\Gamma)$ is called L_∞**-contractive** if $\|T_t\varphi\|_{L_\infty(\Gamma)} \leq \|\varphi\|_{L_\infty(\Gamma)}$ for all $\varphi \in L_2(\Gamma) \cap L_\infty(\Gamma)$ and $t > 0$. If T is positive and L_∞-contractive, we say that T is **submarkovian**.

Proposition 6.1. (a) *The semigroup S^D is submarkovian.*

(b) *The semigroup S is submarkovian.*

Proof. We prove (b). Since by Proposition 5.11 the semigroup S is positive, it remains only to establish L_∞-contractivity. We prove the claim using the criteria in [AE12] Proposition 2.9.

Define $P\colon L_2(\Gamma, \mathbb{R}) \to \{\varphi \in L_2(\Gamma)^+ : \varphi \leq \mathbb{1}_\Gamma\}$ by $P\varphi = (\mathbb{1}_\Gamma \wedge \varphi)^+$. Let $u \in W(\Omega, \mathbb{R})$. Then $(\mathbb{1}_\Omega \wedge u)^+ = (u - (u - \mathbb{1}_\Omega)^+)^+ \in W(\Omega)$ by Proposition 2.15(a). Moreover,

$$\operatorname{Tr}((\mathbb{1}_\Omega \wedge u)^+) = \operatorname{Tr}((u - (u - \mathbb{1}_\Omega)^+)^+) = (\operatorname{Tr} u - (\operatorname{Tr} u - \mathbb{1}_\Gamma)^+)^+ = (\mathbb{1}_\Gamma \wedge \operatorname{Tr} u)^+$$

by Proposition 2.15(d). Since by [GT83] Lemma 7.6 one has that

$$\partial_k((\mathbb{1}_\Omega \wedge u)^+) = \mathbb{1}_{[\mathbb{1} \wedge u > 0]}\partial_k(\mathbb{1}_\Omega \wedge u) = \mathbb{1}_{[u>0]}(\partial_k u - \mathbb{1}_{[u>1]}\partial_k u) = \mathbb{1}_{[0<u\leq 1]}\partial_k u$$

for all $k \in \{1, \ldots, d\}$, it follows that

$$\begin{aligned}
\mathfrak{a}((\mathbb{1}_\Omega \wedge u)^+, u - (\mathbb{1}_\Omega \wedge u)^+) &= \sum_{k,l=1}^{d} \int_\Omega a_{kl}(\partial_k(\mathbb{1}_\Omega \wedge u)^+)\partial_l(u - (\mathbb{1}_\Omega \wedge u)^+) \\
&= \sum_{k,l=1}^{d} \int_\Omega a_{kl}\, \mathbb{1}_{[0<u\leq 1]}(\partial_k u)(\partial_l u - \mathbb{1}_{[0<u\leq 1]}\partial_l u) \\
&= \sum_{k,l=1}^{d} \int_{[0<u\leq 1]} a_{kl}(\partial_k u)(\mathbb{1}_{[u\leq 0]} + \mathbb{1}_{[u>1]})\partial_l u = 0.
\end{aligned}$$

Hence by [AE12] Proposition 2.9 '(ii)⇒(i)' one deduces that $|S_t\varphi| \leq \mathbb{1}_\Gamma$ for all $\varphi \in L_2(\Gamma, \mathbb{R})$ with $|\varphi| \leq \mathbb{1}_\Gamma$ and all $t > 0$.

Now let $\varphi \in L_2(\Gamma, \mathbb{C})$ be such that $|\varphi| \leq \mathbb{1}_\Gamma$ and let $t > 0$. Then for all $\alpha \in [0, 2\pi]$ one has that $\operatorname{Re} e^{i\alpha}\varphi \in L_2(\Gamma, \mathbb{R})$ and $|\operatorname{Re} e^{i\alpha}\varphi| \leq \mathbb{1}_\Gamma$, so $|\operatorname{Re}(e^{i\alpha} S_t\varphi)| = |S_t(\operatorname{Re} e^{i\alpha}\varphi)| \leq \mathbb{1}_\Gamma$. Hence

$$|S_t\varphi| = \sup_{\alpha\in[0,2\pi]} \operatorname{Re}(e^{i\alpha} S_t\varphi) \leq \sup_{\alpha\in[0,2\pi]} |\operatorname{Re}(e^{i\alpha} S_t\varphi)| \leq \mathbb{1}_\Gamma.$$

This proves that S is L_∞-contractive.

The proof of (a) is similar. □

A well-studied property of submarkovian C_0-semigroups on L_2-spaces (over σ-finite measure spaces) is their facility for extrapolation to L_p for $p \in [2, \infty]$ (and then for $p \in [1, 2]$ by duality). Various properties of the semigroup on L_2 such as strong continuity, holomorphy and positivity are inherited by the extrapolation semigroup on L_p, possibly excluding the cases $p = 1$ and $p = \infty$. A summary of such properties can be found at [Are04] Subsection 7.2.2.

Let $p, q \in [1, \infty]$. Given a bounded operator $B \in \mathcal{L}\big(L_2(\Gamma)\big)$ we denote by

$$\|B\|_{p \to q} = \sup\big\{\|B\varphi\|_{L_q(\Gamma)} : \varphi \in L_2(\Gamma) \cap L_p(\Gamma) \text{ and } \|\varphi\|_{L_p(\Gamma)} \le 1\big\} \in [0, \infty]$$

the L_p-L_q norm of B.

Proposition 6.2. (a) *The semigroup S^D extends consistently to a contractive C_0-semigroup $S^{D,(p)}$ on $L_p(\Gamma)$ for all $p \in [1, \infty)$.*

(b) *The semigroup S extends consistently to a contractive C_0-semigroup $S^{(p)}$ on $L_p(\Gamma)$ for all $p \in [1, \infty)$.*

Proof. We prove (a). Note that $\|S_t^D\|_{2\to 2} \le 1$ for all $t > 0$ by Corollary 5.2(a).

Let $p \in [2, \infty)$ and $t > 0$. Since S^D is L_∞-contractive, it follows from Proposition A.10 that

$$\|S_t^D\|_{p\to p} \le \|S_t^D\|_{2\to 2}^{2/p} \, \|S_t^D\|_{\infty\to\infty}^{1-\frac{2}{p}} \le 1.$$

Moreover, $L_2(\Gamma) \cap L_p(\Gamma)$ is dense in $L_p(\Gamma)$, so by Proposition A.3 there exists a unique $S_t^{D,(p)} \in \mathcal{L}\big(L_p(\Gamma)\big)$ such that $S_t^{D,(p)}|_{L_2 \cap L_p} = S_t^D|_{L_2\cap L_p}$ and $\|S_t^{D,(p)}\|_{p\to p} \le 1$. Hence S^D extends consistently to a contractive semigroup $S^{D,(p)} = (S_t^{D,(p)})_{t>0}$ on $L_p(\Gamma)$.

Let $\varphi \in L_2(\Gamma) \cap L_\infty(\Gamma)$. Then by Proposition A.9

$$\begin{aligned}\|(I - S_t^{D,(p)})\varphi\|_{L_p(\Gamma)} &\le \|(I - S_t^D)\varphi\|_{L_2(\Gamma)}^{2/p} \, \|(I - S_t^D)\varphi\|_{L_\infty(\Gamma)}^{1-\frac{2}{p}} \\ &\le 2^{1-\frac{2}{p}} \|\varphi\|_{L_\infty(\Gamma)}^{1-\frac{2}{p}} \|(I - S_t^D)\varphi\|_{L_2(\Gamma)}^{2/p}\end{aligned}$$

for all $t > 0$. Hence $\lim_{t\downarrow 0} \|(I - S_t^{D,(p)})\varphi\|_{L_p(\Gamma)} = 0$ and by density it follows that $S^{D,(p)}$ is a C_0-semigroup on $L_p(\Gamma)$.

By Proposition A.12 the dual semigroup $S^{D^*} = (S_t^{D^*})_{t>0}$ extends consistently to a contractive semigroup $(S^{D^*})^{(1)} = ((S_t^{D^*})^{(1)})_{t>0}$ on $L_1(\Gamma)$. Let $\varphi \in L_1(\Gamma) \cap L_2(\Gamma)$. Since $\sigma(\Gamma) < \infty$, it follows that

$$\|\big(I - (S_t^{D^*})^{(1)}\big)\varphi\|_{L_1(\Gamma)} \le \sigma(\Gamma)^{1/2} \|(I - S_t^{D^*})\varphi\|_{L_2(\Gamma)}$$

for all $t > 0$. Hence by density $(S^{D^*})^{(1)}$ is a C_0-semigroup on $L_1(\Gamma)$. Then by interpolation S^{D^*} extends consistently to a contractive C_0-semigroup on $L_p(\Gamma)$ for all $p \in [1, 2]$. Since the matrix $\big((a_{kl})_{k,l=1}^d\big)^* = (a_{lk})_{k,l=1}^d$ satisfies the same ellipticity condition as $(a_{kl})_{k,l=1}^d$, the semigroup $S^D = \big(S^{D^*}\big)^*$ extends consistently to a contractive C_0-semigroup on $L_p(\Gamma)$ for all $p \in [1, 2]$.

The proof of (b) is similar. □

Note that by duality, the adjoint semigroups S^* and S^{D^*} are submarkovian. Since by Corollary 5.2 the holomorphic semigroups S and S^D are each contractive on a sector, it follows from [Ouh05] Proposition 3.12 together with duality that for all $p \in (1, \infty)$, the extrapolation semigroups $S^{(p)}$ and $S^{D,(p)}$ on $L_p(\Gamma)$ are also holomorphic and contractive on a sector.

Because the form $\mathfrak{a}^D$ is coercive, more than mere contractivity is valid for S^D. Our next result shows that in the limit $|z| \to \infty$, the operators S_z^D converge uniformly to zero at an exponential rate. Recall that $\theta^D \in (0, \frac{\pi}{2}]$ denotes the angle of analyticity for S^D.

Proposition 6.3. *For all $\theta' \in (0, \theta^D)$ there exists a $\delta > 0$ such that*

$$\|S_z^D\|_{2\to 2} \le e^{-\delta \operatorname{Re} z}$$

for all $z \in \Sigma_{\theta'}$.

Proof. Since Tr is continuous, there exists a $c > 0$ such that $\|\operatorname{Tr} v\|_{L_2(\Gamma)} \le c\|v\|_{W^D(\Omega)}$ for all $v \in W^D(\Omega)$. Let $\varphi \in D(\mathcal{N}^D)$. By Proposition 5.1(a) there exists a $u \in W^D(\Omega)$ such that $\operatorname{Tr} u = \varphi$ and $\mathfrak{a}^D(u, v) = (\mathcal{N}^D\varphi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^D(\Omega)$. Since $\mathfrak{a}^D$ is coercive, the choice $v = u$ yields that

$$\operatorname{Re}\big(\mathcal{N}^D\varphi, \varphi\big)_{L_2(\Gamma)} = \operatorname{Re}\mathfrak{a}^D(u) \ge \mu\|u\|^2_{W^D(\Omega)} \ge \frac{\mu}{c^2}\|\varphi\|^2_{L_2(\Gamma)}.$$

Hence

$$\operatorname{Re}\big(\mathcal{N}^D\varphi, \varphi\big)_{L_2(\Gamma)} \ge \frac{\mu}{c^2}\|\varphi\|^2_{L_2(\Gamma)}$$

for all $\varphi \in D(\mathcal{N}^D)$.

Let $\varphi \in D(\mathcal{N}^D)$ and $t > 0$. Then $S_t^D\varphi \in D(\mathcal{N}^D)$ and the map $r \mapsto S_r^D\varphi$ is continuously differentiable on $(0, \infty)$. So Proposition A.8 together with the above provides that

$$\frac{\mathrm{d}}{\mathrm{d}r}\|S_r^D\varphi\|^2_{L_2(\Gamma)} = -2\operatorname{Re}\big(\mathcal{N}^D S_r^D\varphi, S_r^D\varphi\big)_{L_2(\Gamma)} \le \frac{-2\mu}{c^2}\|S_r^D\varphi\|^2_{L_2(\Gamma)}.$$

Write $\delta = 2\mu c^{-2}$. Then

$$\frac{\mathrm{d}}{\mathrm{d}r}\big(e^{\delta r}\|S_r^D\varphi\|^2_{L_2(\Gamma)}\big) = e^{\delta r}\big(\delta\|S_r^D\varphi\|^2_{L_2(\Gamma)} + \frac{\mathrm{d}}{\mathrm{d}r}\|S_r^D\varphi\|^2_{L_2(\Gamma)}\big) \le 0.$$

So the map $r \mapsto e^{\delta r}\|S_r^D\varphi\|^2_{L_2(\Gamma)}$ is decreasing on $(0, \infty)$ and

$$e^{\delta t}\|S_t^D\varphi\|^2_{L_2(\Gamma)} \le \lim_{r\downarrow 0} e^{\delta r}\|S_r^D\varphi\|^2_{L_2(\Gamma)} = \|\varphi\|^2_{L_2(\Gamma)}.$$

Hence one deduces that

$$\|S_t^D\|_{2\to 2} \le e^{-\delta_1 t}$$

for all $t > 0$, where $\delta_1 = \frac{1}{2}\delta$.

Let $\theta' \in (0, \theta^D)$. Then there exist $\theta_0 \in (\theta', \theta^D)$ and $\kappa \in (0, 1)$ such that $\kappa t + is \in \Sigma_{\theta_0}$ for all $t + is \in \Sigma_{\theta'}$. Let $z \in \Sigma_{\theta'}$ and write $z = t + is$. Then

$$\|S_z^D\|_{2\to 2} \le \|S^D_{(1-\kappa)t}\|_{2\to 2}\,\|S^D_{\kappa t + is}\|_{2\to 2} \le e^{-\delta_1(1-\kappa)t}$$

since S^D is contractive on Σ_{θ^D}. This proves the claim. □

By Proposition 6.2(a) together with an interpolation argument, one readily deduces from the above that for all $p \in (1, \infty)$, in the limit $t \to \infty$ the operators $S_t^{D,(p)} \in \mathcal{L}\big(L_p(\Gamma)\big)$ converge uniformly to zero in a similar manner. On the other hand, it is easy to see that S_t does not converge to 0 as $t \to \infty$, since $\mathbb{1}_\Gamma \in \ker \mathcal{N}$ and therefore $S_t \mathbb{1}_\Gamma = \mathbb{1}_\Gamma$ for all $t > 0$. In Proposition 7.7 we prove that S_z instead converges to a rank-one projection on $L_2(\Gamma)$ in the limit $|z| \to \infty$, both uniformly and at an exponential rate.

Next we consider ergodicity. A C_0-semigroup $T = (T_t)_{t>0}$ on $L_2(\Gamma)$ is called **mean ergodic** if the limit

$$P = \lim_{t\to\infty} \frac{1}{t} \int_0^t T_s \, \mathrm{d}s$$

exists in the strong operator topology. We call P the **ergodic projection** of T.

Every bounded C_0-semigroup on a reflexive Banach space X is mean ergodic (see [ABHN01] Corollary 4.3.5). We provide a proof for the case $X = L_2(\Gamma)$.

