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Impact of Transmission on Strategic Behaviour in Electricity Markets

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Abstract

In this thesis, we investigate the impact of transmission on the strategic behaviour of firms competing in deregulated electricity wholesale markets. Assuming uniform-price auctions and locational marginal pricing, we first investigate the properties of the dispatch problem over networks that are constrained and/or lossy. Without loops and losses we derive important results as to how price varies as a function of demand at each node. Whereas, for networks with loops and losses, we discuss the non-convexity of the dispatch problem and show that the optimal value function may be non-convex.

We model the strategic firms as Cournot agents, and discuss how the assumptions surrounding the rationality of the agents can influence the equilibrium outcomes. Under a full-rationality assumption, we prove that over a lossless radial network, with firms owning single generators, the line capacities ensuring that a single-node Cournot equilibrium exists form a convex set. However, in the case of networks with loops or with firms owning multiple generators we provide counter-examples to this convexity.

We investigate the prices coming from a two-node network with a lossy line and find the conditions such that a generator is guaranteed to have a quasi-concave revenue function. We use this result to prove that there exists a pure-strategy equilibrium for a Cournot game over the same network.

In the final chapters we present some applications of this work. We first define a mixed-integer stochastic transmission planning model that uses the set of capacities derived earlier as constraints to ensure that the unconstrained equilibrium exists for all periods. Finally, we examine the effect of carbon charges over a two-node network and find that the imposition of a carbon charge may increase emissions when firms behave strategically over a constrained network.

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Contents

Part I

Background

Chapter 1

Introduction

Over the past two decades, countries and states around the world have deregulated their electricity sectors [9]; this has entailed the corporatisation or privatisation of many previously centrally owned and operated generation and transmission assets. What drove this liberalization is the belief that a competitive market model will deliver better decisions and outcomes than a centrally managed system. A key assumption for these markets to be effective is that the firms in the market would behave in the perfectly competitive manner (offering power at marginal cost) [9]. However, a number of studies have demonstrated that firms are not always behaving in this way (for example [18, 95]). This lack of perfect competition is regularly attributed to the exercise of *market power* (see [95]), which is often present in markets with a small number of large firms and high cost of entry. This market structure enables firms to behave in a strategic rather than competitive manner, which generally results in higher prices for electricity consumers.

One feature of electricity markets that differentiates them from markets for other goods is the presence of a transmission grid. Electricity must flow through this grid to reach customers, while complying with line capacity constraints and the physical laws that govern the power flow. The effect that these constraints can have on the behaviour of firms in the market is not well understood.

In this work, we are interested in the impact that the transmission grid has on the strategic behaviour of firms. We model the firms as Cournot agents, each aiming to maximize its own profit. If there were no transmission grid and all generators and demand were situated at a single node, then we could model this using the classical Cournot game (see [20]), where all producers (generators) and consumers (demand) see a single market price. However, in electricity markets, due to the transmission networks, locational pricing (e.g. nodal pricing) is often used; this allows more appropriate locational price signals to be observed in the market, but makes the modelling of the system more difficult.

With nodal pricing, the specific structure and properties of the transmission grid can affect the opportunities and incentives of the generators and therefore can alter the outcome of the market. Through this work we wish to develop a better understanding of how these transmission grid properties influence the behaviour of firms in the market. These insights may be useful for regulators when considering the impact of proposed grid upgrades on the competition in the market.

The deregulation of electricity markets began in the 1980s in South America. Since then decentralization of electricity sectors has occurred in many countries around the world [85]. Chile in 1982 was the first of these countries to break up their previously centrally managed electricity sector. Four years later in 1986 an *Independent System Operator* (CDEC) was established; from this point onwards generators have been required to submit their plants' marginal costs and availability every hour to the CDEC who determine how much electricity each plant should produce, and compute the marginal price for electricity.

The Chilean reforms were followed in 1990 by England and Wales who established an *electricity pool market* [2]. In this type of market structure, power is offered into the market by generation companies and the demand is satisfied by dispatching the generators in such a way so as to minimize the total cost. From the inception of such a pool market in England and Wales, there was significant criticism about the way the generation companies were structured and the presence of market power. Since this time England and Wales have restructured their electricity market on two further occasions. Norway established its electricity pool market in 1991; since then Sweden, Denmark and Finland have joined forming the *Nord Pool* market.

Brazil began reforms of their power sector in 1996; these reforms were forced, due to there being insufficient funds to pay for the construction of additional capacity to ensure that demand could be met. The reforms aimed to make a competitive power sector with participation from the private sector. In 2004, due to criticisms associated with the previous reforms, the Brazilian electricity system became fully-regulated, with the day-to-day operation of the system centrally managed and auctions held every 3-5 years for long-term electricity contracts to ensure that there is sufficient capacity to meet demand growth.

In the mid 1990s New Zealand and the Australian states of Victoria and New South Wales each began deregulation of their electricity sectors and moved to establish electricity pool markets. Many states in the US soon followed; for example the Pennsylvania-New Jersey-Maryland Interconnection

(PJM), California and New England.

There are some key differences in the design of each of the above markets. These markets have evolved based on the particular requirements of each country, and the lessons learned from electricity markets in other regions. Governments and regulators are tasked with ensuring that these markets deliver the best outcomes for consumers and the economy as a whole. These regulators are able to learn from past experiences from their electricity market or other markets around the world to develop policies which will deliver the desired results. Some of the markets are *energy-only markets* (for example: New Zealand, England and Wales and the Chilean Market); this means that all the generators must recover their capital costs solely from the the power they sell. Other regions provide both energy markets and *capacity markets* (such as PJM and the New England market) [21]; here *capacity-payments* are made to cover the fixed costs of ensuring there is sufficient generation capacity [75].

Since thermal plants often have lengthy start-up and shut-down times, unit commitment is an important feature of electricity markets. In some markets, such as PJM and the New York market this unit commitment is centrally managed (see [89]), whereas in the NZEM and Australian market this is managed internally by the generators.[1]

Another difference between the above markets is the way prices are determined. The two most common ways of pricing power in a pool market are *nodal* and *zonal* pricing. Of the above markets, the NZEM, PJM, and Chilean market all have nodal pricing; this means that the price of (buying or selling) electricity in the wholesale market can differ between any nodes depending on the marginal cost of electricity (including any transport costs due to losses or congestion). Other markets, such as the Australian and Nord Pool markets, have zonal pricing schemes where the grid is divided into a number of zones within which there is a single zonal price.

In this work we will restrict our attention to pool markets with nodal pricing, as is used in the New Zealand wholesale electricity market. Thus we will give a brief background of the New Zealand electricity market and how it is operated. Until 1987, New Zealand's electricity sector was centrally managed by the Ministry of Energy; the generation, transmission investment was run by a government department and the distribution and retailing of electricity was run by locally elected power boards. Decisions on investments in new generation capacity were often political, rather than economic; see Evans and Meade [66].

In 1987, the government at that time started New Zealand on a gradual process of deregulation.

[1] In this thesis, we consider one-shot games, and in this context the importance of unit commitment is limited.

The first stage was to corporatise the New Zealand Electricity Department to become the Electricity Corporation of New Zealand (ECNZ) [9]. It was not until 1994 that any more significant changes occurred, when a new transmission company, Transpower, was established. In 1996, Contact Energy, a new state-owned generation company was formed to compete against ECNZ using existing assets, and later that year a reformed wholesale electricity market was established. In 1999 ECNZ was disestablished and three generating companies were formed: Mighty River Power, Genesis Power and Meridian Energy [9].

After a referendum of self-management of the industry was defeated, the Electricity Commission was established in 2003. Its duties were to oversee the rules and regulation of the market, and oversee Transpower's grid expansion investments.

In its current state, the New Zealand electricity market (NZEM) consists of six main sectors: generation, transmission, distribution, retail, operation and regulation; we discuss these below.

The generation of New Zealand's electricity is dominated by five main generation companies. These consist of three state-owned enterprises: Genesis, Mighty River Power and Meridian; and two privately-owner generators: Contact Energy and Trust Power. According to the New Zealand Ministry of Economic Development [69], in 2008 just over 42,000 GWh of electricity was produced in New Zealand. The five main companies listed above accounted for 94% of this generation.

Examining the fuels used to produce this power: hydro accounted for 52.3% of generation; gas: 23.7%; coal: 10.5%; geothermal: 9.4%; wind: 2.5%; with the remaining technologies each producing less than 1% of the total generation. Looking at these figures, we can see that the New Zealand electricity sector has some key characteristics: it is dominated by a small number of generation firms; and the production is dominated by hydro and gas generation.

Transmission lines are the high-voltage lines which transfer large quantities of power over great distances. Transpower owns and maintains all of New Zealand's transmission assets. Transpower's profits are highly regulated so as to avoid monopolistic behaviour, as discussed by Green in [41]. Since Transpower's profits are linked to their assets' values, before any major investments in transmission are approved they are required to pass the *Grid Investment Test* (GIT) [28]. This entails various alternatives being submitted to the Electricity Commission with a detailed cost-benefit analysis of each. The Electricity Commission then approves the option it deems to be in the best interests of New Zealand.

Distribution lines are local power lines connecting a sub-station to electricity customers. These lines are generally owned by a local distribution company. These companies typically charge residential customers based on their total usage; however for large commercial or industrial customers they have a more complicated tariff structure in place, whereby customers are charged based on a combination of their total usage and peak demand or connection capacity [93].

Electricity retailers typically sell power to consumers at fixed prices, although the prices are periodically adjusted to reflect the long term trends in the spot market. In New Zealand, vertical integration of generation and retail is common because this reduces the risk each company would otherwise face. This is particularly important in New Zealand where year to year changes in lake inflows can drastically affect wholesale electricity prices. For example, during a dry year a retailer without generation assets or contracts would have to purchase electricity from the spot market at high prices, but would then be forced to sell it at a loss to consumers. In fact in 2001, after a prolonged period of high electricity spot prices, one New Zealand retailer (On Energy) was forced to exit the electricity retail business due to inadequate hedge contracts [9].

The operation of the NZEM is controlled by Transpower. Each of the generation companies must submit an offer stack of five tranches of energy (a quantity of electricity at a price) for each half-hour period of the day. The offers for a particular period can be changed up until two hours before that period starts [17]. Transpower solves a *scheduling, pricing and dispatch* (SPD) problem that dispatches the plants with the cheapest offers of electricity in order to meet the demand at minimum cost (see [4]). This optimization problem also defines the optimal prices for electricity at each node in the grid (locational marginal pricing). We discuss the dispatch problem in detail in part II.

In the remainder of part I, we discuss some fundamental concepts of game theory and the computation of a Cournot equilibria; then we discuss electricity markets in more detail and explain how transmission networks can be modelled and how electricity markets are structured.

In part II we present the dispatch problem and discuss how it can be solved; here we are particularly interested in how nodal prices vary as a function of demand at each node. In the case of radial transmission networks we prove some useful mathematical results about how the nodal price varies as a function of demand. Finally, we examine the impact that transmission losses have on the convexity of the dispatch problem.

In part III we introduce the Cournot game over a network and present different ways of modelling the game depending on the assumptions made about the rationality of the generators. We then detail an algorithm that allows us to compute a set of conditions on the properties of the transmission grid to ensure the validity of certain outcomes to the Cournot game. In the last chapter of this part we examine how transmission losses can affect the behaviour of firms in a Cournot game.

In part IV we present some applications of this work. First we look at expansions of the transmission grid and give an example based on the New Zealand market. Then we look at how this work can be useful for modelling the outcomes of changes to energy policy; specifically, we examine a possible outcome of an imposition of a carbon tax.

Chapter 2

Game theory

In this chapter, we will introduce some basic concepts of game theory. First we will define a game, and discuss different ways a game can be structured. We finish with a simple Cournot game and demonstrate how an equilibrium can be computed.

2.1 Definition of a game

A game is a model of how a collection of rational agents may behave in a given setting. Game theoretic models are used to attempt to understand and predict how people or organizations might act in particular situations, as discussed by Fudenburg and Tirole in [32].

When setting up a game, we must first determine the relationship between the agents. In a *cooperative game*, the agents can form binding agreements, whereas in a *non-cooperative game* the agents' decisions are independent. In this work we focus solely on non-cooperative games, which we will apply to model situations where there are multiple agents each acting in their own interests.

A game therefore consists of a set of agents, each agent, i, choosing some strategy, s_i, from its strategy space, $\mathcal{S}_i$. The payoff for agent i is $P_i(s)$, where s is a vector of all agents' strategies.

The final aspect which must be modelled is how the game is played. In a two agent game, if the agents both choose a strategy simultaneously, then the game is simultaneous; otherwise the game is sequential. For larger numbers of agents it is possible to have a mixture of the above game types. For example, for an electricity spot market, modelled as a game, the generators act simultaneously to choose their offers[1], and a system operator then chooses the dispatch quantities

[1] Although, they may change their offers many times before gate closure.

and nodal prices.

2.2 Nash equilibrium

Once a game is defined, it can be used to predict a possible outcome or gain intuition into the nature of the agents' interactions. Under the assumption that all agents are rational and have perfect information, a Nash equilibrium is a point in the joint strategy space of all the agents where no agent can improve its payoff, $P_i(s)$, by unilaterally changing its strategy. That is, each agent, assuming all other agents are fixed, cannot alter its action to increase its payoff.

2.3 Cournot game

The Cournot game, introduced by Cournot in [20], is a non-cooperative game whereby firms in an oligopoly simultaneously choose quantities of some product to produce. The relationship between the market price, p, and demand is determined from a smooth and continuous decreasing demand curve, $D(p)$.

In a Cournot game the firms are assumed to act rationally and also believe that all other firms behave in the same manner. Moreover, each firm, i, chooses its production quantity, q_i, with the aim of maximizing its profit, ρ_i. This is given by its revenue ($q_i \times p$) less some positive, increasing and convex production cost: $C_i(q_i)$.

Consider an example where there are n firms in a particular market. If there is a known demand curve of the form:

$$D(p) = a - bp,$$

where a, $b > 0$, then we can seek a Nash equilibrium for this game as follows. We first find the inverse demand curve; this gives the market price as a function of the firms' production quantities as shown in equation (2.1).

$$\begin{aligned} & \sum_{i=1}^{n} q_i = D\left(p\right), \\ \Rightarrow \quad & \sum_{i=1}^{n} q_i = a - bp, \\ \Rightarrow \quad & p = \frac{a}{b} - \frac{1}{b}\sum_{i=1}^{n} q_i. \end{aligned} \tag{2.1}$$

Since the firms are independent profit maximizers, we can compute firm j's optimal production quantity, q_j^*, as a function of the other firms' decisions. We first observe that firm j's profit function is concave in q_j, as shown below:

$$\rho_j(q) = q_j p - C_j(q_j) \tag{2.2}$$

$$= q_j\left(\frac{a}{b} - \frac{1}{b}\sum_{i=1}^{n} q_i\right) - C_j(q_j). \tag{2.3}$$

For the purposes of illustration, we will ignore upper and lower bounds on production decisions. Hence the maximum profit for firm j can be found from the first-order condition of equation (2.3), given by equation (2.4).

$$\left.\frac{\partial \rho_j}{\partial q_j}\right|_{q_j=q_j^*} = 0, \tag{2.4}$$

$$\Rightarrow \frac{a}{b} - \frac{1}{b}\sum_{i=0,i\neq j}^{n} q_i - \frac{2}{b}q_j^* - \left.\frac{dC_j}{dq_j}\right|_{q_j=q_j^*} = 0,$$

$$\Rightarrow q_j^* = \frac{1}{2}\left(a - \sum_{i=0,i\neq j}^{n} q_i - b\left.\frac{dC_j}{dq_j}\right|_{q_j=q_j^*}\right). \tag{2.5}$$

Equation (2.5) gives a best response function for firm j; this function gives the optimal decision for firm j as a function of all the other firms' production decisions; see [36].

2.3.1 Nash-Cournot equilibrium

Since this Cournot game is simultaneous, all the firms choose a production quantity at the same time. Moreover, since the firms are rational, they anticipate that other firms will respond to their production decisions so as to maximize their own profit. Recall that a Nash equilibrium is a point in the game's decision space where no firm is able to increase its profit by changing only its quantity. This point can be found by simultaneously enforcing all firms' first-order conditions, which is equivalent to solving the following system of simultaneous equations:

$$q_j^e = \frac{1}{2}\left(a - \sum_{i=0,i\neq j}^{n} q_j^e - b\left.\frac{dC_j}{dq_j}\right|_{q_j=q_j^e}\right), \quad j = 1...n, \tag{2.6}$$

where q^e is the vector of production decisions such that no firm has incentive to deviate. Note that, however, we are not guaranteed that there will exist a solution to the above system of equation for all demand functions. To deal with this we increase the strategy space of the firms to allow for mixed strategies as discussed in the next section.

2.3.2 Mixed strategy equilibria

At this point we draw a distinction between pure strategies (as presented above) and mixed strategies. A pure strategy is simply a fixed strategy with no random element; however with a mixed strategy, a firm randomly selects its strategy from either a discrete or continuous probability distribution over the set of pure strategies. Note that for a mixed strategy to be optimal all strategies with positive probability must yield the same payoff, otherwise the firm could improve its payoff by reducing the probability of choosing a less profitable strategy [94].

When no pure strategy equilibrium exists, it is possible to use mixed strategies to compute an equilibrium. Mixed strategy equilibria are much more difficult to compute and the predictive power of such equilibria is questioned [90]. We will examine mixed strategy equilibria in the context of electricity markets in chapter 13.

We will now present a small example to illustrate the computation of a (pure strategy) Nash-Cournot equilibrium.

2.3.3 Example

Consider two firms, with production costs: $C_i(q_i) = c_i q_i$ for $i = 1, 2$. Here the best response functions, given in (2.5), reduce to:

$$q_1^* = \tfrac{1}{2}\left(a - q_2 - bc_1\right), \quad q_2^* = \tfrac{1}{2}\left(a - q_1 - bc_2\right). \tag{2.7}$$

These functions are plotted in figure 2.1 below. The point where the two curves intersect is the Nash equilibrium, since this is where both firms are simultaneously optimizing against each others' decisions.

We could also compute this point algebraically, giving the following Nash equilibrium:

$$q_1^e = \tfrac{1}{3}\left(a - 2bc_1 + bc_2\right), \quad q_2^e = \tfrac{1}{3}\left(a - 2bc_1 + bc_2\right),$$

with an equilibrium market price of:

$$\begin{aligned} p^e &= \frac{a - q_1^e - q_2^e}{b} \\ &= \frac{a + b\left(c_1 + c_2\right)}{3b}. \end{aligned}$$

Note how the equilibrium changes as we alter the marginal cost of each of the firms. If the marginal cost of a single firm is increased, then that firm's production decreases, whereas both its competitor's production and the market price increase.

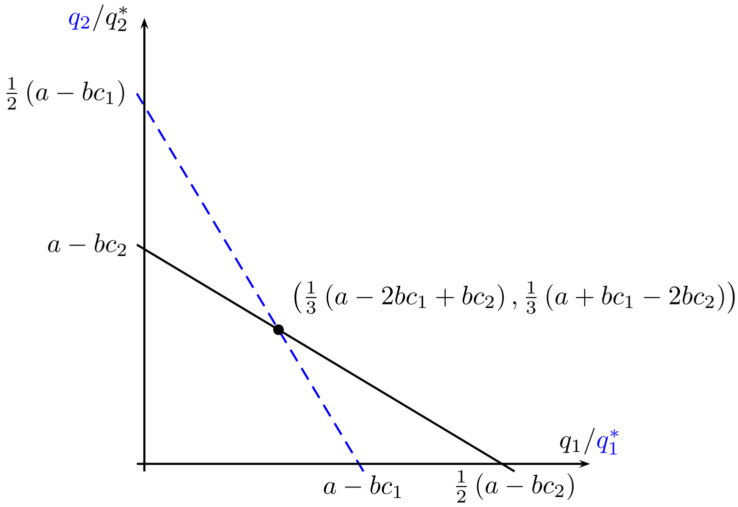

Figure 2.1: Best response functions for two player Cournot game with constant marginal costs.

In this work we extend these basic concepts of game theory to electricity markets. In the electricity market games that we examine, the firms' profit functions are more complex and there is a transmission grid yielding location-based prices. In the next chapter we introduce some fundamental considerations when modelling electricity markets, and later in chapter 7 we provide a background of game theoretic approaches to modelling competition in electricity markets.

Chapter 3

Electricity markets

In this chapter, we will discuss some characteristics of electric power systems and how electricity markets are structured. In the first section, we discuss various power generation technologies, giving pros and cons of different plant and fuel types. We also derive the DC load-flow model to approximate how AC power is routed through a transmission grid. In the subsequent sections, we examine how electricity markets can be structured, and how the market operates in New Zealand.

3.1 Power systems

In this section, we will give an overview of different power generation technologies, and then show how power flow is modelled through a transmission grid.

3.1.1 Power generation

Electricity can be generated from many different technologies, each having their own advantages and disadvantages. Throughout the first half of the twentieth century, the electricity needs for most countries were supplied from thermal plants burning fossil-fuels (coal, gas or oil) or hydroelectric plants [14].

Technologies

Below we will summarise the advantages and disadvantages of the major electricity production technologies.

Thermal In a thermal plant power is produced from steam as flows through a turbine. There are a variety of fuels that can be used to produce the steam. The most common of which are:

Coal Coal-fired power stations are common because of relatively cheap fuel costs (due to the abundance of coal). However, due to its high carbon content, coal plants are facing large cost increases as emission reduction schemes come into force. Furthermore, coal plants are relatively inflexible due to their slow ramp rates, meaning that they are more suited as base-load generators, rather than peaking plants.

Geothermal Thermal plants using heat from geothermal sites are very cheap to run. However the capacity and number of such plants are limited by the prevalence of suitable sites.

Nuclear Nuclear power plants appear to be regaining popularity as emissions reductions are sought. However, the strict safety measures required when developing such a plant means that the capital costs are high (and somewhat independent of the size of the plant); although once the plant is operational, the short-run marginal costs are very low. Hence nuclear plants need to have a large capacity to be economically viable [14].

Gas turbine Gas turbine plants tend to be more flexible than the thermal plants above, and can be used as peaker power plants. Gas turbines can be configured as open-cycle or combined-cycle gas turbines. Open-cycle turbines are more flexible, but have lower efficiency (up to 40%), whereas combined-cycle plants use waste heat from the turbine to produce steam that drives a second turbine; this can improve the efficiency (up to 60%), however reduces the flexibility of the plant. A combined-cycle gas turbine plant produces a little more than a third of the CO_2 emissions per MW that a coal-fired thermal plant emits.

Hydroelectric Hydroelectric power is a major source of electricity in many South American countries as well as in New Zealand. The major advantage of this type of power station is that water used in these stations generally has no cost. Moreover, hydro plants are reasonably flexible, allowing them to adjust their output to meet changes in demand.

The major disadvantage is that a sufficient supply of water is not guaranteed. In the event of a severe drought, there may not be enough water stored in the dams to provide a continuous supply of electricity. This is of particular concern in those countries where hydro power makes up a significant proportion of the total electricity supply; for example in New Zealand, there was a severe drought in 1992 that led to lake levels that were much lower than usual. Due to New Zealand's reliance on hydro power for a significant proportion of its generation, additional LPG generators as well as a conservation campaign were necessary to ensure that rolling-blackouts were avoided [31].

Wind Due to increased uncertainty in future fossil-fuel prices and planned restrictions on CO_2 emissions, there has been renewed interested in wind power over the last ten years [14]. The major advantage of wind power is that the marginal cost of the plant is zero. Due to its low cost, it can offset generation from thermal plants, thereby reducing emissions and the cost of electricity.

The biggest disadvantage of wind turbines is that their generation is uncertain, thus they cannot be relied on to meet peak demand. Therefore to ensure that demand can be met, backup peaking plants will be built (as opposed to, for example, baseload thermal in the 'no wind' case). This backup capacity is very costly, and it could possibly be uneconomical, since it may rarely be required [62].

Furthermore, generation from wind farms can fluctuate significantly; as wind becomes a greater proportion of the electricity supply, these fluctuations can become much more difficult to manage, leading to potential instabilities in the system, if more robust systems are not put in place to handle these fluctuations.

3.1.2 Transmission grid

An electricity grid consists of buses (nodes) and lines (links between nodes). Generators are situated at buses referred to as grid injection points (GIPs), and demand is situated at buses referred to as grid exit points (GXPs). These buses are connected by high-voltage lines which each have reactance and resistance properties and a thermal limit.

AC Power System

Electricity systems use *alternating current* (AC) power; this allows the voltage to be stepped-up easily for transmission over large distances and conversely stepped-down for distribution to consumers. Here we will focus on the power flow in the transmission network. The flow of power between buses connected to the transmission grid is determined based on the physical laws governing power flow and the properties of the transmission lines. Each transmission line has a resistance

r and reactance x, these give an impedance of $z = r + jx$, where $j = \sqrt{-1}$. Hence, we can compute the admittance of a line connecting buses and i and k to be

$$\frac{1}{z_{ik}} = \frac{1}{r_{ik} + jx_{ik}} = \frac{r_{ik} - jx_{ik}}{{r_{ik}}^2 + {x_{ik}}^2}.$$

To solve for the AC power flow over a (directed) transmission network with lines $ik \in \mathcal{A}$, we can define an admittance matrix: $Y = G + jB$, with elements G_{ik} and B_{ik} defined as follows:

$$G_{ik} = \begin{cases} -\frac{r_{ik}}{{r_{ik}}^2+{x_{ik}}^2}, & ik \in \mathcal{A} \; or \; ki \in \mathcal{A}, \\ \sum\limits_{n, in \in \mathcal{A}} \frac{r_{in}}{{r_{in}}^2+{x_{in}}^2}, & i = k, \\ 0, & \text{otherwise}, \end{cases}$$

$$B_{ik} = \begin{cases} \frac{x_{ik}}{{r_{ik}}^2+{x_{ik}}^2}, & ik \in \mathcal{A} \; or \; ki \in \mathcal{A} \\ -\sum\limits_{n, in \in \mathcal{A}} \frac{x_{in}}{{r_{in}}^2+{x_{in}}^2}, & i = k. \\ 0, & \text{otherwise}. \end{cases}$$

The non-linear equations (3.1), below, equate the net-power injected at a bus to the voltages, V, and voltage-angles, θ, at connected buses in the transmission network [82]. These equations are non-linear and the power consists of both *active* power, P, and *reactive* power, Q.

$$\begin{aligned} P_i &= \sum_{k=1}^{N} |V_i|\,|V_k|\,(G_{ik}\cos(\theta_i - \theta_k) + B_{ik}\sin(\theta_i - \theta_k)) \\ Q_i &= \sum_{k=1}^{N} |V_i|\,|V_k|\,(G_{ik}\sin(\theta_i - \theta_k) - B_{ik}\cos(\theta_i - \theta_k)) \end{aligned} \tag{3.1}$$

Due to the non-linearity of these equations, iterative methods such as the Newton-Raphson method are employed to converge to a solution [82]. However, when studying the economics of power-systems, an approximation of these equations is often used; we discuss this in the following section.

DC load-flow approximation

In this work, as we are dealing with electricity markets, rather than the real-time issues of operating a transmission network, we use an approximation of the AC model; this is referred to as a *DC load-flow* approximation. This changes the sinusoidal non-linear equations (3.1) into quadratic equations (or linear equations if transmission losses are ignored) by assuming that the voltages at all buses are equal and voltage-angle differences are small. To convert the AC power equations into the lossless DC load-flow equations, four approximations are used [82, 97]. We first ignore reactive power, leaving us with only the active power equations:

$$P_i = \sum_{k=1}^{N} |V_i|\,|V_k|\,(G_{ik}\cos(\theta_i - \theta_k) + B_{ik}\sin(\theta_i - \theta_k)).$$

We then assume that all bus voltage magnitudes are equal to some arbitrary value; in this case we set $|V_i| = 1$, $\forall i$:

$$P_i = \sum_{k=1}^{N} \left(G_{ik} \cos\left(\theta_i - \theta_k\right) + B_{ik} \sin\left(\theta_i - \theta_k\right)\right).$$

We now assume that the voltage-angle differences are small. Using Taylor series expansions, we can approximate the trigonometric functions as:

$$\begin{aligned} \sin(\alpha) &\approx \alpha, \\ \cos(\alpha) &\approx 1 - \frac{1}{2}\alpha^2, \end{aligned}$$

giving

$$P_i = \sum_{k=1}^{N} \left(G_{ik}\left(1 - \frac{1}{2}\left(\theta_i - \theta_k\right)^2\right) + B_{ik}\left(\theta_i - \theta_k\right)\right).$$

Finally, we must assume that for each line, ik, $r_{ik} << x_{ik}$ this yields $G_{ik} \approx 0$ and $B_{ik} \approx \frac{1}{x_{ik}}$, giving

$$P_i = \sum_{k=1}^{N} \left(\frac{\theta_i - \theta_k}{x_{ik}}\right). \tag{3.2}$$

Equation (3.2) gives an approximation of the relationship between the voltage-angles and the power injections (or offtakes) at each bus. From this equation, we can determine that the power flow on any line ik is equal to:

$$f_{ik} = \frac{\theta_i - \theta_k}{x_{ik}}.$$

This linear relationship between power flow and voltage-angle allows these approximate power flow equations to be solved as a system of linear equations.

Suppose we had a network consisting of a set of nodes, $i \in \mathcal{N}$ and a set of directed arcs $ij \in \mathcal{A}$ (linking nodes i and j). Furthermore, assume that at node i there is some injection of power q_i and some demand d_i and a power flow of f_{ij} on arc ij. For this situation, equation (3.2) becomes:

$$\begin{aligned} &\sum_{j,ij\in\mathcal{A}} f_{ij} - \sum_{j,ji\in\mathcal{A}} f_{ji} = q_i - d_i, \quad && \forall i \in \mathcal{N}, \\ &\tfrac{1}{x_{ij}}\left(\theta_i - \theta_j\right) - f_{ij} = 0, && \forall ij \in \mathcal{A}. \end{aligned} \tag{3.3}$$

The first equation is a node balance equation, equating outflows less inflows to the net injection at the node. The second equation ensures that Kirchhoff's laws are complied with for any loops in the network.

Transmission losses

In the above DC load-flow model, the resistance was assumed to be much less than the reactance, which resulted in a lossless transmission network; this assumption can be relaxed slightly to give quadratic losses. To illustrate this we consider a two-bus setting. Here we approximate the power flow and losses over a line (with resistance r and reactance x) connecting the two buses, as analysed by Philpott in [78]. If we assume that $r << x$ this yields $G_{12} \approx -\frac{r}{x^2}$ (as opposed to 0 above) and $B_{12} \approx \frac{1}{x}$. Therefore the net power injected into bus 1 can be approximated as follows

$$\begin{aligned}
P_1 &= \frac{r}{x^2} + \left(-\frac{r}{x^2} \left(1 - \frac{1}{2} (\theta_1 - \theta_2)^2 \right) + \frac{\theta_1 - \theta_2}{x} \right) \\
&= \frac{r}{2x^2} (\theta_1 - \theta_2)^2 + \frac{\theta_1 - \theta_2}{x} \\
&= \frac{1}{2} r f^2 + f. \qquad (3.4)
\end{aligned}$$

Similarly for bus 2, we have

$$\begin{aligned}
P_2 &= \frac{r}{x^2} - \left(\frac{r}{x^2} \left(1 - \frac{1}{2} (\theta_2 - \theta_1)^2 \right) - \frac{\theta_2 - \theta_1}{x} \right) \\
&= \frac{r}{2x^2} (\theta_1 - \theta_2)^2 + \frac{\theta_2 - \theta_1}{x} \\
&= \frac{1}{2} r f^2 - f. \qquad (3.5)
\end{aligned}$$

From equations (3.4) and (3.5), we can see that if $f + \frac{1}{2} r f^2$ is injected into bus 1, the amount of power received at bus 2 is $f - \frac{1}{2} r f^2$. Therefore the approximate transmission losses are $r f^2$, which are proportional to the average flow on the line squared. We now extend this principle of line losses to the network that we discussed in the previous section; with quadratic losses, equations (3.3) become

$$\begin{aligned}
&\sum_{j, ij \in \mathcal{A}} \left(f_{ij} + \tfrac{1}{2} r_{ij} {f_{ij}}^2 \right) - \sum_{j, ji \in \mathcal{A}} \left(f_{ji} - \tfrac{1}{2} r_{ji} {f_{ji}}^2 \right) = q_i - d_i, \quad \forall i \in \mathcal{N}, \\
&\tfrac{1}{x_{ij}} (\theta_i - \theta_j) - f_{ij} = 0, \qquad \forall ij \in \mathcal{A}.
\end{aligned} \qquad (3.6)$$

Note that due to the quadratic term these equations no longer form a convex set. We will discuss the effect losses has on convexity in more detail in chapter 6.

In the next section, we discuss the electricity market and show how the properties of the transmission grid are important for the computation of electricity prices.

3.2 Electricity Market

In this section we discuss the design and features of electricity markets. Different countries each operate their electricity systems differently. However, regardless of how the electricity systems are operated, the aim is for the system to provide electricity at a low cost to customers.

3.2.1 Market participants

Electricity pool markets consist of six main participants; these are: generation, transmission, distribution, retailers, consumers and the system operator. The system operator organises the operation of the system; they monitor demand on an ongoing basis and instruct the generation plants to produce certain amounts of electricity. The system operator also computes the prices that purchasers should pay for electricity and how much ought to be paid to generators.

In order for the system operator to determine which plants should run at any given time, each plant is required to submit some offer of electricity into the market, detailing how much electricity it is willing to produce and what prices it will charge [43]. In some markets there is also what is known as *demand-side bidding* whereby large consumers of electricity can inform the system operator of the maximum price they will pay for electricity (see [74, 102]).

Because consumption of electricity generally occurs some distance from where it is produced, the electricity is often required to be sent large distances over the transmission grid. The rules governing who can own transmission assets differ depending on the market rules. In some markets, transmission is owned by a regulated monopolist firm, whereas in others, generation firms are allowed to invest in transmission, since the transmission capacity expansion decisions can affect their ability to sell electricity, and hence their profitability [77, 81]. At a local level, once the electricity arrives at a sub-station, it is provided to consumers over local distribution networks. These distribution networks are operated by lines companies, who must ensure their network can meet the local demand.

In electricity markets, most commercial and residential users of electricity buy their power from a retailer. These retailers sell power to consumers at predetermined, fixed prices, while buying that power from the wholesale market. This exposes retailers to considerable risk, for example if spot prices increase unexpectedly. For this reason, the market participants often enter into contracts with each other in order to mitigate the risk that they may otherwise face.

A common way of dealing with this risk is for generation companies to enter into *contracts for differences* (CFDs) on the spot price of electricity. This protects retailers from high electricity prices and generators from low prices. Alternatively, with *vertical integration* firms own generation plants and also sell power to consumers; these firms are called *gentailers* and are able to absorb the risk directly, because they have generation plants which can meet some or all of their obligations to their customers without having to purchase power at the spot price [48].

An important feature of electricity pool markets is that location can affect the price of electricity

[88]. There are two main locational pricing schemes used in electricity markets; these are *nodal pricing* and *zonal pricing.* Both of these pricing schemes aim to ensure that appropriate price signals are observed in the market and additional transmission or generation is built where it is needed. With nodal marginal pricing, the effects of transmission losses and congestion are directly observed in the prices; that is, every node in the network has its own price based on the marginal cost to supply power to that node. With zonal marginal pricing, however, collections of nodes are aggregated into zones and the prices are the same at all nodes in any zone. Ding *et al.* in [23] compare these different pricing methods.

Although locational pricing aims to find the appropriate price for each node (or zone), it can place additional risk on the retailers and generators in the market. For example, if a gentailer has consumers located at a different node than its generation, then they are exposed to *transmission price risk*; that is if prices are high where they do not have any generation capabilities, then they would have to purchase power at the spot price to meet the demand of their customers. To avoid this, in some markets, firms (generators, retailers, or large industrial consumers) can enter into contracts called *financial transmission rights* (FTRs), where a payment is made based on the difference between the prices for a given pair of nodes [64].[1] This type of contract aims to reduce the risk that the market participants face, and allow gentailers to compete in the retail market in regions where they may otherwise face a risk of high prices. Financial transmission rights are not considered in this work; see Zakeri and Downward [101] for an analysis of the impact of FTRs on the behaviour of strategic firms in a New Zealand setting.

3.2.2 Electricity market auction design

Generally, in electricity pool markets, each firm is independently operated and is required to submit a supply function or *offer curve* for each of its generators. This supply function is a non-decreasing function that specifies the quantity of electricity that a plant is willing to produce at a given price, or alternatively the (minimum) price a plant is willing to accept for given quantities of electricity.

Different markets have different restrictions on the form of the offer curve. For example, in New Zealand each plant is restricted to an *offer stack* consisting of five tranches of increasing price [5]; whereas in the California market they have accepted piecewise linear increasing supply functions [72].

In some markets, *demand-side bidding* is allowed. Here large consumers (generally industrial)

[1] This is the simplest form of FTR, known as a balanced, point-to-point FTR.

specify the maximum price they are willing to pay for electricity; demand will therefore reduce as the price becomes high. This adds to the stability of the system when there is a shortage of power.

3.2.3 Welfare in electricity auctions

Dispatch efficiency

In welfare economics, each individual in a market has some welfare function. In an electricity market it is the system operator's aim is to satisfy the demand at the lowest possible cost. If all generators offer in their power at its marginal cost, the total welfare will be maximized by dispatching the lowest cost generation and selling it to the consumers who are willing to pay the most for it. In figure 3.1, we show a generation cost function and demand function, with the shaded region between the curves giving the maximum total welfare.

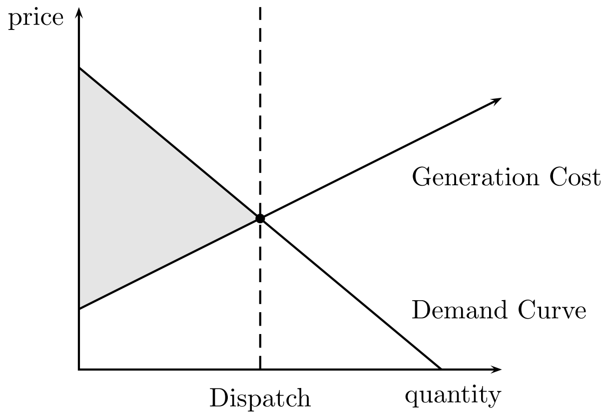

Figure 3.1: Total welfare.

We will now discuss two payment mechanisms for this type of auction: *uniform-price* and *pay-as-bid* [98]. In a uniform-price auction there is a clearing price set where supply and demand meet; this clearing price is paid to the generator for all electricity produced. However, in a pay-as-bid auction the generator is only paid what they bid. A comparison of the revenues of generators under the two payment rules is shown by the area of shaded regions in figure 3.2 below.

From the figure it is clear that, for a given supply function, the revenue earned by a generator in a pay-as-bid auction will be less than (or in some cases equal to) the revenue in a uniform-price auction, therefore generators' offers will change to account for this. In this work, we restrict our analysis to electricity markets with uniform-price auctions.

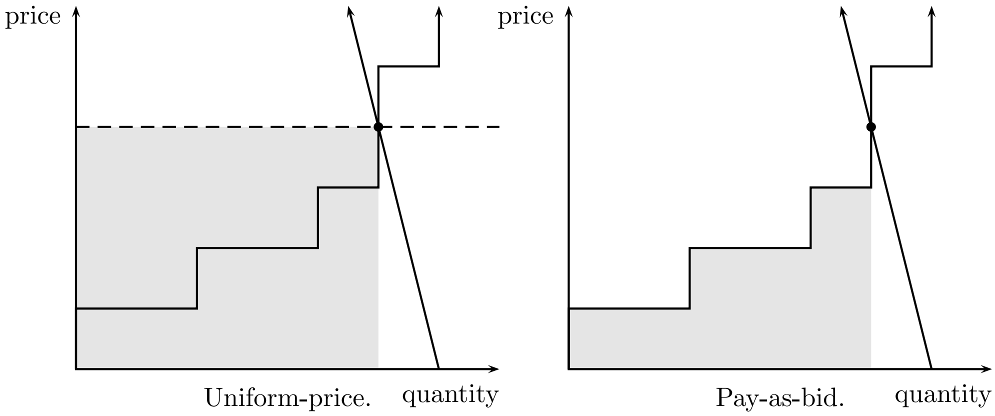

Figure 3.2: Generator revenue comparison.

Market power

One of the key concerns for the regulators of electricity markets is the opportunity of generators to exercise some form of *market power* [71]. Market power is defined to be the ability and incentive for a single generation firm to manipulate the price of electricity (generally this will be by increasing the price, however, it is possible that a firm may have incentive to reduce the price). Recently in New Zealand an investigation by the Commerce Commission resulted in a report by Prof. Frank Wolak [95]. This report found that over the period from 2001 to 2007 generators, at times, exercised market power to increase wholesale prices.[2] Issues of market power can often occur in situations where a supplier is *pivotal* [16]. A pivotal supplier is defined to be any generator without whose capacity, the demand could not be met. This effectively gives that generator an opportunity to charge an arbitrarily high price for a portion of its production. Generally, a generator will only be pivotal during certain periods of the day; to quantify this, a pivotal supplier index is used [35]. However, if generators have a retail arm, or bilateral contracts for energy, then they would have no incentive to reduce their supply below that level. In this situation, the concept of a generator being *net pivotal* is required; in this case the contracting level is also considered when determining whether a firm would have an incentive to exercise market power [95].

In this work, we investigate how transmission can affect firms' abilities to exercise market power. We consider how transmission capacities, loops and losses can influence the behaviour firms when they are seeking to maximize their profits.

[2]In [25] we examine the impact of market restructuring proposed by Wolak.

Part II

Properties of the dispatch problem

Chapter 4

General network without losses

In this chapter, we analyse the dispatch problem for an electricity market over a general network without losses. Given a set of offers from firms, this dispatch problem minimizes the cost of meeting demand (or equivalently maximizes welfare) while meeting all the transmission constraints.

We begin this chapter by discussing the topology of the network. We define the sets of nodes, arcs and loops that comprise the network, and discuss the parameters on the arcs that govern how the power flows. We then introduce the supply functions that generators bid, and the demand that must be satisfied. These elements enable us to formulate the economic dispatch problem.

After we formulate the dispatch problem, we introduce the concept of *KKT regimes.* These regimes correspond to an *active set* of constraints (as described by Nocedal *et al.* in [73]) from the dispatch problem. We use these KKT regimes to reduce the optimality conditions of the dispatch problem to a system of equations (i.e. there are no inequalities), which for certain offer functions can be solved analytically. This allows us to solve the dispatch problem parametrically as a function of demand. These analytical solutions are important later, when we consider Cournot games over networks with loops starting in chapter 9.

Finally, we address issues relating to feasibility of the dispatch problem. For example, we discuss what happens if there is insufficient generation to meet the demand.

4.1 Definitions

In this section, we define the generation offers and the demand as well as the properties of the transmission network.

4.1.1 Transmission network

In section 3.1.2 we introduced a network $(\mathcal{N}, \mathcal{A})$ of nodes $i \in \mathcal{N}$, and directed arcs $ij \in \mathcal{A}$, where i is the tail node and j is the head node. From chapter 3 we know that the power flow on arc ij (denoted f_{ij}) is approximately equal to $\frac{1}{l_{ij}}(\theta_i - \theta_j)$, where l_{ij} is the line reactance.[1] This can be simplified by using a *loop cover*; here we create a set of loops, $k \in \mathcal{L}$, and define a loop matrix, L, by: $L_{ij,k} = 0$ if arc ij is not part of loop k, otherwise it equals either l_{ij} or $-l_{ij}$ depending on the arc's direction in the loop. Lastly we define the capacity of arc ij (in either direction) to be K_{ij}. Example 4.1 belows illustrates these definitions on a six node network with a loop.

Example 4.1. *For the six-node network shown in figure 4.1 below, we define the following sets:*

$$\begin{aligned}
\mathcal{N} &= \{1,2,3,4,5,6\}\,, \\
\mathcal{A} &= \{12,14,25,36,45,65\}\,, \\
\mathcal{L} &= \{1\}\,, \\
\begin{bmatrix} L_{12,1} & L_{14,1} & L_{25,1} & L_{36,1} & L_{45,1} & L_{65,1} \end{bmatrix} &= \begin{bmatrix} l_{12} & -l_{14} & l_{25} & 0 & -l_{45} & 0 \end{bmatrix}.
\end{aligned}$$

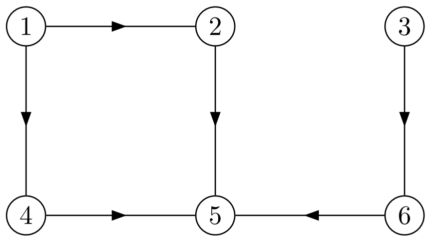

Figure 4.1: Six-node network with a loop.

4.1.2 Bids and demand

At each node, i, there is a known demand, d_i, and a single generator offering a continuously differentiable, strictly increasing supply function $S_i(p)$; the inverse of this, $S_i^{-1}(x)$ is shown in figure 4.2. We denote by x_i the amount of power generated by the generator at node i. The upper and lower bounds of the generator are x_i^+ and x_i^-, respectively.

Observe that the prices coming from this are equivalent to those when we have a demand curve at node i of the form

$$D_i(p) = d_i - S_i(p).$$

[1] Recall that here we are assuming a DC load-flow approximation.

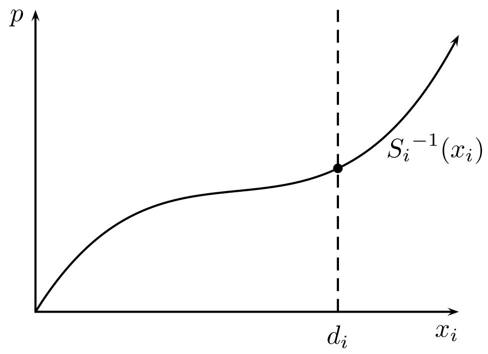

Figure 4.2: Supply function.

4.2 Formulation of dispatch problem

We will now put these elements together to form the *economic dispatch problem*:

$$
\begin{aligned}
\mathrm{P}(d) : \min \quad & \sum_{i \in \mathcal{N}} \int_{x_i^-}^{x_i} S_i^{-1}(\xi)\, .d\xi \\
\\
\text{s.t.} \quad & x_i - \sum_{j, ij \in \mathcal{A}} f_{ij} + \sum_{j, ji \in \mathcal{A}} f_{ji} = d_i && [\pi_i] && \forall i \in \mathcal{N} \\
& \sum_{ij \in \mathcal{A}} L_{ij,k} f_{ij} = 0 && [\lambda_k] && \forall i \in \mathcal{L} \\
& x_i \le x_i^+ && \left[\mu_i^+\right] && \forall i \in \mathcal{N} \\
& x_i \ge x_i^- && \left[\mu_i^-\right] && \forall i \in \mathcal{N} \\
& f_{ij} \le K_{ij} && \left[\eta_{ij}^+\right] && \forall ij \in \mathcal{A} \\
& f_{ij} \ge -K_{ij} && \left[\eta_{ij}^-\right] && \forall ij \in \mathcal{A}.
\end{aligned}
$$

The objective of the above dispatch problem is to minimize the cost of dispatch; specifically, here we minimize the total area under the dispatched supply functions. If the supply functions reflected each generator's true cost, then the dispatch problem would be equivalent to maximizing total welfare (since the demand is inelastic). Note that the first two constraint sets are the DC load-flow equations (3.3). The first constraint set is the node balance constraint; that is, the generation plus the net inflows into a node must equal the demand at that node. The next constraint set ensures that the flows follow Kirchhoff's laws. The inequalities on x enforce the upper and lower bounds on each plant dispatch quantity, and the constraints on f ensure that the flows on the arcs do not violate their thermal capacities.

The dual variable, π_i, associated with the i^{th} node balance constraint gives the rate of change of the optimal value function of $\mathrm{P}(d)$ with respect to a change in demand at node i. Using a locational marginal pricing structure as described in [88], we set the nodal price to be equal to this dual variable. This means that the price at a node (paid to generators and paid by consumers purchasing from the spot market) is defined as the cost to the system of an additional unit of demand at that node.

Since $x_i = S_i(p)$ is an increasing continuous function, we know (from the inverse function theorem in [6]) that $S_i^{-1}(x_i)$ is also an increasing continuous function. Therefore the objective function of the above dispatch problem is a strictly convex function of x. For a given set of parameters, $\mathrm{P}(d)$ can be solved using a nonlinear solver such as `conopt` or `snopt` [1]. However, if we wish to perform parametric analysis on the optimal solution then we need to examine the dispatch problem's KKT conditions. Since the dispatch problem is convex, we know that the KKT conditions, given in (4.1) below, are both necessary and sufficient for the optimality of $\mathrm{P}(d)$; see [57].

$$
\begin{array}{ll}
x_i - \sum_{j,ij\in\mathcal{A}} f_{ij} + \sum_{j,ji\in\mathcal{A}} f_{ij} = d_i & \forall i \in \mathcal{N} \\
\sum_{ij\in\mathcal{A}} L_{ij,k} f_{ij} = 0 & \forall k \in \mathcal{L} \\
S_i^{-1}(x_i) - \pi_i + \mu_i^+ - \mu_i^- = 0 & \forall i \in \mathcal{N} \\
\pi_i - \pi_j + \eta_{ij}^+ - \eta_{ij}^- + \sum_{k\in\mathcal{L}} L_{ij,k}\lambda_k = 0 & \forall ij \in \mathcal{A} \\
0 \le x_i^+ - x_i \perp \mu_i^+ \ge 0 & \forall i \in \mathcal{N} \\
0 \le x_i - x_i^- \perp \mu_i^- \ge 0 & \forall i \in \mathcal{N} \\
0 \le K_{ij} - f_{ij} \perp \eta_{ij}^+ \ge 0 & \forall ij \in \mathcal{A} \\
0 \le K_{ij} + f_{ij} \perp \eta_{ij}^- \ge 0 & \forall ij \in \mathcal{A}.
\end{array}
\tag{4.1}
$$

Due to the orthogonality constraints, the above conditions form a complementarity problem, as discussed by Cottle *et al.* in [19]. An orthogonality condition of the form:

$$0 \le a \perp b \ge 0,$$

is equivalent to

$$a \cdot b = 0, \quad a,\, b \ge 0.$$

These constraints are present in the solution because the constraints that bind at the optimal solution to a convex program depend on the particular parameters of the problem, which are not specified at this point. In the next section we will outline a method for solving the above problem parametrically with respect to the demand, d.

4.2.1 KKT regimes

Here, for an arbitrary demand vector, d, we detail an enumeration method that determines the optimal solution to the dispatch problem parametrically. This method relies on the concept of *KKT* *regimes.* These KKT regimes are constructed by determining *a priori* the active constraints of the dispatch problem, thereby creating a partition of active constraints, as discussed by Berkelaar *et al.* in [8]. For each KKT regime, certain flows and dispatches are forced to be binding, and others are assumed to be unrestricted. For a particular KKT regime, r, the following sets partition $\mathcal{A}$:

$\mathcal{B}_r^+$: the flows for all arcs in this set are fixed at their upper bounds,
$\mathcal{B}_r^-$: the flows for all arcs in this set are fixed at their lower bounds,
$\mathcal{B}_r$: the flows for all arcs in this set are unconstrained,

and these sets partition $\mathcal{N}$:

$\mathcal{D}_r^+$: the generator dispatches for all nodes in this set are fixed at their upper bounds,
$\mathcal{D}_r^-$: the generator dispatches for all nodes in this set are fixed at their lower bounds,
$\mathcal{D}_r$: the generator dispatches for all nodes in this set are unconstrained.

Using the above sets, we can now define a modified dispatch problem for KKT regime r. In this dispatch problem, flows on the arcs and dispatch quantities are set to comply with the conditions of the particular KKT regime, as shown in $\mathrm{P}^r(d)$ below.

$$\mathrm{P}^r(d) : \min \quad \sum_{i\in\mathcal{N}} \int_{x_i^-}^{x_i^r} S_i^{-1}(\xi)\, .d\xi$$

$$\begin{array}{llll}
\text{s.t.} & x_i^r - \sum\limits_{j,ij\in\mathcal{A}} f_{ij}^r + \sum\limits_{j,ji\in\mathcal{A}} f_{ji}^r = d_i & [\pi_i^r] & \forall i \in \mathcal{N} \\
& \sum\limits_{ij\in\mathcal{A}} L_{ij,k} f_{ij}^r = 0 & [\lambda_k^r] & \forall i \in \mathcal{L} \\
& x_i^r = x_i^+ & \left[\mu_i^{r+}\right] & \forall i \in \mathcal{D}_r^+ \\
& x_i^r = x_i^- & \left[\mu_i^{r-}\right] & \forall i \in \mathcal{D}_r^- \\
& f_{ij}^r = K_{ij} & \left[\eta_{ij}^{r+}\right] & \forall ij \in \mathcal{B}_r^+ \\
& f_{ij}^r = -K_{ij} & \left[\eta_{ij}^{r-}\right] & \forall ij \in \mathcal{B}_r^- .
\end{array}$$

Here all the primal and dual variables are given a superscript r to reflect that they correspond to KKT regime r. This modified dispatch problem is still a convex problem, so its KKT conditions are also necessary and sufficient for optimality; these are

$$
\begin{aligned}
& x_i^r - \sum_{j,ij\in\mathcal{A}} f_{ij}^r + \sum_{j,ji\in\mathcal{A}} f_{ij}^r = d_i && \forall i \in \mathcal{N} \\
& \sum_{ij\in\mathcal{A}} L_{ij,k} f_{ij}^r = 0 && \forall k \in \mathcal{L} \\
& S_i^{-1}\left(x_i^r\right) - \pi_i^r + \mu_i^{r+} - \mu_i^{r-} = 0 && \forall i \in \mathcal{N} \\
& \pi_i^r - \pi_j^r + \eta_{ij}^{r+} - \eta_{ij}^{r-} + \sum_{k\in\mathcal{L}} L_{ij,k}\lambda_k^r = 0 && \forall ij \in \mathcal{A} \\
& x_i^r = x_i^+ && \forall i \in \mathcal{D}_r^+ \\
& x_i^r = x_i^- && \forall i \in \mathcal{D}_r^- \\
& f_{ij} = K_{ij} && \forall ij \in \mathcal{B}_r^+ \\
& f_{ij} = -K_{ij} && \forall ij \in \mathcal{B}_r^- . \\
& \mu_i^{r+} = 0 && \forall i \in \mathcal{D}_r \cup \mathcal{D}_r^- \\
& \mu_i^{r-} = 0 && \forall i \in \mathcal{D}_r \cup \mathcal{D}_r^+ \\
& \eta_{ij}^{r-} = 0 && \forall ij \in \mathcal{B}_r \cup \mathcal{B}_r^- \\
& \eta_{ij}^{r-} = 0 && \forall ij \in \mathcal{B}_r \cup \mathcal{B}_r^+ .
\end{aligned}
\tag{4.2}
$$

Note that the KKT conditions in (4.2) are all equality constraints, so depending on the form of the supply functions it may be possible to compute analytic expressions for the solution to the above problem.[2]

Now we will prove a lemma that gives the necessary and sufficient conditions ensuring that an optimal solution to $\mathrm{P}^r(d)$ is also optimal for $\mathrm{P}(d)$.

Lemma 4.2. *Consider the optimal solution to* $\mathrm{P}^r(d)$ *for some KKT regime* r*. This solution is optimal for* $\mathrm{P}(d)$ *if and only if it also satisfies the following conditions:*

$$
\begin{aligned}
& -K_{ij} \le f_{ij}^r \le K_{ij} && \forall ij \in \mathcal{B}_r \\
& x_i^- \le x_i^r \le x_i^+ && \forall i \in \mathcal{D}_r \\
& \eta_{ij}^{r+} \ge 0 && \forall ij \in \mathcal{B}_r^+ \\
& \eta_{ij}^{r-} \ge 0 && \forall ij \in \mathcal{B}_r^- \\
& \mu_i^{r+} \ge 0 && \forall i \in \mathcal{D}_r^+ \\
& \mu_i^{r-} \ge 0 && \forall i \in \mathcal{D}_r^- .
\end{aligned}
\tag{4.3}
$$

Proof. For a solution to solve $\mathrm{P}(d)$, it must satisfy the necessary and sufficient conditions given by (4.1). However, we have an optimal solution to $\mathrm{P}^r(d)$, which, by assumption, must satisfy the conditions given in (4.2).

[2]Of course note that a solution being optimal for $\mathrm{P}^r(d)$ is neither necessary nor sufficient for it to be optimal for $\mathrm{P}(d)$.

Observe that if the conditions given by (4.3) are satisfied by a solution to (4.2), then this solution must also satisfy (4.1). Therefore if the solution satisfies (4.3), then the solution to $\mathrm{P}^r(d)$ is optimal for $\mathrm{P}(d)$.

Moreover, if any of the conditions (4.3) are not satisfied, then either the solution is not feasible for $\mathrm{P}(d)$ or an orthogonality condition of (4.1) is violated. □

The above lemma is important since it gives the conditions which ensure that, for a given regime, the optimal solution to the modified dispatch problem is in fact the optimal solution to the actual dispatch problem. Furthermore, since the optimal solution to $\mathrm{P}(d)$ (so long as one exists) is always consistent with some KKT regime, if we were to enumerate all KKT regimes, then we would find that the optimal dispatch for at least one regime must be optimal for $\mathrm{P}(d)$. This allows us to solve the dispatch problem $\mathrm{P}(d)$ analytically over various ranges of parameters, which we will demonstrate in the next section.

4.3 Price as a function of demand

In this section we will examine how the nodal prices vary with demand when using locational marginal cost pricing.

For a single-node market, since there is one demand parameter and the supply function is increasing, it is straightforward to see that as demand increases so does the price. Over a network, on the other hand, since we have a vector of demands, d, and a vector of prices, π, it is not clear that this relationship still holds. The following theorem extends the single-node result to a network setting.

Theorem 4.3. *The nodal price, π_n, at any node, n, increases with the demand, d_n, at that node.*

Proof. Since the dispatch problem is convex, the cost of dispatch is a convex function of the demand, d. Each nodal price is the dual variable associated with a node balance constraint, hence it is a sub-gradient to the optimal value function of $\mathrm{P}(d)$ expressed as a function of demand at that node. Since this function is strictly convex, the nodal price increases with the demand at that node. □

The above theorem states that at a given node, increasing demand will increase *that* node's price, but does not state what can happen to prices at other nodes. In the next section we show, in the presence of congested loops, that prices at some nodes may be decreasing functions of demand at other nodes.

4.3.1 Spring-washer effect

The *spring-washer effect* as described in [80], refers to the way that nodal prices vary around a constrained loop. Here we will present an example of the spring-washer effect, and demonstrate how prices vary as a function of demand in this type of situation.

Example 4.4. *Consider the three-node loop shown in figure 4.3; here all lines have equal reactance, and line* 23 *has a capacity of* $K_{23} = 20$*, while the other lines have unlimited capacity.*

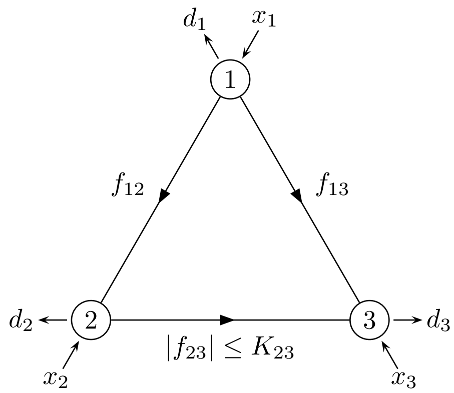

Figure 4.3: Three-node network.

There is a generator at each node, i*, with an offer curve* $S_i(p) = p$*, which has a lower bound of* $x_i^- = 0$ *and an upper bound of* $x_i^+ = \infty$*.*

Suppose that the demand at node 1 is $d_1 = 100$ and the demand at node 2 is $d_2 = 0$. The dispatch problem for this situation as a function of d_3 is then given by

$$\min \quad \tfrac{1}{2}{x_1}^2 + \tfrac{1}{2}{x_2}^2 + \tfrac{1}{2}{x_3}^2$$

$$\begin{array}{lll}
\text{s.t.} & x_1 - f_{12} - f_{13} = 100 & [\pi_1] \\
 & x_2 + f_{12} - f_{23} = 0 & [\pi_2] \\
 & x_3 + f_{13} + f_{23} = d_3 & [\pi_3] \\
 & f_{12} - f_{13} + f_{23} = 0 & [\lambda] \\
 & |f_{23}| \leq 20 & [\eta_{23}^1, \eta_{23}^2] \\
 & x_1,\, x_2,\, x_3 \geq 0 &
\end{array}$$

Using the KKT regime method outlined in the previous section, we can solve the above problem parametrically for the prices.

To illustrate this method, we will consider the KKT regime, r, where:

$$\begin{aligned}\mathcal{B}_r &= \{12, 13, 23\}\,, \\ \mathcal{D}_r &= \{1, 2, 3\}\,.\end{aligned}$$

In this KKT regime, the flows on all arcs and generator dispatches are unconstrained. We can solve the dispatch problem associated with this regime explicitly to give:

$$\begin{array}{ccc} x_1^r = \frac{1}{3}\left(d_3 + 100\right), & x_2^r = \frac{1}{3}\left(d_3 + 100\right), & x_3^r = \frac{1}{3}\left(d_3 + 100\right) \\ \pi_1^r = \frac{1}{3}\left(d_3 + 100\right), & \pi_2^r = \frac{1}{3}\left(d_3 + 100\right), & \pi_3^r = \frac{1}{3}\left(d_3 + 100\right), \\ f_{12}^r = \frac{100}{3}, & f_{13}^r = \frac{1}{3}\left(d_3 - 100\right), & f_{23}^r = \frac{1}{3}d_3, \\ \eta_{23}^{r+} = 0, & \eta_{23}^{r-} = 0, & \lambda^r = 0, \end{array}$$

For this regime to be optimal for $\mathrm{P}(d)$, from lemma 4.2, the solution must satisfy the following additional conditions:

$$\begin{aligned}\tfrac{1}{3}d_3 &\le 20, \\ \tfrac{1}{3}d_3 &\ge -20, \\ \tfrac{1}{3}\left(d_3 + 100\right) &\ge 0.\end{aligned}$$

Therefore, we find that this regime (where all generators generate a positive amount and line 23 is not constrained) is valid for $-60 \le d_3 \le 60$. It can similarly be shown that if $60 < d_3 \le 380$ the line joining nodes 2 and 3 becomes congested at the optimal dispatch; due to this constraint binding, the prices are no longer the same for all nodes. Specifically we would have

$$\begin{aligned}\pi_1 &= \tfrac{1}{3}\left(100 + d_3\right), \\ \pi_2 &= \tfrac{1}{6}\left(380 - d_3\right), \\ \pi_3 &= \tfrac{1}{6}\left(20 + 5d_3\right).\end{aligned}$$

Note that the price at node 2 is decreasing with additional demand at node 3. Finally, if $d_3 > 380$

$$\begin{aligned}\pi_1 &= 160, \\ \pi_2 &= 380 - d_3, \\ \pi_3 &= -60 + d_3.\end{aligned}$$

This is called the spring-washer effect because the prices increase at nodes around the loop until the constraint is reached, at which point the price drops, similar to the shape of a spring-washer.

4.3.2 Infeasible dispatch

Note that due to the bounds on the dispatch quantities, we are not guaranteed that the dispatch is feasible for all demand realisations. For the situation where there is insufficient supply to meet the demand, there must be some form of load shedding or demand rationing as described in [53], whereby certain quantities of demand are not satisfied; this results in a cost to the system based on the *value of lost load* (VOLL) specified. Conversely due to the lower bounds on generation, it is possible that there may be too much supply, which must be disposed of somehow. When either of these situations occurs, the dispatch problem $\mathrm{P}(q)$ becomes infeasible and the nodal prices are therefore undefined.

In order to be able to define the dual variables over all possible realisations of d, we will consider a relaxed version of the dispatch problem that includes an unrestricted variable δ_i in each node balance constraint that allows for load shedding or disposal of electricity at a cost of $\frac{1}{2\epsilon}{\delta_i}^2$.[3] The node balance constraint for node i therefore becomes

$$x_i - \sum_{j,ij\in\mathcal{A}} f_{ij} + \sum_{j,ji\in\mathcal{A}} f_{ij} + \delta_i = d_i.$$

At the limit as $\epsilon \to 0$, we find that the nodal price at any node for which there is load shedding is $+\infty$ and the price at any node for which there is disposal is $-\infty$. We will therefore set the prices to be defined by the dual variables from this relaxed problem when the dispatch problem would otherwise be infeasible.

[3] This is not a price cap like VOLL, instead, $\delta \neq 0$ for any non-zero price. However, later we take the limit as $\epsilon \to 0$ which causes $\delta \to 0$.

Chapter 5

Lossless radial network

In this chapter, we restrict our interest to radial networks[1] without losses. In this context, we prove a number of properties of the dispatch problem, which relate to how price varies as a function of demand. To do this, it is convenient to adopt the relaxed version of the dispatch problem (from section 4.3.2) so that we do not encounter infeasibility. This relaxation means that when the dispatch problem would have otherwise been infeasible, we instead observe very high or low nodal prices.

We first present the dispatch problem for radial networks; because there are no loops, the loop-flow constraints and the corresponding dual variables are no longer present in this formulation. We then proceed to prove a lemma giving necessary and sufficient conditions for a solution to the dispatch problem to be optimal.

In section 5.2, we generalise the concept of KKT regimes. We then prove a number of lemmas relating to nodal prices associated with particular KKT regimes. These results are important for subsequent chapters where we investigate the shape of a generator's residual demand curve.

5.1 Dispatch problem

The dispatch problem, given in chapter 4, required a set of constraints ensuring that the power flow in loops complied with Kirchhoff's laws. However, if there are no loops in the network then these constraints are not required. The relaxed dispatch problem for the case where the network

[1] In radial networks there is a unique path connecting every pair of nodes.

contains no loops is

$$\mathrm{P}(d) : \min \quad \sum_{i\in\mathcal{N}} \left(\int_{x_i^-}^{x_i} S_i^{-1}(\xi)\, d\xi + \tfrac{1}{2\epsilon}{\delta_i}^2 \right)$$

$$\begin{array}{llll} \text{s.t.} & x_i - \sum\limits_{j,ij\in\mathcal{A}} f_{ij} + \sum\limits_{j,ji\in\mathcal{A}} f_{ji} + \delta_i = d_i & [\pi_i] & \forall i \in \mathcal{N} \\ & x_i^- \le x_i \le x_i^+ & [\mu_i^+, \mu_i^-] & \forall i \in \mathcal{N} \\ & -K_{ij} \le f_{ij} \le K_{ij} & [\eta_{ij}^+, \eta_{ij}^-] & \forall ij \in \mathcal{A}. \end{array}$$

Since this problem has linear constraints and a convex objective function, it is equivalent to the following KKT conditions:

$$\begin{aligned} x_i - \sum_{j,ij\in\mathcal{A}} f_{ij} + \sum_{j,ji\in\mathcal{A}} f_{ji} + \delta_i = d_i & \qquad \forall i \in \mathcal{N} \\ \frac{1}{\epsilon}\delta_i - \pi_i = 0 & \qquad \forall i \in \mathcal{N} \\ {S_i}^{-1}(x_i) - \pi_i + \mu_i^+ - \mu_i^- = 0 & \qquad \forall i \in \mathcal{N} \\ \pi_i - \pi_j + \eta_{ij}^+ - \eta_{ij}^- = 0 & \qquad \forall ij \in \mathcal{A} \\ 0 \le x_i - x_i^- \perp \mu_i^- \ge 0 & \qquad \forall i \in \mathcal{N} \\ 0 \le x_i^+ - x_i \perp \mu_i^+ \ge 0 & \qquad \forall i \in \mathcal{N} \\ 0 \le K_{ij} - f_{ij} \perp \eta_{ij}^+ \ge 0 & \qquad \forall ij \in \mathcal{A} \\ 0 \le K_{ij} + f_{ij} \perp \eta_{ij}^- \ge 0 & \qquad \forall ij \in \mathcal{A}. \end{aligned} \tag{5.1}$$

The following lemma redefines the above necessary and sufficient conditions for an optimal solution to $\mathrm{P}(d)$.