Proposition 6.4. *Let $T = (T_t)_{t>0}$ be a bounded C_0-semigroup on $L_2(\Gamma)$. Then T is mean ergodic.*

Proof. There exists an $M > 0$ such that $\|T_t\|_{2\to 2} \le M$ for all $t > 0$. Hence for all $\varphi \in L_2(\Gamma)$ and $t, r > 0$, it follows that

$$\Big\|\frac{1}{r} \int_r^{r+t} T_s\varphi \, \mathrm{d}s\Big\|_{L_2(\Gamma)} \le \frac{1}{r} \int_r^{r+t} \|T_s\varphi\|_{L_2(\Gamma)} \, \mathrm{d}s \le M\|\varphi\|_{L_2(\Gamma)} \frac{1}{r} \int_r^{r+t} \mathrm{d}s = M\|\varphi\|_{L_2(\Gamma)} \frac{t}{r}. \quad (26)$$

For each $n \in \mathbb{N}$ define

$$C_n = \frac{1}{n} \int_0^n T_s \, \mathrm{d}s \in \mathcal{L}\big(L_2(\Gamma)\big).$$

We write

$$\operatorname{fix} T = \bigcap_{t>0} \ker(I - T_t), \qquad \operatorname{fix} T^* = \bigcap_{t>0} \ker(I - T_t^*)$$

and

$$R(I - T) = \bigcup_{t>0} R(I - T_t).$$

We first show that $\operatorname{fix} T$ separates $\operatorname{fix} T^*$. Let $\psi \in \operatorname{fix} T^* \setminus \{0\}$. Then there exists a $\varphi \in L_2(\Gamma) \setminus \{0\}$ such that $(\varphi, \psi)_{L_2(\Gamma)} \neq 0$. Since T is bounded, the sequence $(C_n\varphi)_{n\in\mathbb{N}}$ is bounded in $L_2(\Gamma)$. Passing to a subsequence if necessary, we may assume that there exists a $\pi \in L_2(\Gamma)$ such that $\lim C_n\varphi = \pi$ weakly in $L_2(\Gamma)$. Then by (26) one deduces that

$$\begin{aligned} T_t\pi &= \lim_{n\to\infty} \frac{1}{n} \int_0^n T_{s+t}\varphi \, \mathrm{d}s = \lim_{n\to\infty} \frac{1}{n} \int_t^{n+t} T_s\varphi \, \mathrm{d}s \\ &= \lim_{n\to\infty} \frac{1}{n} \Big(\int_0^n T_s\varphi \, \mathrm{d}s + \int_n^{n+t} T_s\varphi \, \mathrm{d}s - \int_0^t T_s\varphi \, \mathrm{d}s \Big) = \lim_{n\to\infty} C_n\varphi = \pi \end{aligned}$$

weakly for all $t > 0$. Hence $\pi \in \operatorname{fix} T$. Moreover,

$$\begin{aligned} \big(\pi, \psi\big)_{L_2(\Gamma)} = \lim_{n\to\infty} \big(C_n\varphi, \psi\big)_{L_2(\Gamma)} &= \lim_{n\to\infty} \frac{1}{n} \int_0^n \big(\varphi, T_s^*\psi\big)_{L_2(\Gamma)} \, \mathrm{d}s \\ &= \lim_{n\to\infty} \frac{1}{n} \int_0^n \big(\varphi, \psi\big)_{L_2(\Gamma)} \, \mathrm{d}s = \big(\varphi, \psi\big)_{L_2(\Gamma)} \neq 0. \end{aligned}$$

So fix T separates fix T^* and it follows that fix $T^* \cap (\text{fix } T)^\perp = \{0\}$.

Write

$$X = \text{fix } T + \operatorname{span} R(I - T)$$

and let $\psi \in X^\perp$. Clearly $\psi \in (\text{fix } T)^\perp$. Moreover, for all $t > 0$ and $\varphi \in L_2(\Gamma)$ one has that

$$\big(\varphi, (I - T_t^*)\psi\big)_{L_2(\Gamma)} = \big((I - T_t)\varphi, \psi\big)_{L_2(\Gamma)} = 0,$$

so $\psi \in \text{fix } T^*$. Hence $X^\perp = \{0\}$ and $\overline{X} = L_2(\Gamma)$.

Clearly $\lim r^{-1} \int_0^r T_s \pi \, ds = \pi$ for all $\pi \in \text{fix } T$. Moreover, for all $t > 0$ and $\varphi \in L_2(\Gamma)$

$$\begin{aligned}
\frac{1}{r} \int_0^r T_s (I - T_t) \varphi \, ds &= \frac{1}{r} \int_0^r T_s \varphi \, ds - \frac{1}{r} \int_0^r T_{s+t} \varphi \, ds \\
&= \frac{1}{r} \int_0^r T_s \varphi \, ds - \frac{1}{r} \Big(\int_0^r T_s \varphi \, ds + \int_r^{r+t} T_s \varphi \, ds - \int_0^t T_s \varphi \, ds \Big) \\
&= \frac{1}{r} \int_0^t T_s \varphi \, ds - \frac{1}{r} \int_r^{r+t} T_s \varphi \, ds
\end{aligned}$$

for all $r > 0$. Let $\psi \in \operatorname{span} R(I - T)$. Then there exist $k \in \mathbb{N}$, $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{C}$, $t_1, t_2, \ldots, t_k > 0$ and $\varphi_1, \varphi_2, \ldots, \varphi_k \in L_2(\Gamma)$ such that $\psi = \sum_{j=1}^k \alpha_j (I - T_{t_j}) \varphi_j$ and it follows from the above together with (26) that

$$\lim_{r \to \infty} \frac{1}{r} \int_0^r T_s \psi \, ds = \sum_{j=1}^k \alpha_j \Big(\lim_{r \to \infty} \frac{1}{r} \int_0^r T_s (I - T_{t_j}) \varphi_j \, ds \Big) = 0.$$

Consequently one deduces that for all $\varphi \in X$ there exist $\pi \in \text{fix } T$ and $\psi \in \operatorname{span} R(I - T)$ such that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t T_s \varphi \, ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t T_s (\pi + \psi) \, ds = \pi$$

in $L_2(\Gamma)$.

Note that

$$\sup_{t > 0} \Big\| \frac{1}{t} \int_0^t T_s \, ds \Big\|_{2 \to 2} \leq M.$$

Let $\varphi \in L_2(\Gamma)$ and $\varepsilon > 0$. By density there exists a $\psi \in X$ such that $\|\varphi - \psi\|_{L_2(\Gamma)} < \frac{\varepsilon}{4M}$. Let $(r_n)_{n \in \mathbb{N}}$ be a sequence in $(0, \infty)$ with $\lim r_n = \infty$. Then $\lim C_{r_n} \psi$ exists and it follows that

$$\begin{aligned}
\|C_{r_n} \varphi - C_{r_m} \varphi\|_{L_2(\Gamma)} &\leq \|(C_{r_n} - C_{r_m})(\varphi - \psi)\|_{L_2(\Gamma)} + \|(C_{r_n} - C_{r_m}) \psi\|_{L_2(\Gamma)} \\
&\leq 2M \|\varphi - \psi\|_{L_2(\Gamma)} + \|C_{r_n} \psi - C_{r_m} \psi\|_{L_2(\Gamma)} < \varepsilon
\end{aligned}$$

for all $n, m \in \mathbb{N}$ sufficiently large. Hence by completeness the limit $\lim C_{r_n} \varphi$ exists in $L_2(\Gamma)$. Then it follows from a zip argument that the limit $\lim t^{-1} \int_0^t T_s \varphi \, ds$ exists. □

In a Hilbert space, the ergodic projection associated with a contractive C_0-semigroup coincides with the orthogonal projection onto the kernel of its generator.

Proposition 6.5. *Let $T = (T_t)_{t>0}$ be a contractive C_0-semigroup on $L_2(\Gamma)$ with generator $-A$. Then*

$$L_2(\Gamma) = \ker A \oplus \overline{R(A)}$$

and the ergodic projection P of T is equal to the orthogonal projection onto $\ker A$.

Proof. By Proposition 6.4 the semigroup T is mean ergodic. We first show that the map

$$P \colon \varphi \mapsto \lim_{t\to\infty} \frac{1}{t} \int_0^t T_s\varphi \,\mathrm{d}s$$

is a projection from $L_2(\Gamma)$ into $\ker A$. Let $\varphi \in L_2(\Gamma)$. Then it follows from (26) that

$$\begin{aligned} T_t P\varphi &= \lim_{r\to\infty} \frac{1}{r} \int_0^r T_{s+t}\varphi \,\mathrm{d}s = \lim_{r\to\infty} \frac{1}{r} \Big(\int_0^r T_s\varphi \,\mathrm{d}s + \int_r^{r+t} T_s\varphi \,\mathrm{d}s - \int_0^t T_s\varphi \,\mathrm{d}s \Big) \\ &= \lim_{r\to\infty} \frac{1}{r} \int_0^r T_s\varphi \,\mathrm{d}s = P\varphi \end{aligned} \tag{27}$$

for all $t > 0$. So $P^2\varphi = \lim_{t\to\infty} t^{-1} \int_0^t T_s P\varphi \,\mathrm{d}s = P\varphi$ and $\lim_{t\downarrow 0} t^{-1}(I - T_t)P\varphi = 0$. Then $P\varphi \in D(A)$ and $AP\varphi = 0$. Hence P is a projection into $\ker A$.

It follows from the above that $R(P) \subset \ker A$. Conversely, let $\varphi \in \ker A$. Then Proposition A.14 provides that $\varphi - T_t\varphi = \int_0^t T_s A\varphi \,\mathrm{d}s = 0$ for all $t > 0$. So

$$\varphi = \lim_{r\to\infty} \frac{1}{r} \int_0^r \varphi \,\mathrm{d}s = \lim_{r\to\infty} \frac{1}{r} \int_0^r T_s\varphi \,\mathrm{d}s = P\varphi \in R(P)$$

and one therefore deduces that $R(P) = \ker A$.

Next we show that $\ker P = \overline{R(A)}$. It follows from (27) together with the definition of P that $PT_t = T_t P = P$ for all $t > 0$. So $PA\varphi = \lim_{t\downarrow 0} t^{-1} P(I - T_t)\varphi = 0$ for all $\varphi \in D(A)$ and one consequently deduces that $R(A) \subset \ker P$. Since

$$\sup_{t>0} \Big\| \frac{1}{t} \int_0^t T_s \,\mathrm{d}s \Big\|_{2\to 2} \le 1,$$

it follows that $P \in \mathcal{L}\big(L_2(\Gamma)\big)$. So $\ker P$ is closed and $\overline{R(A)} \subset \ker P$. Conversely, let $\psi \in \ker P \cap R(A)^\perp$ and $\varphi \in L_2(\Gamma)$. By Proposition A.13 one has that $\varphi - T_t\varphi = A \int_0^t T_s\varphi \,\mathrm{d}s \in R(A)$ for all $t > 0$. Then $(T_t\varphi, \psi)_{L_2(\Gamma)} = (\varphi, \psi)_{L_2(\Gamma)}$ for all $t > 0$. Hence one deduces that

$$(P\varphi, \psi)_{L_2(\Gamma)} = \lim_{r\to\infty} \frac{1}{r} \int_0^r (T_s\varphi, \psi)_{L_2(\Gamma)} \,\mathrm{d}s = \lim_{r\to\infty} \frac{1}{r} \int_0^r (\varphi, \psi)_{L_2(\Gamma)} \,\mathrm{d}s = (\varphi, \psi)_{L_2(\Gamma)}$$

for all $\varphi \in L_2(\Gamma)$. The choice $\varphi = \psi$ then yields that $\|\psi\|^2_{L_2(\Gamma)} = (P\psi, \psi)_{L_2(\Gamma)} = 0$. Hence $\psi = 0$ and $\ker P = \overline{R(A)}$.

Finally we prove that $\overline{R(A)}^\perp = \ker A$. Let $\varphi \in \overline{R(A)}^\perp$ and $t > 0$. Then $(\varphi, \varphi - T_t\varphi)_{L_2(\Gamma)} = 0$. So $(\varphi, T_t\varphi)_{L_2(\Gamma)} = \|\varphi\|^2_{L_2(\Gamma)}$ and therefore

$$\|\varphi - T_t\varphi\|^2_{L_2(\Gamma)} = \|\varphi\|^2_{L_2(\Gamma)} - 2\operatorname{Re}(\varphi, T_t\varphi)_{L_2(\Gamma)} + \|T_t\varphi\|^2_{L_2(\Gamma)} = \|T_t\varphi\|^2_{L_2(\Gamma)} - \|\varphi\|^2_{L_2(\Gamma)} \le 0$$

since T is contractive. Hence $\varphi = T_t\varphi$ for all $t > 0$ and it follows that $\varphi \in \ker A$. This completes the proof. $\square$

We are now able to prove that the semigroup S is uniformly mean ergodic. We argue as in [EN00] Corollary V.4.8 in order to obtain norm convergence of the Cesàro means.

Theorem 6.6. *The semigroup S is mean ergodic. Moreover,*

$$L_2(\Gamma) = \mathbb{C}\mathbb{1}_\Gamma \oplus R(\mathcal{N}) \tag{28}$$

and the limit

$$P = \lim_{t\to\infty} \frac{1}{t} \int_0^t S_r \,\mathrm{d}r$$

exists in $\mathcal{L}\big(L_2(\Gamma)\big)$.

Proof. We first show that $\mathcal{N}$ has closed range. Let $\lambda > 0$. By Proposition 5.5(b) the operator $(\lambda I + \mathcal{N})^{-1}$ is compact, so by Proposition A.4 the operator $I - \lambda(\lambda I + \mathcal{N})^{-1}$ has closed range. Since

$$\mathcal{N}(\lambda I + \mathcal{N})^{-1} = (\lambda I + \mathcal{N} - \lambda I)(\lambda I + \mathcal{N})^{-1} = I - \lambda(\lambda I + \mathcal{N})^{-1}$$

and $R\big((\lambda I + \mathcal{N})^{-1}\big) = D(\mathcal{N})$, it follows that $R(\mathcal{N}) = R\big(\mathcal{N}(\lambda I + \mathcal{N})^{-1}\big)$ is closed.

By Corollary 5.2(b) the semigroup S is contractive, so S is mean ergodic by Proposition 6.4. Moreover, the orthogonal decomposition (28) follows from Proposition 6.5, since $\ker \mathcal{N} = \mathbb{C}\mathbb{1}_\Gamma$ by Proposition 5.4(b).

We now show that $P = \lim t^{-1} \int_0^t S_r \,\mathrm{d}r$ exists in the uniform operator topology. Using Proposition A.13 we define $V \colon L_2(\Gamma) \to \big(D(\mathcal{N}), \|\cdot\|_{\mathcal{N}}\big)$ by

$$V = \int_0^1 S_r \,\mathrm{d}r,$$

where $\|\cdot\|_{\mathcal{N}}$ is the graph norm on $D(\mathcal{N})$. Then for all $\varphi \in L_2(\Gamma)$

$$\begin{aligned}\|V\varphi\|_{\mathcal{N}} &= \Big\| \int_0^1 S_r\varphi \,\mathrm{d}r \Big\|_{L_2(\Gamma)} + \Big\| \mathcal{N} \int_0^1 S_r\varphi \,\mathrm{d}r \Big\|_{L_2(\Gamma)} = \Big\| \int_0^1 S_r\varphi \,\mathrm{d}r \Big\|_{L_2(\Gamma)} + \|\varphi - S_1\varphi\|_{L_2(\Gamma)} \\ &\le \int_0^1 \|S_r\varphi\|_{L_2(\Gamma)} \,\mathrm{d}r + \|\varphi\|_{L_2(\Gamma)} + \|S_1\varphi\|_{L_2(\Gamma)} \le 3\|\varphi\|_{L_2(\Gamma)}\end{aligned}$$

since S is contractive. So $V \in \mathcal{L}\big(L_2(\Gamma), (D(\mathcal{N}), \|\cdot\|_{\mathcal{N}})\big)$. Since $\mathcal{N}$ has compact resolvent, it follows from Proposition A.5 that the canonical injection $\iota \colon \big(D(\mathcal{N}), \|\cdot\|_{\mathcal{N}}\big) \hookrightarrow L_2(\Gamma)$ is compact. Hence the operator $\widetilde{V} = \iota \circ V \in \mathcal{L}\big(L_2(\Gamma)\big)$ is compact.

It follows from (27) together with the definition of P that $PS_t = S_tP = P$ for all $t > 0$. So

$$P\widetilde{V}(I - S_t)\psi = P(I - S_t)\widetilde{V}\psi = 0 = P(I - S_t)\psi$$

for all $\psi \in L_2(\Gamma)$ and $t > 0$. Let $\varphi \in L_2(\Gamma)$. Then by (28) and the definition of the infinitesimal generator $-\mathcal{N}$, there exist $\pi \in \ker \mathcal{N}$ and $\psi \in D(\mathcal{N})$ such that $\varphi = \pi + \lim_{t\downarrow 0} t^{-1}(I - S_t)\psi$. Clearly $S_t\pi = \pi$ for all $t > 0$, so

$$P\widetilde{V}\varphi = P\widetilde{V}\pi + \lim_{t\downarrow 0} \frac{1}{t} P\widetilde{V}(I - S_t)\psi = P\pi + \lim_{t\downarrow 0} \frac{1}{t} P(I - S_t)\psi = P\varphi$$

and it follows that $P\widetilde{V} = P$. For each $t > 0$ write

$$C_t = \frac{1}{t} \int_0^t S_r \,\mathrm{d}r \in \mathcal{L}\big(L_2(\Gamma)\big).$$

Then

$$C_t\widetilde{V} - P = (C_t - P)\widetilde{V}.$$

Since $\lim C_t = P$ strongly and $\widetilde{V}$ is compact, it follows from Proposition A.6 that

$$\lim_{t\to\infty} \|C_t\widetilde{V} - P\|_{2\to 2} = 0. \tag{29}$$

On the other hand,

$$\begin{aligned}
C_t\widetilde{V} - C_t &= \frac{1}{t} \int_0^t \int_0^1 S_r S_u \,\mathrm{d}u \,\mathrm{d}r - \frac{1}{t} \int_0^t S_r \,\mathrm{d}r \\
&= \frac{1}{t} \int_0^1 \Big(\int_0^t S_{r+u} \,\mathrm{d}r - \int_0^t S_r \,\mathrm{d}r \Big) \mathrm{d}u \\
&= \frac{1}{t} \int_0^1 \Big(\int_0^t S_r \,\mathrm{d}r + \int_t^{t+u} S_r \,\mathrm{d}r - \int_0^u S_r \,\mathrm{d}r - \int_0^t S_r \,\mathrm{d}r \Big) \mathrm{d}u \\
&= \frac{1}{t} \int_0^1 \Big(\int_t^{t+u} S_r \,\mathrm{d}r - \int_0^u S_r \,\mathrm{d}r \Big) \mathrm{d}u
\end{aligned}$$

for all $t > 0$. Hence

$$\begin{aligned}
\|C_t\widetilde{V} - C_t\|_{2\to 2} &\leq \frac{1}{t} \int_0^1 \Big\| \int_t^{t+u} S_r \,\mathrm{d}r - \int_0^u S_r \,\mathrm{d}r \Big\|_{2\to 2} \mathrm{d}u \\
&\leq \frac{1}{t} \int_0^1 \Big(\int_t^{t+u} \|S_r\|_{2\to 2} \,\mathrm{d}r + \int_0^u \|S_r\|_{2\to 2} \,\mathrm{d}r \Big) \mathrm{d}u \\
&\leq \frac{1}{t} \int_0^1 \Big(\int_t^{t+u} \mathrm{d}r + \int_0^u \mathrm{d}r \Big) \mathrm{d}u = \frac{1}{t}
\end{aligned}$$

for all $t > 0$ and it follows that

$$\lim_{t\to\infty} \|C_t\widetilde{V} - C_t\|_{2\to 2} = 0.$$

Then by (29) and the triangle inequality, one deduces that $\lim C_t = P$ uniformly. $\square$

Corollary 6.7. *The ergodic projection P of S is given by*

$$P\varphi = \tfrac{1}{\sigma(\Gamma)} \big(\varphi, \mathbb{1}_\Gamma\big)_{L_2(\Gamma)} \mathbb{1}_\Gamma$$

for all $\varphi \in L_2(\Gamma)$.

Proof. By Proposition 6.5 and Proposition 5.4(b), the ergodic projection P associated with S is equal to the orthogonal projection onto $\mathbb{C}\mathbb{1}_\Gamma$. $\square$

Next we prove that S^D is uniformly mean ergodic. Since $\mathcal{N}^D$ has compact resolvent, it follows as in the first paragraph of the proof of Theorem 6.6 that $R(\mathcal{N}^D)$ is closed. Nevertheless, in the following proof we use Proposition 6.3 and argue as in [Lin74] in order to obtain that $\mathcal{N}^D$ has closed range without using that $(\lambda I + \mathcal{N}^D)^{-1}$ is compact.

Theorem 6.8. *The semigroup S^D is mean ergodic. Moreover, $R(\mathcal{N}^D) = L_2(\Gamma)$ and*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t S_r^D \,\mathrm{d}r = 0$$

in $\mathcal{L}\big(L_2(\Gamma)\big)$.

Proof. By Corollary 5.2(a) the semigroup S^D is contractive, so Proposition 6.4 provides that S^D is mean ergodic. Since $\ker \mathcal{N}^D = \{0\}$ by Proposition 5.4(a), it follows from Proposition 6.5 that the ergodic projection of S^D is equal to the zero operator on $L_2(\Gamma)$ and $\overline{R(\mathcal{N}^D)} = L_2(\Gamma)$. Moreover, by Proposition 6.3 there exists a $\delta > 0$ such that $\|S_t^D\|_{2\to 2} \leq e^{-\delta t}$ for all $t > 0$, so $\lim t^{-1} \int_0^t S_r^D \,\mathrm{d}r = 0$ uniformly.

Since $\mathcal{N}^D$ is injective, there exists a mapping $\big(\mathcal{N}^D\big)^{-1} \colon R(\mathcal{N}^D) \to L_2(\Gamma)$ such that $\big(\mathcal{N}^D\big)^{-1}\mathcal{N}^D = I|_{D(\mathcal{N}^D)}$ and $\mathcal{N}^D\big(\mathcal{N}^D\big)^{-1} = I|_{R(\mathcal{N}^D)}$. Moreover, since

$$\Big\| \int_0^1 S_t^D \,\mathrm{d}t \Big\|_{2\to 2} \leq \int_0^1 e^{-\delta t} \,\mathrm{d}t = \frac{1 - e^{-\delta}}{\delta} < 1,$$

one deduces that the operator $\int_0^1 I - S_t^D \,\mathrm{d}t \in \mathcal{L}\big(L_2(\Gamma)\big)$ is invertible. Let $\psi \in R(\mathcal{N}^D)$ and write $\mathcal{N}^D\varphi = \psi$. Then Proposition A.14 provides that

$$\varphi = \Big(\int_0^1 I - S_t^D \,\mathrm{d}t\Big)^{-1} \int_0^1 (I - S_t^D)\varphi \,\mathrm{d}t = \Big(\int_0^1 I - S_t^D \,\mathrm{d}t\Big)^{-1} \int_0^1 \int_0^t S_r^D \psi \,\mathrm{d}r \,\mathrm{d}t.$$

Hence there exists a $c > 0$ such that

$$\|\big(\mathcal{N}^D\big)^{-1}\psi\|_{L_2(\Gamma)} \leq \Big\|\Big(\int_0^1 I - S_t^D \,\mathrm{d}t\Big)^{-1}\Big\|_{2\to 2} \Big\| \int_0^1 \int_0^t S_r^D \psi \,\mathrm{d}r \,\mathrm{d}t \Big\|_{L_2(\Gamma)} \leq \frac{c}{2} \|\psi\|_{L_2(\Gamma)}$$

for all $\psi \in R(\mathcal{N}^D)$, since S^D is contractive. Then $\big(\mathcal{N}^D\big)^{-1}$ is bounded and densely defined. Moreover, since the operator $\mathcal{N}^D$ is closed it follows that $\big(\mathcal{N}^D\big)^{-1}$ is closed, so $D\big((\mathcal{N}^D)^{-1}\big)$ is closed in $L_2(\Gamma)$ and therefore

$$R(\mathcal{N}^D) = D\big((\mathcal{N}^D)^{-1}\big) = \overline{D\big((\mathcal{N}^D)^{-1}\big)} = \overline{R(\mathcal{N}^D)}$$

as required. □

Our final consideration for this section is irreducibility. Via holomorphy we subsequently obtain strict positivity of the C_0-semigroups S and S^D, a fact proved first for the L_p-setting in [KR81] and then in [MR83] for an arbitrary Banach lattice.

A positive C_0-semigroup $T = (T_t)_{t>0}$ on $L_2(\Gamma)$ is called **irreducible** if for every measurable set $\Gamma_1 \subset \Gamma$ such that

$$T_t L_2(\Gamma_1) \subset L_2(\Gamma_1)$$

for all $t > 0$, it follows that either $\sigma(\Gamma_1) = 0$ or $\sigma(\Gamma \setminus \Gamma_1) = 0$. Here we define $L_2(\Gamma_1) = \{\varphi \in L_2(\Gamma) : \varphi = 0 \ \sigma\text{-a.e. on } \Gamma \setminus \Gamma_1\}$.

Theorem 6.9. *The semigroup S is irreducible. Moreover, for all $\varphi, \psi \in L_2(\Gamma)^+ \setminus \{0\}$,*

$$\big(S_t\varphi, \psi\big)_{L_2(\Gamma)} > 0$$

for all $t > 0$.

Proof. Let $\Gamma_1 \subset \Gamma$ be a measurable set with $\sigma(\Gamma_1) > 0$ and $\sigma(\Gamma \setminus \Gamma_1) > 0$. Let $\varphi \in L_2(\Gamma_1)^+ \setminus \{0\}$ and $\psi \in L_2(\Gamma \setminus \Gamma_1)^+ \setminus \{0\}$. Then by Corollary 6.7

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \big(S_r\varphi, \psi\big)_{L_2(\Gamma)} \, \mathrm{d}r = (P\varphi, \psi)_{L_2(\Gamma)} = \frac{1}{\sigma(\Gamma)} \big(\varphi, \mathbb{1}_\Gamma\big)_{L_2(\Gamma)} \cdot \big(\mathbb{1}_\Gamma, \psi\big)_{L_2(\Gamma)} > 0.$$

Hence there exists a $t > 0$ such that $(S_t\varphi, \psi)_{L_2(\Gamma)} > 0$. Then

$$\sigma\big(\big\{z \in \Gamma \setminus \Gamma_1 : (S_t\varphi)(z) > 0\big\}\big) > 0,$$

so $S_t\varphi \notin L_2(\Gamma_1)$ and S is therefore irreducible.

Next we prove that S is strictly positive. Let $\varphi, \psi \in L_2(\Gamma)^+ \setminus \{0\}$. With a view to contradiction suppose that there exists a $t > 0$ such that $(S_t\varphi, \psi)_{L_2(\Gamma)} = 0$. Let $(t_n)_{n\in\mathbb{N}} \subset (0, t)$ be such that $\lim t_n = 0$ and $\|(I - S_{t_n})\varphi\|_{L_2(\Gamma)} \leq 2^{-n}$ for all $n \in \mathbb{N}$. Without loss of generality we may assume that $t_n \neq t_m$ for all $n \neq m$. Write $\varphi_n = S_{t_n}\varphi$ and $\xi_n = \varphi - \sum_{k=n}^{\infty}(\varphi - \varphi_k)^+$. Then $\xi_n \leq \varphi$ for all $n \in \mathbb{N}$. Moreover,

$$\lim_{n\to\infty} \|\varphi - \xi_n\|_{L_2(\Gamma)} = \lim_{n\to\infty} \Big\| \sum_{k=n}^{\infty} (\varphi - \varphi_k)^+ \Big\|_{L_2(\Gamma)} = 0,$$

so $\lim \xi_n^+ = \varphi^+ = \varphi$ in $L_2(\Gamma)$. For all $n, m \in \mathbb{N}$ with $m \geq n$, one obtains that

$$\xi_n \leq \xi_m \leq \varphi - (\varphi - \varphi_m)^+ = \varphi \wedge \varphi_m \leq \varphi_m.$$

Let $n \in \mathbb{N}$ and define $f_n \colon (0, \infty) \to \mathbb{C}$ by $f_n(r) = (S_r\xi_n^+, \psi)_{L_2(\Gamma)}$. Then it follows from Corollary 5.2(b) that f_n is analytic. Since by Proposition 5.11 the semigroup S is positive, one deduces that

$$0 \leq (S_{t-t_m}\xi_n^+, \psi)_{L_2(\Gamma)} \leq (S_{t-t_m}\varphi_m, \psi)_{L_2(\Gamma)} = (S_{t-t_m}S_{t_m}\varphi, \psi)_{L_2(\Gamma)} = (S_t\varphi, \psi)_{L_2(\Gamma)} = 0$$

for all $m \geq n$. Then $f_n(t - t_m) = 0$ for all $m \geq n$, so $(t - t_m)_{m\in\mathbb{N}, m\geq n}$ is a sequence of distinct zeros of f_n with limit $\lim t - t_m = t \in (0, \infty)$. Hence $f_n = 0$. This is for all $n \in \mathbb{N}$. Therefore

$$(S_r\varphi, \psi)_{L_2(\Gamma)} = \lim_{n\to\infty} (S_r\xi_n^+, \psi)_{L_2(\Gamma)} = \lim_{n\to\infty} f_n(r) = 0$$

for all $r > 0$, which contradicts the first paragraph. □

We conclude with the irreducibility of S^D in the case where the coefficients (a_{kl}) are symmetric. Since the form $\mathfrak{a}^D$ is then symmetric, the Dirichlet-to-Neumann operator $\mathcal{N}^D$ is self-adjoint and the semigroup S^D therefore consists of self-adjoint operators on $L_2(\Gamma)$. We follow the approach used in [AE15] in order to deduce that S^D is irreducible. By the positivity of S^D and the following domination estimate, the task is reduced to having to prove that the semigroup $(e^{-t\mathcal{N}_R^D})_{t>0}$ is irreducible.

Proposition 6.10. *Let $R \geq R_0$. Then*

$$0 \leq e^{-t\mathcal{N}_R^D}\varphi \leq S_t^D \varphi$$

for all $\varphi \in L_2(\Gamma)^+$ and $t > 0$.

Proof. Let $\varphi \in L_2(\Gamma)^+$. Since by Proposition 2.15(d) the operator Tr is positive, it follows from (22) together with Propositions 4.6(a) and (b) that

$$0 \leq (\lambda I + \mathcal{N}_R^D)^{-1}\varphi \leq (\lambda I + \mathcal{N}^D)^{-1}\varphi$$

for all $\lambda \geq 0$. Hence

$$0 \leq e^{-t\mathcal{N}_R^D}\varphi = \lim_{n\to\infty}(I + \frac{t}{n}\mathcal{N}_R^D)^{-n}\varphi \leq \lim_{n\to\infty}(I + \frac{t}{n}\mathcal{N}^D)^{-n}\varphi = S_t^D\varphi$$

for all $t > 0$. $\square$

In order to obtain irreducibility of the semigroup $(e^{-t\mathcal{N}_R^D})_{t>0}$ on $L_2(\Gamma)$, we consider the semigroup generated by particular realisation in $L_2(\Omega_R)$ of the elliptic operator $-\sum\partial_l(a_{kl}\partial_k)$. The appropriate form domain is the following characterisation of $W_R^D(\Omega)$ appearing in Lemma 4.1.

Let $R \geq R_0$ and define

$$V_R = \{u \in H^1(\Omega_R) : \mathbb{1}_{\partial B_R}\mathrm{Tr}_{\Omega_R}u = 0\}.$$

Then $H_0^1(\Omega_R) \subset V_R \subset H^1(\Omega)$ and V_R is dense in $L_2(\Omega_R)$. We assume that $a_{kl} = a_{lk}$ for all $k, l \in \{1, \ldots, d\}$. Let $\beta \in L_\infty(\Gamma, \mathbb{R})$. Define the sesquilinear form $\mathfrak{a}_R^\beta \colon V_R \times V_R \to \mathbb{C}$ by

$$\mathfrak{a}_R^\beta(u, v) = \sum_{k,l=1}^d \int_{\Omega_R} a_{kl}(\partial_k u)\overline{\partial_l v} - \int_\Gamma \beta\, \mathrm{Tr}_R u\, \overline{\mathrm{Tr}_R v}.$$

Then $\mathfrak{a}_R^\beta$ is continuous, elliptic, symmetric and densely defined in $L_2(\Omega_R)$, so there exists a lower-bounded self-adjoint operator A_R^β in $L_2(\Omega_R)$ associated with $\mathfrak{a}_R^\beta$. It follows from [Ouh05] Lemma 1.25 that

$$\overline{D(A_R^\beta)}^{H^1(\Omega_R)} = V_R$$

and since Ω_R has Lipschitz boundary, by [EE87] Theorem V.4.17 one deduces that the embedding $V_R \hookrightarrow L_2(\Omega_R)$ is compact. Therefore by Proposition A.5 the operator A_R^β has compact resolvent.