Lemma 5.1. *Suppose that (x, f, δ) satisfies the node balance constraints of $\mathrm{P}(d)$. Then (x, f, δ) solves $\mathrm{P}(d)$ if and only if for each $ij \in \mathcal{A}$ at least one of the following conditions is satisfied:*

- $\pi_i \ge \pi_j$ *and* $f_{ij} = -K_{ij}$*, or*
- $\pi_i \le \pi_j$ *and* $f_{ij} = K_{ij}$*, or*
- $\pi_i = \pi_j$ *and* $-K_{ij} \le f_{ij} \le K_{ij}$*,*

and for each $i \in \mathcal{N}$ at least one of the following conditions is satisfied:

- $\pi_i \ge S^{-1}(x_i)$ *and* $x_i = x_i^+$*, or*
- $\pi_i \le S^{-1}(x_i)$ *and* $x_i = x_i^-$*, or*

- $\pi_i = S^{-1}(x_i)$ *and* $x_i^- \leq x_i \leq x_i^+$.

Proof. The result follows from the fact that P(d) is a convex optimization problem, rendering the conditions (5.1) necessary and sufficient for optimality of P(d). If there exists an instance of x, f and δ satisfying the conditions of the lemma then (x, f, δ) will be optimal if and only if, for all $i \in \mathcal{N}$,

$$\begin{aligned} S_i^{-1}(x_i) - \pi_i + \mu_i^+ - \mu_i^- &= 0, \\ 0 \leq x_i - x_i^- &\perp \mu_i^- \geq 0, \\ 0 \leq x_i^+ - x_i &\perp \mu_i^+ \geq 0, \end{aligned} \tag{5.2}$$

and for all $ij \in \mathcal{A}$,

$$\begin{aligned} \pi_i - \pi_j + \eta_{ij}^+ - \eta_{ij}^- &= 0 \\ 0 \leq K_{ij} - f_{ij} &\perp \eta_{ij}^+ \geq 0 \\ 0 \leq K_{ij} + f_{ij} &\perp \eta_{ij}^- \geq 0. \end{aligned} \tag{5.3}$$

Since $x_i^+ > x_i^-$, the three conditions in (5.2) can be rewritten as,

- $\pi_i \geq S^{-1}(x_i)$ and $x_i = x_i^+$, or
- $\pi_i \leq S^{-1}(x_i)$ and $x_i = x_i^-$, or
- $\pi_i = S^{-1}(x_i)$ and $x_i^- \leq x_i \leq x_i^+$.

Similarly, the three conditions in (5.3) can be written as,

- $\pi_i \geq \pi_j$ and $f_{ij} = -K_{ij}$, or
- $\pi_i \leq \pi_j$ and $f_{ij} = K_{ij}$, or
- $\pi_i = \pi_j$ and $-K_{ij} \leq f_{ij} \leq K_{ij}$.

The above conditions together with the node balance constraints of P(d) are therefore necessary and sufficient for optimality of P(d). Hence if the above conditions are satisfied then we have an optimal solution to P(d), as the node balance constraints are satisfied by assumption. □

5.2 Price as a function of demand at node n

In this section, we establish some properties of the price at an arbitrary but fixed node n, as a function of demand at that node. We will prove that if d^* is some vector of demands such that no line is constrained and no dispatch quantity, x_i, is at a bound at the optimal solution to $\mathrm{P}(d^*)$ then for any vector of demands d with $d_n > d_n^*$ and $d_i = d_i^*$, $\forall i \in \mathcal{N} \setminus \{n\}$, the nodal price π_n is given by the maximum price over a particular set of KKT regimes. Conversely for any vector of demands d such that $d_n < d_n^*$ the nodal price π_n is given by the minimum price over another set of KKT regimes.

To establish these results, it is convenient to adopt the convention throughout this section that all arcs in the radial network are directed towards node n; this adjusted set of arcs will be denoted $\mathcal{A}_n$. Furthermore, for every arc $ij \in \mathcal{A}_n$, we define $\mathcal{N}_n^i$ and $\mathcal{A}_n^i$ to consist of all nodes and arcs in the subtree rooted at node i not including arc ij, respectively. This naming convention is demonstrated for a particular arc $ij \in \mathcal{A}_n$ in figure 5.1 below.

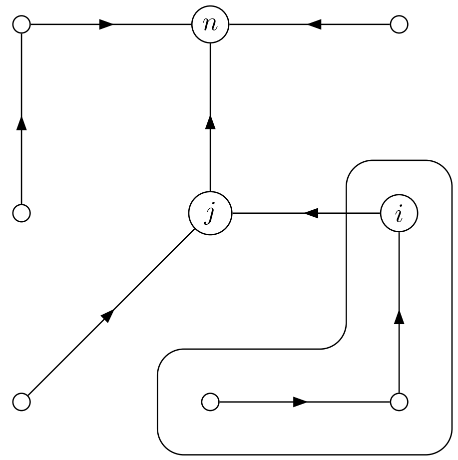

Figure 5.1: The subtree rooted at node i (encircled) consists of nodes $\mathcal{N}_n^i$ and arcs $\mathcal{A}_n^i$.

Now, we consider a modification of the concept of KKT regimes introduced in chapter 4. Here, not only do we set line flows and dispatches to be at upper bound, lower bound or unrestricted, but we also allow some to be determined by the dispatch problem. We therefore introduce two additional sets corresponding to KKT regime r:

- $\mathcal{B}_r^\times$: the flows for all arcs in this set are bounded and determined by the dispatch problem,
- $\mathcal{D}_r^\times$: the generator dispatches for all nodes in this set are bounded and determined by the dispatch problem.

Hence $\mathcal{A}_n$ is now partitioned by $\{\mathcal{B}_r, \mathcal{B}_r^+, \mathcal{B}_r^-, \mathcal{B}_r^\times\}$ and $\mathcal{N}$ is partitioned by $\{\mathcal{D}_r, \mathcal{D}_r^+, \mathcal{D}_r^-, \mathcal{D}_r^\times\}$. Furthermore, the regimes that we consider in this section are of a particular form: a KKT regime, r, pertaining to node n is determined by choosing a subtree $\mathcal{T}_r$ of the network rooted at node n.

We denote the nodes within $\mathcal{T}_r$ by $\mathcal{N}_r$ and all the arcs within $\mathcal{T}_r$ are assumed to be unconstrained, hence all such arcs are in $\mathcal{B}_r$.

The network is therefore decomposed into $\mathcal{T}_r$ and several other subtrees, each rooted at the tail node of an arc with its head node in $\mathcal{N}_r$. The set of arcs that link these subtrees to $\mathcal{T}_r$ are at either upper or lower bound, therefore $\mathcal{B}_r^+ \cup \mathcal{B}_r^-$ comprise all such arcs. All remaining arcs are in $\mathcal{B}_r^\times$.

For the dispatch quantities, all dispatches at nodes in $\mathcal{N}\backslash\mathcal{N}_r$ are in $\mathcal{D}_r^\times$, whereas all dispatches at nodes in $\mathcal{N}_r$ may be in one of $\mathcal{D}_r$, $\mathcal{D}_r^+$ or $\mathcal{D}_r^-$.

For each node, n, we denote by $\mathcal{R}_n$ the set of all such KKT regimes, r, with $\mathcal{T}_r$ rooted at node n. Furthermore, we define:

$$\mathcal{R}_n^+ = \left\{r \in \mathcal{R}_n \,|\, \mathcal{B}_r^- = \mathcal{D}_r^- = \emptyset\right\}, \tag{5.4}$$

$$\mathcal{R}_n^- = \left\{r \in \mathcal{R}_n \,|\, \mathcal{B}_r^+ = \mathcal{D}_r^+ = \emptyset\right\}.[2] \tag{5.5}$$

In the following example, we consider a 9-node radial network; here we depict a KKT regime, $r \in \mathcal{R}_5^+$.

Example 5.2. *Figure 5.2 shows a network with the subtree $\mathcal{T}_r$ associated with regime r highlighted.*

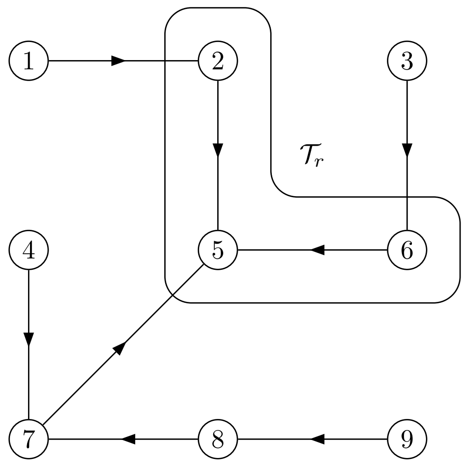

Figure 5.2: A KKT regime rooted at node 5. Here $\mathcal{D}_r = \{2, 5, 6\}$.

From this figure and the definition of a KKT regime above, we know that $\mathcal{N}_r = \{2, 5, 6\}$, $\mathcal{B}_r = \{25, 65\}$, $\mathcal{B}_r^+ \cup \mathcal{B}_r^- = \{12, 36, 75\}$, $\mathcal{B}_r^\times = \{47, 87, 98\}$. *We also designate* $\mathcal{D}_r = \{2, 5, 6\}$, *which implies that* $\mathcal{D}_r^+ \cup \mathcal{D}_r^- = \emptyset$, *and* $\mathcal{D}_r^\times = \{1, 3, 4, 7, 8, 9\}$.

[2] Thus $\mathcal{R}_n^{+(-)}$ consists only of KKT regimes with dispatches at upper (lower) bound and lines congested towards (away from) node n.

Since we have extended the KKT regime concept, we must now introduce a modified dispatch problem, $\mathrm{P}^r(d)$. This problem forces the solution to be consistent with KKT regime r; it is defined as:

$$\mathrm{P}^r(d): \min \quad \sum_{i\in\mathcal{N}} \left(\int_{x_i^-}^{x_i} S_i^{-1}(\xi)\, d\xi + \tfrac{1}{2\epsilon} \delta_i^{r2} \right)$$

$$\begin{array}{llll}
\text{s.t.} & x_i^r - \sum\limits_{j,ij\in\mathcal{A}_n} f_{ij}^r + \sum\limits_{j,ji\in\mathcal{A}_n} f_{ji}^r + \delta_i^r = d_i & [\pi_i^r] & \forall i \in \mathcal{N} \\
& -K_{ij} \le f_{ij}^r \le K_{ij} & \left[\eta_{ij}^{r+}, \eta_{ij}^{r-}\right] & \forall ij \in \mathcal{B}_r^\times \\
& x_i^- \le x_i^r \le x_i^+ & \left[\mu_i^{r+}, \mu_i^{r-}\right] & \forall i \in \mathcal{D}_r^\times \\
& x_i^r = x_i^+ & \left[\eta_{ij}^{r+}\right] & \forall i \in \mathcal{D}_r^+. \\
& x_i^r = x_i^- & \left[\eta_{ij}^{r-}\right] & \forall i \in \mathcal{D}_r^-. \\
& f_{ij}^r = K_{ij} & \left[\mu_i^{r+}\right] & \forall ij \in \mathcal{B}_r^+. \\
& f_{ij}^r = -K_{ij} & \left[\mu_i^{r-}\right] & \forall ij \in \mathcal{B}_r^-.
\end{array}$$

Note that the objective function and the node balance constraints, above, are identical to those of $\mathrm{P}(d)$. The difference is in the inequality constraints; the variables associated with sets superscripted with $+$ or $-$ are forced to be at upper or lower bound respectively; whereas the flows on all arcs in $\mathcal{B}_r$ and the dispatch quantities at all nodes in $\mathcal{D}_r$ are unbounded. In what follows, we present a lemma giving necessary and sufficient conditions for which the optimal solution to $\mathrm{P}^r(d)$ above is the same as the optimal solution to the problem $\mathrm{P}(d)$ defined earlier in this chapter.

Lemma 5.3. *Consider the optimal solution to* $\mathrm{P}^r(d)$*. This is also the optimal solution to* $\mathrm{P}(d)$ *if and only if:*

$$\begin{array}{rl}
-K_{ij} \le f_{ij}^r \le K_{ij} & \forall ij \in \mathcal{B}_r \\
x_i^- \le x_i^r \le x_i^+ & \forall i \in \mathcal{D}_r \\
\pi_i^r \le \pi_j^r & \forall ij \in \mathcal{B}_r^+ \\
\pi_i^r \ge \pi_j^r & \forall ij \in \mathcal{B}_r^- \\
\pi_i^r \le S^{-1}\left(x_i^-\right) & \forall i \in \mathcal{D}_r^- \\
\pi_i^r \ge S^{-1}\left(x_i^+\right) & \forall i \in \mathcal{D}_r^+.
\end{array}$$

Proof. As in the case of lemma 4.2, for the optimal solution to $\mathrm{P}^r(d)$ to be feasible for $\mathrm{P}(d)$ all the primal constraints must be met. Since the node balance constraints are present in $\mathrm{P}^r(d)$, we only need to satisfy the following additional constraints for feasibility

$$\begin{array}{rl}
-K_{ij} \le f_{ij}^r \le K_{ij} & \forall ij \in \mathcal{B}_r \\
x_i^- \le x_i^r \le x_i^+ & \forall i \in \mathcal{D}_r.
\end{array}$$

If the above conditions are satisfied, and the solution also satisfies

$$\begin{aligned} \pi_i^r &\leq \pi_j^r && \forall ij \in \mathcal{B}_r^+ \\ \pi_i^r &\geq \pi_j^r && \forall ij \in \mathcal{B}_r^- \\ \pi_i^r &\leq S^{-1}\left(x_i^-\right) && \forall i \in \mathcal{D}_r^- \\ \pi_i^r &\geq S^{-1}\left(x_i^+\right) && \forall i \in \mathcal{D}_r^+, \end{aligned}$$

by lemma 5.1 this solution must be an optimal for $\mathrm{P}(d)$. Therefore the above conditions are sufficient for the optimality for $\mathrm{P}(d)$. Furthermore, it is clear that if any of these constraints were violated that the KKT conditions given in (5.1) would not be satisfied, meaning that the solution would not be optimal for $\mathrm{P}(d)$, hence the constraints are also necessary. □

In the next lemma we derive an equation which provides a relationship between price and demand in any unconstrained part of the network at the optimal solution of $\mathrm{P}^r(d)$. This enables the price at all nodes to be computed from the optimal primal variables.

Lemma 5.4. *Consider the optimal solution to* $\mathrm{P}^r(d)$ *for some arbitrary but fixed vector of demands,* d*. Let* $\mathcal{T}^1$ *be a subtree of the network with nodes* $\mathcal{N}^1$ *and arcs* $\mathcal{A}^1$*, within which no arc is congested. For every arc,* $ij \in \mathcal{A}_n \backslash \mathcal{A}^1$*, with either its head or tail in* $\mathcal{T}^1$*, set:*

$ij \in \mathcal{B}^+$ *if and only if arc* ij *is directed towards* $\mathcal{T}^1$*,*
$ij \in \mathcal{B}^-$ *if and only if arc* ij *is directed away from* $\mathcal{T}^1$*.*

For every node $i \in \mathcal{N}^1$*, set* $i \in \mathcal{D}^1$ *if and only if the dispatch at node* i *is not at a bound. Here the relationship between the nodal price at all nodes in* $\mathcal{T}^1$ *and the demand is given by:*

$$\sum_{i \in \mathcal{D}^1} S_i\left(\pi^1\right) + \sum_{i \in \mathcal{N}^1} \epsilon \pi^1 = \sum_{i \in \mathcal{N}^1} d_i - \sum_{ij \in \mathcal{B}^+} f_{ij}^r + \sum_{ij \in \mathcal{B}^-} f_{ij}^r - \sum_{i \in \mathcal{N}^1 \backslash \mathcal{D}^1} x_i^r.$$

Proof. Consider the situation stated in the lemma. The optimal solution to $\mathrm{P}^r(d)$ must satisfy the following constraint for all nodes

$$x_i^r - \sum_{j, ij \in \mathcal{A}_n} f_{ij}^r + \sum_{j, ji \in \mathcal{A}_n} f_{ji}^r + \delta_i^r = d_i.$$

Now we sum over this constraint for all $i \in \mathcal{N}^1$, yielding

$$\sum_{i \in \mathcal{N}^1} x_i^r + \sum_{i \in \mathcal{N}^1} \delta_i^r = \sum_{i \in \mathcal{N}^1} d_i - \sum_{ij \in \mathcal{B}^+} f_{ij}^r + \sum_{ij \in \mathcal{B}^-} f_{ij}^r. \tag{5.6}$$

From the optimality conditions of $\mathrm{P}^r(d)$ we know that

$$S^{-1}(x_i) = \pi_i^r, \quad \forall i \in \mathcal{D}^1,$$

$$\frac{1}{\epsilon}\delta_i^r = \pi_i^r, \quad \forall i \in \mathcal{N}^1,$$

and also that

$$\pi_i^r = \pi^1, \quad \forall i \in \mathcal{N}^1.$$

Thus from equation (5.6) we have

$$\sum_{i\in\mathcal{D}^1} S_i\left(\pi^1\right) + \sum_{i\in\mathcal{N}^1} \epsilon\pi^1 = \sum_{i\in\mathcal{N}^1} d_i - \sum_{ij\in\mathcal{B}^+} f_{ij}^r + \sum_{ij\in\mathcal{B}^-} f_{ij}^r - \sum_{i\in\mathcal{N}^1\backslash\mathcal{D}^1} x_i^r.$$

as required. □

Consider the optimal solution to the modified dispatch problem $\mathrm{P}^r(d)$; from lemma 5.4, we know that the price at node n, π_n^r, is governed by the following equation

$$\sum_{i\in\mathcal{D}_r} S_i\left(\pi_n^r\right) + \sum_{i\in\mathcal{N}_r} \epsilon\pi_n^r = \sum_{i\in\mathcal{N}_r} d_i - \sum_{ij\in\mathcal{B}_r^+} K_{ij} + \sum_{ij\in\mathcal{B}_r^-} K_{ij} - \sum_{i\in\mathcal{D}_r^-} x_i^- - \sum_{i\in\mathcal{D}_r^+} x_i^+. \tag{5.7}$$

The following lemma shows that for a given KKT regime, r, the nodal price at all nodes in $\mathcal{T}_r$ is increasing with demand at node n.

Lemma 5.5. *Consider equation* (5.7)*, for some KKT regime rooted at node n. The nodal price at all nodes in $\mathcal{N}_r$, π_n^r is increasing with demand, d_n, at node n.*[3]

Proof. To prove this we differentiate both sides of equation (5.7) with respect to d_n, giving

$$\frac{d}{dd_n}\left(\sum_{i\in\mathcal{D}_r} S_i\left(\pi_n^r\right) + \sum_{i\in\mathcal{N}_r} \epsilon\pi_n^r\right) = 1.$$

Using the chain rule yields

$$\frac{d\pi_n^r}{dd_n}\sum_{i\in\mathcal{D}_r} S_i{}'\left(\pi_n^r\right) + \sum_{i\in\mathcal{N}_r} \epsilon = 1,$$

which can be rearranged to give

$$\frac{d\pi_n^r}{dd_n} = \frac{1}{\sum_{i\in\mathcal{D}_r} S_i{}'\left(\pi_n^r\right) + \sum_{i\in\mathcal{N}_r} \epsilon}. \tag{5.8}$$

Since $S_i{}'\left(\pi_n^r\right)$ and ϵ are positive, therefore $d\pi_n^r/dd_n$ is positive, as required. □

Now we will show that if there exists some vector $d = d^*$ for which no line or dispatch is at a bound, that increasing (decreasing) the n^{th} component of d above d_n^* will cause a unique change in flow towards (away from) node n.

[3]Observe that within each KKT regime the nodal price is a differentiable function of demand.

Lemma 5.6. *Suppose for some vector $d = d^*$ that no line or dispatch is at a bound in the optimal solution to the problem* $\mathrm{P}(d)$*. Suppose now that for some node n, d_n is increased. Then as d_n increases π_n is increasing, the flow in every line is increasing in the direction of node n, and the dispatch at each node is also increasing.*

Proof. Note that initially the network contains no congested lines and no dispatch at a bound. By lemma 5.3, the optimal solution to $\mathrm{P}(d)$ is equivalent to the optimal solution to $\mathrm{P}^{r_1}(d)$ where KKT regime r_1 is defined such that $\mathcal{D}_{r_1} = \mathcal{N}_{r_1} = \mathcal{N}$. Therefore, at least until any flows or dispatches reach their bounds, by lemma 5.5, we know that as d_n increases the nodal prices at all nodes increase. Since at optimality

$$\pi_i^{r_1} = S_i^{-1}(x_i), \quad \forall i \in \mathcal{D}_{r_1},$$

each dispatch quantity x_i must increase as d_n increases. Because the network has a radial structure, the changes in line flows are unique and non-negative in the direction of node n.

Suppose that at $d_n = \bar{d}_n$ some bound is reached; there are two possibilities here: either a flow f_{ij} reaches its capacity K_{ij} or a dispatch x_i reaches its upper bound x_i^+. In the case that the flow f_{ij} reaches its capacity, suppose that $\pi_i = \bar{\pi}_i$ when $d_n = \bar{d}_n$; this would define a new KKT regime $r_2 \in \mathcal{R}_n$ with the following properties:

$$\mathcal{N}_{r_2} = \mathcal{N}_{r_1} \backslash \mathcal{N}_n^i,$$

$$\mathcal{B}_{r_2}^+ = \left(\mathcal{B}_{r_1}^+ \cup \{ij\}\right) \backslash \mathcal{A}_n^i, \quad \mathcal{B}_{r_2}^- = \emptyset, \quad \mathcal{B}_{r_2} = \mathcal{B}_{r_1} \backslash \left(\mathcal{A}_n^i \cup \{ij\}\right), \quad \mathcal{B}_{r_2}^\times = \mathcal{B}_{r_1}^\times \cup \mathcal{A}_n^i,$$

$$\mathcal{D}_{r_2}^+ = \mathcal{D}_{r_1}^+ \backslash \mathcal{N}_n^i, \quad \mathcal{D}_{r_2}^- = \emptyset, \quad \mathcal{D}_{r_2} = \mathcal{D}_{r_1} \backslash \mathcal{N}_n^i, \quad \mathcal{D}_{r_2}^\times = \mathcal{D}_{r_1}^\times \cup \mathcal{N}_n^i.$$

As d_n increases further, until some other bound is reached, by lemma 5.5, we know that the price for all nodes $j \in \mathcal{N}_{r_2}$ continues to increase; whereas for nodes $i \in \mathcal{N} \backslash \mathcal{N}_{r_2}$ the price remains fixed at $\bar{\pi}_i$. Therefore the KKT regime r_2 continues to satisfy the conditions of lemma 5.3, hence the optimal solution to $\mathrm{P}^{r_2}(d)$ is optimal for $\mathrm{P}(d)$.

On the other hand, if a dispatch x_i were to hit its upper bound, this would define new a KKT regime $r_2 \in \mathcal{R}_n$, identical to r_1 except:

$$\mathcal{D}_{r_2}^+ = \mathcal{D}_{r_1}^+ \cup \{i\}, \quad \mathcal{D}_{r_2} = \mathcal{D}_{r_1} \backslash \{i\}.$$

Here as d_n increases, until some other bound is reached, by lemma 5.5, we know that the price at all nodes $i \in \mathcal{N}_{r_2}$ continues to increase, which satisfies the optimality condition that $\pi_i^{r_2} > S_i(x_i^+)$. Hence by lemma 5.3 the solution to $\mathrm{P}^{r_2}(d)$ is optimal for $\mathrm{P}(d)$.

As d_n increases we obtain a sequence of KKT regimes r_1, r_2, $r_3, \ldots$ with $\mathcal{D}_{r_{k+1}} \subset \mathcal{D}_{r_k}$, so we may apply this argument recursively for each k to yield the result. $\square$

Corollary 5.7. *Suppose for some vector $d = d^*$ that no line or dispatch is at a bound in the optimal solution to problem* $\mathrm{P}(d)$*. Suppose now that at node n, d_n is decreased. Then as d_n decreases π_n is decreasing, the flow in every line is non-increasing in the direction of node n, and the dispatch at each node is also decreasing.*

Proof. The proof of this result is analogous to the result proved for lemma 5.6 above. $\square$

If we have an initial vector d^* such that at the solution to $\mathrm{P}(d^*)$ there are no lines or dispatches at capacity, then lemma 5.6 and corollary 5.7 imply that by increasing d_n, only KKT regimes $r \in \mathcal{R}_n^+$ (see equation (5.4)) are possible; conversely for $d_n < d_n^*$, only regimes $r \in \mathcal{R}_n^-$ (see equation (5.5)) are possible.

Theorem 5.14, presented at the end of this section, states that if you initially have an unconstrained network and then the demand at node n is increased, then the price at that node is given by the maximum price from KKT regimes in $\mathcal{R}_n^+$; additionally corollary 5.15 states that if the demand at node n were decreased then price at that node would be given by the minimum price from KKT regimes in $\mathcal{R}_n^-$. These are the key results of this chapter. In order to prove these results, however, we must first present several technical lemmas relating to the optimal solution corresponding to a KKT regime $r \in \mathcal{R}_n^+$. These lemmas each present a scenario whereby we have a solution to the modified dispatch $\mathrm{P}^r(d)$ for some KKT regime, however, the solution is not optimal for $\mathrm{P}(d)$. For each, we prove that there exists another KKT regime $s \in \mathcal{R}_n^+$ for which the price at node n is higher.

Initially we prove that if at the optimal solution to $\mathrm{P}(d)$ no line is congested away from node n, then for any $r \in \mathcal{R}_n^+$ at the optimal solution to $\mathrm{P}^r(d)$ there is no line congested away from node n.

Lemma 5.8. *Suppose at the optimal solution to* $\mathrm{P}(d)$ *that no line is congested away from node n and no dispatch at lower bound. For any KKT regime $r \in \mathcal{R}_n^+$, the optimal solution to* $\mathrm{P}^r(d)$ *will yield $f_{ij}^r > -K_{ij}$ for all $ij \in \mathcal{B}_r^\times$ and $x_i^r > x_i^-$ for all $i \in \mathcal{D}_r^\times$.*

Proof. For any KKT regime $r \in \mathcal{R}_n^+$, consider an arc ij that is a member of the set $\mathcal{B}_r^+$. From the definition of KKT regimes that are elements of $\mathcal{R}_n^+$, we know that node j is in subtree $\mathcal{T}_r$. Furthermore, we know that the subtree rooted at node i not including the arc ij consists of nodes $\mathcal{N}_n^i$ and arcs $\mathcal{A}_n^i$.

Note that the flow out of this subtree through the arc ij in the optimal solution to $\mathrm{P}(d)$ must satisfy, $f_{ij}^r \leq K_{ij}$. So by choosing KKT regime r, where the flow along arc ij is set to equal K_{ij}, we are effectively increasing demand at node i, and by lemma 5.6, we know that the change in flow on the arcs in $\mathcal{A}_n^i$ must be in the direction of node i, and all dispatches must be non-decreasing. Hence, as there is no line congested away from node n and no dispatch at lower bound in the optimal solution to $\mathrm{P}(d)$, this will remain the case for any regime that forces an increase in flow on any line ij. □

We use the above result in the following lemma, in which we will show that if for some KKT regime in $\mathcal{R}_n^+$, the flow on a line is from an expensive node to a cheap node, then there must exist another KKT regime in $\mathcal{R}_n^+$ yielding a higher price at node n.[4]

Lemma 5.9. *Suppose we have an optimal solution to* $\mathrm{P}^r(d)$, *for some* $r \in \mathcal{R}_n^+$ *and there exists an arc* lm *in* $\mathcal{B}_r^+$ *giving* $\pi_l^r > \pi_m^r$. *There exists a KKT regime* $s \in \mathcal{R}_n^+$ *such that the solution to* $\mathrm{P}^s(d)$ *yields a price at node* m: $\pi_m^s > \pi_m^r$.

Proof. From equation (5.7), we know that initially the price π_m^r must satisfy the following equation:

$$\sum_{i \in \mathcal{D}_r} S_i\left(\pi_m^r\right) + \sum_{i \in \mathcal{N}_r} \epsilon \pi_m^r = \sum_{i \in \mathcal{N}_r} d_i - K_{lm} - \sum_{ij \in \mathcal{B}_r^+ \setminus \{lm\}} K_{ij} - \sum_{i \in \mathcal{D}_r^+} x_i^+. \tag{5.9}$$

We now construct a new KKT regime, $s \in \mathcal{R}_n^+$ based on the optimal solution to $\mathrm{P}^r(d)$. We define $\mathcal{N}_s$ to consist of all nodes connected to node l via an unconstrained path (including node l) and all nodes in $\mathcal{N}_r$, as shown in figure 5.3. We know from lemma 5.8 that at the optimal solution to $\mathrm{P}^r(d)$ no lines are congested away from node n, and also that no dispatch is at its lower bound, therefore we set $\mathcal{B}_s^- = \mathcal{D}_s^- = \emptyset$. Hence $\mathcal{B}_s^+$ must contain all arcs connecting $\mathcal{N}_s$ to other subtrees in the network. Furthermore, we define $\mathcal{D}_s^+$ to contain all nodes in $\mathcal{D}_r^+$ and also all nodes $\mathcal{N}_s \cup \mathcal{D}_r^\times$ that were at upper bound at the optimal solution to $\mathrm{P}^r(d)$.

Now we will examine the price at node l for KKT regime r. From lemma 5.5, the price π_l^r must satisfy the following equation:

$$\sum_{i \in \mathcal{D}_s \setminus \mathcal{D}_r} S_i\left(\pi_l^r\right) + \sum_{i \in \mathcal{N}_s \setminus \mathcal{N}_r} \epsilon \pi_l^r = \sum_{i \in \mathcal{N}_s \setminus \mathcal{N}_r} d_i + K_{lm} - \sum_{ij \in \mathcal{B}_s^+ \setminus \mathcal{B}_r^+} K_{ij} - \sum_{i \in \mathcal{D}_s^+ \setminus \mathcal{D}_r^+} x_i^+. \tag{5.10}$$

Adding equations (5.9) and (5.10) gives

$$\sum_{i \in \mathcal{D}_r} S_i\left(\pi_m^r\right) + \sum_{i \in \mathcal{D}_s \setminus \mathcal{D}_r} S_i\left(\pi_l^r\right) + \sum_{i \in \mathcal{N}_r} \epsilon \pi_m^r + \sum_{i \in \mathcal{N}_s \setminus \mathcal{N}_r} \epsilon \pi_l^r = \sum_{i \in \mathcal{N}_s} d_i - \sum_{ij \in \mathcal{B}_s^+} K_{ij} - \sum_{i \in \mathcal{D}_s^+} x_i^+.$$

[4] Recall that the prices of all nodes within the unconstrained subtree rooted at node n are equal.

Again from equation (5.7), we have that the nodal price π_m^s satisfies

$$\sum_{i\in\mathcal{D}_s} S_i\left(\pi_m^s\right) + \sum_{i\in\mathcal{N}_s} \epsilon\pi_m^s = \sum_{i\in\mathcal{N}_s} d_i - \sum_{ij\in\mathcal{B}_s^+} K_{ij} - \sum_{i\in\mathcal{D}_s^+} x_i^+.$$

Hence we have that,

$$\sum_{i\in\mathcal{D}_s} S_i\left(\pi_m^s\right) + \sum_{i\in\mathcal{N}_s} \epsilon\pi_m^s = \sum_{i\in\mathcal{D}_r} S_i\left(\pi_m^r\right) + \sum_{i\in\mathcal{D}_s\backslash\mathcal{D}_r} S_i\left(\pi_l^r\right) + \sum_{i\in\mathcal{N}_r} \epsilon\pi_m^r + \sum_{i\in\mathcal{N}_s\backslash\mathcal{N}_r} \epsilon\pi_l^r.$$

Recall that $\epsilon > 0$ and

$$S_i{}'\left(p\right) > 0, \quad \forall i \in \mathcal{N}, \tag{5.11}$$

and as $\pi_l^r > \pi_m^r$ this means that

$$\begin{aligned}\sum_{i\in\mathcal{D}_s} S_i\left(\pi_m^s\right) + \sum_{i\in\mathcal{N}_s} \epsilon\pi_m^s \quad &> \quad \sum_{i\in\mathcal{D}_r} S_i\left(\pi_m^r\right) + \sum_{i\in\mathcal{D}_s\backslash\mathcal{D}_r} S_i\left(\pi_m^r\right) + \sum_{i\in\mathcal{N}_r} \epsilon\pi_m^r + \sum_{i\in\mathcal{N}_s\backslash\mathcal{N}_r} \epsilon\pi_m^r \\ &= \quad \sum_{i\in\mathcal{D}_s} S_i\left(\pi_m^r\right) + \sum_{i\in\mathcal{N}_s} \epsilon\pi_m^r.\end{aligned}$$

From above and (5.11) we have that

$$\pi_m^s > \pi_m^r,$$

as required. □

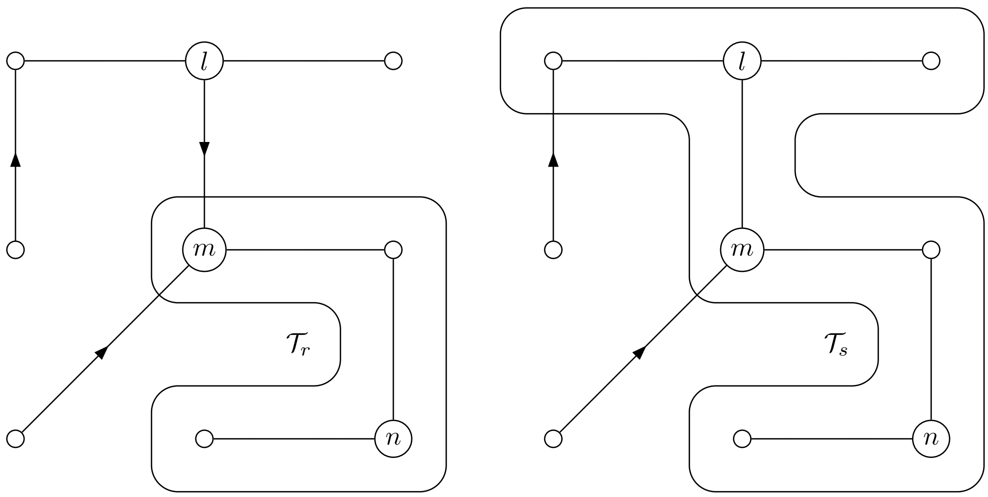

Figure 5.3: A radial network showing KKT regimes r and s respectively (see lemma 5.9).