Lemma 6.11. *Let $R \geq R_0$. Then the following are valid.*

(a) *Let $\beta \in L_\infty(\Gamma, \mathbb{R})$. Then the semigroup $(e^{-tA_R^\beta})_{t>0}$ on $L_2(\Omega_R)$ is positive and irreducible.*

(b) *Let $\beta_1, \beta_2 \in L_\infty(\Gamma, \mathbb{R})$ with $\beta_1 \leq \beta_2$. Then*

$$0 \leq e^{-tA_R^{\beta_1}}u \leq e^{-tA_R^{\beta_2}}u$$

for all $u \in L_2(\Omega_R)$ with $u \geq 0$ and all $t > 0$.

Proof. We first prove (a). Since $(\operatorname{Re} u)^+ \in V_R$ for all $u \in V_R$ and the coefficients (a_{kl}) are real-valued, [Ouh05] Theorem 4.2 provides that $(e^{-tA_R^\beta})_{t>0}$ is positive.

Let $E \subset \Omega_R$ be a measurable set and suppose that $\mathbb{1}_E V_R \subset V_R$. Since $C_c^\infty(\Omega_R) \subset V_R \subset H^1(\Omega_R)$, it follows from [Are06] Proposition 11.1.2 that either $|E| = 0$ or $|\Omega_R \setminus E| = 0$. Let $u, v \in V_R$ and suppose that $\operatorname{supp} u \cap \operatorname{supp} v = \varnothing$. Then $\operatorname{supp} \operatorname{Tr}_R u \cap \operatorname{supp} \operatorname{Tr}_R v = \varnothing$, so $\mathfrak{a}_R^\beta(u, v) = 0$ and by [Ouh05] Corollary 2.11 the semigroup $(e^{-tA_R^\beta})_{t>0}$ is irreducible.

We now prove (b). Since $\beta_1 \le \beta_2$, it follows that $\mathfrak{a}_R^{\beta_1}(u, v) \ge \mathfrak{a}_R^{\beta_2}(u, v)$ for all $u, v \in V_R$ with $u, v \ge 0$. Moreover, by (a) the semigroups $(e^{-tA_R^{\beta_1}})_{t>0}$ and $(e^{-tA_R^{\beta_2}})_{t>0}$ are positive, so [Ouh96] Theorem 3.7 provides that $(e^{-tA_R^{\beta_2}})_{t>0}$ dominates $(e^{-tA_R^{\beta_1}})_{t>0}$. □

In the following lemma we use the Krein–Rutman theorem together with the preceding result in order to infer the existence of an eigenfunction of A_R^β whose trace is strictly positive a.e. on Γ.

Lemma 6.12. *Let $R \ge R_0$ and $\beta \in L_\infty(\Gamma, \mathbb{R})$. Denote by λ_1 the smallest eigenvalue of A_R^β. Suppose that $u \in D(A_R^\beta)$ with $u > 0$ is an eigenfunction of A_R^β corresponding to λ_1. Then $(\operatorname{Tr}_R u)(z) > 0$ for a.e. $z \in \Gamma$.*

Proof. Note that since $(e^{-tA_R^\beta})_{t>0}$ is positive and A_R^β has compact resolvent, the existence of an element $u \in D(A_R^\beta)$ such that $u > 0$ and $A_R^\beta u = \lambda_1 u$ follows from the Krein–Rutman theorem [BKR17] Theorem 12.15. By Lemma 6.11(a) the semigroup $(e^{-tA_R^\beta})_{t>0}$ is irreducible, so Proposition A.16 provides that $u(x) > 0$ for a.e. $x \in \Omega_R$. Then $\operatorname{Tr}_R u \ge 0$.

Define
$$\Lambda = \{z \in \Gamma : (\operatorname{Tr}_R u)(z) = 0\}$$
and write $\beta_1 = \beta + \mathbb{1}_\Lambda$. Then $\beta_1 \in L_\infty(\Gamma, \mathbb{R})$ and
$$\mathfrak{a}_R^{\beta_1}(u, v) = \mathfrak{a}_R^\beta(u, v) = (A_R^\beta u, v)_{L_2(\Omega_R)} = (\lambda_1 u, v)_{L_2(\Omega_R)}$$
for all $v \in V_R$. Hence $u \in D(A_R^{\beta_1})$ and $A_R^{\beta_1} u = \lambda_1 u$. Since $u > 0$ and $A_R^{\beta_1}$ is self-adjoint with compact resolvent, it follows from Proposition 6.11(a) together with Proposition A.18 that λ_1 is the smallest eigenvalue of $A_R^{\beta_1}$. Moreover, by Proposition 6.11(b) one obtains that
$$0 \le e^{-tA_R^\beta} w \le e^{-tA_R^{\beta_1}} w$$
for all $w \in L_2(\Omega_R)$ with $w \ge 0$ and all $t > 0$. Then $A_R^\beta = A_R^{\beta_1}$ by Proposition A.17. Hence
$$\int_\Gamma (\beta_1 - \beta) \operatorname{Tr}_R w \, \overline{\operatorname{Tr}_R v} = \mathfrak{a}_R^\beta(w, v) - \mathfrak{a}_R^{\beta_1}(w, v) = 0$$
for all $w, v \in V_R$. So $\beta = \beta_1$ σ-a.e., since $\operatorname{Tr}_R V_R$ is dense in $L_2(\Gamma)$. Therefore $\sigma(\Lambda) = \int_\Gamma \beta_1 - \beta = 0$ and the claim follows. □

Theorem 6.13. *Suppose that $a_{kl} = a_{lk}$ for all $k, l \in \{1, \dots, d\}$. Then the semigroup S^D is irreducible.*

Proof. Let $R \ge R_0$. We shall prove that the semigroup $(e^{-t\mathcal{N}_R^D})_{t>0}$ on $L_2(\Gamma)$ is irreducible. Since by Proposition 5.11 the semigroup S^D is positive, the claim then follows from Proposition 6.10 together with [Ouh05] Theorem 2.9.

Let $\Gamma_1 \subset \Gamma$ be a measurable set such that $\sigma(\Gamma_1) > 0$ and suppose that $e^{-t\mathcal{N}_R^D} L_2(\Gamma_1) \subset L_2(\Gamma_1)$ for all $t > 0$. Then $T = \left(e^{-t\mathcal{N}_R^D}|_{L_2(\Gamma_1)}\right)_{t>0}$ is a C_0-semigroup on the closed subspace $L_2(\Gamma_1)$ of $L_2(\Gamma)$, by [EN00] Paragraph I.5.12. Let $-\mathcal{N}_1$ denote the generator of T. Then

$$D(\mathcal{N}_1) = \left\{\varphi \in D(\mathcal{N}_R^D) \cap L_2(\Gamma_1) : \mathcal{N}_R^D \varphi \in L_2(\Gamma_1)\right\}$$

and $\mathcal{N}_1 = \mathcal{N}_R^D|_{D(\mathcal{N}_1)}$. Moreover, $\mathcal{N}_1$ is a self-adjoint operator in $L_2(\Gamma_1)$ with compact resolvent. Let μ_1 denote the smallest eigenvalue of $\mathcal{N}_1$. By the Krein–Rutman theorem there exists a $\varphi \in D(\mathcal{N}_1)$ such that $\varphi > 0$ and $\mathcal{N}_1\varphi = \mu_1\varphi$. Then by Lemma 4.1 one deduces that there exists a $u \in V_R$ such that $\mathrm{Tr}_R u = \varphi$ and

$$\mathfrak{a}_R^D(u, v) = (\mathcal{N}_R^D \mathrm{Tr}_R u, \mathrm{Tr}_R v)_{L_2(\Gamma)} = (\mathcal{N}_1 \mathrm{Tr}_R u, \mathrm{Tr}_R v)_{L_2(\Gamma)} = (\mu_1 \mathrm{Tr}_R u, \mathrm{Tr}_R v)_{L_2(\Gamma)} \tag{30}$$

for all $v \in V_R$. Since $\left(\mathrm{Tr}_{\Omega_R}(u^-)\right)|_\Gamma = \mathrm{Tr}_R(u^-) = \varphi^- = 0$ and $(\mathrm{Tr}_{\Omega_R} u)|_{\partial B_R} = 0$, it follows that $\mathrm{Tr}_{\Omega_R}(u^-) = 0$ and $u^- \in H_0^1(\Omega_R) \subset V_R$. Then the choice $v = u^-$ in (30) yields that $\mu \int_{\Omega_R} |\nabla(u^-)|^2 \leq \mathfrak{a}_R^D(u^-) = 0$, so u^- is constant and it follows that $u^- = 0$. Therefore $u \geq 0$ and since $\varphi \neq 0$, one obtains that $u > 0$.

Now consider the form $\mathfrak{a}_R^\beta$ on V_R with $\beta = \mu_1 \mathbb{1}_\Gamma$. Then it follows from (30) that $\mathfrak{a}_R^\beta(u, v) = 0$ for all $v \in V_R$. So $u \in D(A_R^\beta)$ and $A_R^\beta u = 0$. By Proposition 6.11(a) together with Proposition A.18, one obtains that u corresponds to the smallest eigenvalue of A_R^β. Then $\varphi(z) = (\mathrm{Tr}_R u)(z) > 0$ for a.e. $z \in \Gamma$, by Lemma 6.12. Since $\varphi \in L_2(\Gamma_1)$, by definition $\varphi = 0$ a.e. on $\Gamma \setminus \Gamma_1$ and therefore $\sigma(\Gamma \setminus \Gamma_1) = 0$. $\square$

Corollary 6.14. *Suppose that $a_{kl} = a_{lk}$ for all $k, l \in \{1, \ldots, d\}$. Let $\varphi, \psi \in L_2(\Gamma)^+ \setminus \{0\}$. Then*

$$\left(S_t^D \varphi, \psi\right)_{L_2(\Gamma)} > 0$$

for all $t > 0$.

Proof. Since Corollary 5.2(a) provides that the semigroup S^D is holomorphic, by an argument similar to the second paragraph of the proof of Theorem 6.9, it suffices to verify that there exists a $t > 0$ such that $(S_t^D \varphi, \psi)_{L_2(\Gamma)} > 0$.

Suppose to the contrary that $(S_t^D \varphi, \psi)_{L_2(\Gamma)} = 0$ for all $t > 0$. Write $\Gamma_1 = \Gamma \setminus \operatorname{supp} \psi$. Then $\sigma(\Gamma \setminus \Gamma_1) > 0$. Moreover,

$$(\varphi, \psi)_{L_2(\Gamma)} = \lim_{t \downarrow 0} \left(S_t^D \varphi, \psi\right)_{L_2(\Gamma)} = 0.$$

So $\varphi \in L_2(\Gamma_1)$ and since $\sigma(\operatorname{supp} \varphi) > 0$, it follows that $\sigma(\Gamma_1) > 0$. Hence there exists an $r > 0$ such that $S_r^D \varphi \notin L_2(\Gamma_1)$, since S^D is irreducible. Then $\sigma(\{z \in \operatorname{supp} \psi : (S_r^D \varphi)(z) > 0\}) > 0$ and it follows that $(S_r^D \varphi, \psi)_{L_2(\Gamma)} > 0$, a contradiction. $\square$

7 The heat kernel on $\Sigma_\theta \times \Gamma \times \Gamma$

Our final endeavour is to examine the heat kernel associated with the Dirichlet-to-Neumann operator. We prove joint continuity of the kernel on $\Sigma_\theta \times \Gamma \times \Gamma$, a result that seems yet to appear in the literature, even for the case $a_{kl} = \delta_{kl}$. It is well known that for second-order elliptic operators, the asymptotic behaviour of the heat kernel depends on the nature of the specified boundary conditions. This observation remains valid in the case of the Dirichlet-to-Neumann operator.

In this section we prove the following. Recall that $\theta^D, \theta^N \in (0, \frac{\pi}{2}]$ denote the angles of analyticity for the holomorphic semigroups S^D and S, respectively.

Theorem 1.3. (a) *There exists a continuous function $K^D \colon \Sigma_{\theta^D} \times \Gamma \times \Gamma \to \mathbb{C}$ such that*

$$(S_z^D \varphi)(w_1) = \int_\Gamma K_z^D(w_1, w_2)\varphi(w_2)\, \mathrm{d}w_2$$

for all $w_1 \in \Gamma$, $\varphi \in L_1(\Gamma)$ and $z \in \Sigma_{\theta^D}$.

(b) *The map $z \mapsto K_z^D(w_1, w_2)$ is analytic on Σ_{θ^D} for all $w_1, w_2 \in \Gamma$.*

(c) *For all $\theta' \in (0, \theta^D)$ there exist $c, \delta > 0$ such that*

$$\|K_z^D\|_{L_\infty(\Gamma\times\Gamma)} \leq c\,(\operatorname{Re} z)^{-(d-1)} e^{-\delta \operatorname{Re} z}$$

for all $z \in \Sigma_{\theta'}$.

Theorem 1.4. (a) *There exists a continuous function $K \colon \Sigma_{\theta^N} \times \Gamma \times \Gamma \to \mathbb{C}$ such that*

$$(S_z \varphi)(w_1) = \int_\Gamma K_z(w_1, w_2)\varphi(w_2)\, \mathrm{d}w_2$$

for all $w_1 \in \Gamma$, $\varphi \in L_1(\Gamma)$ and $z \in \Sigma_{\theta^N}$.

(b) *The map $z \mapsto K_z(w_1, w_2)$ is analytic on Σ_{θ^N} for all $w_1, w_2 \in \Gamma$.*

(c) *For all $\theta' \in (0, \theta^N)$ there exists a $c > 0$ such that*

$$\|K_z\|_{L_\infty(\Gamma\times\Gamma)} \leq c\,(1 \wedge \operatorname{Re} z)^{-(d-1)}$$

for all $z \in \Sigma_{\theta'}$.

Existence of the heat kernel follows from ultracontractivity of the semigroup, which we verify first. We then prove the above and subsequently obtain that the semigroups S and S^D leave $C(\Gamma)$ invariant, before deducing that S decays uniformly and at an exponential rate to its associated ergodic projection P. Using this result we prove that for all $\theta' \in (0, \theta^N)$, the family $(K_z)_{z\in\Sigma_{\theta'}}$ converges uniformly to $\frac{1}{\sigma(\Gamma)}\mathbb{1}_{\Gamma\times\Gamma}$ in the limit $|z| \to \infty$.

Throughout this section we fix $s = \frac{2(d-1)}{d-2} > 2$.

Lemma 7.1. *There exists a $c > 0$ such that*

$$\|\operatorname{Tr} u\|_{L_s(\Gamma)} \leq c\|u\|_{W(\Omega)}$$

for all $u \in W(\Omega)$.

Proof. Since Ω_{R_0} is a Lipschitz domain, it follows from [Neč12] Theorem 2.4.2 that the map $\mathrm{Tr}_{\Omega_{R_0}} \colon H^1(\Omega_{R_0}) \to L_s(\partial\Omega_{R_0})$ is bounded. Hence by Proposition 2.11(d) there exists a $c > 0$ such that

$$\begin{aligned}\|\mathrm{Tr}\, u\|_{L_s(\Gamma)} &= \|\mathrm{Tr}_{R_0}(u|_{\Omega_{R_0}})\|_{L_s(\Gamma)} \le \|\mathrm{Tr}_{\Omega_{R_0}}(u|_{\Omega_{R_0}})\|_{L_s(\partial\Omega_{R_0})} \\ &\le \|\mathrm{Tr}_{\Omega_{R_0}}\|_{\mathcal{L}(H^1(\Omega_{R_0}), L_s(\partial\Omega_{R_0}))} \|u|_{\Omega_{R_0}}\|_{H^1(\Omega_{R_0})} \\ &\le \|\mathrm{Tr}_{\Omega_{R_0}}\|_{\mathcal{L}(H^1(\Omega_{R_0}), L_s(\partial\Omega_{R_0}))} \Big(\int_\Omega |\nabla u|^2 + \int_{\Omega_{R_0}} |u|^2 \Big)^{1/2} \le c\|u\|_{W(\Omega)}\end{aligned}$$

for all $u \in W(\Omega)$. □

Lemma 7.2. (a) *$S_t^D L_2(\Gamma) \subset D(\mathcal{N}^D)$ for all $t > 0$. Moreover, there exists a $c > 0$ such that*

$$\|\mathcal{N}^D S_t^D\|_{2\to 2} \le ct^{-1}$$

for all $t > 0$.

(b) *$S_t L_2(\Gamma) \subset D(\mathcal{N})$ for all $t > 0$. Moreover, there exists a $c > 0$ such that*

$$\|\mathcal{N} S_t\|_{2\to 2} \le ct^{-1}$$

for all $t > 0$.

Proof. We prove (a). Let $t > 0$ and $\varphi \in L_2(\Gamma)$. By Corollary 5.2(a) the semigroup S^D is holomorphic, so the map $r \mapsto S_r^D \varphi$ is differentiable on $(0, \infty)$. Then the limit

$$\lim_{h\downarrow 0} \frac{1}{h}\big(I - S_h^D\big) S_t^D \varphi = \lim_{h\downarrow 0} \frac{1}{h}\big(S_t^D \varphi - S_{t+h}^D \varphi\big)$$

exists and therefore $S_t^D \varphi \in D(\mathcal{N}^D)$.

Note that by Corollary 5.2(a) there exists a $\theta \in (0, \frac{\pi}{2}]$ such that S^D is contractive and holomorphic on the sector $\Sigma_\theta \subset \mathbb{C}$. Let $t > 0$ and write $r = \frac{1}{2} t \sin\theta$. Define $\gamma \colon [0, 2\pi] \to \Sigma_\theta$ by $\gamma(s) = t + re^{is}$. Then

$$-\mathcal{N}^D S_t^D = \frac{\mathrm{d}}{\mathrm{d}t} S_t^D = \frac{1}{2\pi i} \int_\gamma \frac{S_z^D}{(z-t)^2} \, \mathrm{d}z$$

by Cauchy's formula and it follows that

$$\|\mathcal{N}^D S_t^D\|_{2\to 2} \le \frac{1}{2\pi} \int_0^{2\pi} \frac{\|S_{t+re^{is}}^D\|_{2\to 2}}{|re^{is}|} \, \mathrm{d}s \le \frac{1}{r} = \frac{2}{t\sin\theta}$$

as required.

The proof of (b) is similar. □

The semigroups S and S^D are ultracontractive.

Proposition 7.3. (a) *There exists a $c > 0$ such that*

$$\|S_t^D\|_{2\to s} \le ct^{-1/2}$$

for all $t > 0$.

(b) *There exists a $c > 0$ such that*

$$\|S_t\|_{2\to s} \le c t^{-1/2}$$

for all $t \in (0, 1]$.

Proof. We first prove (a). Let $c > 0$ be as in Lemma 7.1. Note that by Lemma 7.2(a) there exists a $c_1 > 0$ such that $\|\mathcal{N}^D S_t^D\|_{2\to 2} \le c_1 t^{-1}$ for all $t > 0$. Let $t > 0$ and $\varphi \in L_2(\Gamma)$. Then Lemma 7.2(a) provides that $S_t^D \varphi \in D(\mathcal{N}^D)$, so by Proposition 5.1(a) there exists a $u \in W^D(\Omega)$ such that $\operatorname{Tr} u = S_t^D \varphi$ and $\mathfrak{a}^D(u, v) = (\mathcal{N}^D S_t^D \varphi, \operatorname{Tr} v)_{L_2(\Gamma)}$ for all $v \in W^D(\Omega)$. Since the form $\mathfrak{a}^D$ is coercive, the choice $v = u$ yields that

$$\|S_t^D \varphi\|_{L_s(\Gamma)}^2 \le c^2 \|u\|_{W^D(\Omega)}^2 \le \frac{c^2}{\mu} \operatorname{Re} \mathfrak{a}^D(u) = \frac{c^2}{\mu} \operatorname{Re} \big(\mathcal{N}^D S_t^D \varphi, S_t^D \varphi\big)_{L_2(\Gamma)}.$$

Therefore

$$\|S_t^D \varphi\|_{L_s(\Gamma)}^2 \le \frac{c^2}{\mu} \|\mathcal{N}^D S_t^D \varphi\|_{L_2(\Gamma)} \, \|S_t^D \varphi\|_{L_2(\Gamma)} \le \frac{c^2}{\mu} c_1 t^{-1} \|\varphi\|_{L_2(\Gamma)}^2$$

since S^D is contractive.

We now prove (b). By Lemma 7.2(b) there exists a $c_2 > 0$ such that $\|\mathcal{N} S_t\|_{2\to 2} \le c_2 t^{-1}$ for all $t > 0$. Let $t \in (0, 1]$ and $\varphi \in L_2(\Gamma)$. By an argument similar to the above together with Lemma 3.1, one deduces that there exists a $c > 0$ such that

$$\begin{aligned}\|S_t \varphi\|_{L_s(\Gamma)}^2 &\le c\big(\operatorname{Re} \big(\mathcal{N} S_t \varphi, S_t \varphi\big)_{L_2(\Gamma)} + \|S_t \varphi\|_{L_2(\Gamma)}^2\big) \\ &\le c(c_2 t^{-1} + 1) \|\varphi\|_{L_2(\Gamma)}^2 \le c(c_2 + 1) t^{-1} \|\varphi\|_{L_2(\Gamma)}^2\end{aligned}$$

as required. □

Lemma 7.4. (a) *There exists a $c > 0$ such that*

$$\|S_t^D\|_{2\to\infty} \le c t^{-(d-1)/2}$$

for all $t > 0$.

(b) *There exists a $c > 0$ such that*

$$\|S_t\|_{2\to\infty} \le c(1 \wedge t)^{-(d-1)/2}$$

for all $t > 0$.

Proof. We first prove (a). Let $c > 0$ be as in Proposition 7.3(a) and let $p \in [2, \infty)$. Since by Proposition 6.1(a) the semigroup S^D is L_∞-contractive, by Proposition A.10 one has

$$\|S_t^D\|_{p\to \frac{ps}{2}} \le \|S_t^D\|_{2\to s}^{2/p} \, \|S_t^D\|_{\infty\to\infty}^{1-\frac{2}{p}} \le c^{2/p} t^{-1/p} \tag{31}$$

for all $t > 0$.

For each $n \in \mathbb{N}_0$ write $t_n = \frac{s-1}{s} s^{-n} > 0$ and $p_n = 2(\frac{s}{2})^n \in [2, \infty)$. Then $\sum_{n=0}^\infty t_n = 1$ and $\sum_{n=0}^\infty \frac{1}{p_n} = \frac{s}{2(s-2)} = \frac{d-1}{2}$. Moreover, $\lim p_n = \infty$ and since $p_{n+1} = \frac{p_n s}{2}$, it follows from (31) that

$$\|S_t^D\|_{p_n\to p_{n+1}} \le c^{2/p_n} t^{-1/p_n}$$

for all $t > 0$ and all $n \in \mathbb{N}_0$. Note that

$$\log t_n^{-1/p_n} = -2^{-1}\Big(\frac{s}{2}\Big)^{-n} \log\Big(\frac{s-1}{s} s^{-n}\Big)$$
$$= 2^{-1}\Big(\log \frac{s}{s-1}\Big)\Big(\frac{2}{s}\Big)^n + 2^{-1}(\log s) n \Big(\frac{2}{s}\Big)^n > 0$$

for all $n \in \mathbb{N}_0$. Therefore $\sum_{n=0}^{\infty} \log t_n^{-1/p_n} < \infty$ and it follows that there exists a $c_1 > 0$ such that $\prod_{n=0}^{\infty} t_n^{-1/p_n} = c_1$. Hence by the semigroup property

$$\|S_t^D\|_{2\to\infty} = \|S^D_{\sum t t_n}\|_{2\to\infty} \leq \prod_{n=0}^{\infty} \|S^D_{t t_n}\|_{p_n \to p_{n+1}} \leq \prod_{n=0}^{\infty} c^{2/p_n} (t t_n)^{-1/p_n} = c_2 t^{-(d-1)/2}$$

for all $t > 0$, where $c_2 = c^{d-1} c_1$.

We now prove (b). Let $c > 0$ be as in Proposition 7.3(b). Since S is contractive, it follows that for all $t > 1$ one has

$$\|S_t\|_{2\to s} \leq \|S_1\|_{2\to s} \|S_{t-1}\|_{2\to 2} \leq c.$$

Therefore $\|S_t\|_{2\to s} \leq c t^{-1/2} e^t$ for all $t > 0$. Write $T_t = e^{-t} S_t$. Then $\|T_t\|_{2\to s} \leq c t^{-1/2}$ for all $t > 0$. Hence by an argument similar to the proof of (a), one deduces that there exists a $c_1 > 0$ such that $e^{-t}\|S_t\|_{2\to\infty} = \|T_t\|_{2\to\infty} \leq c_1 t^{-(d-1)/2}$ for all $t > 0$. Then for all $t \in (0,1]$

$$\|S_t\|_{2\to\infty} \leq c_1 e t^{-(d-1)/2} = c_2 t^{-(d-1)/2},$$

where $c_2 = c_1 e$. By the contractivity of S once again, it follows that for all $t > 1$

$$\|S_t\|_{2\to\infty} \leq \|S_1\|_{2\to\infty} \|S_{t-1}\|_{2\to 2} \leq c_2.$$

This proves the claim. $\square$

We are now able to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. We first show that S^D maps $L_2(\Gamma)$ into $C(\Gamma)$. Let $t > 0$ and $\varphi \in L_2(\Gamma)$. Then $S^D_{2t}\varphi \in D(\mathcal{N}^D)$ by Lemma 7.2(a). Let $\lambda > 0$ and write $\psi = \mathcal{N}^D S^D_{2t}\varphi + \lambda S^D_{2t}\varphi$. By Proposition 5.1(a) there exists a $u \in W^D(\Omega)$ such that $\operatorname{Tr} u = S^D_{2t}\varphi$ and

$$\mathfrak{a}^D(u, v) + \lambda \int_\Gamma \operatorname{Tr} u \, \overline{\operatorname{Tr} v} = \int_\Gamma \psi \, \overline{\operatorname{Tr} v}$$

for all $v \in W^D(\Omega)$. Then $u = B^D_\lambda \psi$. Moreover, $\psi = S^D_t \mathcal{N}^D S^D_t \varphi + \lambda S^D_{2t}\varphi \in L_\infty(\Gamma)$ since $S^D_t \in \mathcal{L}\big(L_2(\Gamma), L_\infty(\Gamma)\big)$ by Lemma 7.4(a). Hence $u \in C(\overline{\Omega})$ by Theorem 1.1(a) and it follows that $S^D_{2t}\varphi = \operatorname{Tr} u \in C(\Gamma)$. Then $S^D_t L_2(\Gamma) \subset C(\Gamma)$ for all $t > 0$. Now let $z \in \Sigma_{\theta^D}$. Then there exists a $t > 0$ such that $z - t \in \Sigma_{\theta^D}$, so

$$S^D_z L_2(\Gamma) = S^D_t S^D_{z-t} L_2(\Gamma) \subset S^D_t L_2(\Gamma) \subset C(\Gamma).$$

Therefore $S^D_z L_2(\Gamma) \subset C(\Gamma)$ for all $z \in \Sigma_{\theta^D}$.

Let $t > 0$. In order to prove Statements (a) and (b) we shall show that for all $z \in \Sigma_{\theta^D}$, the operator S^D_{4t+z} has a kernel $K^D_{4t+z} \colon \Gamma \times \Gamma \to \mathbb{C}$ such that the map $(z, w_1, w_2) \mapsto K^D_{4t+z}(w_1, w_2)$

is continuous on $\Sigma_{\theta^D} \times \Gamma \times \Gamma$, and for all $w_1, w_2 \in \Gamma$ the map $z \mapsto K^D_{4t+z}(w_1, w_2)$ is analytic on Σ_{θ^D}. Then the claim follows from the fact that Σ_{θ^D} is open in $\mathbb{C}$.

By Lemma 7.4(a) the operator S^D_t is bounded from $L_2(\Gamma)$ into $C(\Gamma)$, so the Riesz representation theorem provides that for all $w_1 \in \Gamma$ there exists a $k^{(1)}_{w_1} \in L_2(\Gamma)$ such that

$$(S^D_t \varphi)(w_1) = (\varphi, k^{(1)}_{w_1})_{L_2(\Gamma)}$$

for all $\varphi \in L_2(\Gamma)$. Then

$$\|k^{(1)}_{w_1}\|_{L_2(\Gamma)} = \sup_{\|\varphi\|_{L_2(\Gamma)} \le 1} |(S^D_t \varphi)(w_1)| \le \|S^D_t\|_{L_2(\Gamma) \to C(\Gamma)} < \infty$$

for all $w_1 \in \Gamma$, so $\sup_{w_1 \in \Gamma} \|k^{(1)}_{w_1}\|_{L_2(\Gamma)} < \infty$. Moreover, since $S^D_t \varphi \in C(\Gamma)$ for all $\varphi \in L_2(\Gamma)$, it follows that the map $w_1 \mapsto k^{(1)}_{w_1}$ is weakly continuous from Γ into $L_2(\Gamma)$. By Lemma 7.4(a) together with duality one has that $S^D_t \in \mathcal{L}\big(L_1(\Gamma), L_\infty(\Gamma)\big)$, so by the Dunford–Pettis theorem [DP40] Theorem 2.2.5 the operator S^D_t has a kernel in $L_\infty(\Gamma \times \Gamma)$. Since $\sigma(\Gamma) < \infty$, it follows that S^D_t is Hilbert–Schmidt and therefore compact. Hence the map $w_1 \mapsto S^D_t k^{(1)}_{w_1}$ is continuous from Γ into $L_2(\Gamma)$. By duality one similarly deduces that for all $w_2 \in \Gamma$ there exists a $k^{(2)}_{w_2} \in L_2(\Gamma)$ such that

$$(S^{D*}_t \varphi)(w_2) = (\varphi, k^{(2)}_{w_2})_{L_2(\Gamma)}$$

for all $\varphi \in L_2(\Gamma)$, and that the map $w_2 \mapsto S^{D*}_t k^{(2)}_{w_2}$ is continuous from Γ into $L_2(\Gamma)$. Moreover, for all $T \in \mathcal{L}\big(L_2(\Gamma)\big)$ and $\varphi \in L_2(\Gamma)$ one has that

$$\begin{aligned}(S^D_t T S^D_t \varphi)(w_1) &= (T S^D_t \varphi, k^{(1)}_{w_1})_{L_2(\Gamma)} = (\varphi, S^{D*}_t T^* k^{(1)}_{w_1})_{L_2(\Gamma)} \\ &= \int_\Gamma \varphi(w_2) \overline{(S^{D*}_t T^* k^{(1)}_{w_1})(w_2)} \, \mathrm{d}w_2 \\ &= \int_\Gamma \big(T k^{(2)}_{w_2}, k^{(1)}_{w_1}\big)_{L_2(\Gamma)} \varphi(w_2) \, \mathrm{d}w_2\end{aligned}$$

for all $w_1 \in \Gamma$, since

$$(S^{D*}_t T^* k^{(1)}_{w_1})(w_2) = \big(T^* k^{(1)}_{w_1}, k^{(2)}_{w_2}\big)_{L_2(\Gamma)} = \big(k^{(1)}_{w_1}, T k^{(2)}_{w_2}\big)_{L_2(\Gamma)}$$

for all $w_2 \in \Gamma$.