Lemma 5.10. *Suppose we have an optimal solution to* $\mathrm{P}^r(d)$, *for some* $r \in \mathcal{R}_n^+$ *and there exists an arc* $lm \in \mathcal{B}_r$ *giving* $f_{lm}^r > K_{lm}$. *There exists a KKT regime* $s \in \mathcal{R}_n^+$ *such that the solution to* $\mathrm{P}^s(d)$ *yields a price at node* n: $\pi_n^s > \pi_n^r$.

Proof. We will first define a new KKT regime $s \in \mathcal{R}_n^+$, based on regime r. Consider the tree $\mathcal{T}_r$ corresponding to KKT regime r, and split this into two subtrees by removing arc lm. We define $\mathcal{T}_s$ to be the subtree containing node n; KKT regime s (illustrated in figure 5.4) therefore has the following properties:

$$\mathcal{N}_s = \mathcal{N}_r \backslash \mathcal{N}_n^l,$$

$$\mathcal{B}_s^+ = \left(\mathcal{B}_r^+ \cup \{lm\}\right) \backslash \mathcal{A}_n^l, \quad \mathcal{B}_s^- = \emptyset, \quad \mathcal{B}_s = \mathcal{B}_r \backslash \left(\mathcal{A}_n^l \cup \{lm\}\right), \quad \mathcal{B}_s^\times = \mathcal{B}_r^\times \cup \mathcal{A}_n^l,$$

$$\mathcal{D}_s^+ = \mathcal{D}_r^+ \backslash \mathcal{N}_n^l, \quad \mathcal{D}_s^- = \emptyset, \quad \mathcal{D}_s = \mathcal{D}_r \backslash \mathcal{N}_n^l, \quad \mathcal{D}_s^\times = \mathcal{D}_r^\times \cup \mathcal{N}_n^l.$$

Here we have created a new KKT regime s, for which equation (5.7) implicitly defines the price at all nodes in $\mathcal{N}_s$; this equation is

$$\sum_{i \in \mathcal{D}_s} S_i\left(\pi_n^s\right) + \sum_{i \in \mathcal{N}_s} \epsilon \pi_n^s = \sum_{i \in \mathcal{N}_s} d_i - \sum_{ij \in \mathcal{B}_s^+} K_{ij} - \sum_{i \in \mathcal{D}_s^+} x_i^+, \tag{5.12}$$

whereas for KKT regime r, from lemma 5.4 (pertaining to the subtree $\mathcal{T}_s$) we have that π_n^r must satisfy the following equation:

$$\sum_{i \in \mathcal{D}_s} S_i\left(\pi_n^r\right) + \sum_{i \in \mathcal{N}_s} \epsilon \pi_n^r = \sum_{i \in \mathcal{N}_s} d_i - \sum_{ij \in \mathcal{B}_s^+ \backslash \{lm\}} K_{ij} - f_{lm}^r - \sum_{i \in \mathcal{D}_s^+} x_i^+. \tag{5.13}$$

Subtracting the equation (5.13) from (5.12) becomes:

$$\sum_{i \in \mathcal{D}_s} \left[S_i\left(\pi_n^s\right) - S_i\left(\pi_n^r\right)\right] + \sum_{i \in \mathcal{N}_s} \epsilon\left[\pi_n^s - \pi_n^r\right] = -K_{lm} + f_{lm}^r.$$

As $f_{lm}^r > K_{lm}$, $S_i'(p) > 0$ and $\epsilon > 0$, from above we have that $\pi_n^s > \pi_n^r$, as required. $\square$

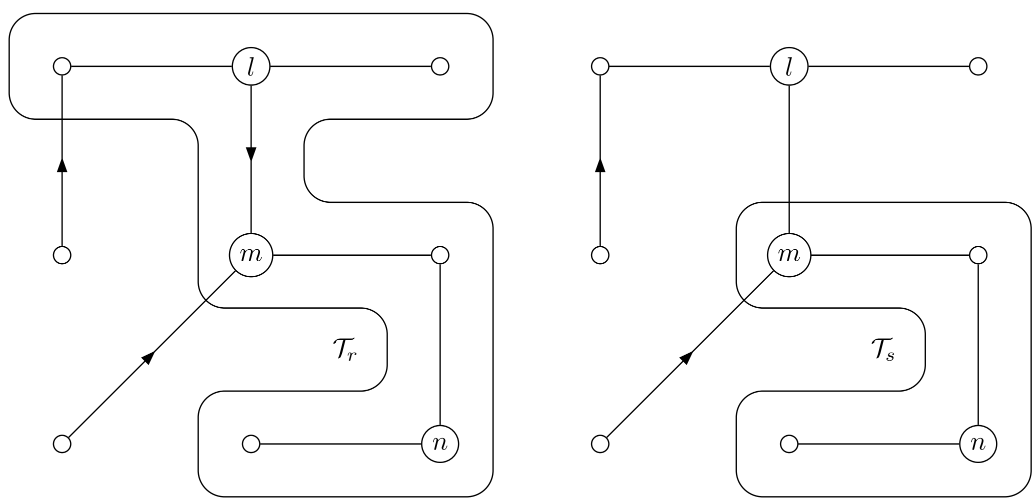

Figure 5.4: A radial network showing KKT regimes r and s respectively (see lemma 5.10).

Lemma 5.11. *Suppose we have an optimal solution to* $\mathrm{P}^r(d)$*, for some* $r \in \mathcal{R}_n^+$ *and there exists a dispatch at node* $k \in \mathcal{D}_r$ *giving* $x_k^r > x_k^+$*. There exists a KKT regime* $s \in \mathcal{R}_n^+$ *such that the solution to* $\mathrm{P}^s(d)$ *yields a price at node* n*:* $\pi_n^s > \pi_n^r$*.*

Proof. For KKT regime r, from equation (5.7) we have the following implicit function for the nodal price at node n:

$$\sum_{i \in \mathcal{D}_r} S_i(\pi_n^r) + \sum_{i \in \mathcal{N}_r} \epsilon \pi_n^r = \sum_{i \in \mathcal{N}_r} d_i - \sum_{ij \in \mathcal{B}_r^+} K_{ij} - \sum_{i \in \mathcal{D}_r^+} x_i^+. \tag{5.14}$$

Now construct a new KKT regime s, which is identical to regime r except that

$$\mathcal{D}_s^+ = \mathcal{D}_r^+ \cup \{k\}, \quad \text{and} \quad \mathcal{D}_s = \mathcal{D}_r \backslash \{k\}.$$

Again from equation (5.7), the price at node n in this regime is implicitly defined by:

$$\sum_{i \in \mathcal{D}_s} S_i(\pi_n^s) + \sum_{i \in \mathcal{N}_s} \epsilon \pi_n^s = \sum_{i \in \mathcal{N}_s} d_i - \sum_{ij \in \mathcal{B}_s^+} K_{ij} - \sum_{i \in \mathcal{D}_s^+} x_i^+. \tag{5.15}$$

Subtracting (5.15) from (5.14) gives

$$\sum_{i \in \mathcal{D}_s} \left[S_i(\pi_n^r) - S_i(\pi_n^s)\right] + \sum_{i \in \mathcal{N}_s} \epsilon \left[\pi_n^r - \pi_n^s\right] + S_k(\pi_n^r) = x_k^+,$$

which can be rewritten as:

$$\sum_{i \in \mathcal{D}_s} \left[S_i(\pi_n^s) - S_i(\pi_n^r)\right] + \sum_{i \in \mathcal{N}_s} \epsilon \left[\pi_n^s - \pi_n^r\right] = x_k^r - x_k^+.$$

As $x_k^r > x_k^+$, $S_i{}'(p) > 0$ and $\epsilon > 0$, from above we have that $\pi_n^s > \pi_n^r$, as required. □

Lemma 5.12. *Suppose we have an optimal solution to* $\mathrm{P}^r(d)$*, for some* $r \in \mathcal{R}_n^+$ *and there exists some* $k \in \mathcal{D}_r^+$ *yielding* $\pi_n^r < S_k{}^{-1}(x_k^+)$*. There exists a KKT regime* $s \in \mathcal{R}_n^+$ *such that the solution to* $\mathrm{P}^s(d)$ *yields a price at node* n*:* $\pi_n^s > \pi_n^r$*.*

Proof. For KKT regime r, from equation (5.7) we have the following implicit function for the nodal price at node n:

$$\sum_{i \in \mathcal{D}_r} S_i(\pi_n^r) + \sum_{i \in \mathcal{N}_r} \epsilon \pi_n^r = \sum_{i \in \mathcal{N}_r} d_i - \sum_{ij \in \mathcal{B}_r^+} K_{ij} - \sum_{i \in \mathcal{D}_r^+} x_i^+. \tag{5.16}$$

Now construct a new KKT regime s, which is identical to regime r except that

$$\mathcal{D}_s^+ = \mathcal{D}_r^+ \backslash \{k\}, \quad \text{and} \quad \mathcal{D}_s = \mathcal{D}_r \cup \{k\}.$$

Again from equation (5.7), the price at node n in this regime is implicitly defined by:

$$\sum_{i \in \mathcal{D}_s} S_i(\pi_m^s) + \sum_{i \in \mathcal{N}_s} \epsilon \pi_n^s = \sum_{i \in \mathcal{N}_s} d_i - \sum_{ij \in \mathcal{B}_s^+} K_{ij} - \sum_{i \in \mathcal{D}_s^+} x_i^+. \tag{5.17}$$

Subtracting (5.16) from (5.17) gives

$$\sum_{i \in \mathcal{D}_r} \left[S_i\left(\pi_n^s\right) - S_i\left(\pi_n^r\right)\right] + \sum_{i \in \mathcal{N}_r} \epsilon\left[\pi_n^s - \pi_n^r\right] + S_k\left(\pi_n^s\right) - x_k^+ = 0,$$

As $\pi_n^r < {S_k}^{-1}\left(x_k^+\right)$ and ${S_i}'(p) > 0$, we have that

$$S_k\left(\pi_n^r\right) < x_k^+,$$

hence

$$\sum_{i \in \mathcal{D}_r} \left[S_i\left(\pi_n^s\right) - S_i\left(\pi_n^r\right)\right] + \sum_{i \in \mathcal{N}_r} \epsilon\left[\pi_n^s - \pi_n^r\right] + S_k\left(\pi_n^r\right) - S_k\left(\pi_n^s\right) > 0,$$

from above, (and as $\epsilon > 0$) we have that $\pi_n^s > \pi_n^r$, as required. □

We now prove a simple lemma in which we define a modified dispatch problem that has the same optimal solution as $\mathrm{P}(d)$ under certain conditions.

Lemma 5.13. *Suppose at the optimal solution to* $\mathrm{P}(d)$*, that* $f_{ij} > -K_{ij}, \forall ij \in \mathcal{A}_n$ *and* $x_i > x_i^-, \forall i \in \mathcal{N}$*; this solution is equivalent to the optimal solution to* $\mathrm{P}'(d)$*, given below*

$$\begin{aligned}
\mathrm{P}'(d) = \min \quad & \textstyle\sum_{i \in \mathcal{N}} \int_{x_i^-}^{x_i} S_i^{-1}(\xi)\, d\xi \\
\\
\text{s.t.} \quad & x_i - \textstyle\sum_{j, ij \in \mathcal{A}_n} f_{ij} + \sum_{j, ji \in \mathcal{A}_n} f_{ji} = d_i \quad [\pi_i] \quad \forall i \in \mathcal{N} \\
& f_{ij} \leq K_{ij} \qquad \forall ij \in \mathcal{A}_n. \\
& x_i \leq x_i^+ \qquad \forall i \in \mathcal{N}
\end{aligned}$$

Proof. By assumption, the constraints $f_{ij} \geq -K_{ij}$ and $x_i \geq x_i^-$ in $\mathrm{P}(d)$ are not binding. It is clear that if a constraint to a convex problem is not active at the optimal solution, the removal of the constraint has no bearing on the optimal solution. Therefore we can create a problem $\mathrm{P}'(d)$ without this constraint, which has the same optimal solution. □

The KKT conditions for the modified dispatch problem, $\mathrm{P}'(d)$ are

$$\begin{aligned}
& x_i - \textstyle\sum_{j, ij \in \mathcal{A}_n} f_{ij} + \sum_{j, ji \in \mathcal{A}_n} f_{ji} = d_i && \forall i \in \mathcal{N} \\
& 0 \leq \pi_i - S_i^{-1}(x_i) \ \perp\ x_i^+ - x_i \geq 0 && \forall i \in \mathcal{N} \\
& 0 \leq \pi_j - \pi_i \ \perp\ K_{ij} - f_{ij} \geq 0 && \forall ij \in \mathcal{A}_n.
\end{aligned} \tag{5.18}$$

The next theorem and corollary are the main result of this chapter. They state that if the network is initially in an unconstrained state, and the demand at node n is increased (decreased) then the nodal price at node n is given by the maximum (minimum) price over all KKT regimes $r \in \mathcal{R}_n^+$ ($r \in \mathcal{R}_n^-$).

Theorem 5.14. *Suppose for some vector of demands $d = d^*$ that all arcs are unconstrained and no dispatch quantity is at a bound at the optimal solution to* P(d)*. Suppose now that d_n, the demand at node n, is increased. Then for any value of $d_n \geq d_n^*$ the nodal price at node n is given by*

$$\pi_n^* = \max_{r \in \mathcal{R}_n^+} \pi_n^r,$$

where π_n^r is the nodal price at node n associated with KKT regime r.

Proof. Since there are only a finite number of decompositions in $\mathcal{R}_n^+$, we know that the maximum nodal price π_n^* is attained for some KKT regime. Furthermore, since the economic dispatch problem is a strictly convex quadratic program it has a unique optimal primal solution, and so there is a unique vector of prices, π satisfying (5.1). As $d_n \geq d_n^*$ and $d_i = d_i^*$, $\forall i \in \mathcal{N} \setminus \{n\}$, from lemma 5.6 we know that solution to (5.1) must correspond to some KKT regime in $\mathcal{R}_n^+$. Furthermore, from lemma 5.13, we know that the optimal solution to P(d) is identical to the solution to (5.18).

We will proceed by demonstrating that any KKT regime $r \in \mathcal{R}_n^+$ that does not satisfy the conditions (5.18), does not yield $\pi_n^r = \pi_n^*$. Thus, we will have that any regime, $r \in \mathcal{R}_n^+$, yielding $\pi_n^r = \pi_n^*$ must satisfy (5.18). Hence from above, we will have that the (unique) optimal solution to P(d) must satisfy $\pi_n = \pi_n^*$.

Suppose that we have a KKT regime $r \in \mathcal{R}_n^+$ with $\pi_n^r = \pi_n^*$. From lemma 5.3, if the optimal solution to P$^r(d)$ does not satisfy conditions (5.1) then it must be the case that at least one of the following is true:

(A) there is a congested arc $lm \in \mathcal{B}_r^+$ where $\pi_l^r > \pi_m^r$,

(B) the capacity of an arc in $\mathcal{B}_r$ is exceeded,

(C) a dispatch, x_k^r in $\mathcal{D}_r$ exceeds its capacity, or

(D) the dispatch, x_k^r at node $k \in \mathcal{D}_r^+$ is at capacity, but $\pi_n^r < S_k^{-1}\left(x_k^+\right)$.

However, as it is sufficient to satisfy the conditions (5.18), we will ignore the lower bound constraints on the flows on arcs in $\mathcal{B}_r$ and dispatches at nodes in $\mathcal{D}_r$.

For (A), from lemma 5.9, we know there exists a KKT regime $s \in \mathcal{R}_n^+$ giving $\pi_n^s > \pi_n^r$, which violates the assumption that π_n^r is maximal over all KKT regimes in $\mathcal{R}_n^+$.

Now consider case (B). From lemma 5.10, we know there exists a KKT regime $s \in \mathcal{R}_n^+$ giving $\pi_n^s > \pi_n^r$, which violates the assumption that π_n^r is maximal over all KKT regimes in $\mathcal{R}_n^+$.

For (C), from lemma 5.11, we know there exists a KKT regime $s \in \mathcal{R}_n^+$ giving $\pi_n^s > \pi_n^r$, which violates the assumption that π_n^r is maximal over all KKT regimes in $\mathcal{R}_n^+$.

Finally, for (D), from lemma 5.12, we know there exists a KKT regime $s \in \mathcal{R}_n^+$ giving $\pi_n^s > \pi_n^r$, which violates the assumption that π_n^r is maximal over all KKT regimes in $\mathcal{R}_n^+$.

From above, if we have a KKT regime in $\mathcal{R}_n^+$ that does not satisfy the conditions (5.18), there exists another KKT regime in $\mathcal{R}_n^+$ yielding a higher price at node n. Due to there being a unique solution to P(d) and a finite number of KKT regimes, the KKT regime corresponding to the highest price at node n yields the nodal price associated with P(d). □

Corollary 5.15. *Suppose that for some vector of injections $d = d^*$ that all arcs are unconstrained at the optimal dispatch. Suppose now that d_n, the demand at node n, is decreased. Then for any value of $d_n \leq d_n^*$ the nodal price at node n is given by*

$$\pi_n^* = \min_{r \in \mathcal{R}_n^-} \pi_n^r,$$

where π_n^r is the nodal price at node n associated with KKT regime r.

Proof. The proof of this result is analogous to that proved for theorem 5.14 above. □

The above results are important when we examine Cournot games over radial networks in chapter 11.

5.3 Linear supply functions

Suppose the generators each submit linear supply functions $S_i(p) = a_i p, \forall i \in \mathcal{N}$. Then the nodal price at node n for some KKT regime $r \in \mathcal{R}_n$ given by equation (5.7) simplifies to

$$\pi_n^r = \frac{\sum\limits_{i \in \mathcal{N}_r} d_i - \sum\limits_{ij \in \mathcal{B}_r^+} K_{ij} + \sum\limits_{ij \in \mathcal{B}_r^-} K_{ij} - \sum\limits_{i \in \mathcal{D}_r^-} x_i^- - \sum\limits_{i \in \mathcal{D}_r^+} x_i^+}{\sum\limits_{i \in \mathcal{D}_r} a_i + \sum\limits_{i \in \mathcal{N}_r} \epsilon},$$

this nodal price is a linear function of d.

If there exists some demand vector $d = d^*$ such that no line is constrained and no dispatch is at a bound at the optimal solution to P(d^*), then from theorem 5.14 and corollary 5.15 we know that if the i^{th} component of d^* is reduced, then the optimal nodal price, as a function of d_i, is the pointwise minimum of a set of linear functions (yielding a concave function). Whereas if the i^{th} component of d^* is increased, then the nodal price, as a function of d_i, is the pointwise maximum of a set of linear functions (yielding a convex function). An illustration of such a function of d_i is shown in figure 5.5.

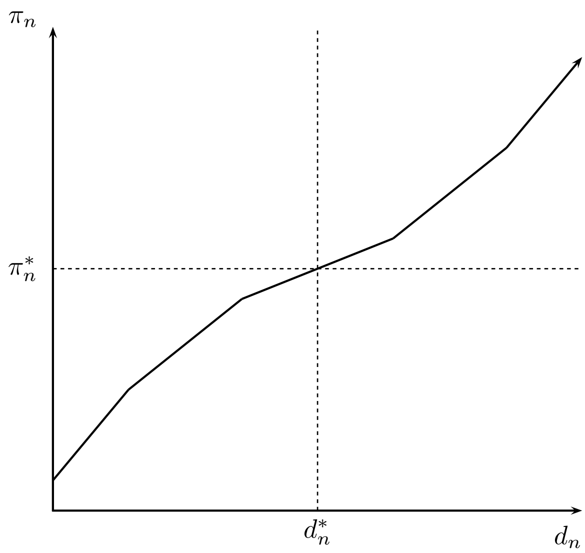

Figure 5.5: Price as a function of demand at node n.

Chapter 6

Transmission losses

In this chapter, we examine losses in transmission networks. We consider different approximations for these losses and examine how these approximations can affect the prices from the resulting dispatch problems.

We will show that with quadratic losses, the dispatch problem ceases to be a convex program. However, under certain conditions, we prove that there exists a convex program equivalent to the dispatch problem with quadratic losses (with the same optimal solution). Alternatively, when no such equivalent convex program exists, we demonstrate that the objective value function of the dispatch problem may cease to be a convex function of the demand at each node. In this situation, the price at a node may decrease with additional demand at the same node.

This chapter is laid out as follows. We first present two methods of approximating quadratic losses, and discuss how nodal prices are affected by a lossy line. We then prove a theorem which gives a set of conditions ensuring that the dispatch problem with losses is equivalent to a convex program. Finally, we discuss the effect of losses on networks with loops. The results of this chapter are important when we consider Cournot games over lossy networks in chapter 14.

6.1 Line loss approximations

There are various ways to approximate losses in a DC load flow model, of which we will consider two methods in this chapter. The first approximation, as used by Borenstein *et al.* in [12] and Hobbs *et al.* in [46], is to consider losses to be a quadratic function of the amount of power that is sent, i.e.

$$f_r = f_s - r{f_s}^2, \qquad \text{(a)}$$

where f_r is the power received and f_s is the power sent. With this method, if the amount of power sent exceeds $f_s = \frac{1}{2r}$ the total amount of electricity received begins to decrease, as shown in figure 6.1. Moreover, note that from the above quadratic equation we can compute that the maximum power that can be received is $f_r = \frac{1}{4r}$. Later, we will show that this approximation is poor and should only be used when the power flow on the line is known to be small.

A better approximation is to assume that the amount of power lost is proportional to the average flow on a line squared, as presented in chapter 3, and used in [22, 76]. Specifically, if we have two nodes joined by a line and $f_s = f + \frac{1}{2}rf^2$ is sent from node 1 along the line, then the amount arriving at node 2 will be $f_r = f - \frac{1}{2}rf^2$, where f is the average flow. Note that the total loss in this case is rf^2, hence the losses are proportional to the average of what is sent and what arrives; whereas for approximation (a) above, it is proportional to the square of the sent power.

For the second approximation, we will compute the amount of electricity arriving at a node as a function of what is sent. First, solving for f as a function of f_s gives

$$f = \frac{-1 \pm \sqrt{1 + 2rf_s}}{r}.$$

We can ignore the solution giving $f < 0$, since this is not physically possible. Solving for f_r then gives

$$f_r = \frac{2\left(\sqrt{1 + 2rf_s} - 1\right)}{r} - f_s. \qquad \text{(b)}$$

Under this approximation, when $f_s = \frac{3}{2r}$ the marginal contribution of any electricity sent down the line is zero, and at this point the amount of power received is $f_r = \frac{1}{2r}$ and the average flow on the line is $f = \frac{1}{r}$. Figure 6.1 shows plots of the loss approximations, compared with the lossless case.

The two approximations seem to be quite similar for $0 \leq f_s \leq \frac{1}{4r}$, however, figure 6.1 only shows the quantities of power sent and arriving, not the marginal losses. Since the nodal prices are based on the marginal cost of suppling power to a node, the optimal solution to the dispatch problem may be vary considerably depending on the loss approximation used. Figure 6.2 compares the derivative of the power that arrives with respect to what is sent for the two approximations.

From figure 6.2, we can see that, in fact, if the amount of power sent exceeds $\frac{1}{10r}$, then the marginal losses of the two approximations (and hence the nodal prices) begin to differ. For the remainder of this section we will model losses using approximation (b).

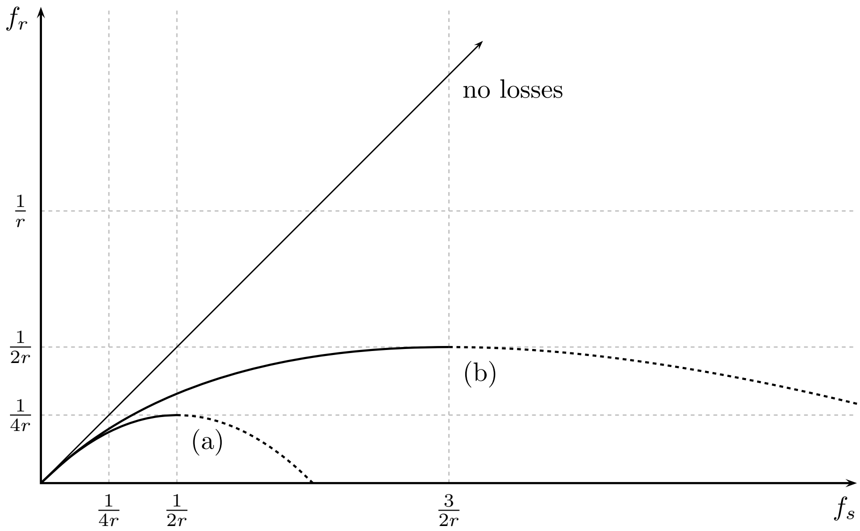

Figure 6.1: Approximations of line losses.

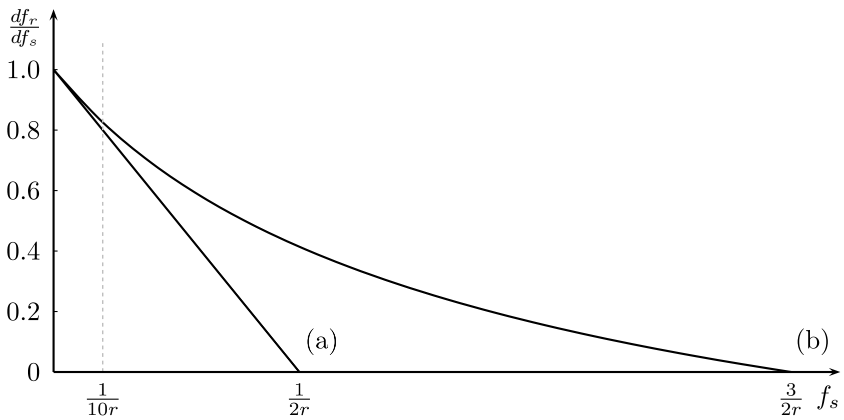

Figure 6.2: Marginal power received as a function of power sent.

6.2 Economic dispatch problem

We will now define an altered dispatch problem that includes losses of the type described in method (b) above:

$$
\begin{aligned}
\mathrm{P}\,(d) : \min \quad & \sum_{i \in \mathcal{N}} \int_{x_i^-}^{x_i} S_i^{-1}(\xi)\,.d\xi \\
\text{s.t.} \quad & x_i - \sum_{j, ij \in \mathcal{A}} \left(f_{ij} + \tfrac{1}{2} r_{ij} {f_{ij}}^2 \right) + \sum_{j, ji \in \mathcal{A}} \left(f_{ji} - \tfrac{1}{2} r_{ji} {f_{ji}}^2 \right) = d_i \quad [\pi_i] \quad \forall i \in \mathcal{N} \\
& \sum_{ij \in \mathcal{A}} L_{ij,k} f_{ij} = 0 \quad [\lambda_k] \quad \forall k \in \mathcal{L} \\
& -x_i^- \le x_i \le x_i^+ \quad \left[\mu_{ij}^+, \mu_{ij}^-\right] \quad \forall i \in \mathcal{N} \\
& -K_{ij} \le f_{ij} \le K_{ij} \quad \left[\eta_{ij}^+, \eta_{ij}^-\right] \quad \forall ij \in \mathcal{A}
\end{aligned}
$$

The only change in this dispatch problem from the one introduced in chapter 4 is that the node balance constraints now contain the quadratic losses. Since the node balance constraints are equality constraints, the presence of these quadratic terms means that the dispatch problem no longer has a convex feasible region.

Due to this dispatch problem having a non-convex feasible region, its KKT conditions (as they are first order conditions) are not sufficient for optimality, as discussed in [73]. This means that a solution that satisfies the KKT conditions of a non-convex optimization problem may be a local minimum, saddle point or even a local maximum. Thus the task of computing the optimal solution to the dispatch problem is more difficult than when dealing with the lossless transmission case.

However, so long as P(d) satisfies a *constraint qualification*, the KKT conditions are necessary for the optimal solution to P(d), or in order words, the optimal solution is a KKT point [57]. From the *weak reverse* constraint qualification, discussed in [37], we have that if a mathematical program satisfies the following conditions:

(1) there exists some solution x_0 in the interior of the feasible region,

(2) the equality constraints are pseudolinear (both pseudoconvex and pseudoconcave at all interior solutions)[1], and

(3) the inequality constraints are pseudoconcave at all interior solutions[2],

then the optimal solution satisfies the KKT conditions.

From the formulation of P(d) above, we can see that the inequality constraints are linear, hence condition (3) is satisfied. The loop-flow constraints are affine so they comply with condition (2). The node balance constraints are not linear, but they are each monotonic with respect to x; and so long as $K_{ij} < \frac{1}{r_{ij}}$ the constraints are monotonic with respect to f_{ij}. Therefore the node balance constraints are pseudolinear, complying with condition (2). Finally, we require that there exists some interior point; this is not true for all choices of parameters. Therefore we will assume that the parameters are chosen such that there is an interior point in the set of feasible solutions. For example, if the upper bounds on fringes were sufficiently large, then the demand at each node could be satisfied without any power transfer between nodes; this would be an interior solution.

[1] From [65] we know that a function of a single variable is a pseudolinear function if it is either strictly increasing or strictly decreasing.

[2] Note that all concave functions are pseudoconcave; see [65].

The KKT conditions of P(d) are given by (6.1) below.

$$
\begin{aligned}
& x_i - \sum_{j,ij\in\mathcal{A}} \left(f_{ij} + \tfrac{1}{2} r_{ij} {f_{ij}}^2\right) + \sum_{j,ji\in\mathcal{A}} \left(f_{ji} - \tfrac{1}{2} r_{ji} {f_{ji}}^2\right) = d_i && \forall i \in \mathcal{N} \\
& \sum_{ij\in\mathcal{A}} l_{ijk} f_{ij} = 0 && \forall k \in \mathcal{L} \\
& S_i^{-1}(x_i) - \pi_i + \mu_i^+ - \mu_i^- = 0 && \forall i \in \mathcal{N} \\
& \pi_i (1 + r_{ij} f_{ij}) - \pi_j (1 - r_{ij} f_{ij}) + \eta_{ij}^+ - \eta_{ij}^- + \sum_{k\in\mathcal{L}} L_{ij,k} \lambda_k = 0 && \forall ij \in \mathcal{A} \\
& 0 \le x_i^+ - x_i \perp \mu_i^+ \ge 0 && \forall i \in \mathcal{N} \\
& 0 \le x_i - x_i^- \perp \mu_i^- \ge 0 && \forall i \in \mathcal{N} \\
& 0 \le K_{ij} - f_{ij} \perp \eta_{ij}^+ \ge 0 && \forall ij \in \mathcal{A} \\
& 0 \le K_{ij} + f_{ij} \perp \eta_{ij}^- \ge 0 && \forall ij \in \mathcal{A}.
\end{aligned}
\tag{6.1}
$$

Since P(d) satisfies a constraint qualification, these conditions are necessary (but not sufficient) for an optimal solution to P(d).

In the next section, we will discuss how we can avoid the non-convexity issues caused by quadratic losses, in order to find the optimal solution of the dispatch problem.

6.2.1 Convexity of dispatch problem

Consider the problem $\mathrm{P}^C(d)$; this is a relaxed version of the actual dispatch problem, as discussed previously by Neuhoff *et al.* in [70]. Here the equality node balance constraints have been replaced by inequalities. Thus, the feasible region of this altered problem is now convex. This means that the KKT conditions of $\mathrm{P}^C(d)$ are both necessary and sufficient for global optimality [57].

$$
\begin{aligned}
\mathrm{P}^C(d): \min \quad & \sum_{i\in\mathcal{N}} \int_{x_i^-}^{x_i} S_i^{-1}(\xi)\, .d\xi \\
\text{s.t.} \quad & x_i - \sum_{j,ij\in\mathcal{A}} \left(f_{ij} + \tfrac{1}{2} r_{ij} {f_{ij}}^2\right) + \sum_{j,ji\in\mathcal{A}} \left(f_{ji} - \tfrac{1}{2} r_{ji} {f_{ji}}^2\right) \ge d_i && [\pi_i] && \forall i \in \mathcal{N} \\
& \sum_{ij\in\mathcal{A}} L_{ij,k} f_{ij} = 0 && [\lambda_k] && \forall k \in \mathcal{L} \\
& -x_i^- \le x_i \le x_i^+ && [\mu_{ij}^+, \mu_{ij}^-] && \forall i \in \mathcal{N} \\
& -K_{ij} \le f_{ij} \le K_{ij} && [\eta_{ij}^+, \eta_{ij}^-] && \forall ij \in \mathcal{A}.
\end{aligned}
$$

Now we will present a theorem that gives sufficient conditions for a solution to P(d) to be globally optimal.

Theorem 6.1. *Consider a solution that satisfies the KKT conditions of* $\mathrm{P}^C(d)$ *with* $\pi_i > 0$, $\forall i \in \mathcal{N}$*. This solution is the (global) optimal solution for* P(d).

Proof. Since the solution satisfies the KKT conditions of $\mathrm{P}^C(d)$, it is the optimal solution for $\mathrm{P}^C(d)$. Moreover, since $\pi_i > 0$, $\forall i \in \mathcal{N}$, the node balance constraints must be binding. Therefore this solution is feasible for P(d). The result follows from $\mathrm{P}^C(d)$ being a relaxation of P(d). □

From the theorem above, we know that if we have a solution to the KKT conditions of $\mathrm{P}^C(d)$ then so long as all the nodal prices are positive, the solution is optimal for P(d). This result is useful for analysing radial networks with losses; here nodal prices are positive (so long as offer prices, for positive quantities, are positive and the problem is feasible). In chapter 14, we consider Cournot games over a radial network with quadratic losses and investigate how the prices coming from such a dispatch problem can affect the concavity of generators' revenue functions, and the outcomes of the Cournot games.