Now let $z \in \Sigma_{\theta^D}$ and define $K^D_{4t+z} \colon \Gamma \times \Gamma \to \mathbb{C}$ by

$$K^D_{4t+z}(w_1, w_2) = \big(S^D_z S^D_t k^{(2)}_{w_2}, S^{D*}_t k^{(1)}_{w_1}\big)_{L_2(\Gamma)}.$$

Then $\|K^D_{4t+z}\|_{L_\infty(\Gamma \times \Gamma)} < \infty$, since $\sup_{w_1 \in \Gamma} \|k^{(1)}_{w_1}\|_{L_2(\Gamma)} < \infty$ and $\sup_{w_2 \in \Gamma} \|k^{(2)}_{w_2}\|_{L_2(\Gamma)} < \infty$. Write $T = S^D_t S^D_z S^D_t \in \mathcal{L}\big(L_2(\Gamma)\big)$. Then for all $\varphi \in L_2(\Gamma)$

$$\begin{aligned}(S^D_{4t+z} \varphi)(w_1) &= (S^D_t T S^D_t \varphi)(w_1) \\ &= \int_\Gamma \big(T k^{(2)}_{w_2}, k^{(1)}_{w_1}\big)_{L_2(\Gamma)} \varphi(w_2) \, \mathrm{d}w_2 \\ &= \int_\Gamma \big(S^D_t S^D_z S^D_t k^{(2)}_{w_2}, k^{(1)}_{w_1}\big)_{L_2(\Gamma)} \varphi(w_2) \, \mathrm{d}w_2\end{aligned}$$

$$= \int_\Gamma K^D_{4t+z}(w_1, w_2)\varphi(w_2)\,\mathrm{d}w_2$$

for all $w_1 \in \Gamma$. Moreover, by Lemma 7.4(a) together with duality one obtains that $S^D_{3t+z} \in \mathcal{L}\big(L_1(\Gamma), L_2(\Gamma)\big)$, as S^D is contractive on Σ_{θ^D}. So $S^D_{4t+z}\varphi \in C(\Gamma)$ for all $\varphi \in L_1(\Gamma)$ and by density one deduces that

$$(S^D_{4t+z}\varphi)(w_1) = \int_\Gamma K^D_{4t+z}(w_1, w_2)\varphi(w_2)\,\mathrm{d}w_2$$

for all $w_1 \in \Gamma$ and $\varphi \in L_1(\Gamma)$. Finally since the maps $w_2 \mapsto S^D_t k^{(2)}_{w_2}$ and $w_1 \mapsto S^{D*}_t k^{(1)}_{w_1}$ are continuous from Γ into $L_2(\Gamma)$, and by Corollary 5.2(a) the map $z \mapsto S^D_z$ is continuous from Σ_{θ^D} into $\mathcal{L}\big(L_2(\Gamma)\big)$, it follows that the map

$$(z, w_1, w_2) \mapsto \big(S^D_z S^D_t k^{(2)}_{w_2}, S^{D*}_t k^{(1)}_{w_1}\big)_{L_2(\Gamma)} = K_{4t+z}(w_1, w_2)$$

is continuous on $\Sigma_{\theta^D} \times \Gamma \times \Gamma$. Moreover, since $S^D_t k^{(2)}_{w_2} \in L_2(\Gamma)$ and $S^{D*}_t k^{(1)}_{w_1} \in L_2(\Gamma)$ for all $w_1, w_2 \in \Gamma$, by Corollary 5.2(a) once again one obtains that the map $z \mapsto K^D_{4t+z}(w_1, w_2)$ is analytic on Σ_{θ^D} for all $w_1, w_2 \in \Gamma$. This proves Statements (a) and (b).

We now prove (c). Note that by Lemma 7.4(a) together with duality, there exists a $c > 0$ such that

$$\|S^D_t\|_{2\to\infty} \le ct^{-(d-1)/2}$$

and

$$\|S^D_t\|_{1\to 2} \le ct^{-(d-1)/2}$$

for all $t > 0$. Let $\theta' \in (0, \theta^D)$. Then there exist $\theta_0 \in (\theta', \theta^D)$ and $\kappa \in (0,1)$ such that $\kappa t + is \in \Sigma_{\theta_0}$ for all $t + is \in \Sigma_{\theta'}$. Moreover, by Proposition 6.3 there exists a $\delta > 0$ such that $\|S^D_z\|_{2\to 2} \le e^{-\delta \operatorname{Re} z}$ for all $z \in \Sigma_{\theta_0}$. Let $z \in \Sigma_{\theta'}$ and write $z = t + is$. Then

$$\begin{aligned}
\|S^D_z\|_{1\to\infty} &\le \|S^D_{\frac12(1-\kappa)t}\|_{2\to\infty}\, \|S^D_{\kappa t+is}\|_{2\to 2}\, \|S^D_{\frac12(1-\kappa)t}\|_{1\to 2} \\
&\le c^2\big(\tfrac12(1-\kappa)t\big)^{-(d-1)} e^{-\delta\kappa t} \\
&= c_1(\operatorname{Re} z)^{-(d-1)} e^{-\delta_1 \operatorname{Re} z}
\end{aligned}$$

where $c_1 = c^2\, 2^{d-1}(1-\kappa)^{-(d-1)}$ and $\delta_1 = \delta\kappa$. Hence by Statement (a)

$$\begin{aligned}
\|K^D_z\|_{L_\infty(\Gamma\times\Gamma)} &= \sup_{w_1\in\Gamma} \|K^D_z(w_1, \cdot)\|_{C(\Gamma)} \\
&= \sup\Big\{\Big|\int_\Gamma K^D_z(w_1, w_2)\varphi(w_2)\,\mathrm{d}w_2\Big| : \varphi \in L_1(\Gamma), \|\varphi\|_{L_1(\Gamma)} \le 1 \text{ and } w_1 \in \Gamma\Big\} \\
&= \sup\big\{|(S^D_z\varphi)(w_1)| : \varphi \in L_1(\Gamma), \|\varphi\|_{L_1(\Gamma)} \le 1 \text{ and } w_1 \in \Gamma\big\} \\
&= \sup\big\{\|S^D_z\varphi\|_{L_\infty(\Gamma)} : \varphi \in L_1(\Gamma) \text{ and } \|\varphi\|_{L_1(\Gamma)} \le 1\big\} \\
&= \|S^D_z\|_{1\to\infty} \le c_1(\operatorname{Re} z)^{-(d-1)} e^{-\delta_1 \operatorname{Re} z}
\end{aligned}$$

as required. □

Proof of Theorem 1.4. Statements (a) and (b) follow from arguments analogous to the proofs of the corresponding statements in Theorem 1.3. We prove (c). Let $\theta' \in (0, \theta^N)$. Then there exist $\theta_0 \in (\theta', \theta^N)$ and $\kappa \in (0,1)$ such that $\kappa t + is \in \Sigma_{\theta_0}$ for all $t + is \in \Sigma_{\theta'}$. Arguing as in the proof of Theorem 1.3(c), one deduces from Lemma 7.4(b) that there exists a $c > 0$ such that

$$\|S_z\|_{1\to\infty} \leq c\big(1 \wedge \tfrac{1}{2}(1-\kappa)\operatorname{Re} z\big)^{-(d-1)} \leq c_1\big(1 \wedge \operatorname{Re} z\big)^{-(d-1)}$$

for all $z \in \Sigma_{\theta'}$, where $c_1 = c\, 2^{d-1}(1-\kappa)^{-(d-1)}$. The estimate for $\|K_z\|_{L_\infty(\Gamma\times\Gamma)}$ then follows as in the proof of Theorem 1.3(c). $\square$

The semigroups S and S^D leave $C(\Gamma)$ invariant.

Corollary 7.5. (a) *Let $z \in \Sigma_{\theta^D}$. Then $S^D_z C(\Gamma) \subset C(\Gamma)$.*

(b) *Let $z \in \Sigma_{\theta^N}$. Then $S_z C(\Gamma) \subset C(\Gamma)$.*

Proof. In the above we have proved that S^D and S map $L_2(\Gamma)$ into $C(\Gamma)$, so the corollary follows from the fact that $C(\Gamma) \subset L_2(\Gamma)$. $\square$

By the proof of Theorem 1.3, the semigroup S^D consists of compact operators on $L_2(\Gamma)$. Because S^D is ultracontractive, this property is inherited by the extrapolation semigroup $S^{D,(p)}$ on $L_p(\Gamma)$ for all $p \in [1,\infty]$. Indeed, as S^D is contractive on both $L_1(\Gamma)$ and $L_\infty(\Gamma)$ and is continuous from $L_1(\Gamma)$ into $L_\infty(\Gamma)$, an interpolation argument yields that S^D maps $L_p(\Gamma)$ continuously into $L_q(\Gamma)$ for all $p, q \in [1,\infty]$ with $p \leq q$. Since $\sigma(\Gamma) < \infty$, it then follows from the factorisation

$$L_p(\Gamma) \overset{S^{D,(p)}_{t/3}}{\to} L_\infty(\Gamma) \hookrightarrow L_2(\Gamma) \overset{S^D_{t/3}}{\to} L_2(\Gamma) \overset{S^D_{t/3}}{\to} L_\infty(\Gamma) \hookrightarrow L_p(\Gamma)$$

that $S^{D,(p)}_t$ is compact for all $t > 0$. One similarly deduces that the semigroup $S^{(p)}$ consists of compact operators on $L_p(\Gamma)$ for all $p \in [1,\infty]$.

We denote by K^D and K the kernels as introduced in Theorems 1.3 and 1.4. The domination of S^D by S yields an analogous relationship between the kernels for real time.

Proposition 7.6. *Let $t > 0$. Then*

$$0 \leq K^D_t(w_1, w_2) \leq K_t(w_1, w_2)$$

for a.e. $w_1, w_2 \in \Gamma$.

Proof. Note first that $K_t, K^D_t \in L_\infty(\Gamma\times\Gamma)$ are positive by [AB94] Proposition 1.9(a), since by Proposition 5.11 the semigroups S and S^D are positive. Write $\widetilde{S}_t = S_t - S^D_t$. Then $\widetilde{S}_t L_2(\Gamma)^+ \subset L_2(\Gamma)^+$ by the domination estimate in Proposition 5.11. Write $\widetilde{K}_t = K_t - K^D_t \in L_\infty(\Gamma\times\Gamma)$. Then

$$\begin{aligned}(\widetilde{S}_t\varphi)(w_1) &= \int_\Gamma K_t(w_1,w_2)\varphi(w_2)\,\mathrm{d}w_2 - \int_\Gamma K^D_t(w_1,w_2)\varphi(w_2)\,\mathrm{d}w_2 \\ &= \int_\Gamma \widetilde{K}_t(w_1,w_2)\varphi(w_2)\,\mathrm{d}w_2\end{aligned}$$

for all $w_1 \in \Gamma$ and $\varphi \in L_1(\Gamma)$. Hence $\widetilde{K}_t$ is positive by [AB94] Proposition 1.9(a). $\square$

By Proposition 6.3 the semigroup S^D converges in the norm topology to its ergodic projection $0 \in \mathcal{L}\big(L_2(\Gamma)\big)$ at an exponential rate. The semigroup S and its associated ergodic projection $P\colon \varphi \mapsto \frac{1}{\sigma(\Gamma)}(\varphi, \mathbb{1}_\Gamma)_{L_2(\Gamma)}\mathbb{1}_\Gamma$ exhibit similar behaviour.

Proposition 7.7. *For all $\theta' \in (0, \theta^N)$ there exist $c, \delta > 0$ such that*

$$\|S_z - P\|_{2\to 2} \le c e^{-\delta \operatorname{Re} z}$$

for all $z \in \Sigma_{\theta'}$.

Proof. Note that by Proposition 5.11 and Theorem 6.9 the semigroup S is positive and irreducible. Let $t > 0$. Since $\sigma(\Gamma) < \infty$ and by Lemma 7.4(b) one has $S_t \in \mathcal{L}\big(L_1(\Gamma), L_\infty(\Gamma)\big)$, it follows from the Dunford–Pettis theorem that S_t is compact. Moreover, duality together with Proposition 5.4(b) provides that $\ker \mathcal{N}^* = \ker \mathcal{N} = \mathbb{C}\mathbb{1}_\Gamma$. Hence by [Are08] Theorem 4.5 there exist $c, \delta > 0$ such that

$$\|S_t - P\|_{2\to 2} \le c e^{-\delta t}$$

for all $t > 0$.

It follows from (27) together with the definition of P that $S_t P = P S_t = P$ for all $t > 0$. Define $f\colon \Sigma_{\theta^N} \to \mathcal{L}\big(L_2(\Gamma)\big)$ by $f(z) = S_z P - P S_z$. Then $f|_{(0,\infty)} = 0$. Moreover, by Corollary 5.2(b) the function f is holomorphic on the connected open set $\Sigma_{\theta^N} \subset \mathbb{C}$, so $f = 0$ on Σ_{θ^N}. By a similar argument, one deduces that the map $z \mapsto S_z P - P$ is identically zero on Σ_{θ^N}. Therefore $S_z P = P S_z = P$ for all $z \in \Sigma_{\theta^N}$.

Let $\theta' \in (0, \theta^N)$. Then there exist $\theta_0 \in (\theta', \theta^N)$ and $\kappa \in (0, 1)$ such that $\kappa t + is \in \Sigma_{\theta_0}$ for all $t + is \in \Sigma_{\theta'}$. Let $z \in \Sigma_{\theta'}$ and write $z = t + is$. Then

$$S_z - P = S_{\kappa t+is} S_{(1-\kappa)t} - S_{\kappa t+is} P = S_{\kappa t+is}(S_{(1-\kappa)t} - P)$$

and

$$\|S_z - P\|_{2\to 2} \le \|S_{\kappa t+is}\|_{2\to 2}\, \|S_{(1-\kappa)t} - P\|_{2\to 2} \le c e^{-\delta(1-\kappa)t}$$

since S is contractive on Σ_{θ^N}. This proves the claim. □

It follows from Theorem 1.3(c) that for all $\theta' \in (0, \theta^D)$, the family $(K^D_z)_{z\in\Sigma_{\theta'}}$ converges uniformly to zero in the limit $|z| \to \infty$. Our final result implies that for all $\theta' \in (0, \theta^N)$, the family $(K_z)_{z\in\Sigma_{\theta'}}$ converges to $\frac{1}{\sigma(\Gamma)}$ in a similar manner.

Theorem 7.8. *For all $\theta' \in (0, \theta^N)$ there exist $c, \delta > 0$ such that*

$$\|K_z - \tfrac{1}{\sigma(\Gamma)}\|_{L_\infty(\Gamma\times\Gamma)} \le c(1 \wedge \operatorname{Re} z)^{-(d-1)} e^{-\delta \operatorname{Re} z}$$

for all $z \in \Sigma_{\theta'}$.

Proof. Note that by Corollary 6.7, the ergodic projection P associated with S extends to an operator on $L_1(\Gamma)$ defined by $P\varphi = \frac{1}{\sigma(\Gamma)}(\int_\Gamma \varphi)\mathbb{1}_\Gamma$, where we continue to denote by P the extension to $L_1(\Gamma)$. Then Theorem 1.4(a) provides that for all $z \in \Sigma_{\theta^N}$,

$$((S_z - P)\varphi)(w_1) = \int_\Gamma K_z(w_1, w_2)\varphi(w_2)\,\mathrm{d}w_2 - \tfrac{1}{\sigma(\Gamma)}\mathbb{1}_\Gamma(w_1)\int_\Gamma \varphi(w_2)\mathbb{1}_\Gamma(w_2)\,\mathrm{d}w_2$$

$$= \int_\Gamma K_z(w_1,w_2)\varphi(w_2) - \tfrac{1}{\sigma(\Gamma)} \mathbb{1}_\Gamma(w_1)\varphi(w_2)\mathbb{1}_\Gamma(w_2)\, \mathrm{d}w_2$$

$$= \int_\Gamma \big(K_z - \tfrac{1}{\sigma(\Gamma)} \mathbb{1}_{\Gamma\times\Gamma}\big)(w_1,w_2)\varphi(w_2)\, \mathrm{d}w_2$$

for all $w_1 \in \Gamma$ and $\varphi \in L_1(\Gamma)$. Moreover, by Lemma 7.4(b) together with duality, there exists a $c_0 > 0$ such that

$$\|S_t\|_{2\to\infty} \le c_0(1 \wedge t)^{-(d-1)/2}$$

and

$$\|S_t\|_{1\to 2} \le c_0(1 \wedge t)^{-(d-1)/2}$$

for all $t > 0$.