However, in networks with loops, the non-convexity of the dispatch problem is an issue. In networks with loops, negative prices are possible even with positive offer prices (as demonstrated by the spring-washer effect in section 4.3). A negative price at a node reflects that an increase in demand at that node will reduce the cost of dispatch. These potential negative prices make it impossible to invoke theorem 6.1 above, to replace the dispatch problem with a equivalent convex problem. In the next section we will use an example to explore these issues in detail.

6.2.2 Loops and losses

When we are dealing with networks with loops and quadratic losses, the KKT conditions of the dispatch problem, given earlier in (6.1), are necessary, but not sufficient for a solution to be optimal (so long as the aforementioned constraint qualification is satisfied). These conditions form a non-linear complementarity problem. This problem can have multiple solutions (representing local maxima, minima and saddle-points for P(q)) and it can therefore be very difficult to find the global minimum of P(q). In fact, the optimal value function for P(q) may cease to be a convex function of demand; this means that prices may in fact be decreasing with increased demand at a node. The following example examines these issues.

Example 6.2. *Suppose we have an electricity network consisting of a single three-node loop, as shown in figure 6.3. For simplicity, assume that all lines have equal reactances, and only line* 23 *has a thermal capacity,* 100. *Furthermore, line* 13 *will be the only lossy line with a quadratic loss coefficient of* r_{13}.

There is one generator at node 1 *offering power at price,* $p_1 = 10$ *and one at node* 2 *offering power at price,* $p_2 = 50$. *The demand at nodes* 1 *and* 2 *are fixed at* $d_1 = 0$, $d_2 = 2000$ *respectively, whereas the demand at node* 3 *will be treated as a parameter,* d_3.

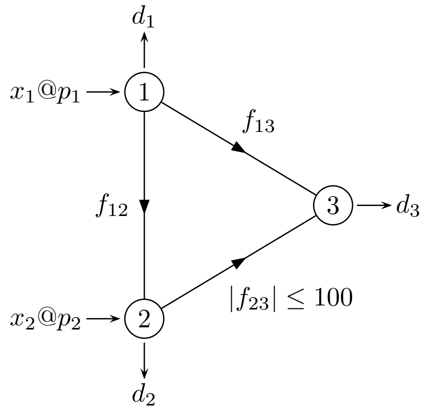

Figure 6.3: Three-node loop with quadratic losses.

The dispatch problem for this scenario is therefore:

$$
\begin{aligned}
P(d_3): \min \quad & 10x_1 + 50x_2 \\
\text{s.t.} \quad & x_1 - f_{12} - f_{13} - \tfrac{1}{2} r_{13} {f_{13}}^2 = 0 && [\pi_1] \\
& x_2 + f_{12} - f_{23} = 2000 && [\pi_2] \\
& f_{13} + f_{23} - \tfrac{1}{2} r_{13} {f_{13}}^2 = d_3 && [\pi_3] \\
& f_{12} - f_{13} + f_{23} = 0 && [\lambda_1] \\
& |f_{23}| \leq 100 && [\eta_{23}^1, \eta_{23}^2] \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

If we initially consider the situation where $r_{13} = 0$, then we actually have a lossless network. The (optimal) cost of dispatch as a function of d_3 is

$$
\begin{aligned}
C(d_3) &= 10x_1 + 50x_2 \\
&= \begin{cases} \text{infeasible}, & d_3 < -150, \\ 88000 - 30d_3, & -150 \leq d_3 \leq 1700, \\ 20000 + 10d_3, & 1700 \leq d_3 \leq 2300, \\ \text{infeasible}, & d_3 > 2300. \end{cases}
\end{aligned}
$$

The above optimal value function is piecewise linear and convex. On the other hand, when the lossy line is reinstated with $r_{13} = \frac{1}{5000}$, the cost function becomes

$$\begin{aligned} C(d_3) &= 10x_1 + 50x_2 \\ &= \begin{cases} \text{infeasible}, & d_3 < -150.4, \\ -10000 + 2000\sqrt{2400 - d_3} - 10d_3, & -150.4 \le d_3 \le 1376, \\ 225000 - 300\sqrt{4625 - d_3} - d_3, & 1376 \le d_3 \le 1816, \\ \text{infeasible}, & d_3 > 1816. \end{cases} \end{aligned}$$

From the plot of this function, given in figure 6.4, it is clear that for $-150.4 \le d_3 \le 1376$ the optimal value function is concave. This observation can be verified from the sign of the second derivative of the expression above. Note that as the optimal value function is decreasing in d_3 this implies a negative price at node 3. Furthermore, since this function is concave, the nodal price is in fact decreasing with demand at the associated node. This provides a counter-example showing that theorem 4.3 does not extend to networks with quadratic losses.

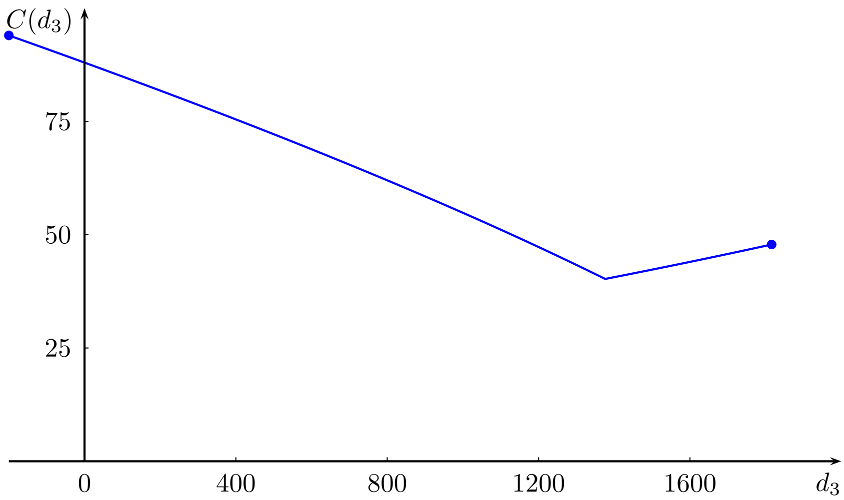

Figure 6.4: Optimal value function, $C(d_3)$, with $r_{13} = \frac{1}{5000}$.

Note that, in the example above, the optimal value function is concave with respect to the demand at node 3 when the price at node 3 is negative. A question that arises from this is whether this behaviour can occur at a node when its price is positive. In fact, it can. In the example below we augment the network by adding an additional loop. Over this network, we demonstrate that the optimal value function can be concave and increasing with respect to changing demand at node 4.

Example 6.3. *Suppose we have an electricity network consisting of two loops, as shown in figure 6.5. Again, we will assume that all lines have equal reactances, and lines* 23 *and* 35 *have thermal capacities,* $K_{23} = 100$ *and* $K_{35} = 500$*. Finally, as in the previous example, line* 13 *will be the only lossy line, with a quadratic loss coefficient of* r_{13}*.*

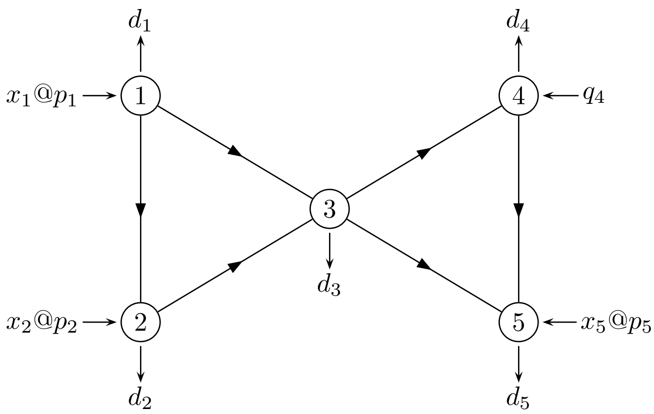

Figure 6.5: Five nodes with two loops.

Nodes 1, 2 *and* 5 *each have a single generator offering electricity at prices* $p_1 = 10$, $p_2 = 50$, $p_5 = 150$, *respectively. The demand at nodes* 1, 2, 3, 5 *are fixed at* $d_1 = 0$, $d_2 = 2000$, $d_3 = 0$ *and* $d_5 = 500$ *respectively, whereas the demand are node* 4 *will be treated as a parameter,* d_4.

This dispatch problem can be written as the following optimization problem:

$$
\begin{aligned}
P\left(q_4\right): \min \quad & 10x_1 + 50x_2 + 150x_5 \\
\\
\text{s.t.} \quad & x_1 - f_{12} - f_{13} - \tfrac{1}{2} r_{13} {f_{13}}^2 = 0 && [\pi_1] \\
& x_2 + f_{12} - f_{23} = 2000 && [\pi_2] \\
& f_{13} + f_{23} - f_{34} - f_{35} - \tfrac{1}{2} r_{13} {f_{13}}^2 = 0 && [\pi_3] \\
& f_{34} - f_{45} = d_4 && [\pi_4] \\
& x_5 + f_{35} + f_{45} = 500 && [\pi_5] \\
& f_{12} - f_{13} + f_{23} = 0 && [\lambda_1] \\
& f_{34} - f_{35} + f_{45} = 0 && [\lambda_2] \\
& |f_{23}| \leq 100 && \left[\eta_{23}^1, \eta_{23}^2\right] \\
& |f_{35}| \leq 500 && \left[\eta_{35}^1, \eta_{35}^2\right] \\
& x_1, x_2, x_5 \geq 0.
\end{aligned}
$$

For the instance where $d_1 = 0$, $d_2 = 3000$, $d_3 = 0$, $d_4 = 1500$, $d_5 = 500$, $K_{23} = 100$, $K_{35} = 250$, $p_1 = 10$, $p_2 = 50$ and $p_3 = 150$, we will investigate the dispatch problem's optimal value as a function of d_4, the demand at node 4.

If we consider the situation where $r_{13} = \frac{1}{5000}$, the optimal value as a function of d_4 is

$$\begin{aligned} C(d_4) &= 10x_1 + 50x_2 + 150x_5 \\ &= \begin{cases} -15000 + 2000\sqrt{1900 - d_4} - 10d_4, & 0 \le d_4 \le 500, \\ -55000 + 1000\sqrt{6600 - 2d_4} + 70d_4, & 500 \le d_4 \le 1252, \\ 180000 - 1500\sqrt{15500 - 2d_4} + 70d_4, & 1252 \le d_4 \le 2132, \\ \text{infeasible}, & d_4 > 2132. \end{cases} \end{aligned}$$

This function, plotted in figure 6.6, consists of three pieces: the first and second are concave, and the third is convex.

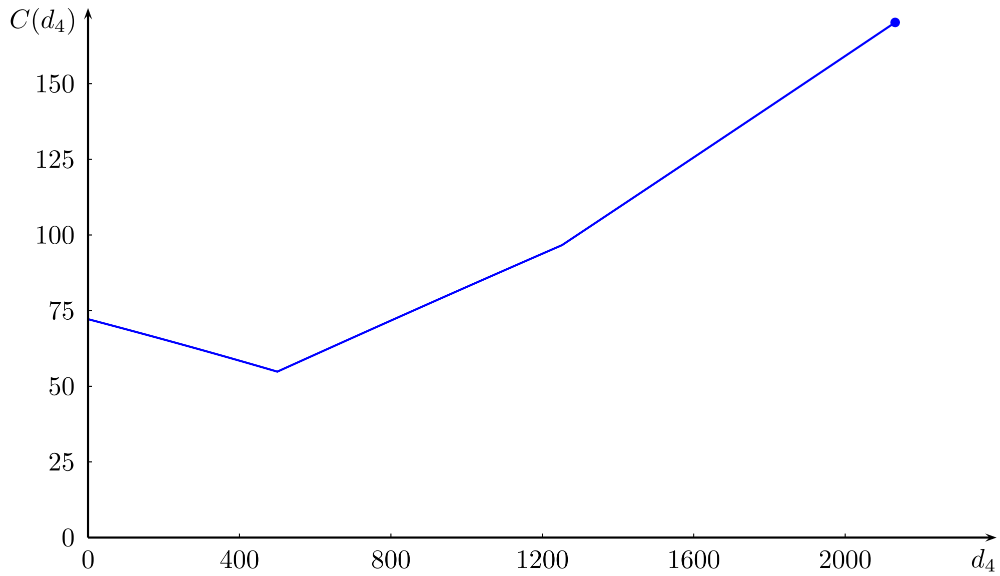

Figure 6.6: Optimal value function, $C(d_4)$.

The subgradients of this function with respect to d_4 give the nodal price at node 4. This is given by the following piecewise point to set mapping:

$$\begin{aligned} \pi_4 &= 10x_1 + 50x_2 + 150x_5 \\ &= \begin{cases} -10 - \frac{1000}{\sqrt{1900 - d_4}}, & 0 \le d_4 < 500, \\ \left[-10 - 50\sqrt{\frac{2}{7}}, 70 - 25\sqrt{\frac{2}{7}}\right], & d_4 = 500, \\ 70 - \frac{1000}{\sqrt{6600-2d_4}}, & 500 < d_4 < 1252, \\ \left[\frac{435}{8}, \frac{1580}{19}\right], & d_4 = 1252, \\ 70 + \frac{1500}{\sqrt{15500-2d_4}}, & 1252 < d_4 < 2132, \\ \left[\frac{4460}{53}, \infty\right), & d_4 = 2132. \end{cases} \end{aligned}$$

This point to set mapping is plotted in figure 6.7 below. Here we can see that the price is in fact decreasing with d_4, up until the demand reaches 500, at which point the price jumps up but is still decreasing in d_4. When d_4 reaches 1252, the price jumps up again, but the price is now increasing with demand. Finally at $d_4 = 2132$ the price becomes unbounded as we approach infeasibility.

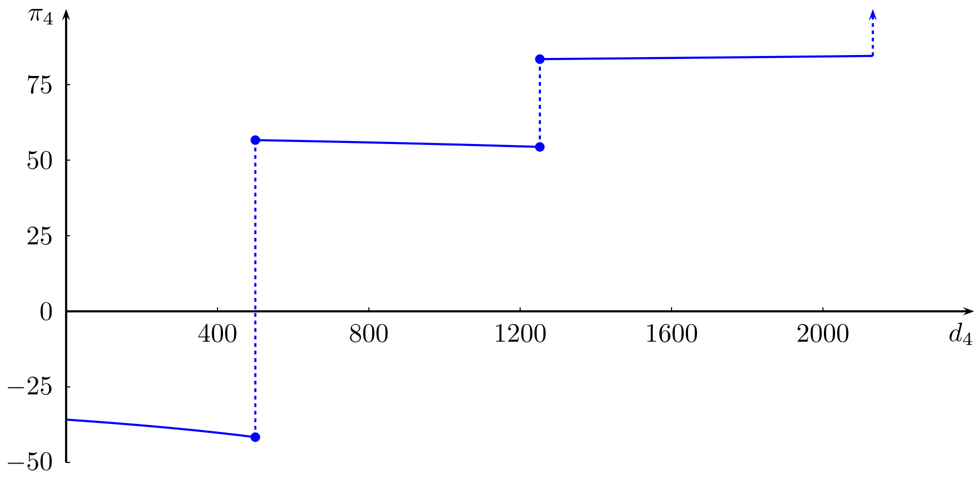

Figure 6.7: Price at node 4 as a function of d_4.

We proved in theorem 6.1 that so long as all the nodal prices are positive at the optimal solution to $\mathrm{P}^C(d)$, then that solution is also optimal for the original dispatch problem with the node balance equation being equality-constrained. Since $P^C(d)$ is a convex problem, we know that the nodal price at a node must be increasing with demand at that node. The above examples show that if there is at least one negative price in the network the optimal value function may cease to be concave. This non-concavity yields the result that the nodal price at a node may, in fact, decrease when demand is increased at that node.

In such a situation, the incentives of the participants may be counter-intuitive. A generator may be able to increase its nodal price by producing more, whereas a consumer may be able to reduce the price at its node by increasing its load. Due to the non-convexity of the dispatch problem, and the possibility of these peculiar incentives, we do not consider networks with both loops and losses, in the remainder of this work.

Part III

Cournot games in electricity markets with transmission

Chapter 7

Modelling strategic behaviour

In this chapter, we discuss modelling of competition between generators as Cournot games. We begin with a comprehensive review of the how these models are used and discuss the issues that are faced. We then discuss how we set up the Cournot game over a transmission network and how price elasticity is modelled in the market. Finally we present the strategy space and payoff functions for the firms competing in the the market.

7.1 Background

Over the past two decades, the power industries in many countries have been deregulated. They have transitioned from the traditionally centrally managed operations by the creation of electricity markets and the introduction of competition.

Due to the high cost of building generation plants and connecting them to the grid, electricity markets typically feature a limited number of competing firms. In economics this type of market structure is often referred to as an *oligopoly*. With this type of market structure, generators may have incentive to behave in a strategic manner in order to increase their respective profits.

In electricity markets, risk is an important factor that all market participants must consider. Due to uncertainty in future demand growth and changes in availability of supply[1] there are constant fluctuations in electricity prices. In order for market participants to have some certainty about their revenue and costs, generators and retailers will often enter into bilateral contracts for differences (CFDs). These contracts serve to reduce the exposure of generators or retailers to the volatility of

[1] For example, due to changing gas prices and reservoir levels or unexpected outages.

the wholesale prices. Each generator may wish enter into a contract with a electricity retailer at any node. Alternatively, in countries such as New Zealand where *vertical integration* (firms owning generation plants and having retail customers) is common, this can be modelled as an implicit contract equal to the size of the retail load.

One of the most basic game-theoretic models of oligopolies is the Cournot game [20]; in an electricity market setting, this consists of generators choosing quantities of power to inject into the grid at their node, and the price being determined from the dispatch problem discussed in chapter 4. Due to their relative tractability as compared with more complicated models such as supply function games (for example [42, 56]), Cournot models are commonly used to model the strategic behaviour of generators within electricity pool markets, for example [12, 70, 99].[2]

Typically Cournot models require demand elasticity in order for price to vary as a function of the production. However, in wholesale electricity markets, demand is relatively inelastic, because a large proportion of the load is on fixed-price contracts (for example residential and commercial customers) and is not exposed to the wholesale spot price; although as smart-metering technology becomes more wide-spread, and electricity retailer pricing structures change, demand will become more price-responsive [13]. However, industrial users exposed to the spot price may have incentive to reduce their load when prices are high [87]. Alternatively the elasticity may come about from a *competitive fringe*, which is an aggregation of offers in the market from generally small generators who are not behaving in a strategic manner [11].

Rosen in [84] gives conditions for the existence and uniqueness of pure-strategy equilibria in continuous games. For single-node electricity markets (with no transmission lines) Rosen's conditions can be shown to be satisfied for linear fringe supply functions, guaranteeing that there exists a unique equilibrium.

This work is focused on the impact that transmission networks have on the outcomes of these Cournot games. When the classic single price Cournot model is extended to electricity markets with nodal pricing, the assumptions that are made can greatly affect the outcomes of the model. Yao, Oren and Adler discuss in [99] different types of Cournot equilibrium models that can be constructed by altering the assumptions that are made about the rationality of generators with respect to transmission.

Arguably the most realistic of these is the *full-rationality* assumption. In this setting, generators

[2]The use of a Cournot model will typically result in higher prices than a similarly defined supply function model; this is because supply function offers increase the apparent demand elasticity in the market, whereas Cournot offers do not.

anticipate how their injection affects the congestion of transmission lines, and hence their prices. In the full-rationality setting, the generators act (simultaneously) as leaders, each choosing a quantity of electricity to inject into the grid at their node. The generators when making their injection decision, anticipate the prices that will be determined by the system operator, who acts a follower in the game.

An early contribution to the understanding of this model was provided by Borenstein, Bushnell and Stoft in [12]; here the authors give examples of Cournot games over a two-node network with linear demand functions. They show, in a symmetric two-node scenario, for a sufficiently large capacity on the line, that there is no flow on the line at equilibrium. However, they demonstrate that there exists an interval of positive line capacities for which there are no pure-strategy equilibria. This result is important as it highlights that a line does not necessarily need to be utilised in order for it to facilitate competition between generators in different parts of the network. Furthermore, in asymmetrical situations (where demand is not the same at both nodes), they show that depending on the parameters of the model, there may either be a unique Nash-Cournot equilibrium, multiple equilibria, or no equilibria.

The results of Borenstein *et al.* arise because in the full-rationality Cournot model with transmission, the profits of generators, as a function of their own injection, cease to be concave and can often have multiple local maxima. This occurs because the generators are aware of how congestion in the transmission network will affect their nodal prices. The shapes of these profit functions can mean that pure strategy Nash equilibria are not guaranteed to exist or to be unique (see for example [99]).

The development of models for solving equilibrium models over transmission networks is an area of research gaining considerable attention as market regulators try to reduce the opportunity for generators to exploit their positions in the market. The equilibria to these games are given by the simultaneous solution to a set of bi-level optimization models. These can be modelled as equilibrium problems with equilibrium constraints (EPECs), as discussed by Hu and Ralph in [50]. Here each generator maximizes its profit by solving a mathematical program with equilibrium constraints (MPEC) [47]; these problems are non-convex because of generators' abilities to cause lines to become congested. In a Cournot game, the generators act simultaneously, so the solution to the EPEC is given by the simultaneous solution to the MPECs for all of the generators. This problem is inherently difficult to solve, because the first order conditions are not sufficient to guarantee global optimality for each MPEC. The solutions computed by Hu and Ralph (using first-order and second-order conditions) are referred to as local Nash equilibria, but are not necessarily true Nash-

Cournot equilibria, because each generator is at some local (but not necessarily global) optimum.

Due to the issues of existence and uniqueness of equilibria under the assumption of full-rationality, Yao *et al.* in [99] introduce bounded-rationality Cournot models, where generators do not anticipate the effect that they may have on congestion in the transmission grid. In their work they consider two bounded-rationality models, one where the generators assume that power flows on the lines are fixed, and one where they assume that the price difference between nodes is fixed. For each of these models, with linear demand curves, they prove that there exists a unique equilibrium regardless of the parameters of the model. Another bounded rationality model was proposed by Barquin and Vazquez in [7]. In this model, an assumption is made about the state of all lines in the network (whether or not they are congested and in which direction) and each of the generators believes that this state will remain unchanged regardless of changes to their injection. An iterative approach is then employed to find a state of the network for which the optimal state of the network is consistent with the generators' beliefs. We discuss these models in more detail later in chapter 8.

The main focus of the subsequent chapters is to analyse the full-rationality model and derive conditions guaranteeing that pure-strategy equilibria exist. Borenstein *et al.* in [12] perform this analysis for a two-node network, deriving the conditions on the transmission line ensuring existence of particular types of equilibria. In this work, we extend this idea to more general networks and demand functions and derive conditions on the parameters of the game which ensure that a particular equilibrium exists. The difficulty in deriving such conditions is due to the combinatorial nature of problems involving congested transmission networks; in the worst case, the number of ways that the grid can become congested increases exponentially with the number of lines in the network. However, we show that when dealing with radial networks the problem simplifies in such a way that the set of line capacities that are necessary and sufficient for the existence of an unconstrained Nash-Cournot equilibrium is a convex polyhedral set.

In the next section we will discuss the Cournot game in detail, giving the players of the game and their payoff functions.

7.2 Cournot model

In this section, we will present the details of the Cournot model upon which many of the subsequent chapters rely. We will first discuss how we model a competitive fringe, which creates price elasticity in the market. This elasticity (whether it be from demand-side bidding or a competitive fringe) is required in a Cournot game since these bids or offers set the nodal prices. We will then discuss the behaviour and payoff functions of the strategic firms: first in the case where each generator is independent; and then we augment this formulation to allow for contracts for differences and also allow for firms that own multiple generation plants.

7.2.1 Fringe generation

In the formulation of the dispatch problem given in chapter 4, it was assumed that there was one generator at each node, $i \in \mathcal{N}$, offering a supply function: $S_i(p)$. In order to create a Cournot model, we will assume that each such generator represents a competitive fringe, and $S_i(p)$ is the combined fringe supply function for node i. The fringe supply functions are therefore fixed throughout the game and provide price elasticity at each node.

7.2.2 Strategic firms

Here we will introduce the strategic generators that inject into the network. Each strategic generator attempts to maximize its own profit. The set $\mathcal{G}$ consists of all such generators in the game; moreover, we define $\mathcal{G}_i$ to be the set of generators located at node i, where $\mathcal{G}_i$, for all i in $\mathcal{N}$ are disjoint subsets of $\mathcal{G}$, (each generator is located at exactly one node). The vector V gives the capacities of all the generators, hence V_g is the capacity for generator g. In the Cournot game, each generator, $g \in \mathcal{G}_i$ chooses a quantity $q_g \in [0, V_g]$ to inject at node i in order to maximize its profit. For simplicity, we treat this as an injection at that node in the dispatch problem (as opposed to an offer at \$0).[3]

[3] So long as nodal prices are positive, an injection of power at a node is equivalent to offering that same amount of power at price \$0.

Dispatch problem

With the inclusion of strategic generators, the lossless dispatch problem, introduced in chapter 4, becomes

$$\mathrm{P}\left(q\right): \min \quad \sum_{i \in \mathcal{N}} \int_{x_i^-}^{x_i} S_i^{-1}\left(\xi\right).d\xi$$

$$\begin{array}{llll} \text{s.t.} & x_i - \sum\limits_{j,ij \in \mathcal{A}} f_{ij} + \sum\limits_{j,ji \in \mathcal{A}} f_{ji} = d_i - \sum\limits_{g \in \mathcal{G}_i} q_g & [\pi_i] & \forall i \in \mathcal{N} \\ & \sum\limits_{ij \in \mathcal{A}} L_{ij,k} f_{ij} = 0 & [\lambda_k] & \forall i \in \mathcal{L} \\ & x_i \le x_i^+ & \left[\mu_i^+\right] & \forall i \in \mathcal{N} \\ & x_i \ge x_i^- & \left[\mu_i^-\right] & \forall i \in \mathcal{N} \\ & f_{ij} \le K_{ij} & \left[\eta_{ij}^+\right] & \forall ij \in \mathcal{A} \\ & f_{ij} \ge -K_{ij} & \left[\eta_{ij}^-\right] & \forall ij \in \mathcal{A}. \end{array}$$

Note that we have presented the dispatch problem, $\mathrm{P}(q)$, as a function of the vector of injections, q as opposed to the vector of demands d. We are now interested in how the nodal prices, π, vary as functions of the strategic generators' generation levels. In particular, since the generators are maximizing their profits, we will examine how the nodal price varies as a function of injection at that node. We will now present a simple lemma, allowing us to apply the results of part II (which relate to changes in demand) to changes in injection quantity.

Lemma 7.1. *Consider the injection level q_g for each generator $g \in \mathcal{G}$. Suppose now that for some $g \in \mathcal{G}_i$ the injection is increased by δ. This is equivalent to instead decreasing the demand at node i by δ.*

Proof. The right-hand side of the node balance equations are given by

$$d_i - \sum_{g \in \mathcal{G}_i} q_g. \tag{7.1}$$

Therefore the optimal solution to $\mathrm{P}(q)$ is equivalent to the optimal solution to $\mathrm{P}(\hat{d})$, where the elements of $\hat{d}$ are given by expression (7.1). It is therefore clear that any increase in q_g is mathematically identical to a reduction in demand at the corresponding node. □

Attributes of strategic generators

So far we have defined the sets $\mathcal{G}_i$ that consist of all the generators at each node, i. Because these generators are strategic, they each attempt to maximize their respective profit functions.

In the simplest case, the profit of a particular generator is given by its revenue minus its costs. Each generator is paid the nodal price for its entire injection at the node; hence the revenue for generator g at node i is defined as:

$$R_g = q_g \pi_i(q_g),$$

where π_i is written as a function of q_g, assuming all other generators inject some fixed amount. We will define the production costs for each generator, g, to be given by a non-negative, non-decreasing, convex function: $C_g(q_g)$. Therefore we have that

$$C_g(q_g) \geq 0, \quad C'_g(q_g) \geq 0, \quad C''_g(q_g) \geq 0, \quad \forall g \in \mathcal{G}.$$

From above we find the profit for generator g to be

$$\rho_g(q_g) = q_g \pi_i(q_g) - C_g(q_g).$$

Contracting We can formulate a slightly different profit function if we allow generators to enter into contracts for differences with retailers or large industrial customers (or if the firm is a vertically-integrated gentailer). A contract between a generator, $g \in \mathcal{G}_n$, and a retailer at node i will specify a quantity of electricity, q^C_{gi}, and a price π^C_{gi}. This means that the retailer, needing to purchase electricity for its customers, faces a fixed price of electricity for the contracted quantity. The profit function of generator g therefore becomes

$$\rho_g(q_g) = q_g \pi_n\left(q_g\right) - C_g\left(q_g\right) + \sum_{i \in \mathcal{N}} q^C_{gi}\left(\pi^C_{gi} - \pi_i\right),$$

note that since q^C_{gi} and π^C_{gi} are both fixed quantities, this means that the value of $q^C_{gi}\pi^C_{gi}$ is not affected by the generator's choice of output q_g; hence the particular contract prices do not influence the generator's behaviour. Therefore, to keep the expression simple, we exclude the contract revenue from the generator's profit function, yielding

$$\rho_g(q_g) = q_g \pi_n\left(q_g\right) - C_g\left(q_g\right) - \sum_{i \in \mathcal{N}} q^C_{gi}\pi_i.$$

Firms owning multiple plants So far we have considered the situation where each generator is an individual entity, focused solely on optimizing its own profit. However, in most markets, there are a few dominant firms owning a number of plants, possibly at multiple nodes in the grid. To model this situation, we define the set $\mathcal{F}$ to consist of all the firms in the market; moreover, we define the set $\mathcal{G}^f$ to consist of all the generators owned by firm f. In the case that there is no contracting, we have the following profit function for firm f:

$$\rho_f\left(q\right) = \sum_{i \in \mathcal{N}} \left[\sum_{g \in \mathcal{G}_i \cap \mathcal{G}^f} \left(q_g \pi_i - C_g\left(q_g\right)\right) \right].$$

The profit function above merely sums over the profit corresponding to each generator owned by firm f at each node. Note that each π_i is a function of all $q_g \in \mathcal{G}^f$. Moreover, if we allow for contracts for differences, the revenue function becomes:

$$\rho_f(q) = \sum_{i \in \mathcal{N}} \left[\sum_{g \in \mathcal{G}_i \cap \mathcal{G}^f} \left(q_g \pi_i - C_g(q_g) \right) \right] - \sum_{i \in \mathcal{N}} q_{fi}^C \pi_i, \tag{7.2}$$

where q_{fi}^C is the quantity of contracted demand of firm f at node i.

In the subsequent chapters we will discuss the existence of Cournot equilibria using the above profit functions.

Chapter 8

Rationality of the generators

In this chapter we will discuss how assumptions about the strategic generators' rationality can influence the outcome of a Cournot oligopoly over a transmission network. Specifically, we will compare four Cournot models. The first is referred to as the *full-rationality* game; here each generator anticipates the effect its decision has on the congestion in the underlying transmission network, and hence on its nodal price. We will also consider three types of *bounded-rationality*; in these Cournot models, the generators do not anticipate how their actions may influence, either (i) the flows on the lines, (ii) the price premia between nodes or (iii) congestion in the network.

In the previous chapter, we discussed how the full-rationality assumption potentially leads to issues relating to lack of existence or uniqueness of pure-strategy Cournot equilibria. In this chapter, we will give some examples of these issues and demonstrate that with certain assumptions about the rationality of strategic generators these issues can be avoided. However, we will show that although we may have a unique equilibrium in a bounded-rationality game, this equilibrium may not necessarily make sense. Specifically, by limiting the rationality of the generators with respect to transmission, many of the important characteristics that transmission introduces are being ignored. For example, under certain bounded-rationality assumptions it is possible to have find a duopoly solution when generators are separated by a line with capacity 0, or conversely find that generators act as monopolists even when joined by a line of infinite capacity. These outcomes cannot occur when generators have full-rationality.[1]

This chapter is laid out as follows: we first present the full-rationality Cournot game over a

[1] In [86] Ryan *et al.* contrast the full-rationality and bounded-rationality assumptions when firms' fuel costs are modelled endogenously by way of a fuel network.

transmission grid, and show how each generator can model their objective in the game as a bi-level optimization problem and give some simple examples; we then present the three bounded-rationality models, in which the generators either assume that the line flows, the price-premia between nodes or state of the network are fixed. For each of these models we will show how equilibrium outcomes are affected by the assumptions made.

8.1 Full rationality

In this section, we will analyze the full-rationality Cournot game over a transmission network. In this context, full-rationality means that each of the generators is aware of exactly how its injection decision will influence nodal prices coming from the dispatch problem. For each generator, this can be modelled as a bi-level game, i.e. if generator g at node n has a profit function of the form:

$$\rho_g\left(q_g, q_{-g}\right) = q_g \pi_n - C_g\left(q_g\right),$$

the bi-level problem that generator n would optimize would be:

$$\begin{aligned} \rho_g^*\left(q_{-g}\right) = \max \quad & \rho_g\left(q_g, q_{-g}\right) = q_g \pi_n - C_g\left(q_g\right) \\ \text{s.t.} \quad & \left\{x, f, \pi, \lambda, \eta^+, \eta^-, \mu^+, \mu^-\right\} \text{ optimally solves P}\left(q\right) \\ & 0 \leq q_g \leq V_g. \end{aligned}$$

In the above problem, the constraint ensuring that the dispatch problem is optimally solved is equivalent to the KKT conditions given by (4.1). Due to the orthogonality constraints in these KKT conditions, each generator's profit maximization problem has a non-convex constraint set. More specifically, this problem can be classified as a *mathematical problem with equilibrium constraints* (MPEC); these problems may have multiple local maxima, meaning that first order conditions would no longer be sufficient for optimality. We will now demonstrate this issue for a two-node symmetric situation, in example 8.1 below.

Example 8.1. *We consider a network consisting of two-nodes with a single line of capacity K_{12}. Each of the nodes has a demand of 100 and there is a fringe at each node offering identical supply functions, $S_i\left(p\right) = p$ with no upper or lower bounds. Also at each node, i, there is a strategic generator injecting some quantity, q_i, with a constant marginal cost of 0.*

This yields the following dispatch problem:

$$
\begin{aligned}
\min \quad & \tfrac{1}{2}{x_1}^2 + \tfrac{1}{2}{x_2}^2 \\
\\
\text{s.t.} \quad & x_1 - f_{12} = 100 - q_1 \\
& x_2 + f_{12} = 100 - q_2 \\
& |f_{12}| \leq K.
\end{aligned}
$$

Without loss of generality, we will consider this problem from generator 1's point of view. Generator 1 wishes to find the optimum quantity to inject. This problem (parameterised by q_2, the injection of generator 2) can be written as the following MPEC:

$$
\begin{aligned}
\rho_1^* \left(q_2\right) = \max \quad & \rho_1 \left(q_1, q_2\right) \\
\\
\text{s.t.} \quad & \pi_1 - f_{12} = 100 - q_1 \\
& \pi_2 + f_{12} = 100 - q_2 \\
& -\pi_1 + \pi_2 + \eta_{12}^{+} - \eta_{12}^{-} = 0 \\
& 0 \leq K - f_{12} \ \perp \ \eta_{12}^{+} \geq 0 \\
& 0 \leq K + f_{12} \ \perp \ \eta_{12}^{-} \geq 0 \\
& q_n \geq 0.
\end{aligned}
$$

We will now demonstrate the potential for the above problem to have multiple local maxima. For the purposes of illustration we set $q_2 = 70$ and plot $\rho_1 \left(q_1, 70\right)$ for various values of the line capacity K; these plots are shown in figure 8.1.

From these plots it is clear that the KKT conditions would not be sufficient to determine the optimal solution. If $K = 0$ or $K = \infty$ the profit functions are concave, however for line sizes between these two extremes is it possible that the profit function will contain multiple local maxima.

In order to solve the resulting MPEC, one would either need use a solver such as `NLPEC` [34], or alternatively recast the problem as an integer program by replacing each orthogonality constraint with a set of linear constraints and a binary variable. For example, if we had an orthogonality constraint of the form:

$$0 \leq a \ \perp \ b \geq 0,$$

it can be equivalently represented by the following set of constraints:

$$
\begin{aligned}
& 0 \leq a \leq Mz \\
& 0 \leq b \leq M\left(1 - z\right) \\
& z \in \left\{0, 1\right\},
\end{aligned}
$$

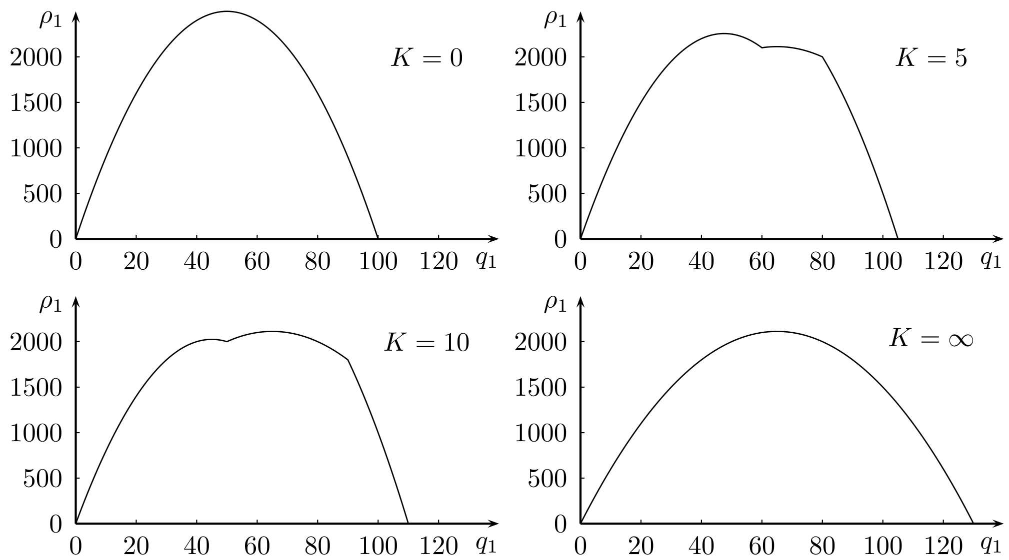

Figure 8.1: Profit as function of injection, q_1.

where M is a large positive number.

Unfortunately, the issue of the objective function being non-concave is more significant than solely making the task of optimizing a generator's profit function more difficult. This non-concavity can lead to the existence of multiple equilibria or even the lack of existence of pure-strategy equilibria. In the next section, we will discuss some examples yielding these outcomes; these examples are based on those by Borenstein *et al.* in [12].

8.1.1 Cournot equilibria

In a Cournot game setting, the lack of concavity of generators' profit functions can affect the existence and uniqueness of pure-strategy equilibria. In the full-rationality setting, the presence of small transmission capacities has the potential to allow generators to exercise market power by withholding and congesting lines toward their node. This may mean that an unconstrained equilibrium, which would have existed if the line capacity had been infinitely large, might not exist if the capacity were sufficiently small.

Here we will give examples of situations in a two-node setting where there are different outcomes depending on the parameters of the market. These examples are based on examples by Borenstein *et al.* in [12].

There is a network consisting of two-nodes with a single line of capacity K. The demand at node 1 is δd and the demand at node 2 is $(1-\delta)\,d$, for some $\delta \in [0,1]$. There is a fringe at node 1 offering a supply function, $S_1(p) = \alpha a p$ with no upper or lower bounds, and a fringe at node 2 offering $S_2(p) = (1-\alpha)\,ap$. Also at each node, i, is a strategic generator injecting some quantity, q_i, with a constant marginal cost of 0. This situation has a dispatch problem of the form:

$$\begin{aligned}
\min \quad & \tfrac{1}{2\alpha a}{x_1}^2 + \tfrac{1}{2(1-\alpha)a}{x_2}^2 \\
\\
\text{s.t.} \quad & x_1 - f_{12} = \delta d - q_1 \\
& x_2 + f_{12} = (1-\delta)\,d - q_2 \\
& |f_{12}| \leq K.
\end{aligned} \tag{8.1}$$

Since the above dispatch problem only has one transmission line, we can, in fact, solve for the nodal prices parametrically as functions of q_1 and q_2 (as discussed in chapter 4). These nodal prices are:

$$\pi_1(q_1,q_2) = \begin{cases} \frac{1}{a\alpha}\left(d\delta - K - q_1\right), & (1-\alpha)\,q_1 - \alpha q_2 \geq d\,(\delta-\alpha) - K, \\ \frac{1}{a}\left(d - q_1 - q_2\right), & d\,(\delta-\alpha) - K \leq (1-\alpha)\,q_1 - \alpha q_2 \leq d\,(\delta-\alpha) + K, \\ \frac{1}{a\alpha}\left(d\delta + K - q_1\right), & (1-\alpha)\,q_1 - \alpha q_2 \geq d\,(\delta-\alpha) + K, \end{cases}$$

$$\pi_2(q_1,q_2) = \begin{cases} \frac{1}{a(1-\alpha)}\left(d\,(1-\delta) + K - q_2\right), & (1-\alpha)\,q_1 - \alpha q_2 \geq d\,(\delta-\alpha) - K, \\ \frac{1}{a}\left(d - q_1 - q_2\right), & d\,(\delta-\alpha) - K \leq (1-\alpha)\,q_1 - \alpha q_2 \leq d\,(\delta-\alpha) + K, \\ \frac{1}{a(1-\alpha)}\left(d\,(1-\delta) - K - q_2\right), & (1-\alpha)\,q_1 - \alpha q_2 \geq d\,(\delta-\alpha) + K. \end{cases}$$

Example 8.2. *Now suppose we set $d = 2$, $a = 2$, $\delta = 0.5$ and $\alpha = 0.5$. This yields a symmetric situation.*

We can plot the optimal injection for each generator as a function of the other generator's injection. These are the best response curves for the game. If we consider the situation where $K = \infty$ we are effectively dealing with a single node situation, as discussed in chapter 2; the best response curves for this are shown in figure 8.2 (i) (in these plots the best response for generators 1 and 2 are shown in red and blue, respectively). Alternatively, if we consider the other extreme, by setting $K = 0$, we have two isolated markets, so neither generator's injection affects the other; this yields two local monopolies, the best response curves for this situation are given in figure 8.2 (ii). (Note that here the optimal response is independent of the other generator's action.)

Now suppose we choose different values for the line capacity, in this situation we may or may not have an equilibrium. The best response curves for the case where $K = 0.2$ and $K = 0.05$ are shown in figure 8.3. We see that for $K = 0.2$, we have an equilibrium with quantities identical to the equilibrium in the case with $K = \infty$, however, there is much more complicated interaction between the generators. We will examine generator 2's best response curve in detail below:

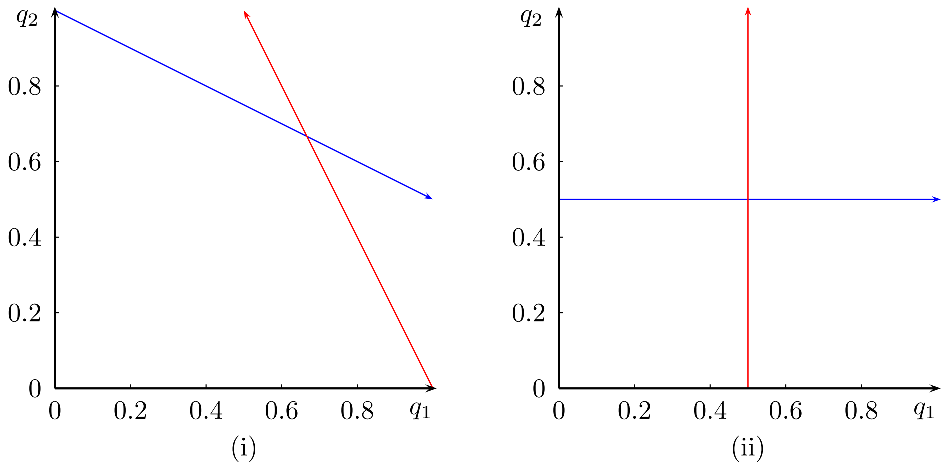

Figure 8.2: Best Response Curves: $K = \infty$ (left); $K = 0$ (right).

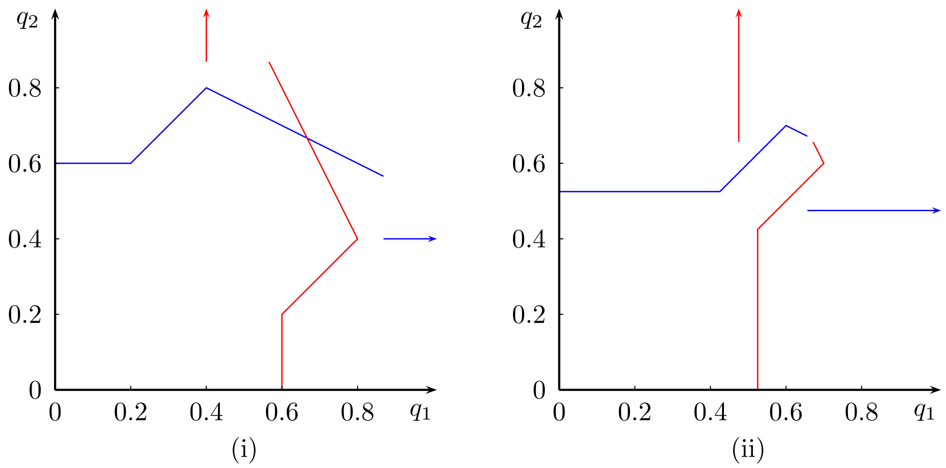

Figure 8.3: Best Response Curves: $K = 0.2$ (left); $K = 0.05$ (right).

- If generator 1 withholds, the best response for generator 2 is to cause the line to congest from node 2 to node 1 by choosing a monopolistic strategy for the demand at node 2 plus the exports.
- As generator 1 increases its injection, the best response quantity begins to increase as generator 2 seeks to ensure the line remains (just) congested.
- Upon further increase of q_1, generator 2 responds as a duopolist with the line uncongested.
- Finally as q_1 approaches 1, the best response of generator 2 is to allow the line to congest towards node 2 by playing a monopolistic strategy over the demand at node 2 less the imports.

For $K = 0.05$, we have no pure-strategy equilibrium as the best response curves do not intersect. As the line capacity is small, there is incentive for each generator to withhold as a response to the other generator's single-node equilibrium quantity. This is shown by the discontinuity in each generator's best response curve. Borenstein *et al.* in [12] show that in fact for any capacity $K \geq 1 - \frac{2}{3}\sqrt{2} \approx 0.0572$, we are guaranteed that the single-node equilibrium will exist, and for any capacity less than that, no pure-strategy Cournot equilibrium will exist. One of the key contributions of this result is that it demonstrated that even though the equilibrium flow on the line is 0 (due to symmetry), a positive capacity is required to ensure a duopoly outcome. Later, we will see that this result does not hold for the bounded-rationality models.

Let us now turn our attention to an asymmetric situation. Here we will demonstrate that multiple equilibria are also possible in this context.

Example 8.3. *Consider again the two-node situation with its dispatch problem given by* (8.1). *In this case we set* $K = 0.2$, $\delta = 0.7$ *and* $\alpha = 0.7$.

This gives the best response curves in figure 8.4 (i); here there exists no equilibria. However, keeping K and δ fixed and setting $\alpha = 0.9$ the game changes considerably. From the best response curves shown in figure 8.4 (ii), we find that there are two Cournot equilibria, one being the single node equilibrium found in previous examples, and the other being a congested equilibrium, with node 1 being an exporter of power, and node 2 being an importer of power.

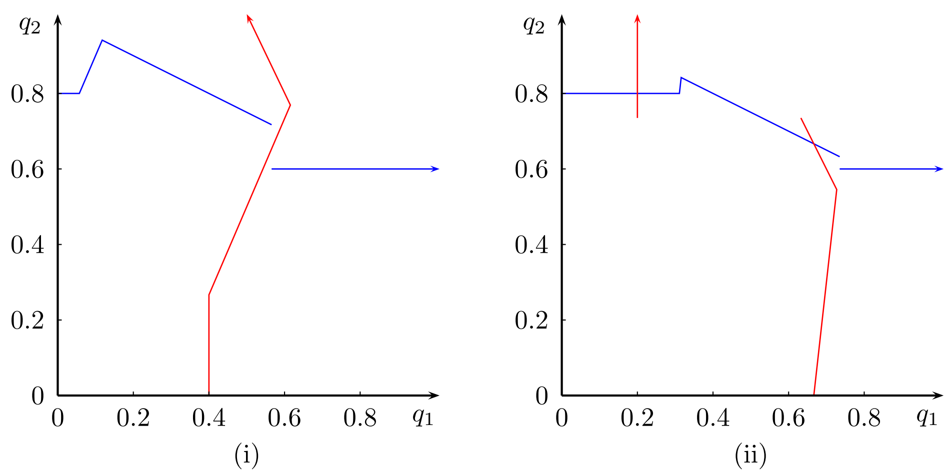

Figure 8.4: Best Response Curves.

The examples, above, show that depending on the relative sizes of demand at either node (δ), the relative fringe slopes (α), and the size of the line (K), there may be a single pure-strategy equilibrium, two equilibria or no equilibrium. The figures below give regions in $\{\delta, \alpha\}$ space, supporting the different equilibria types for particular values of $\frac{K}{d}$. Specifically, the red region is defined by pairs of δ and α for which the single-node equilibrium exists; the blue region is defined by pairs of δ and α for which a congested equilibrium exists with the power flowing from node 2 to 1; and finally the green region is defined by pairs of δ and α for which a congested equilibrium exists with the power flowing from node 1 to 2. Figure 8.5 (i) shows these regions for $\frac{K}{d} = 0.1$, and figure 8.5 (ii) shows these regions for $\frac{K}{d} = 0.25$. From these figures, we can see that as the line capacity as a proportion of total demand increases, the region supporting the single-node equilibrium becomes larger; whereas the regions supporting the congested equilibria become smaller.[2]

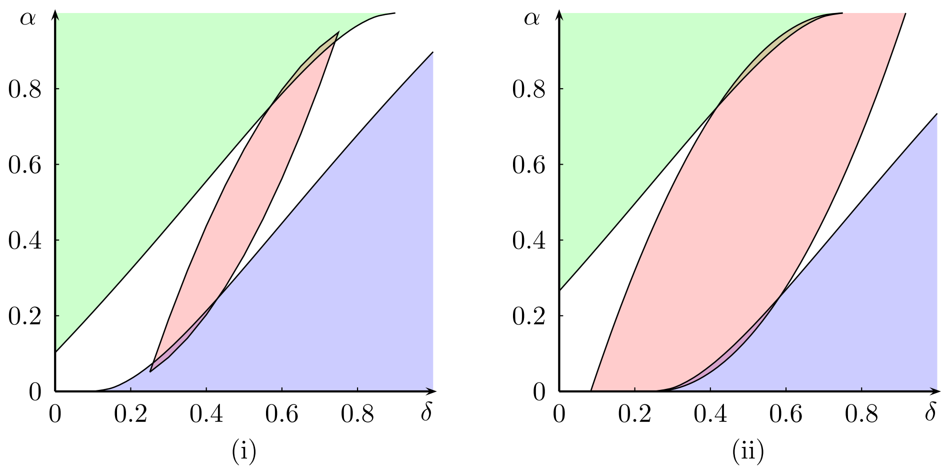

Figure 8.5: Full rationality.

The above diagrams show the different cases we discussed in the examples earlier in this section. The white region represents parameters for which there is no equilibrium, and where the regions overlap, there are multiple equilibria. Note that the regions supporting multiple equilibria are relatively small. This is because, the demands and fringe slopes need to support both a congested, local monopoly equilibrium, and a duopoly equilibrium.

Moreover, in this chapter we are ignoring the ownership of retail contracts and FTRs (see, e.g., [52, 54]); the presence of these would shift the boundaries of the different equilibrium types. We will discuss equilibria in the presence of contracting in subsequent chapters.

8.2 Bounded rationality

In the previous section, we showed that in the full-rationality paradigm we are not guaranteed uniqueness or existence of pure-strategy equilibria. This is a concern when incorporating the equi-

[2] For values of $\frac{K}{d} < 0.0572$ the single-node equilibrium does not exist.

librium result as part of a larger problem. An approach to avoid these complications is outlined by Yao, Oren and Adler in [99]. Here the generators have bounded-rationality; this means information about how their action (the quantity they inject) affects the transmission in the network is hidden from the generators. Yao *et al.* present two bounded-rationality models: one where the flows on the lines are fixed; and one where the differences between nodal prices are fixed. We will discuss the details and implications of the two models below.

8.2.1 Fixed flows

In the first of the bounded-rationality models by Yao *et al.*, each of the generators assumes that the power flows on all the lines are unaffected by its injection quantity, and that the flows are instead computed simultaneously by the system operator. Therefore, in the game, each generator $g \in \mathcal{G}_i$ for each node $i \in \mathcal{N}$ optimizes the following problem:

$$\begin{aligned} \max \quad & q_g \pi_i - C_g(q_g) \\ \text{s.t.} \quad & q_g + S_i(\pi_i) = d_i - \sum_{h \in \mathcal{G}_i \setminus \{g\}} q_h + \sum_{j, ij \in \mathcal{A}} f_{ij} - \sum_{j, ji \in \mathcal{A}} f_{ji} \\ & 0 \le q_g \le V_g, \end{aligned}$$

Note that in the above problem, that q_h, $\forall h \in \mathcal{G}_i \setminus \{g\}$ are fixed, and so are the flows f_{ij}, $\forall ij \in \mathcal{A}$. The quantities, q_g are determined by each of the generators, while simultaneously the system operator determines the optimal power flows, f_{ij} from the following dispatch problem:

$$\begin{aligned} \mathrm{P}(q) : \min \quad & \sum_{i \in \mathcal{N}} \int_{x_i^L}^{x_i} S_i^{-1}(\xi) \, .d\xi \\ \text{s.t.} \quad & x_i - \sum_{j, ij \in \mathcal{A}} f_{ij} + \sum_{j, ji \in \mathcal{A}} f_{ji} = d_i - q_i \quad & [\pi_i] \quad & \forall i \in \mathcal{N} \\ & \sum_{ij \in \mathcal{A}} l_{ijk} f_{ij} = 0 & [\lambda_k] \quad & \forall k \in \mathcal{L} \\ & f_{ij} \le K_{ij} & [\eta_{ij}^+] \quad & \forall ij \in \mathcal{A} \\ & -f_{ij} \le K_{ij} & [\eta_{ij}^-] \quad & \forall ij \in \mathcal{A}. \end{aligned}$$

In the case where the supply functions are linear and increasing, $S_i(p) = a_i p$, where $a_i > 0$. Yao *et al.* prove that this game can be written as a linear complementarity problem of the form:

$$0 \le Mq + y \perp q \ge 0, \tag{8.2}$$

where M is a positive definite matrix. Therefore by theorem 3.1.6 of [19], we are guaranteed that there is a unique solution to the LCP (8.2), above; hence there is a unique pure-strategy equilibrium for this game.

In the symmetric two-node case, discussed in example 8.2, solving for the equilibrium under the assumption of fixed-flow bounded rationality will result in two local monopolies, regardless of the size of the line. Since the generators believe that the flow into or out of each node is fixed, they see no opportunity to compete with generators located at other nodes.

In more general instances of this type of game (with perhaps multiple firms competing at each node), each node effectively becomes a distinct oligopoly, and there is no competition between generators located at different nodes. Since generators only compete with generators at their own node, this form of bounded-rationality will tend to result in equilibria with less competition (and higher prices) than might otherwise be predicted.

We now will compare these bounded-rationality equilibria with those coming from the full-rationality game. Using the two-node model with dispatch problem (8.1), figure 8.6 gives regions in $\{\delta, \alpha\}$ space, supporting the different equilibria types for particular values of $\frac{K}{d}$. The red region is defined by pairs of δ and α for which the line is not congested at equilibrium; the blue region is defined by pairs of δ and α for which a congested equilibrium exists with the power flowing from node 2 to 1; and the green region is defined by pairs of δ and α for which a congested equilibrium exists with the power flowing from node 1 to 2. Figure 8.6 (i) shows these regions for $\frac{K}{d} = 0.1$, and figure 8.6 (ii) shows these regions for $\frac{K}{d} = 0.25$.

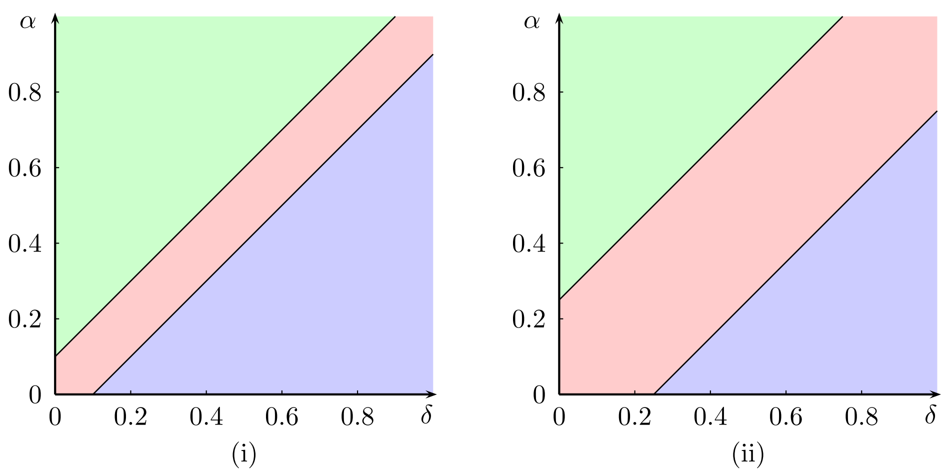

Figure 8.6: Bounded rationality: flows fixed.

Note that in this paradigm, there is no overlap of the regions and the regions fill the space. This corroborates with the result that there exists a unique equilibrium to the game in this setting.

8.2.2 Fixed price premia

Another form of bounded rationality presented by Yao *et al.* in [99] fixes price-premia, which are the differences between the nodal prices and a reference price. Without loss of generality we can define the reference price to be the price at node 1; this gives the price premia for node i as

$$\phi_i = \pi_i - \pi_1. \tag{8.3}$$

In this form of bounded-rationality, generators see fixed price premia, but the reference price varies as a function of their injection. To ensure feasibility, we require that the total amount of electricity generated by the fringes must equal the total demand less the amount of electricity injected by the strategic generators; this gives the following constraint:

$$\sum_{i\in\mathcal{N}} S_i\left(\pi_1 + \phi_i\right) = \sum_{i\in\mathcal{N}} d_i - \sum_{h\in\mathcal{G}} q_h.$$

Therefore in the case of linear supply functions ($S_i(p) = a_i p$) each generator must solve the following profit maximization problem:

$$\begin{aligned}
\max \quad & q_g\left(\pi_1 + \phi_i\right) - C_g\left(q_g\right) \\
\\
\text{s.t.} \quad & \textstyle\sum_{i\in\mathcal{N}} a_i\left(\pi_1 + \phi_i\right) + q_g = \sum_{i\in\mathcal{N}} d_i - \sum_{h\in\mathcal{G}\setminus\{g\}} q_h \\
& 0 \le q_g \le V_g.
\end{aligned}$$

As in the fixed-flow bounded-rationality game, here the system operator solves the dispatch problem simultaneously; however, instead of determining the flows, it determines the price premia, defined in equation (8.3). Yao *et al.* prove that this game also has a unique equilibrium [99].

A major drawback of this rationality assumption can be seen from the result of the symmetric two-node case, given in example 8.2. This form of bounded-rationality results in a duopoly equilibrium, irrespective of the size of the line (even if $K = 0$, which should result in two distinct markets).

Once again we can compare these bounded-rationality equilibria with those coming from the full-rationality game. In figure 8.7 we give regions in $\{\delta, \alpha\}$ space, supporting the different equilibria types for particular values of $\frac{K}{d}$. Again, the red region is defined by pairs of δ and α for which the line is not congested at equilibrium (the prices at both nodes are the same); the blue region is defined by pairs of δ and α for which a congested equilibrium exists with the power flowing from node 2 to 1; and the green region is defined by pairs of δ and α for which a congested equilibrium exists with the power flowing from node 1 to 2. Figure 8.7 (i) shows these regions for $\frac{K}{d} = 0.1$, and figure 8.7 (ii) shows these regions for $\frac{K}{d} = 0.25$.

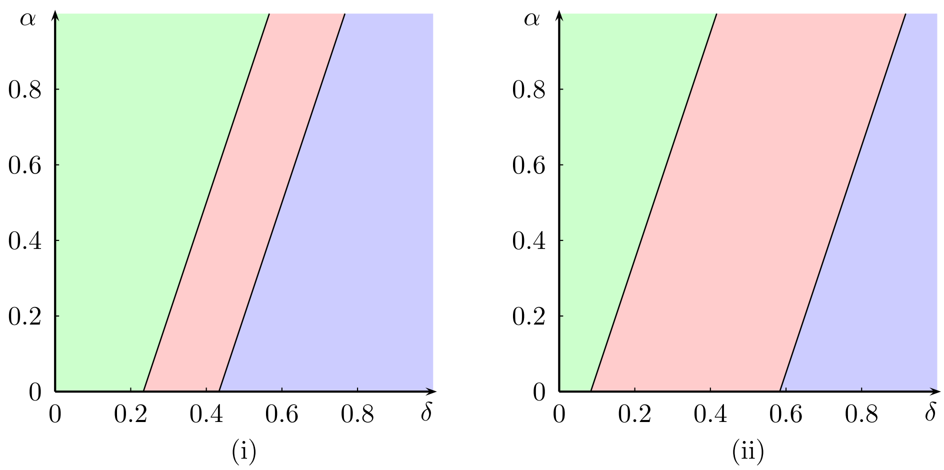

Figure 8.7: Bounded rationality: price-premia fixed.

Note that there is no overlap of the regions and the regions fill the space. This again reinforces the fact that there exists a unique equilibrium to the game in this setting.

8.2.3 Fixed network state

The final bounded-rationality approach is that of Barquin and Vazquez in [7]. Here generators do not believe that their injection can alter the state of the lines in the network (whether they are congested and in which direction). Given a state of the network, the generators simultaneously choose injections q to maximize their respective profits. An equilibrium to this game is said to exist if the system operator, given the injections q, chooses the same state of the network.

Generator g at node i has the following profit maximization problem:

$$\begin{aligned} \max \quad & q_g \pi_i^r (q) - C_g (q_g) \\ \\ \text{s.t.} \quad & 0 \le q_g \le V_g, \end{aligned}$$

where $\pi_i^r (q)$ is the price as a function of q, corresponding to the state of the network, r.

Barquin and Vazquez present an iterative approach to compute these bounded rationality equilibria, whereby generators first come to an equilibrium for a fixed network state, and then the system operator solves the dispatch problem and the network state is updated. This process is repeated until we converge to a point at which no player (generator or system operator) wishes to deviate. Note that it is possible that no such equilibrium exists, in this case the algorithm will cycle. Conversely, however, if the algorithm cycles, this does not necessarily guarantee that no equilibrium exists.

Note that the difference between these equilibria and full-rationality equilibria is that here the generators do not consider changing the state of lines to improve their profit, which means that each may be only at a local optimum in the full-rationality setting.[3]

We can see how equilibria from this rationality setting relate to equilibria in the other rationality paradigms by looking at which types of equilibria exist for the various parameters of the game. In figure 8.8 we give regions in $\{\delta, \alpha\}$ space, supporting the different equilibria types for two particular values of $\frac{K}{d}$. Again, the red region is defined by pairs of δ and α for which the line is not congested at equilibrium (the prices at both nodes are the same); the blue region is defined by pairs of δ and α for which a congested equilibrium exists with the power flowing from node 2 to 1; and the green region is defined by pairs of δ and α for which a congested equilibrium exists with the power flowing from node 1 to 2. Figure 8.8 (i) shows these regions for $\frac{K}{d} = 0.1$, and figure 8.8 (ii) shows these regions for $\frac{K}{d} = 0.25$.

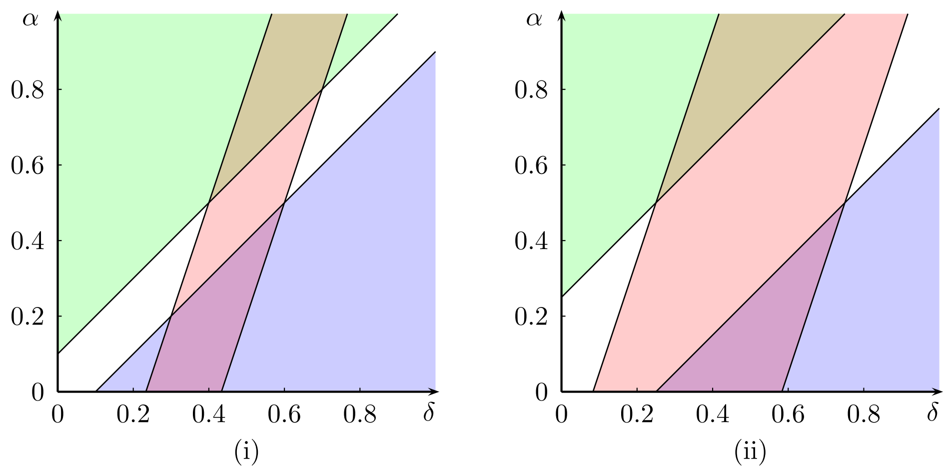

Figure 8.8: Bounded rationality: network state fixed.

From these diagrams, we can see that in this setting the existence of multiple equilibria or conversely no equilibrium are common. Note that the unconstrained equilibria coming from this model are identical to those from the fixed-premia setting, and the congested equilibria coming from this model are identical to those from the fixed-flow setting. Moreover, since the generators in these equilibria are locally optimizing (as they do not consider the effects of changing the congestion) the full-rationality regions from figure 8.5 are subsets of the regions above.

In the next chapter, we present the concept of *candidate equilibria* that is similar to this equilibrium

[3] This kind of assumption may be appropriate in situations where firms have limited information regarding the shape of their residual demand curves.

paradigm, however, we use it to determine the conditions for existence of full-rationality equilibria. Subsequent chapters focus solely on the full-rationality paradigm.

Chapter 9

Equilibria in full-rationality setting

In this chapter, we present the different types of equilibria that can occur in Cournot games over transmission networks and discuss how they can be computed. In the previous chapter, we gave examples over small networks where the assumptions of full-rationality led to a lack of existence of pure-strategy equilibrium or potentially multiple equilibria. Here we will present methods for computing these in a more general setting.

This chapter is laid out as follows. We first discuss pure-strategy and mixed-strategy equilibria; we then present two algorithms for computing pure-strategy equilibria, and discuss their strengths and weaknesses. Finally we introduce the concept of *candidate equilibria*, and prove that in the case of linear fringe supply functions, for each KKT regime (discussed in chapter 4), there exists exactly one candidate equilibrium, which can be computed from a linear complementarity problem.

9.1 Nature of equilibria

In this section, we discuss the different types of equilibria that can occur in Cournot games over transmission networks; these are pure-strategy equilibria and mixed-strategy equilibria.

9.1.1 Pure-strategy equilibria

In pure-strategy equilibria, each player has chosen a single strategy which maximizes its payoff function. In terms of electricity markets, each generator has chosen an injection which maximizes its profit (or each firm chooses a vector of injections for its generators). As demonstrated in chapter 8, in the full-rationality setting, at equilibrium, each generator is globally maximizing its own profit with full knowledge of how the its action affects the congestion in the transmission and its nodal price. However, we showed that, due to non-convexity in each generator's profit function, multiple equilibria, or even lack of existence of pure-strategy equilibria were possible outcomes.