Let $\theta' \in (0, \theta^N)$. Then there exist $\theta_0 \in (\theta', \theta^N)$ and $\kappa \in (0,1)$ such that $\kappa t + is \in \Sigma_{\theta_0}$ for all $t + is \in \Sigma_{\theta'}$. Moreover, by Proposition 7.7 there exist $c, \delta > 0$ such that

$$\|S_z - P\|_{2\to 2} \le c e^{-\delta \operatorname{Re} z}$$

for all $z \in \Sigma_{\theta_0}$. Let $z \in \Sigma_{\theta'}$ and write $z = t + is$. Then

$$\begin{aligned} S_z - P &= S_{\frac12(1-\kappa)t} S_{\kappa t + is} S_{\frac12(1-\kappa)t} - P \\ &= S_{\frac12(1-\kappa)t} S_{\kappa t + is} S_{\frac12(1-\kappa)t} - S_{\frac12(1-\kappa)t} P S_{\frac12(1-\kappa)t} \\ &= S_{\frac12(1-\kappa)t} (S_{\kappa t + is} - P) S_{\frac12(1-\kappa)t} \end{aligned}$$

and it follows that

$$\begin{aligned} \|S_z - P\|_{1\to\infty} &\le \|S_{\frac12(1-\kappa)t}\|_{2\to\infty}\, \|S_{\kappa t+is} - P\|_{2\to 2}\, \|S_{\frac12(1-\kappa)t}\|_{1\to 2} \\ &\le c_0^2 c (1 \wedge \tfrac12(1-\kappa)t)^{-(d-1)} e^{-\delta\kappa t} \\ &\le c_1 (1 \wedge \operatorname{Re} z)^{-(d-1)} e^{-\delta_1 \operatorname{Re} z} \end{aligned}$$

where $c_1 = c_0^2 c\, 2^{d-1}(1-\kappa)^{-(d-1)}$ and $\delta_1 = \delta\kappa$. Hence

$$\begin{aligned} \|K_z - \tfrac{1}{\sigma(\Gamma)}\|_{L_\infty(\Gamma\times\Gamma)} &= \sup_{w_1\in\Gamma} \|\big(K_z - \tfrac{1}{\sigma(\Gamma)}\mathbb{1}_{\Gamma\times\Gamma}\big)(w_1,\cdot)\|_{C(\Gamma)} \\ &= \sup\big\{|((S_z - P)\varphi)(w_1)| : \varphi \in L_1(\Gamma), \|\varphi\|_{L_1(\Gamma)} \le 1 \text{ and } w_1 \in \Gamma\big\} \\ &= \|S_z - P\|_{1\to\infty} \le c_1(1 \wedge \operatorname{Re} z)^{-(d-1)} e^{-\delta_1 \operatorname{Re} z} \end{aligned}$$

as claimed. □

A Appendix

In this section we gather various auxiliary facts from functional analysis.

Sobolev spaces

The following result is known as the Neumann-type Poincaré inequality.

Proposition A.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then there exists a $c > 0$ such that*

$$\int_\Omega |u - \langle u \rangle|^2 \le c \int_\Omega |\nabla u|^2$$

for all $u \in H^1(\Omega)$, where $\langle u \rangle = \frac{1}{|\Omega|} \int_\Omega u$.

Proof. See [GP05] Theorem 2.5.21 and Remark 2.5.15. □

Proposition A.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and let $B \subset \mathbb{R}^d$ be an open set with $\Omega \subset B$ and $|B| < \infty$. Then there exists a bounded operator $E \colon H^1(\Omega) \to H^1(B)$ such that $(Eu)|_\Omega = u$ and $E\mathbb{1}_\Omega = \mathbb{1}_B$.*

Proof. By [AF03] Theorem 5.24 (see also [Ste70] Chapter VI) there exists a bounded operator $E_0 \colon H^1(\Omega) \to H^1(\mathbb{R}^d)$ such that $(E_0 u)|_\Omega = u$. Define $E_1 \colon H^1(\Omega) \to H^1(B)$ by $E_1 u = (E_0 u)|_B$. Then $E_1 \in \mathcal{L}\big(H^1(\Omega), H^1(B)\big)$ and $(E_1 u)|_\Omega = u$. Define $P \colon H^1(\Omega) \to \mathbb{C}\mathbb{1}_\Omega$ by

$$Pu = \frac{1}{|\Omega|}\Big(\int_\Omega u\mathbb{1}_\Omega\Big)\mathbb{1}_\Omega.$$

Then $P \in \mathcal{L}\big(H^1(\Omega)\big)$. Define $E \colon H^1(\Omega) \to H^1(B)$ by

$$Eu = E_1(u - Pu) + \frac{1}{|\Omega|}\Big(\int_\Omega u\mathbb{1}_\Omega\Big)\mathbb{1}_B.$$

Then $E \in \mathcal{L}\big(H^1(\Omega), H^1(B)\big)$ and

$$(Eu)|_\Omega = u - Pu + \frac{1}{|\Omega|}\Big(\int_\Omega u\mathbb{1}_\Omega\Big)\mathbb{1}_\Omega = u.$$

Moreover,

$$E\mathbb{1}_\Omega = E_1(\mathbb{1}_\Omega - \mathbb{1}_\Omega) + \frac{1}{|\Omega|}\Big(\int_\Omega \mathbb{1}_\Omega\Big)\mathbb{1}_B = \mathbb{1}_B$$

as required. □

Banach spaces

Proposition A.3. *Let X be a normed linear space and Y a Banach space. Let D denote a dense subspace of X and let $T \in \mathcal{L}(D, Y)$. Then there exists a unique $\widetilde{T} \in \mathcal{L}(X, Y)$ such that $\widetilde{T}|_D = T$ and $\|\widetilde{T}\|_{\mathcal{L}(X,Y)} = \|T\|_{\mathcal{L}(D,Y)}$.*

Proof. The case $T = 0$ is trivial. Write $M = \|T\|_{\mathcal{L}(D,Y)} > 0$ and let $\varepsilon > 0$. Let $x, x_1, x_2, \ldots \in D$ and suppose that $\lim x_n = x$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ one has that $\|x_n - x_m\|_X < \frac{\varepsilon}{M}$. Hence

$$\|Tx_n - Tx_m\|_Y \leq M\|x_n - x_m\|_X < \varepsilon$$

for all $n, m \geq N$. Then $(Tx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y and is therefore convergent, since Y is complete.

Define $\widetilde{T}\colon X \to Y$ by $\widetilde{T}x = \lim Tx_n$. Let $x \in X$ and $(x_n)_{n\in\mathbb{N}}, (w_n)_{n\in\mathbb{N}} \subset D$ be such that $\lim x_n = x$ and $\lim w_n = x$. Then the sequences $(Tx_n)_{n\in\mathbb{N}}$ and $(Tw_n)_{n\in\mathbb{N}}$ are Cauchy in Y and it follows that there exist $y, z \in Y$ such that $\lim Tx_n = y$ and $\lim Tw_n = z$. Then

$$\|y - z\|_Y = \lim_{n\to\infty} \|Tx_n - Tw_n\| \leq M \lim_{n\to\infty} \|x_n - w_n\|_X = 0,$$

so $y = z$. Therefore $\widetilde{T}$ is well-defined. Note that for all $x \in X$, by density there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in D such that $\lim x_n = x$. Moreover, it is easy to see that $\widetilde{T}$ is linear and $\widetilde{T}|_D = T$.

Next we prove that $\|\widetilde{T}\|_{\mathcal{L}(X,Y)} = \|T\|_{\mathcal{L}(D,Y)}$. Clearly

$$\sup_{x\in X\setminus\{0\}} \frac{\|\widetilde{T}x\|_Y}{\|x\|_X} \geq \|T\|_{\mathcal{L}(D,Y)}$$

since $D \subset X$ and $\widetilde{T}|_D = T$. Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D such that $\lim x_n = x$. Then

$$\|\widetilde{T}x\|_Y = \lim_{n\to\infty} \|Tx_n\|_Y \leq \|T\|_{\mathcal{L}(D,Y)} \lim_{n\to\infty} \|x_n\|_X = \|T\|_{\mathcal{L}(D,Y)}\|x\|_X,$$

so $\|\widetilde{T}\|_{\mathcal{L}(X,Y)} \leq \|T\|_{\mathcal{L}(D,Y)}$ and equality follows.

Finally we prove that $\widetilde{T}$ is unique. Let $\widehat{T} \in \mathcal{L}(X,Y)$ and suppose that $\widehat{T}|_D = T$. Let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in D such that $\lim x_n = x$. Then

$$\widehat{T}x = \lim_{n\to\infty} \widehat{T}x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} \widetilde{T}x_n = \widetilde{T}x$$

and it follows that $\widehat{T} = \widetilde{T}$. □

Proposition A.4. *Let T be a compact operator on a Banach space X and let $\lambda \in \mathbb{C}\setminus\{0\}$. Then $\lambda I - T$ has closed range.*

Proof. Since T is compact it follows that $\dim\ker(\lambda I - T) < \infty$ and there exists a closed subspace M of X such that $X = \ker(\lambda I - T) \oplus M$. Write $S = (\lambda I - T)|_M$. Then $S \in \mathcal{L}(M, X)$ is injective and $R(S) = R(\lambda I - T)$.

Next we show that there exists an $r > 0$ such that

$$r\|x\| \leq \|Sx\| \tag{32}$$

for all $x \in M$. Suppose to the contrary that for each $n \in \mathbb{N}$ there exists an $x_n \in M$ such that $\|Sx_n\| < \frac{1}{n}\|x_n\|$. Without loss of generality we may assume that $\|x_n\| = 1$ for all $n \in \mathbb{N}$. Then $\lim Sx_n = 0$ and the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded. Since T is compact, by passing to a subsequence if necessary we may assume that there exists a $y \in X$ such that $\lim Tx_n = y$. Hence

$$\lim_{n\to\infty} \lambda x_n = \lim_{n\to\infty} (S+T)x_n = y.$$

Then $y \in M$, since $(\lambda x_n)_{n\in\mathbb{N}} \subset M$. Therefore $Sy = \lambda \lim Sx_n = 0$, so $y = 0$ and it follows that $0 = \|y\| = \lim |\lambda| \|x_n\| = |\lambda| > 0$, a contradiction.

Now let $y \in \overline{R(\lambda I - T)} = \overline{R(S)}$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in M such that $\lim Sx_n = y$. So $(Sx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and it follows from (32) that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in M. Since M is closed there exists an $x \in M$ such that $\lim x_n = x$ and because S is bounded it follows that $y = \lim Sx_n = Sx \in R(S) = R(\lambda I - T)$. $\square$

Proposition A.5. *Let A be a closed operator on a Banach space X with $\rho(A) \neq \varnothing$. Then A has compact resolvent if and only if the canonical injection $\iota\colon (D(A), \|\cdot\|_A) \hookrightarrow X$ is compact, where $\|\cdot\|_A$ is the graph norm on $D(A)$.*

Proof. Suppose first that A has compact resolvent. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $(D(A), \|\cdot\|_A)$ and let $\lambda \in \rho(A)$. For each $n \in \mathbb{N}$ write $y_n = (\lambda I - A)x_n$. Then there exists an $M > 0$ such that

$$\|y_n\|_X \leq (|\lambda| + 1)\|x_n\|_A \leq (|\lambda| + 1)M$$

for all $n \in \mathbb{N}$. Since the operator $(\lambda I - A)^{-1}\colon X \to X$ is compact, it follows that the sequence $(x_n)_{n\in\mathbb{N}} = \big((\lambda I - A)^{-1} y_n\big)_{n\in\mathbb{N}}$ is contained in a subset of X with compact closure, and therefore admits a convergent subsequence in X.

Conversely, suppose that the injection ι is compact and let $\lambda \in \rho(A)$. Since

$$-A(\lambda I - A)^{-1} = (\lambda I - A + \lambda I)(\lambda I - A)^{-1} = I + \lambda(\lambda I - A)^{-1},$$

it follows that

$$\begin{aligned}\|(\lambda I - A)^{-1}x\|_A &= \|(\lambda I - A)^{-1}x\|_X + \|x + \lambda(\lambda I - A)^{-1}x\|_X \\ &\leq \big(1 + (1 + |\lambda|)\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)}\big)\|x\|_X\end{aligned}$$

for all $x \in X$. Hence $(\lambda I - A)^{-1} \in \mathcal{L}\big(X, (D(A), \|\cdot\|_A)\big)$ and it follows that $(\lambda I - A)^{-1} = \iota \circ (\lambda I - A)^{-1}\colon X \to X$ is compact. $\square$

Hilbert spaces

Proposition A.6. *Let H be a Hilbert space and let $T, T_1, T_2, \ldots \in \mathcal{L}(H)$. Suppose that $\lim T_n = T$ strongly and let $K \in \mathcal{L}(H)$ be compact. Then $\lim T_n K = TK$ uniformly.*

Proof. Suppose to the contrary that there exists a $\delta > 0$ such that for each $N \in \mathbb{N}$, there exist $n \geq N$ and $x \in H$ with $\|x\| \leq 1$ such that $\|(T - T_n)Kx\| \geq \delta$. Then there exist $n_1 < n_2 < \ldots$ and a sequence $(x_k)_{k\in\mathbb{N}}$ in H such that $\|x_k\| \leq 1$ and $\|(T - T_{n_k})Kx_k\| \geq \delta$ for all $k \in \mathbb{N}$. Passing to a subsequence if necessary, we may assume that $(x_k)_{k\in\mathbb{N}}$ converges weakly in H. Then since K is compact, there exists a $y \in H$ such that $\lim Kx_k = y$. So

$$\begin{aligned}\|(T - T_{n_k})Kx_k\| &\leq \|(T - T_{n_k})y\| + \|(T - T_{n_k})(Kx_k - y)\| \\ &\leq \|(T - T_{n_k})y\| + \|T - T_{n_k}\|_{\mathcal{L}(H)}\|Kx_k - y\|.\end{aligned}$$

By the uniform boundedness principle, the sequence $(T_{n_k})_{k\in\mathbb{N}}$ is bounded in $\mathcal{L}(H)$. Hence $\delta \leq \lim \|(T - T_{n_k})Kx_k\| = 0$, a contradiction. $\square$

Proposition A.7. *Let H_1 and H_2 be Hilbert spaces. Let $T, T_1, T_2, \ldots \in \mathcal{L}(H_1, H_2)$ and suppose that T is compact. Suppose that for all $x, x_1, x_2, \ldots \in H_1$ with $\lim x_n = x$ weakly in H_1, it follows that $\lim T_n x_n = Tx$ in H_2. Then $\lim T_n = T$ in $\mathcal{L}(H_1, H_2)$.*

Proof. Suppose to the contrary that there exists an $\delta > 0$ such that for each $N \in \mathbb{N}$, there exist $n \geq N$ and $x \in H_1$ with $\|x\|_{H_1} \leq 1$ such that $\|(T - T_n)x\|_{H_2} \geq \delta$. Then there exist $n_1 < n_2 < \ldots$ and a sequence $(x_k)_{k\in\mathbb{N}}$ in H_1 such that $\|(T - T_{n_k})x_k\|_{H_2} \geq \delta$ for all $k \in \mathbb{N}$. Since $(x_k)_{k\in\mathbb{N}}$ is bounded in H_1, there exist a subsequence $(x_{k_l})_{l\in\mathbb{N}}$ of $(x_k)_{k\in\mathbb{N}}$ and a $y \in H_1$ such that $\lim x_{k_l} = y$ weakly in H_1. Then $\lim T_{n_{k_l}} x_{k_l} = Ty$ in H_2. Moreover, since T is compact it follows that $\lim T x_{k_l} = Ty$ in H_2. Then $\delta \leq \lim \|(T - T_{n_{k_l}})x_{k_l}\|_{H_2} = 0$, a contradiction. $\square$

The following product rule is well known.

Proposition A.8. *Let H be a Hilbert space. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2 = 2\,\mathrm{Re}\,\big(u'(t), u(t)\big)$$

for all $u \in C^1((0,\infty), H)$.

Proof. Observe that

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\big(\|u(t+h)\|^2 - \|u(t)\|^2\big) &= \lim_{h\to 0}\frac{1}{h}\Big(\big(u(t+h) - u(t), u(t)\big) + \big(u(t), u(t+h) - u(t)\big)\Big) \\
&= \big(u'(t), u(t)\big) + \big(u(t), u'(t)\big) \\
&= \big(u'(t), u(t)\big) + \overline{\big(u'(t), u(t)\big)} \\
&= 2\,\mathrm{Re}\,\big(u'(t), u(t)\big)
\end{aligned}$$

as required. $\square$

L_p-spaces

Proposition A.9. *Let X denote a measure space. Let $p, q, r \in [1, \infty]$ and $\theta \in [0,1]$ be such that $p \leq r \leq q$ and*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

Let $u \in L_p(X) \cap L_q(X)$. Then $u \in L_r(X)$ and the estimate

$$\|u\|_{L_r(X)} \leq \|u\|_{L_p(X)}^{\theta}\|u\|_{L_q(X)}^{1-\theta}$$

is valid.