9.1.2 Mixed-strategy equilibria

When there exists no pure-strategy equilibrium, in order to find an equilibrium we must turn to *mixed-strategies*. As discussed in chapter 2, a mixed-strategy for a player consists of a distribution over a set of equally profitable strategies. In terms of an electricity market game, a generator will randomly choose an injection based on the probability distribution corresponding to its mixed-strategy. Glicksberg [38], has proved that in any game where the strategies of the players are compact and their payoff functions are continuous there exists some mixed-strategy equilibrium [32]. We discuss mixed-strategies in more detail for a two-node network in chapter 13.

9.2 Computing equilibria

In this section, we discuss techniques that can be used to compute Nash equilibria in the full-rationality setting.

With fully-rational firms, the Cournot game used to model competition between generators is actually a Stackelberg type game, as discussed by Bjørndal *et al.* in [10]. The game proceeds as follows: first the generators choose their injection quantities (simultaneously), and based on those injections the system operator determines the optimal flows and nodal prices. In other words, the generators act as simultaneous leaders, all anticipating the prices from the optimal dispatch (determined by the system operator who is a follower). To compute the Nash-Cournot equilibrium, a bi-level program must be solved for all generators simultaneously (see for example [50, 10]). A typical way to solve this type of problem is to embed the optimal dispatch conditions as constraints in the generators' profit maximization problems to form *mathematical programs with equilibrium constraints* (MPECs), as discussed by Luo *et al.* in [63]. The profit maximization problem for an arbitrary generator, g, at node n is therefore:

$$\begin{aligned} \max \quad & q_g \pi_n - C_g\left(q_g\right) \\ \text{s.t.} \quad & \{x, \pi, \eta^+, \eta^-, \mu^+, \mu^-, \lambda\} \text{ solves } \mathrm{P}\left(q\right) \\ & 0 \leq q_g \leq V_g. \end{aligned} \tag{9.1}$$

Unfortunately, due to the inequality constraints in the dispatch problem, each generator's constraint set contains orthogonality conditions, which are non-convex. This non-convexity means that a solution satisfying the KKT conditions of a generator's profit maximization problem in (9.1) is not necessarily the globally optimal solution to the problem. In other words, the KKT conditions are no longer sufficient for optimality.

We will now present two numerical methods for computing equilibria in this setting. The first involves formulating the game as an *equilibrium problem with equilibrium constraints* (EPEC) and solving a complementarity problem. The second is to use a *sequential best response technique* to converge to an equilibrium.

9.2.1 Equilibrium problems with equilibrium constraints

When we model a set of players simultaneously maximizing their payoff functions by solving MPECs, we have an equilibrium problem with equilibrium constraints, (see [50, 83]). In a full-rationality Cournot game over a transmission grid, each generator solves the MPEC given by (9.1), and the simultaneous solution to these profit maximization problems is the solution to an EPEC.

These EPECs are inherently difficult to solve; this is because, as mentioned above, the optimization problem given by (9.1) has a non-convex constraint set, and therefore could potentially have multiple local optima. Ralph and Hu in [50] and Ralph and Smeers in [83] discuss numerical approaches to solving EPECs in the context of electricity markets with transmission. In these Ralph *et al.* introduce the concept of *local Nash equilibria*. A local Nash equilibrium is similar to a Nash equilibrium, but rather than each generator choosing a strategy that globally optimizes its payoff, instead some players may be only locally optimal. They demonstrate that every local Nash equilibrium is a solution to the mixed complementarity problem formed from the KKT conditions of (9.1) for each generator held simultaneously. This complementarity problem is also referred to as the *All-KKT system* by Hu in [49].

This All-KKT system can be solved using the `PATH` solver (see [30]). Once a solution is computed it would then need to be verified to determine whether it is a *global* Nash-Cournot equilibrium, or indeed a *local* Nash equilibrium.

Although this method may yield a Nash equilibrium, if no Nash equilibrium is found we are not guaranteed that one does not exist (conversely if no equilibrium exists we cannot confirm this). Similarly, if multiple Nash equilibria exist, we are not guaranteed to find all of them.

9.2.2 Sequential best response

An alternate method for computing Nash equilibria is called *sequential best response* (this is sometimes referred to as *fictitious play* or *tatonnement* [94]). There are a number of ways that this method can be implemented (see for example [32, 94]), but generally players solve their profit maximization problem sequentially, assuming that the other players are fixed at their previous value. This process is repeated until the strategies converge to a fixed-point, which is an equilibrium to the game. For the Cournot game that we are dealing with, the process can be carried out as follows:

i. Choose some vector of injections $q^{(0)}$; this is the initial vector at iteration $i = 0$.

ii. For each generator $g \in \mathcal{G}$, compute $q_g^{(i+1)} = q_g$ from the optimal solution to (9.1), having set: $q_h = q_h^{(i)}$, $\forall h \in \mathcal{G} \backslash \{g\}$.

iii. If $\left\| q^{(i+1)} - q^{(i)} \right\| < \epsilon$ (where ϵ is some convergence tolerance) then end, otherwise set $i = i+1$, and go to step ii.

This procedure is relatively straightforward to implement, however some global optimization technique (such as an MPEC solver or integer programming) must be employed to find the solution to (9.1) for each iteration[1]. Any fixed-point found from the above procedure is guaranteed to be an equilibrium to the game (since at such a point all generators are optimal). However this method is not guaranteed to converge to an equilibrium if one exists; moreover, if multiple equilibria exist running this algorithm from multiple starting points will not necessarily be able to find all equilibria.

Although both the EPEC method and the sequential best response technique for computing equilibria in this setting may find Nash-Cournot equilibria for full-rationality Cournot games over transmission networks, if no equilibrium is found then we are not guaranteed that one does not exist. Furthermore, they do not provide much insight into what the underlying reasons are for why

[1]Since this optimization step is for a single firm, it would be possible to do this, for example, in a supply function context.

there may be no equilibria. In the next section, we will discuss the concept of *candidate equilibria*, which provide more understanding into the form of equilibria that can exist.

9.3 Candidate equilibria

Due to the issues with non-convexity and because the methods discussed above are not guaranteed to find all equilibria, in this work we take a different approach to computing equilibria. In this section we will introduce the concept of candidate equilibria. In the context of transmission networks, these candidate equilibria satisfy some necessary conditions for them to be equilibria, but those conditions are not sufficient. Here we will define the candidate equilibria, and in the following chapters we present algorithms and conditions that verify whether the candidate equilibria are in fact bona fide equilibria.

The dispatch problem, $\mathrm{P}(q)$ defined in chapter 7 presents many challenges when attempting to compute equilibria. Due to the bounds on the fringe dispatch quantities and the capacities on the lines, the generators can often encounter non-concave profit functions, as discussed in chapter 8. To try to better understand what is happening in these models, we will use the concept of KKT regimes, introduced in chapter 4, to divide this dispatch problem into a set of simpler problems.

For each KKT regime, r, we can define a candidate equilibrium. Recall that for a particular KKT regime we modify the dispatch problem by setting certain flows and fringes to be at upper or lower bound and the leave the remaining unconstrained, as described in section 4.2.1. The prices from this modified dispatch problem, $P^r(q)$, are seen by all the strategic generators and used to compute an equilibrium.

In this modified game, each generator now maximizes:

$$
\begin{aligned}
\max \quad & q_g \pi_n - C_g\left(q_g\right) \\
\text{s.t.} \quad & \{x, \pi, \eta^+, \eta^-, \mu^+, \mu^-, \lambda\} \text{ solves } \mathrm{P}^r\left(q\right) \\
& 0 \leq q_g \leq V_g.
\end{aligned}
\tag{9.2}
$$

Note that since $\mathrm{P}^r(q)$ does not have any inequality constraints, the constraints of (9.2) do not contain any orthogonality constraints.

In what follows we will set up this game, and prove that in the case of linear supply functions and quadratic costs, there exists an unique equilibrium for this game.

9.3.1 Existence and uniqueness of candidate equilibria

Consider the situation where the fringe supply functions are of the form $S_i(p) = a_i p$, $\forall i \in \mathcal{N}$, where $a_i > 0$ and the cost function of each generator, g, is an increasing quadratic function of the generation level: $C_g(q_g) = u_g q_g + v_g {q_g}^2$. In this case, for a particular congestion regime, r, we will show that there exists a unique equilibrium for the game.

We will first consider the existence and uniqueness of the *unconstrained candidate equilibrium* (for which no line nor fringe is constrained) where each generator is independently operated, and then present a more general proof for any KKT regime allowing for multiple ownership and contracts for differences.

Unconstrained candidate equilibrium

We will first demonstrate the computation of the candidate equilibrium for the KKT regime where no line is congested, and no fringe is at a bound. For this KKT regime, U, we have $\mathcal{B}_U^+ = \mathcal{B}_U^- = \mathcal{D}_U^+ = \mathcal{D}_U^- = \emptyset$.

First note that, without constraints on the line flows or the fringes, the dispatch problem, $\mathrm{P}^U(q)$ simplifies to the following:

$$\begin{aligned} \min \quad & \sum_{i\in\mathcal{N}} \tfrac{1}{2a_i} x_i^2 \\ \text{s.t.} \quad & \sum_{i\in\mathcal{N}} x_i = \sum_{i\in\mathcal{N}} d_i - \sum_{g\in\mathcal{G}} q_g, \quad \left[\pi^U\right] \end{aligned}$$

where π^U is the nodal price at every node. This is merely as extension of the classic Cournot model to allow for multiple nodes. From the KKT conditions of the above dispatch problem π^U can be shown to be

$$\pi^U = \frac{\sum\limits_{i\in\mathcal{N}} d_i - \sum\limits_{g\in\mathcal{G}} q_g}{\sum\limits_{i\in\mathcal{N}} a_i}.$$

For the unconstrained regime, the profit for generator $g \in \mathcal{G}$ is therefore given by

$$\rho_g = q_g \left(\frac{\sum\limits_{i\in\mathcal{N}} d_i - \sum\limits_{h\in\mathcal{G}} q_h}{\sum\limits_{i\in\mathcal{N}} a_i} \right) - u_g q_g - v_g {q_g}^2. \tag{9.3}$$

Hence we have the following profit maximization problem for generator g

$$\begin{aligned} \max \quad & q_g \left(\frac{\sum_{i\in\mathcal{N}} d_i - \sum_{h\in\mathcal{G}} q_h}{\sum_{i\in\mathcal{N}} a_i} \right) - u_g q_g - v_g {q_g}^2 \\ \text{s.t.} \quad & 0 \le q_g \le V_g. \end{aligned}$$

This can be formulated as an LCP, yielding

$$0 \leq u_g + 2v_g q_g - \frac{\sum\limits_{i\in\mathcal{N}} d_i - \sum\limits_{h\in\mathcal{G}} q_h - q_g}{\sum\limits_{i\in\mathcal{N}} a_i} + \nu_g \quad \perp \quad q_g \geq 0,$$

$$0 \leq V_g - q_g \quad \perp \quad \nu_g \geq 0,$$

where ν_g is the Lagrange multiplier on generator g's capacity constraint.

Above are the optimality conditions for a single generator. If these conditions are held simultaneously for all $g \in \mathcal{G}$, a larger LCP is formed, as given by (9.4). The solution to this new LCP is therefore a Nash-Cournot equilibrium for when there are no capacity restrictions in the dispatch problem. Note that q, V, u and v are vectors containing components q_g, V_g, u_g and v_g, $\forall g \in \mathcal{G}$, respectively.

$$0 \leq \begin{bmatrix} N & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} q \\ \nu \end{bmatrix} - \begin{bmatrix} y \\ V \end{bmatrix} \perp \begin{bmatrix} q \\ \nu \end{bmatrix} \geq 0, \tag{9.4}$$

where

$$\begin{aligned} N &= \frac{I_{|\mathcal{G}|} + J_{|\mathcal{G}|}}{\sum\limits_{i\in\mathcal{N}} a_i} + \operatorname{diag}(2v), \qquad (9.5) \\ y &= \frac{\sum\limits_{i\in\mathcal{N}} d_i}{\sum\limits_{i\in\mathcal{N}} a_i} - u, \end{aligned}$$

with $I_{|\mathcal{G}|}$ and $J_{|\mathcal{G}|}$ being the identity matrix and a square matrix of ones respectively, each of size $|\mathcal{G}|$. Also note that u and v are vectors consisting of elements u_g and v_g respectively.

Lemma 9.1. *Matrix N, given in equation (9.5) is positive definite[2], so long as $v_g \geq 0$, $\forall g \in \mathcal{G}$.*

Proof. Consider matrix N, given in equation (9.5). It is the (scaled) sum of three matrices: an identity matrix I, which is clearly positive definite; a square matrix of ones, J, which from proposition 1 in [59] we know is positive semi-definite; and finally a diagonal matrix with elements of the vector v on the diagonal. It is clear that so long as $v_g \geq 0$, $\forall g \in \mathcal{G}$, diag$(v)$ is positive semi-definite. Hence N is a positive definite matrix. □

Since N is a positive definite matrix, the LCP given above can be written as the following strictly convex quadratic program

$$\min \quad \tfrac{1}{2} q^T N q - y^T q$$

$$\text{s.t.} \quad 0 \leq q \leq V.$$

[2] The matrix A is positive definite if and only if $x^T A x > 0$, for every non-zero vector, x.

The above quadratic program clearly has a unique solution, which is the candidate equilibrium corresponding to the unconstrained KKT regime, U.

In the next section we generalize this result to show that there is a unique candidate equilibrium associated with every KKT regime, while allowing firms to own multiple generators and have contracts for differences.

General candidate equilibria

We now will prove that there exists a unique candidate equilibrium for each KKT regime. We define the price at each node, i, for KKT regime, r, to be π_i^r; these prices are the dual variables corresponding to the node balance constraints of the dispatch problem for regime r, $P^r(q)$.

With linear supply functions, the dispatch problem $P^r(q)$ is a convex quadratic program with no inequality constraints; hence its KKT conditions form a linear system of equations, with generator injections on the right-hand side. Therefore the optimal objective value, which we will define as $\phi^r(q)$, can be written as the following quadratic function of the generators' injections, q:

$$\phi^r(q) = \frac{1}{2} q^T M q - b^T q + c,$$

where M is a positive semi-definite matrix. Therefore, since the nodal price at a given node is defined by the rate of change of the cost of dispatch with respect to demand at that node, we can define the nodal price at node i at which there is some generator, g, in terms of ϕ^r as follows:

$$\pi_i^r = \frac{\partial \phi^r}{\partial d_i} = -\frac{\partial \phi^r}{\partial q_g} = -Mq + b. \tag{9.6}$$

With equation (7.2), this gives the profit maximization problem for firm f as:

$$\begin{aligned} \max \quad & \rho_f(q) = \sum_{g \in \mathcal{G}_f} \left[-q_g \tfrac{\partial \phi^r}{\partial q_g} - u_g q_g - v_g {q_g}^2 - \sum_{i \in \mathcal{N}} q_{fi}^C \tfrac{\partial \phi^r}{\partial d_i} \right] \\ \text{s.t.} \quad & 0 \leq q_g \leq V_g, \qquad \forall g \in \mathcal{G}_f. \end{aligned}$$

Since M is a positive semi-definite matrix, the above problem is convex, hence its optimal solution is equivalent to the solution to its KKT conditions, which constitute the following LCP:

$$\begin{aligned} 0 \leq u_g + 2v_g q_g + \sum_{h \in \mathcal{G}_f} q_h \frac{\partial^2 \phi^r}{\partial q_h \partial q_g} + \frac{\partial \phi^r}{\partial q_g} + \nu_g + \sum_{i \in \mathcal{N}} q_{fi}^C \frac{\partial^2 \phi^r}{\partial d_i \partial q_g} \quad & \perp \quad q_g \geq 0, \quad \forall g \in \mathcal{G}_f, \\ 0 \leq V_g - q_g \quad & \perp \quad \nu_g \geq 0, \quad \forall g \in \mathcal{G}_f. \end{aligned}$$

At the Nash equilibrium for this game, all firms must be simultaneously maximizing their profit. We can compute this point by solving the above complementarity problem for all firms simultaneously.

This can be modelled by the following augmented LCP:

$$\begin{aligned} 0 \le u_g + 2v_g q_g + \sum_{h\in\mathcal{G}_f} q_h \frac{\partial^2 \phi^r}{\partial q_h \partial q_g} + \frac{\partial \phi^r}{\partial q_g} + \nu_g + \sum_{i\in\mathcal{N}} q^C_{fi} \frac{\partial^2 \phi^r}{\partial d_i \partial q_g} \quad &\perp \quad q_g \ge 0, \quad \forall g\in\mathcal{G}_f, \quad \forall f\in\mathcal{F}, \\ 0 \le V_g - q_g \quad &\perp \quad \nu_g \ge 0, \quad \forall g\in\mathcal{G}_f, \quad \forall f\in\mathcal{F}. \end{aligned}$$

From (9.6), this LCP can be written in matrix vector form as:

$$0 \le \begin{bmatrix} N & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} q \\ \nu \end{bmatrix} - \begin{bmatrix} y \\ V \end{bmatrix} \perp \begin{bmatrix} q \\ \nu \end{bmatrix} \ge 0$$

where

$$\begin{aligned} N &= \operatorname{diag}(2v) + \sum_{f\in\mathcal{F}} T^f + M, \\ y &= b - u - \sum_{f\in\mathcal{F}} S^f, \end{aligned} \tag{9.7}$$

with the elements of T^f: t^f_{ij}, defined as follows:

$$t^f_{gh} = \begin{cases} m_{gh}, & \{g,h\}\in\mathcal{G}_f, \\ 0, & otherwise, \end{cases}$$

and the elements of S^f: s^f_g, defined as:

$$s^f_g = \begin{cases} \sum_{i\in\mathcal{N}} q^C_{fi} \frac{\partial^2 \phi^r}{\partial d_i \partial q_g}, & g\in\mathcal{G}_f, \\ 0, & otherwise. \end{cases}$$

Lemma 9.2. *The matrix N, given in equation* (9.7), *is positive definite so long as $v_g > 0$, $\forall g \in \mathcal{G}$.*

Proof. Consider matrix N, given in equation (9.7). It is the sum the following matrices: a positive semi-definite matrix M; the matrix T^f for all $f\in\mathcal{F}$; and finally a diagonal matrix with elements of the vector v on the diagonal.

First recall that M is a positive semi-definite matrix. Hence, for each $f\in\mathcal{F}$, T^f must be a positive semi-definite matrix.

It is clear that so long as $v_g > 0$, $\forall g\in\mathcal{G}$, diag$(2v)$ is positive definite. Hence N must be a positive definite matrix. $\square$

The LCP above can be written as the following strictly convex quadratic program:

$$\min \quad \tfrac{1}{2} q^T N q - y^T q$$

$$\text{s.t.} \quad 0 \le q \le V,$$

which has a unique solution [19]. The solution to this problem is the candidate equilibrium for regime r. Since the equilibrium problem is equivalent to a convex quadratic program, it can be computed by any convex QP solver.

We have shown, above, that for each KKT regime there exists a unique equilibrium to a modified Cournot game, which is a candidate equilibrium for the full-rationality game. In the next chapter we will derive further conditions which ensure that this candidate equilibrium is a bona-fide equilibrium. However, first we will present an example of the computation of a candidate equilibrium over a three-node network for a particular KKT regime.

Example 9.3. *Here we will demonstrate the computation of a candidate equilibrium for a three node network, with all the lines having the same reactance; this situation is shown in figure 9.1. There is one generator at each node, which (for simplicity) we have assumed has no production costs, no contracts and no generation capacity. We will compute the candidate equilibrium for the KKT regime where the line joining nodes 2 and 3 is at its upperbound.*

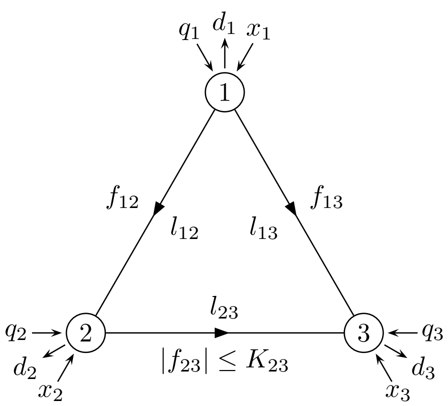

Figure 9.1: Three-node network.

We first present the dispatch problem for this KKT regime:

$$\min \quad \tfrac{1}{2a_1}{x_1}^2 + \tfrac{1}{2a_2}{x_2}^2 + \tfrac{1}{2a_3}{x_3}^2$$

$$\begin{aligned} \text{s.t.} \quad & -x_1 + f_{12} + f_{13} = q_1 - d_1 \quad [\pi_1] \\ & -x_2 - f_{12} + K_{23} = q_2 - d_2 \quad [\pi_2] \\ & -x_3 - f_{13} - K_{23} = q_3 - d_3 \quad [\pi_3] \\ & f_{12} - f_{13} + K_{23} = 0. \end{aligned}$$

Suppose now that the demand at the nodes are $d_1 = 100$, $d_2 = 100$ and $d_3 = 200$, and the fringe supply functions are $S_i(p) = p$ for all nodes (without upper or lower bounds). In this situation the nodal prices as functions of the injections at each node are:

$$\begin{aligned}
\pi_1 &= \tfrac{1}{3}\left(400 - q_1 - q_2 - q_3\right), \\
\pi_2 &= \tfrac{1}{6}\left(500 - 2q_1 - 5q_2 + q_3\right) + \tfrac{3}{2}K_{23}, \\
\pi_3 &= \tfrac{1}{6}\left(1100 - 2q_1 + q_2 - 5q_3\right) - \tfrac{3}{2}K_{23}.
\end{aligned}$$

Suppose we have three firms, each owning one of the generators. For example, firm 1, owning the generator at node 1, has the following profit maximization problem:

$$\begin{aligned}
\max \quad & \rho_1\left(q_1\right) = \tfrac{1}{3}q_g\left(400 - q_1 - q_2 - q_3\right) \\
\\
\text{s.t.} \quad & q_1 \geq 0.
\end{aligned}$$

Hence we can formulate the resulting game as the following LCP (as demonstrated above):

$$0 \leq \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{3} & -\frac{1}{6} \\ \frac{1}{3} & -\frac{1}{6} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} - \begin{bmatrix} \frac{400}{3} \\ \frac{500}{6} + \frac{3}{2}K_{23} \\ \frac{1100}{6} - \frac{3}{2}K_{23} \end{bmatrix} \perp \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \geq 0.$$

If, for example, $K_{23} = 20$ the above LCP can be solved to give the following candidate equilibrium:

$$q_1 = 142.86, \quad q_2 = 46.23, \quad q_3 = 68.05,$$

with prices

$$\pi_1 = 47.62, \quad \pi_2 = 38.53, \quad \pi_3 = 56.71.$$

This candidate equilibrium is not necessarily an equilibrium to the original game. It satisfies the condition that every generator is at some local optimum point in their decision space, however, two additional requirements must be satisfied for it to be an equilibrium in the full-rationality game. The first is that we require this point to be optimal for the original dispatch problem; that is if the generators each inject the quantities above, the optimal solution to the dispatch problem must correspond to the chosen KKT regime. The second is that each generator must be globally optimizing their profit functions, that is there can be no incentive to deviate from the candidate equilibrium by changing which lines are congested. In the next chapter we will discuss these additional conditions in more detail to enable us to verify under what circumstances candidate equilibria are in fact bona fide equilibria in the full-rationality game.

Chapter 10

Existence of equilibria over general networks

In this chapter, we derive conditions which guarantee that there exists particular Nash-Cournot equilibria in the full-rationality setting. Of particular interest to us is how the capacities of the transmission lines and bounds on fringes may affect the existence of equilibria. In chapter 8, we showed for two-node networks that, due to generators exploiting a line with a small capacity, there may not exist any pure-strategy equilibria. In this chapter, we extend the principles behind those examples to more general networks and revenue functions.

Specifically, we detail an algorithm that defines the sets of line capacities and fringe bounds guaranteeing the existence of particular candidate equilibria. Following this, we give examples demonstrating the implementation of this algorithm.

This chapter is laid out as follows: we first introduce the concept of *consistent* candidate equilibria, and discuss the incentives for firms to deviate from an equilibrium. We then detail an algorithm that enables the derivation of set of conditions ensuring that a particular candidate equilibrium is valid. Following this, we discuss the implementation of the algorithm and provide some simple examples.

10.1 Existence of equilibria

In this section we will derive the conditions which guarantee that a candidate equilibrium (from the previous chapter) is in fact a bona-fide equilibrium in the full-rationality setting. Recall that a point in a game's joint decision space is an equilibrium if (and only if) each player is maximizing its payoff with respect to its decision [94]. Therefore in order for a candidate equilibrium to be a true equilibrium, we require that each firm is maximizing its profit (globally), and that the candidate equilibrium complies with the optimality conditions of the dispatch problem. We detail these conditions in the following sections.

10.1.1 Consistent candidate equilibria

In order for a candidate equilibrium, q^r, associated with KKT regime r to be valid, the solution to the dispatch problem, $\mathrm{P}(q^r)$ must be consistent with the requirement of KKT regime r. Specifically, the vector of candidate equilibrium injections, q^r, must be such that the conditions given in lemma 4.2 are satisfied.[1] If the candidate equilibrium, q^r, complies with these conditions, then we call this a *consistent* candidate equilibrium. Otherwise, the candidate equilibrium is not consistent and hence cannot be a valid equilibrium to the game.

10.1.2 Incentive to deviate

If the candidate equilibrium is consistent, then we must check whether there is an incentive for any firm to deviate to a different KKT regime.

Suppose that for some KKT regime, r, we have obtained a consistent candidate equilibrium with injections q^r. From the way candidate equilibria are computed (discussed in chapter 9), we know there is no incentive to deviate within regime r. Therefore in order for q^r to be a bona fide equilibrium, we require that there is no incentive for any generator to deviate to another regime. One way to check this is to solve the problem given by (9.1), for each firm with all other firms' generators fixed at the candidate equilibrium solution, q^r. This problem is a mathematical program with equilibrium constraints (MPEC). If the maximum profit is attained at $q_g^* = q_g^r, \forall g \in \mathcal{G}_f$ for all firms $f \in \mathcal{F}$, then the candidate equilibrium is a bona fide equilibrium (see [83])

However, if we wish to investigate how the existence of a particular equilibrium depends on pa-

[1] Recall that these conditions ensure that the implicit congestion assumptions made by choosing a KKT regime comply with the optimality conditions of the dispatch problem.

rameters of the problems, we need to take a different approach. Here we must use a parametric approach to solve this problem, by considering deviations to each KKT regime separately. Therefore for each KKT regime that may be deviated to, we must consider the maximum profit each firm can attain within that regime. For some firm $f \in \mathcal{F}$ deviating to KKT regime s, this is given by the following maximization problem:

$$\rho_f^{s*}(q_{-g}) = \max \quad \sum_{i\in\mathcal{N}} \left[\sum_{g\in\mathcal{G}_i\cap\mathcal{G}^f} \left(q_g \pi_i^s - C_g\left(q_g\right)\right) \right] - \sum_{i\in\mathcal{N}} q_{fi}^C \pi_i^s$$

$$\text{s.t.} \qquad 0 \le q_g \le V_g, \quad \forall g \in \mathcal{G}^f$$

$$\text{Constraints given by (4.2) and (4.3) for KKT regime } s.$$

Note that $\pi^s(q)$ is uniquely determined by the solution to $\mathrm{P}^s(q)$, the dispatch problem for KKT regime s.

Since this problem is convex with linear constraints, the optimal solution will be the solution to its KKT conditions. If there exists some KKT regime, s, such that $\rho_f^s(q_{-n}^r) > \rho_f^{r*}$ then there will be an incentive for firm f to deviate from the candidate equilibrium for KKT regime r. Therefore for each regime, s, and firm, f, we will derive necessary and sufficient conditions, on the parameters of the game, yielding $\rho_f^s(q_{-n}^r) > \rho_f^{r*}$. The complement of the union of these conditions on the parameters gives the set of conditions guaranteeing that no firm has any incentive to deviate from the candidate equilibrium.

Finally, we must ensure that the equilibrium is consistent, therefore we take the intersection of the conditions, above, with those from lemma 4.2. The set of fringe bounds and line capacities formed by these conditions are necessary and sufficient to support the existence of this particular Cournot equilibrium (the candidate equilibrium corresponding to KKT regime r).

We will now formally present the algorithm that derives the conditions ensuring that a particular candidate equilibrium is in fact an equilibrium to the full-rationality Cournot game.

10.1.3 Algorithm

Algorithm 10.1. *This algorithm details the procedure that finds the conditions under which a candidate equilibrium for KKT regime r is guaranteed to be a bona-fide equilibrium:*

Step 1: Compute the candidate equilibrium, q^r, for KKT regime r.

Step 2: Compute the conditions given in lemma 4.2 which ensure that KKT regime r is the optimal regime for $\mathrm{P}(q^r)$.

Step 3: For each firm, f, and KKT regime, r, find the conditions for which firm f has incentive to deviate from the candidate equilibrium for KKT regime r.

Step 4: Take the complement of the union of the conditions found in step 3; these give the conditions ensuring that no generator has incentive to deviate from the candidate equilibrium.

Step 5: Take the intersection of the conditions found in step 2, with the conditions found in step 4; together these guarantee that the candidate equilibrium is consistent and that there is no incentive for any generator to deviate to another regime.

10.1.4 Implementation

The above algorithm has been implemented algebraically using `C#.net` and `Mathematica` [44, 96]. Using `C#.net` the mathematical description of a network is constructed, and the KKT conditions for the dispatch problem are generated. These KKT conditions are passed into `Mathematica` to be solved algebraically for each possible orthogonality outcome. Lastly, inequalities defining the conditions ensuring existence of equilibria are generated.

This procedure is demonstrated below in an example with a three-node network, with linear fringe supply functions.

Example 10.2. *Consider the three-node network shown in figure 10.1.*

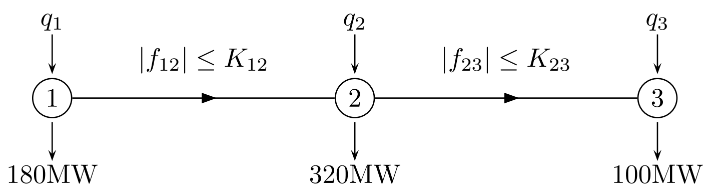

Figure 10.1: Three-node radial network.

For this network we define the following fringe supply functions:

$$S_1(p) = S_2(p) = S_3(p) = p.$$

The demand at the nodes are $d_1 = 180$, $d_2 = 320$, $d_3 = 100$ as shown in figure 10.1. Furthermore, we will assume that the generators have no costs and no capacities; that is

$$C_1(q) = C_2(q) = C_3(q) = 0, \quad V_1 = V_2 = V_3 = \infty.$$

Finally, we will assume that the upper and lower bounds on each of the fringes are $+\infty$ and $-\infty$, respectively; hence we will only consider KKT regimes whereby the fringes are not at bound, yielding

$$\mathcal{D}_r^+ = \mathcal{D}_r^- = \emptyset, \quad \forall r \in \mathcal{R}.$$

We will first consider the unconstrained candidate equilibrium. This is associated with KKT regime 1, with the following sets: $\mathcal{B}_1^+ = \emptyset$, $\mathcal{B}_1^- = \emptyset$. We can therefore find nodal prices as functions of the injections (from the KKT conditions (4.1)) to be:

$$\pi_1^1 = \pi_2^1 = \pi_3^1 = 200 - \frac{q_1 + q_2 + q_3}{3}.$$

Thus the profit maximization problem for each generator, i, is:

$$\max_{q_i \geq 0} \quad q_i \left(200 - \frac{q_1 + q_2 + q_3}{3}\right).$$

The candidate equilibrium can therefore be found to be:

$$q_1^1 = q_2^1 = q_3^1 = 150,$$

with nodal prices

$$\pi_1^1 = \pi_2^1 = \pi_3^1 = 50,$$

candidate equilibrium flows

$$f_{12}^1 = 20, \; f_{23}^1 = -100, \tag{10.1}$$

and profits

$$\rho_1^1 = \rho_2^1 = \rho_3^1 = 7500.$$

The next step is to find the conditions which ensure that the candidate equilibrium is consistent. Following (10.1), the conditions in lemma 4.2 become:

$$K_{12} \geq 20, \; K_{23} \geq 100.$$

Now we must find the conditions ensuring that no generator has incentive to deviate from the unconstrained equilibrium. To do this, we enumerate all possible deviations for each generator. These are the regimes that a generator may have incentive to deviate to:

$r = 2$: $\mathcal{B}_2^+ = \{12\}$, $\mathcal{B}_2^- = \emptyset$,

$r = 3$: $\mathcal{B}_3^+ = \emptyset$, $\mathcal{B}_3^- = \{23\}$,

$r = 4$: $\mathcal{B}_4^+ = \{12\}$, $\mathcal{B}_4^- = \{23\}$,

$r=5$: $\mathcal{B}_5^+=\emptyset$, $\mathcal{B}_5^-=\{12\}$,

$r=6$: $\mathcal{B}_6^+=\{23\}$, $\mathcal{B}_6^-=\emptyset$,

$r=7$: $\mathcal{B}_7^+=\{12,23\}$, $\mathcal{B}_7^-=\emptyset$,

$r=8$: $\mathcal{B}_8^+=\emptyset$, $\mathcal{B}_8^-=\{12,23\}$,

$r=9$: $\mathcal{B}_9^+=\{23\}$, $\mathcal{B}_9^-=\{12\}$.