Proof. Without loss of generality we may assume that $\theta \in (0,1)$. Since $u \in L_p(X)$ it follows that

$$\Big(\int_X \big|\,|u|^{p\frac{r\theta}{p}}\big|^{\frac{p}{r\theta}}\Big)^{\frac{r\theta}{p}} = \|u\|_{L_p(X)}^{r\theta} < \infty.$$

So $|u|^{p\frac{r\theta}{p}} \in L_{\frac{p}{r\theta}}(X)$. One similarly deduces that $|u|^{q\frac{r(1-\theta)}{q}} \in L_{\frac{q}{r(1-\theta)}}(X)$. Then since $\frac{r\theta}{p} + \frac{r(1-\theta)}{q} = 1$, Hölder's inequality provides that

$$\int_X |u|^r = \int_X |u|^{p\frac{r\theta}{p}} |u|^{q\frac{r(1-\theta)}{q}} \leq \|u\|_{L_p(X)}^{r\theta} \|u\|_{L_q(X)}^{r(1-\theta)}$$

as required. □

Proposition A.10 (Riesz–Thorin). *Let X denote a σ-finite measure space and let $p_1, p_2, q_1, q_2 \in [1, \infty]$ and $\theta \in [0,1]$ be such that*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2} \quad \textit{and} \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Let $T \colon L_{p_1}(X) + L_{p_2}(X) \to L_{q_1}(X) + L_{q_2}(X)$ be such that $T|_{L_{p_1}(X)} \in \mathcal{L}\big(L_{p_1}(X), L_{q_1}(X)\big)$ and $T|_{L_{p_2}(X)} \in \mathcal{L}\big(L_{p_2}(X), L_{q_2}(X)\big)$. Then $T|_{L_p(X)} \in \mathcal{L}\big(L_p(X), L_q(X)\big)$ and the estimate

$$\|T\|_{p \to q} \leq \|T\|_{p_1 \to q_1}^{\theta} \|T\|_{p_2 \to q_2}^{1-\theta}$$

is valid.

Proof. See [LZ12] Theorem 3.16. □

Proposition A.11. *Let $U \subset \mathbb{R}^d$ be an open set. Let $u \in L_2(U)$ and suppose that there exists a $c > 0$ such that*

$$\Big| \int_U u\overline{v} \,\Big| \leq c\|v\|_{L_\infty(U)}$$

for all $v \in L_2(U) \cap L_\infty(U)$. Then $u \in L_1(U)$ and $\|u\|_{L_1(U)} \leq c$.

Proof. Suppose to the contrary that $\int_U |u| = \infty$. Then

$$\sup\Big\{ \int_K |u| : K \subset U \text{ is compact} \Big\} = \infty.$$

Write $\mathcal{T} = \{\chi \in C_c^\infty(U) : \chi \geq 0 \text{ and } \|\chi\|_{L_\infty(U)} \leq 1\}$. Then [HR79] Theorem 12.14 provides that

$$\sup_{\chi \in \mathcal{T}} \int_U |u|\chi = \infty.$$

Let $\chi \in \mathcal{T}$ and write $v = (\operatorname{sgn} u)\chi$. Then $v \in L_2(U) \cap L_\infty(U)$ and $\|v\|_{L_\infty(U)} \leq 1$, so

$$\int_U |u|\chi = \int_U u(\overline{\operatorname{sgn} u})\chi = \Big| \int_U u\overline{v} \Big| \leq c.$$

Therefore $\int_U |u|\chi \leq c$ for all $\chi \in \mathcal{T}$, a contradiction. Hence $u \in L_1(U)$. Moreover, [HR79] Theorem 12.13 provides that

$$\|u\|_{L_1(U)} = \sup\Big\{ \Big| \int_U u\overline{\chi} \Big| : \chi \in C_c^\infty(U) \text{ and } \|\chi\|_{L_\infty(U)} \leq 1 \Big\}$$

and it therefore follows from the hypothesis that $\|u\|_{L_1(U)} \leq c$. □

The following is a particular case of [EL17] Lemma 2.1.

Proposition A.12. *Let $U \subset \mathbb{R}^d$ be an open set and let $T \in \mathcal{L}\big(L_2(U)\big)$. Suppose that there exists a $c > 0$ such that $\|Tu\|_\infty \le c\|u\|_\infty$ for all $u \in L_2(U) \cap L_\infty(U)$. Then there exist unique operators $\widehat{T} \in \mathcal{L}\big(L_1(U)\big)$ and $\widetilde{T} \in \mathcal{L}\big(L_\infty(U)\big)$ such that $\widehat{T}|_{L_1 \cap L_2} = T^*|_{L_1 \cap L_2}$ and $\widetilde{T}|_{L_2 \cap L_\infty} = T|_{L_2 \cap L_\infty}$. Moreover, $\widetilde{T} = \big(\widehat{T}\big)^*$ and $\|\widetilde{T}\|_{\mathcal{L}(L_\infty(U))} \le c$.*

Proof. Let $u \in L_1(U) \cap L_2(U)$. Then

$$|(T^*u, v)| = |(u, Tv)| \le \|u\|_1 \|Tv\|_\infty \le c\,\|u\|_1 \|v\|_\infty$$

for all $v \in L_2(U) \cap L_\infty(U)$. Hence $T^*u \in L_1(U)$ and $\|T^*u\|_1 \le c\|u\|_1$ by Proposition A.11. Then $T^*|_{L_1 \cap L_2}$ is bounded from $(L_1(U) \cap L_2(U), \|\cdot\|_1)$ into $L_1(U)$ and since $L_1(U) \cap L_2(U)$ is dense in $L_1(U)$, it follows from Proposition A.3 that there exists a unique $\widehat{T} \in \mathcal{L}\left(L_1(U)\right)$ such that $\widehat{T}|_{L_1 \cap L_2} = T^*|_{L_1 \cap L_2}$.

Define $\widetilde{T} \colon L_\infty(U) \to L_\infty(U)$ by $\widetilde{T} = \big(\widehat{T}\big)^*$. Let $u \in L_2(U) \cap L_\infty(U)$. Then

$$\langle \widetilde{T}u, v\rangle = \langle u, \widehat{T}v\rangle \ = \langle u, T^*v\rangle \ = \langle Tu, v\rangle$$

for all $v \in L_1(U) \cap L_2(U)$. So $\langle \widetilde{T}u, v\rangle = \langle Tu, v\rangle$ first for all $v \in L_1(U) \cap L_2(U)$ and then for all $v \in L_1(U)$ by density. Hence $\widetilde{T}|_{L_2 \cap L_\infty} = T|_{L_2 \cap L_\infty}$.

We now show that the operator $\widetilde{T}$ is unique. Let $\overline{T} \in \mathcal{L}\left(L_\infty(U)\right)$ and suppose that $\overline{T}|_{L_2 \cap L_\infty} = T|_{L_2 \cap L_\infty}$. Let $u \in L_\infty(U)$. Since $L_2(U) \cap L_\infty(U)$ is w^*-dense in $L_\infty(U)$, there exists a net $(u_\alpha)_{\alpha \in I}$ in $L_2(U) \cap L_\infty(U)$ such that $\lim u_\alpha = u$ in $(L_\infty(U), w^*)$. Then

$$\langle \overline{T}u, v\rangle = \lim_{\alpha \in I} \langle \overline{T}u_\alpha, v\rangle = \lim_{\alpha \in I} \langle Tu_\alpha, v\rangle = \lim_{\alpha \in I} \langle \widetilde{T}u_\alpha, v\rangle = \langle \widetilde{T}u, v\rangle$$

for all $v \in L_1(U)$, so $\overline{T}u = \widetilde{T}u$ and $\widetilde{T}$ is consequently unique. Finally, let $u \in L_\infty(U)$. Then

$$|\langle \widetilde{T}u, v\rangle| = |\langle u, T^*v\rangle| \le c\,\|u\|_\infty\,\|v\|_1$$

for all $v \in L_1(U) \cap L_2(U)$. So $|\langle \widetilde{T}u, v\rangle| \le c\,\|u\|_\infty\,\|v\|_1$ first for all $v \in L_1(U) \cap L_2(U)$ and then for all $v \in L_1(U)$ by density. Therefore $\|\widetilde{T}\|_{\mathcal{L}(L_\infty(U))} \le c$ as required. □

C_0-semigroups

Proposition A.13. *Let $T = (T_t)_{t>0}$ be a C_0-semigroup on a Banach space X with generator $-A$. Let $t > 0$. Then $\int_0^t T_s x \, \mathrm{d}s \in D(A)$ and*

$$A \int_0^t T_s x \, \mathrm{d}s = x - T_t x$$

for all $x \in X$.

Proof. Let $x \in X$ and $h > 0$. Then

$$\begin{aligned}\frac{1}{h}(I - T_h) \int_0^t T_s x \, \mathrm{d}s &= \frac{1}{h}\Big(\int_0^t T_s x \, \mathrm{d}s - \int_0^t T_{s+h} x \, \mathrm{d}s\Big)\\ &= \frac{1}{h}\Big(\int_0^t T_s x \, \mathrm{d}s - \int_h^{t+h} T_s x \, \mathrm{d}s\Big)\end{aligned}$$

$$= \frac{1}{h}\Big(\int_0^t T_s x \,\mathrm{d}s - \Big(\int_0^t T_s x \,\mathrm{d}s + \int_t^{t+h} T_s x \,\mathrm{d}s - \int_0^h T_s x \,\mathrm{d}s \Big) \Big)$$
$$= \frac{1}{h}\Big(\int_0^h T_s x \,\mathrm{d}s - \int_t^{t+h} T_s x \,\mathrm{d}s \Big)$$
$$= \frac{1}{h}\Big(\int_0^h T_s x \,\mathrm{d}s - T_t \int_0^h T_s x \,\mathrm{d}s \Big)$$
$$= (I - T_t) \frac{1}{h} \int_0^h T_s x \,\mathrm{d}s.$$

Since the map $s \mapsto T_s x$ is continuous, it follows that

$$\lim_{h \downarrow 0} \frac{1}{h}(I - T_h) \int_0^t T_s x \,\mathrm{d}s = (I - T_t) \lim_{h \downarrow 0} \frac{1}{h} \int_0^h T_s x \,\mathrm{d}s = (I - T_t)x$$

as required. □

Proposition A.14. *Let $T = (T_t)_{t>0}$ be a C_0-semigroup on a Banach space X with generator $-A$. Let $x, y \in X$. Then $x \in D(A)$ and $Ax = y$ if and only if*

$$\int_0^t T_s y \,\mathrm{d}s = x - T_t x$$

for all $t > 0$

Proof. Suppose first that $x \in D(A)$ and $Ax = y$. Let $t > 0$. Then by the fundamental theorem of calculus

$$x - T_t x = \int_0^t -\frac{\mathrm{d}}{\mathrm{d}s} T_s x \,\mathrm{d}s = \int_0^t A T_s x \,\mathrm{d}s = \int_0^t T_s A x \,\mathrm{d}s = \int_0^t T_s y \,\mathrm{d}s$$

as required.

Conversely, one has that

$$\lim_{t \downarrow 0} \frac{1}{t}(I - T_t)x = \lim_{t \downarrow 0} \frac{1}{t} \int_0^t T_s y \,\mathrm{d}s = y$$

and the claim follows. □

The domain of the generator of a C_0-semigroup is maximal in the following sense.

Proposition A.15. *Suppose that $-A$ and $-B$ each generate a C_0-semigroup on a Banach space X. If $A \subset B$, then $A = B$.*

Proof. There exists an $\omega \in \mathbb{R}$ such that $\{\lambda \in \mathbb{C} : \operatorname{Re} \lambda > \omega\} \subset \rho(-A) \cap \rho(-B)$. Choose $\lambda \in \rho(-A) \cap \rho(-B)$. Then $\lambda I + A \subset \lambda I + B$. Let $x \in D(B)$. Since $\lambda I + A$ is surjective, there exists a $y \in D(A)$ such that $(\lambda I + A)y = (\lambda I + B)x$. By assumption $D(A) \subset D(B)$, so $y \in D(B)$ and

$$(\lambda I + B)y = (\lambda I + A)y = (\lambda I + B)x.$$

Hence $(\lambda I + B)(x - y) = 0$. Since $\lambda I + B$ is injective, it follows that $x = y \in D(A)$. □

Proposition A.16. *Let $U \subset \mathbb{R}^d$ be an open set and suppose that $-A$ generates an irreducible C_0-semigroup on $L_2(U)$. Suppose that $u \in D(A)$ with $u > 0$ is an eigenvector of A. Then $u(x) > 0$ for a.e. $x \in U$.*

Proof. See [BKR17] Proposition 14.12(a) and Example 10.16(b). □

The following particular case of [AB92] Theorem 1.3 was proved in [AE15] Proposition 5.13.

Proposition A.17. *Let $U \subset \mathbb{R}^d$ be an open set and let A and B be two lower-bounded self-adjoint operators in $L_2(U)$ with compact resolvent. Suppose that $(e^{-tA})_{t>0}$ is irreducible and that*

$$0 \leq e^{-tA}u \leq e^{-tB}u \tag{33}$$

for all $u \in L_2(U)$ with $u \geq 0$ and all $t > 0$. Suppose further that the smallest eigenvalues of A and B are equal. Then $A = B$.

Proof. Without loss of generality we may assume that the smallest eigenvalues of A and B are zero. Since by hypothesis $(e^{-tA})_{t>0}$ and $(e^{-tB})_{t>0}$ are positive and A and B have compact resolvent, it follows from the Krein–Rutman theorem [BKR17] Theorem 12.15 that there exist $u_1, u_2 \in L_2(U)$ with $u_1, u_2 > 0$ such that $u_1 \in \ker A$ and $u_2 \in \ker B$. Hence $e^{-tA}u_1 = u_1$ and $e^{-tB}u_2 = u_2$ for all $t > 0$, by Proposition A.14. Moreover, since $(e^{-tA})_{t>0}$ is irreducible, it follows from (33) together with [Ouh05] Theorem 2.9 that $(e^{-tB})_{t>0}$ is irreducible. So $u_1(x) > 0$ and $u_2(x) > 0$ for a.e. $x \in U$, by Proposition A.16.

Let $t > 0$. Then $e^{-tB}u_1 - e^{-tA}u_1 \geq 0$ by (33). Moreover, it follows from the self-adjointness of the generators that the semigroups consist of self-adjoint operators, so

$$(e^{-tB}u_1 - e^{-tA}u_1, u_2)_{L_2(U)} = (u_1, e^{-tB}u_2)_{L_2(U)} - (e^{-tA}u_1, u_2)_{L_2(U)} = 0.$$

Since $u_2(x) > 0$ a.e. on U, it follows that $e^{-tB}u_1 - e^{-tA}u_1 = 0$. Let $u \in L_2(U)$ be such that $u > 0$. Then $e^{-tB}u - e^{-tA}u \geq 0$ and

$$(e^{-tB}u - e^{-tA}u, u_1)_{L_2(U)} = (u, e^{-tB}u_1 - e^{-tA}u_1)_{L_2(U)} = 0.$$

Hence $e^{-tB}u - e^{-tA}u = 0$ and one deduces that $e^{-tA}u = e^{-tB}u$ for all $u \in L_2(U)$ with $u \geq 0$ and all $t > 0$. Then $(e^{-tA})_{t>0} = (e^{-tB})_{t>0}$ by linearity and it follows that $A = B$. □

Proposition A.18. *Let $U \subset \mathbb{R}^d$ be an open set and let A be a self-adjoint operator in $L_2(U)$ with compact resolvent. Suppose that $-A$ generates a positive irreducible C_0-semigroup on $L_2(U)$. Let λ denote an eigenvalue of A corresponding to an eigenfunction $u \in D(A)$ of A with $u > 0$. Then λ is the smallest eigenvalue of A.*

Proof. Let λ_1 denote the smallest eigenvalue of A. Without loss of generality we may assume that $\lambda_1 = 0$. So $\lambda \geq 0$. Since $(e^{-tA})_{t>0}$ is positive and A has compact resolvent, it follows from the Krein–Rutman theorem that there exists a $u_1 \in \ker A$ with $u_1 > 0$. Moreover, since $(e^{-tA})_{t>0}$ is irreducible Proposition A.16 provides that $u_1(x) > 0$ for a.e. $x \in U$. Then $(u, u_1)_{L_2(U)} > 0$ and

$$\lambda(u, u_1)_{L_2(U)} = (Au, u_1)_{L_2(U)} = (u, Au_1)_{L_2(U)} = 0,$$

so $\lambda = 0$. □

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