Since this is a repetitive process, we will only demonstrate this procedure for generator 2 for $r=2$. For $r=2$ with $q_1^1=q_3^1=150$, the nodal prices can be calculated to be:

$$\pi_1^2=30+K_{12}, \quad \pi_2^2=\pi_3^3=\tfrac{1}{2}\left(270-K_{12}-q_2\right).$$

Therefore we have the following convex profit maximization problem:

$$\begin{aligned}
\max \quad & \rho_2^2\left(q_2\right)=q_2\times\tfrac{1}{2}\left(270-K_{12}-q_2\right)\\
\\
\text{s.t.} \quad & q_2\leq 210-3K_{12}\\
& q_2\leq 370-K_{12}+2K_{23}\\
& q_2\geq 370-K_{12}-2K_{23}\\
& q_2\geq 0.
\end{aligned}$$

The constraints of the above problem ensure that if generator 2 injects q_2 then regime 2 will be optimal. For example, since $\{1,2\}\in\mathcal{B}_2^+$ then $\pi_1^2\leq\pi_2^2$; this gives

$$30+K_{12}\leq\frac{1}{2}\left(270-K_{12}-q_2\right),$$

which can be rearranged to yield

$$q_2\leq 210-3K_{12}.$$[2]

If the optimal value of the above problem exceeds the profit attained in the candidate equilibrium for generator 2 then there is incentive for that generator to deviate from the candidate equilibrium. Since the line capacities are not specified at this stage, we must solve the problem parametrically. To do this we will consider the following potential KKT points for the above problem:

1. $q_2=\frac{1}{2}\left(270-K_{12}\right)$,

2. $q_2=210-3K_{12}$,

[2]The other constraints in the above optimization problem ensure that the flow on line 23 does not exceed its capacity.

	Feasibility Conditions	Optimality Conditions	Incentive Condition
1.	$K_{12} \leq 30$ $K_{12} - 4K_{23} \leq 470$ $K_{12} + 4K_{23} \geq 470$ $K_{12} \leq 270$	$\frac{1}{2}(270 - K_{12} - 2q_2) = 0$	$\frac{1}{8}(270 - K_{12})^2 > 7500$
2.	$K_{12} + K_{23} \geq -80$ $K_{12} - K_{23} \leq -80$ $K_{12} \leq 70$	$K_{12} \geq 30$	$(210 - 3K_{12})(30 + K_{12}) > 7500$
3.	$K_{12} + K_{23} \leq -80$ $K_{23} \geq 0$ $K_{12} - 2K_{23} \geq 370$	$K_{12} - 4K_{23} \geq 470$	$(370 - K_{12} + 2K_{23})(-50 - K_{23}) > 7500$
4.	$K_{12} - K_{23} \leq -80$ $K_{23} \geq 0$ $K_{12} + 2K_{23} \leq 370$	$K_{12} + 4K_{23} \leq 470$	$(370 - K_{12} - 2K_{23})(K_{23} - 50) > 7500$
5.	$K_{12} \leq 70$ $K_{12} - 2K_{23} \leq 370$ $K_{12} + 2K_{23} \geq 370$	$K_{12} \geq 270$	$0 > 7500$

Table 10.1: Deviation conditions for generator 2.

3. $q_2 = 370 - K_{12} + 2K_{23}$,

4. $q_2 = 370 - K_{12} - 2K_{23}$,

5. $q_2 = 0$.

From these potential KKT points we can derive a set of feasibility, optimality and incentive conditions for each of the above potential solutions. These conditions must be met to ensure that the solution is feasible, optimal, and yields a profit higher than the candidate equilibrium; these are given in table 10.1.

We now plot the feasibility conditions and optimality conditions for each of the five possible solutions. The feasibility conditions are shown in figure 10.2 (i), and in figure 10.2 (ii) we plot the intersection of these conditions with the incentive conditions. These regions give the pairs of K_{12} and K_{23} such that there is incentive for generator 2 to deviate into KKT regime 2 from the candidate equilibrium corresponding to regime 1. Particularly, the blue region corresponds to KKT regime 1, the green region to KKT regime 2 and the red region to KKT regime 4.

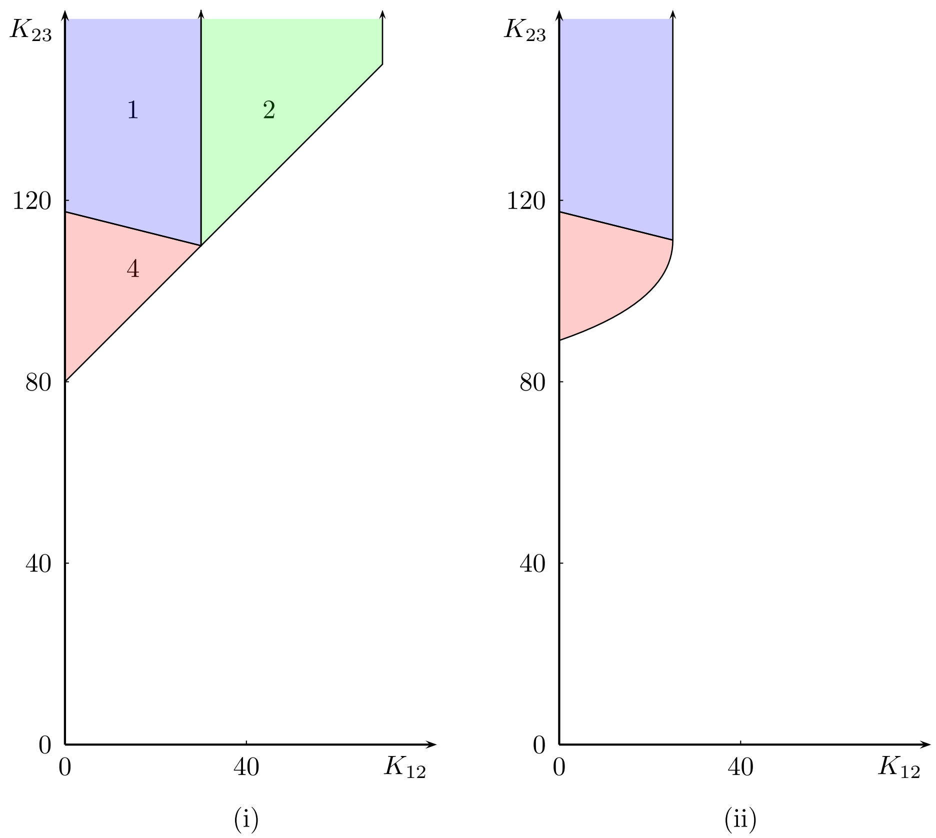

Figure 10.2: Deviation set for generator 2.

Since we are interested in determining the set of capacities for which there is feasible incentive to deviate to a particular regime, the optimality conditions are actually redundant. If a solution is feasible and yields a profit higher than the candidate equilibrium then that is sufficient for there to be incentive to deviate. This simplifies the characterisation of the deviation set for each generator and KKT regime.

We can employ the same method for all KKT regimes and all generators to find a deviation set for each possible deviation. The complement of the union of these sets gives the conditions on the capacities of the two lines which ensure that the unconstrained equilibrium exists. The region defined by these conditions is shaded blue in figure 10.3 below. Furthermore, we can determine the set of conditions on the line capacities that ensure the existence of constrained equilibria. The region shaded green gives the line capacities ensuring the existence of an equilibrium with KKT regime 4. The region shaded red gives capacities supporting an equilibrium with KKT regime 3. Finally the region shaded yellow gives capacities supporting an equilibrium with KKT regime 5. For the other KKT regimes, given a pair of line capacities, there exists at least one generator who would have incentive to deviate from the candidate equilibrium to another KKT regime.

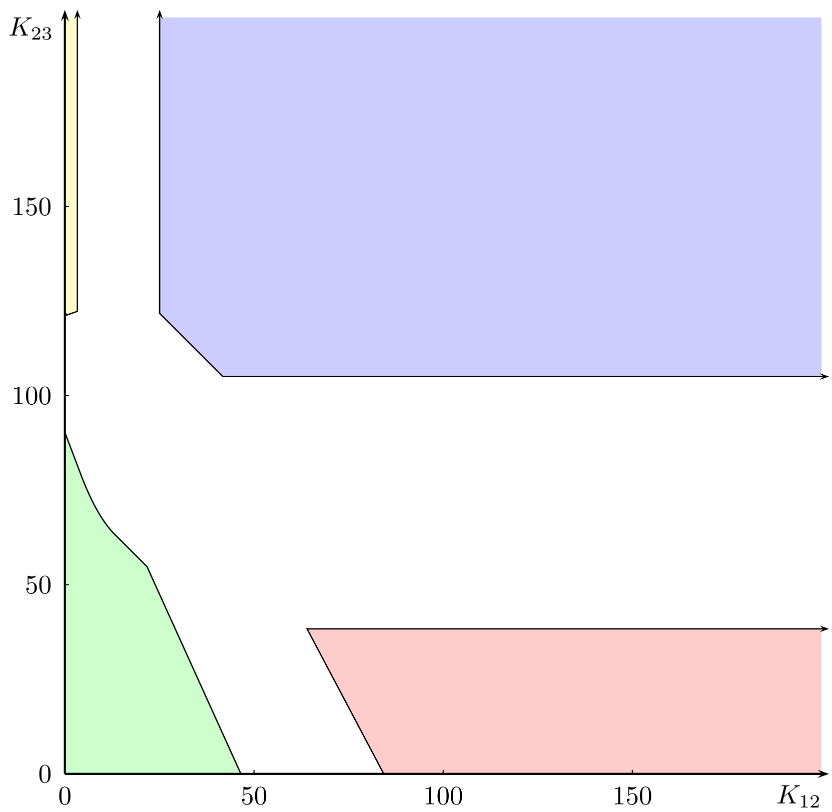

Figure 10.3: Three nodes: conditions for different equilibria.

The regions depicted in figure 10.3 are a key result of this work. They show specifically how the types of equilibria are affected by the parameters of the game (in this case, the line capacities). Later, in chapter 15 we consider the question of transmission investments, where the decisions are the line capacities. In this context, being able to a priori determine the line capacities that give various Cournot equilibria may be a useful tool.

To complete this chapter, we will consider another example; this time the network is a three-node loop and we derive the conditions on the line capacities ensuring that an unconstrained equilibrium exists.

10.1.5 Loops

We will now consider a network containing a loop. Now there is no longer a unique path electricity must take between every pair of nodes, so Kirchhoff's laws must be applied to determine a unique power flow across lines in the network.

First consider a three-node linear network, with infinite capacity on both arcs, shown in figure

10.4 (i). Since both lines can support an arbitrarily large power flow, we know that there exists an unconstrained equilibrium over this network. Now, if we were to create a new line connecting the two end nodes then a loop would be formed, as shown in 10.4 (ii). Since the flows on the lines must follow Kirchhoff's laws, at equilibrium there will be some flow on the new line. Therefore if the line is created with a limited capacity and that capacity is not sufficient to support the flow in the previous equilibrium, the existence of that unconstrained equilibrium will be precluded. This is the most basic example of how the presence of loops can give counter-intuitive results.

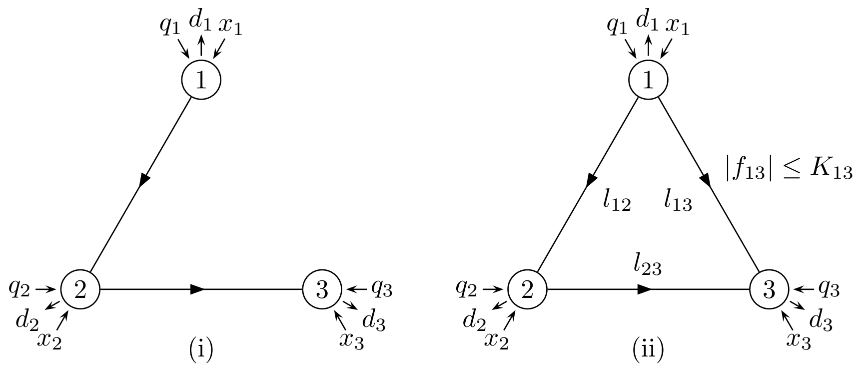

Figure 10.4: Three node network; creating a loop.

In what follows, we will compute a set of line capacities ensuring that the unconstrained candidate equilibrium is valid for a three-node loop grid.

Consider a three-node loop, as shown in figure 10.5, with zero costs for all generators. We can calculate the unconstrained equilibrium quantities to be:

$$q_1^U = q_2^U = q_3^U = \frac{d_1 + d_2 + d_3}{4}.$$

Set generators 2 and 3 to these equilibrium quantities, and consider possible deviations for generator 1. In order to derive the necessary and sufficient conditions for the existence of the unconstrained equilibrium, we must consider the incentive to deviate to, and feasibility of, each particular KKT regime. We will show, for a particular example over a three-node loop, that the non-deviation set for generator 1 is non-convex.

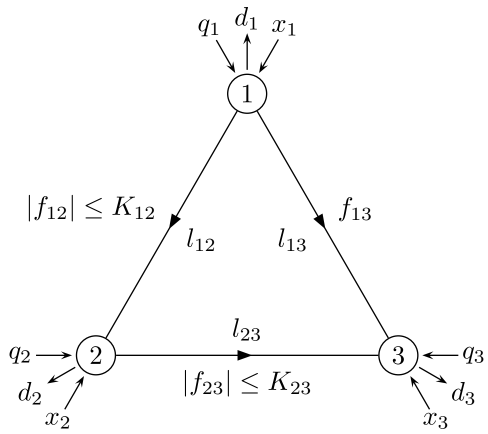

Figure 10.5: Three-node network; two lines have capacity constraints.

Assuming linear fringes of the form $S_i(p) = a_i p$, the dispatch problem for the three-node loop situation, shown in figure 10.5, is given below:

$$\min \quad \tfrac{1}{2a_1}{x_1}^2 + \tfrac{1}{2a_2}{x_2}^2 + \tfrac{1}{2a_3}{x_3}^2$$

$$\begin{aligned}
& q_1 + x_1 - f_{12} - f_{13} = d_1 && [-\pi_1] \\
& q_2 + x_2 + f_{12} - f_{23} = d_2 && [-\pi_2] \\
& q_3 + x_3 + f_{13} + f_{23} = d_3 && [-\pi_3] \\
& l_{12} f_{12} - l_{13} f_{13} + l_{23} f_{23} = 0 && [\lambda] \\
& |f_{12}| \le K_{12} && \left[\eta_{12}^{+}, \eta_{12}^{-}\right] \\
& |f_{23}| \le K_{23} && \left[\eta_{23}^{+}, \eta_{23}^{-}\right].
\end{aligned}$$

Since the above problem is convex, we can replace it with its KKT equivalent system, shown below

$$
\begin{aligned}
q_1 + a_1\pi_1 - f_{12} - f_{13} &= d_1 \\
q_2 + a_2\pi_2 + f_{12} - f_{23} &= d_2 \\
q_3 + a_3\pi_3 + f_{13} + f_{23} &= d_3 \\
l_{12}f_{12} - l_{13}f_{13} + l_{23}f_{23} &= 0 \\
\pi_1 - \pi_2 + l_{12}\lambda + \eta_{12}^{+} - \eta_{12}^{-} &= 0 \\
\pi_1 - \pi_3 - l_{13}\lambda &= 0 \\
\pi_2 - \pi_3 + l_{23}\lambda + \eta_{23}^{+} - \eta_{23}^{-} &= 0 \\
0 \leq K_{12} - f_{12} \quad &\perp \quad \eta_{12}^{+} \geq 0 \\
0 \leq K_{12} + f_{12} \quad &\perp \quad \eta_{12}^{-} \geq 0 \\
0 \leq K_{23} - f_{23} \quad &\perp \quad \eta_{23}^{+} \geq 0 \\
0 \leq K_{23} + f_{23} \quad &\perp \quad \eta_{23}^{-} \geq 0.
\end{aligned}
$$

As discussed earlier, the KKT conditions above can be solved parametrically by considering different KKT regimes that define the congestion on the lines; the five KKT regimes are:

1. no lines congested,
2. $f_{12} = -K_{12}$, f_{23} not congested,
3. $f_{23} = K_{23}$, $f_{12} = -K_{12}$,
4. $f_{23} = -K_{23}$, $f_{12} = -K_{12}$,
5. $f_{23} = K_{23}$, f_{12} not congested.

For each of these cases, it is possible to find the feasibility and incentive conditions for deviation (as computed in the previous example); these conditions give a set of capacities such that generator 1 has incentive to deviate from the unconstrained equilibrium. For this example, we set the following fringe properties

$$a_1 = 1.44, \quad a_2 = 3.4, \quad a_3 = 1.44.$$

The demands are set to be

$$d_1 = 3.04, \quad d_2 = 0.76, \quad d_3 = 1.5,$$

and the line reactances are set to be

$$l_{12} = 0.178, \quad l_{13} = 0.104, \quad l_{23} = 0.104.$$

The unconstrained candidate equilibrium corresponds to KKT regime 1, above. Therefore we will find the conditions on line capacities such that it is profitable for generator to deviate to each of the KKT regimes 2–5. These can be computed in a similar manner to that in the previous example. The resulting conditions for each of the 4 KKT regimes are given by the shaded regions in the plots in figure 10.6.

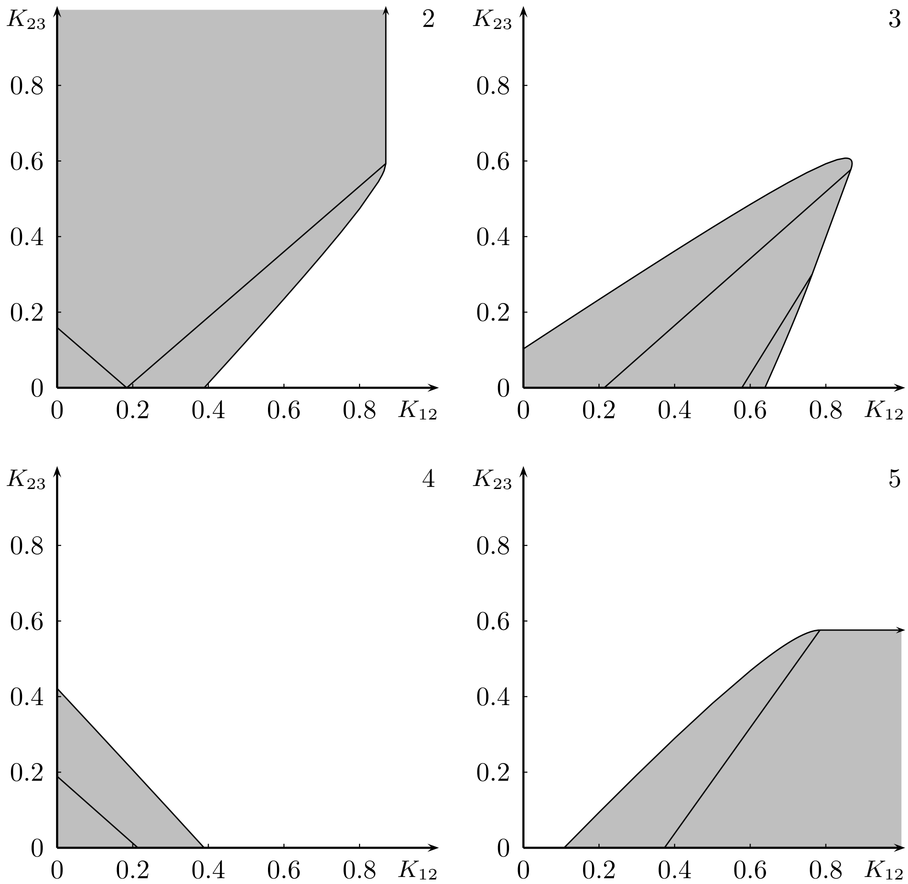

Figure 10.6: Deviation sets for generator 1 for KKT regimes 2–5.

The complement of the union of all these conditions gives the set of capacities which ensure that generator 1 has no incentive to deviate. This is the non-deviation set for generator 1 and is given by the unshaded region of figure 10.7 (i), below. Note that if we zoom in on the corner of this set, as shown in 10.7 (ii), we can see that the non-deviation set for this generator is not convex.

In the next chapter, we will prove that for radial networks with each generator independently owned, the conditions, ensuring that the unconstrained equilibrium is valid, form a convex polyhedral set. The above example shows that these results are not valid if there are loops in the network.

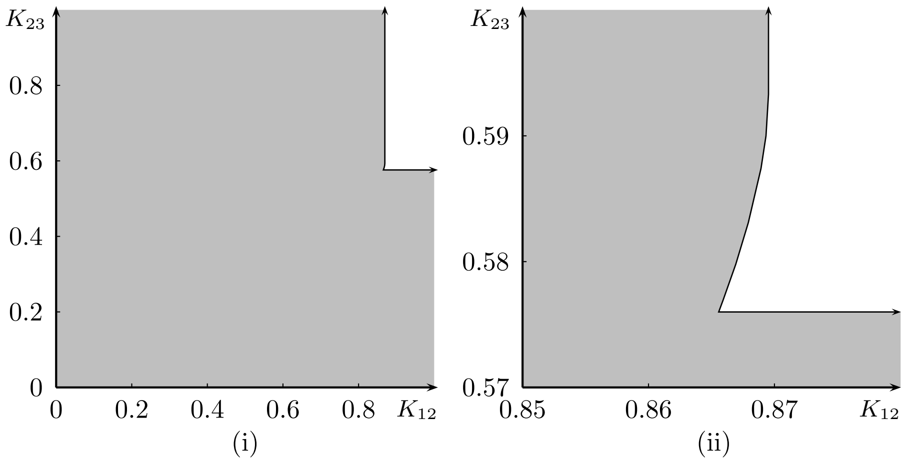

Figure 10.7: Non-deviation set for generator 1 in a three-node loop (unshaded region).

Chapter 11

Unconstrained equilibria over radial networks

In this chapter we turn our attention to the specific case of the existence of the unconstrained equilibrium over radial transmission networks. The algorithm presented in the previous chapter to compute the set that guarantees the existence of various candidate equilibria is combinatorial in nature, and therefore becomes very computationally expensive as the size of the network and number of generators grow. Here, by restricting our interest to radial networks, we will derive a number of results, based on the shape of the residual demand curve discussed in chapter 5.[1]

We start this chapter by discussing radial networks where the fringe supply functions are smooth increasing functions and where the generation costs are convex increasing functions. We prove that in this case the conditions on the line capacities and fringe bounds that are necessary to ensure the existence of an unconstrained Cournot equilibrium form a convex set. We then restrict ourselves to linear supply functions and quadratic costs, and derive the specific conditions ensuring the existence of an unconstrained equilibrium in this case. Finally we present some simple examples of the application of these conditions.

[1]In this chapter, we extend the results of Downward *et al.* [26] to more general fringe supply functions and convex generation costs.

11.1 General supply curves and cost functions

In this section we are interested in the existence of an unconstrained equilibrium in the presence of smooth increasing fringe supply functions, as specified in section 4.1.2.[2] We will assume that each generator is an independent profit maximizer, and it has some convex cost function, $C_g(q_g)$, of the form defined in section 7.2.2.[3]

Furthermore, we allow each generator, g, to have a contract for differences with some retailer at its own node, n; the volume of this contract is defined as q_g^C. Therefore, ignoring the fixed value of the contract, the profit function for generator g is

$$\rho_g(q_g) = \left(q_g - q_g^C\right)\pi_n(q) - C_g(q_g).$$

In the following sections, we will show that the necessary and sufficient conditions guaranteeing the existence of any unconstrained equilibrium are given by a convex set in terms of the line capacities and fringe bounds.

11.1.1 Unconstrained equilibrium

We will first determine a set of necessary conditions for an unconstrained equilibrium. This is achieved by ignoring the presence of line capacities and fringe bounds, which in effect reduces the game to one where all the generators and demand are located at a single node. All generators, therefore will receive the same nodal price, $\pi^U(q)$, where q is a vector of all generator injections. We will now define the equilibrium for this unconstrained game.

Recall that a set of strategies is a Nash equilibrium if all players are maximizing their respective profit functions, given the actions of the other players. Hence, each generator, g, must optimize the following:

$$\begin{aligned} \max \quad & \left(q_g - q_g^C\right)\pi^U(q) - C_g(q_g) \\ \text{s.t.} \quad & 0 \le q_g \le V_g. \end{aligned} \tag{11.1}$$

The following complementarity problem, comprises the KKT conditions for the above problem for each generator, g.

$$\begin{aligned} 0 \le \left(q_g^C - q_g^U\right)\left.\frac{\partial \pi^U}{\partial q_n}\right|_{q=q^U} - \pi^U\left(q^U\right) + C_g'\left(q_g^U\right) + \nu_g \quad \perp \quad q_g^U \ge 0, \quad \forall g \in \mathcal{G}, \\ 0 \le V_g - q_g^U \quad \perp \quad \nu_g \ge 0, \quad \forall g \in \mathcal{G}. \end{aligned} \tag{11.2}$$

[2] Recall that these functions must also be invertible.

[3] Recall that the cost function must be a smooth, increasing, convex function.

From [57], we know that the complementarity problem, above, provides necessary conditions for optimality of the problem given in (11.1) for each generator. Therefore, any solution to the game where all generators simultaneously solve (11.1) must be a solution to (11.2).

Because we have not restricted the fringe supply functions to be concave, the generators are potentially facing non-convex profit functions, hence we are not guaranteed that there will exist an unconstrained candidate equilibrium for the game, or if one does exist that it is unique. However, for the purposes of this work, we suppose that there exists a unique equilibrium for the game above. We will refer to this as the unconstrained candidate equilibrium, q^U.

In the following sections, we will derive the necessary and sufficient conditions for the existence of this unconstrained equilibrium.

11.1.2 Existence of unconstrained equilibrium

Suppose now that we reintroduce the line capacities, K_{ij} and the upper and lower bounds on the fringe dispatch, x_i^+ and x_i^- respectively. We are interested in determining the conditions on the line capacities and the bounds on the fringe which guarantee that the unconstrained q^U remains an equilibrium[4].

For the KKT regime, U, associated with the unconstrained equilibrium, we have $\mathcal{B}_U = \mathcal{A}_n$ and $\mathcal{D}_U = \mathcal{N}$ (that is, all lines and fringes in KKT regime U are not at bound). Hence, from lemma 4.2, we know that the conditions ensuring that the unconstrained equilibrium is consistent are:

$$\begin{aligned} -K_{ij} \leq f_{ij}^U \leq K_{ij} \quad & \forall ij \in \mathcal{A}_n, \\ -x_i^- \leq x_i^U \leq x_i^+ \quad & \forall i \in \mathcal{N}. \end{aligned} \tag{11.3}$$

The above conditions ensure that the unconstrained candidate equilibrium strategies, q^U, do not cause any constraints to bind in the dispatch problem. Now to determine whether this candidate equilibrium is in fact a bona fide equilibrium, we must consider whether any generator has incentive to congest any line or constrain any fringe by changing its injection decision.

We approach this problem by considering each generator's incentive to deviate individually. For generator g situated at node n: we fix $q_h = q_h^U$ for all generators, $h \neq g$. To simplify notation we define $\pi_n(q_g)$ to be the nodal price at node n as a function of q_g, where all other generators' injections are fixed at their respective unconstrained candidate equilibrium quantities, q^U.

[4] That is, the presence of the bounds does not make the candidate equilibrium inconsistent, nor does it give any generator incentive to deviate.

Therefore the profit function of generator g at node n becomes

$$\rho_g(q_g) = \left(q_g - q_g^C\right)\pi_n(q_g) - C_g(q_g). \tag{11.4}$$

We will now prove that the conditions on the line capacities and fringe bounds ensuring that a generator g has no incentive to deviate from q_g^U form a convex set. To show this, we will first determine whether there are any intervals to which a generator has no incentive to deviate; the following lemmas and corollaries specify these intervals.

Lemma 11.1. *Suppose that, at the unconstrained candidate equilibrium, generator g at node n injects $q_g^U \geq q_g^C$. In the presence of line capacities and bounds on the fringes, it is never profitable to deviate from this equilibrium by injecting $q_g > q_g^U$.*

Proof. To prove this we consider two possibilities: either $q_g^U = V_g$ or $q_g^U < V_g$. If $q_g^U = V_g$ then the generator is at capacity, so it is clearly not possible to deviate by injecting more than q_g^U. Alternatively, suppose $q_g^U < V_g$ and let the generator deviate by injecting some $q_g > q_g^U$. From corollary 5.15 we know that

$$\pi_n(q_g) = \min_{r \in \mathcal{R}_n^+} \pi_n^r(q_g). \tag{11.5}$$

This implies that, in particular,

$$\pi_n(q_g) \leq \pi_n^U(q_g), \tag{11.6}$$

where $\pi_n^U(q_g)$ is the nodal price at node n pertaining to the KKT regime for which no lines are constrained and no fringes are at bounds.

The profit for generator g is given by equation (11.4). Therefore, with inequality (11.6) and the fact that $q_g \geq q_g^C$ we have

$$\rho_g(q_g) \leq \left(q_g - q_g^C\right)\pi_n^U(q_g) - C_g(q_g) = \rho_g^U(q_g). \tag{11.7}$$

Now, as q_g^U is the injection of generator g in the unconstrained candidate equilibrium it must be a global maximizer for the unconstrained profit function; with this and the inequality in expression (11.7) we have

$$\rho_g(q_g) \leq \rho_g^U(q_g) \leq \rho_g^U\left(q_g^U\right),$$

therefore there can be no incentive to deviate by injecting more than q_g^U. □

Corollary 11.2. *Suppose that, at the unconstrained candidate equilibrium, generator g at node n, injects $q_g^U \leq q_g^C$. In the presence of line capacities and bounds on the fringes, it is never profitable to deviate from this equilibrium by injecting $q_g < q_g^U$.*

Proof. The proof of this result is analogous to that proved in lemma 11.1 above. □

Lemma 11.3. *Consider an increasing convex function $f(x)$, satisfying the following condition*

$$f'(x_1) \leq y. \tag{11.8}$$

For $x_0 \leq x_1$ the following inequality holds

$$f(x_1) - f(x_0) \leq y(x_1 - x_0).$$

Proof. As $f(x)$ is an increasing convex function we can derive the following inequality

$$f(x_1) - (x_1 - x_0) f'(x_1) \leq f(x_0).$$

Since $x_0 \leq x_1$, from (11.8) we have

$$f(x_1) - y(x_1 - x_0) \leq f(x_0),$$

which implies

$$f(x_1) - f(x_0) \leq y(x_1 - x_0),$$

as required. □

Lemma 11.4. *Suppose that, at the unconstrained candidate equilibrium, generator g at node n injects $q_g^U \geq q_g^C$. In this situation it is never profitable to deviate by injecting some $q_g \leq q_g^C$.*

Proof. At the unconstrained candidate equilibrium, generator g's profit is

$$\rho_g^U = \left(q_g^U - q_g^C\right) \pi_n^U\left(q_g^U\right) - C_g\left(q_g^U\right). \tag{11.9}$$

Now suppose that generator g decreases its injection to $q_g \leq q_g^C$; its profit would now be given by

$$\rho_g(q_g) = \left(q_g - q_g^C\right) \pi_n(q_g) - C_g(q_g). \tag{11.10}$$

From theorem 4.3[5] and the orthogonality condition (11.2), we know that if $q_g^U \geq q_g^C$, then at the unconstrained equilibrium

$$\pi_n^U\left(q_g^U\right) \geq C_g'\left(q_g^U\right),$$

so long as $q_g^U > 0$. However, if $q_g^U = q_g^C = 0$ then it is easy to verify the statement of this lemma (since q_g cannot be negative).

Since $C_g(q_g)$ is a convex function and $q_g \leq q_g^U$, we therefore have, from lemma 11.3, that

$$\left(q_g^U - q_g\right) \pi_n^U\left(q_g^U\right) - C_g\left(q_g^U\right) + C_g(q_g) \geq 0,$$

[5] This theorem states that price at a node is an increasing function of the demand at the node.

which can be rearranged to give

$$\left(q_g^U - q_g^C\right)\pi_n^U\left(q_g^U\right) - C_g\left(q_g^U\right) \geq \left(q_g - q_g^C\right)\pi_n^U\left(q_g^U\right) - C_g\left(q_g\right).$$

From lemma 5.6 we know that as a generator withholds its nodal price increases. This gives

$$\pi_n\left(q_g\right) \geq \pi_n^U\left(q_g^U\right).$$

So from above and the fact that $q_g < q_g^C$, we have that

$$\left(q_g^U - q_g^C\right)\pi_n^U\left(q_g^U\right) - C_g\left(q_g^U\right) \geq \left(q_g - q_g^C\right)\pi_n\left(q_g\right) - C_g\left(q_g\right),$$

which with equations (11.9) and (11.10) gives

$$\rho_g^U\left(q_g^U\right) \geq \rho_g\left(q_g\right),$$

as required. □

This proves that if generator g, at the unconstrained candidate equilibrium, injected greater than (or equal to) its total contracted load then there is no incentive for the generator to deviate from the candidate equilibrium by injecting a quantity less than (or equal to) its total contracted load at its node.

Corollary 11.5. *Suppose at the unconstrained candidate equilibrium, generator g at node n injects $q_g^U \leq q_g^C$. In this situation it is never profitable to deviate by injecting some $q_g \geq q_g^C$.*

Proof. The proof of this result is analogous to that proved in lemma 11.4 above. □

From the preceding lemmas we know that if generator g injects more than (or equal to) its contracted quantity in the unconstrained equilibrium, there is no incentive to deviate to a point outside the following interval:

$$q_g \in \left(q_g^C, q_g^U\right).$$

Conversely if generator g injects less than (or equal to) its contracted quantity in the unconstrained equilibrium, there is no incentive to deviate to a point outside of the interval:

$$q_g \in \left(q_g^U, q_g^C\right).$$

Hence if $q_n^U = q_n^C$ there is no incentive to deviate.

Now we will consider different KKT regimes (as discussed in chapter 5) and derive conditions on the capacities on the lines and bounds on the fringes that guarantee the existence of the unconstrained equilibrium.

Consider the KKT regime $r \in \mathcal{R}_n$. Recall that the sets $B_r^{+(-)}$ and $D_r^{+(-)}$ contain arcs or fringes assumed to be at their upper (lower) bounds, respectively. Given these definitions, we can now define

$$Z^r = \overbrace{\sum_{ij \in \mathcal{B}_r^+} K_{ij} - \sum_{ij \in \mathcal{B}_r^+} K_{ij}}^{\text{Line Capacities}} + \overbrace{\sum_{i \in \mathcal{D}_r^-} x_i^- + \sum_{i \in \mathcal{D}_r^+} x_i^+}^{\text{Fringe Bounds}}. \tag{11.11}$$

The profit function of generator g within this regime is given by:

$$\rho_g^r\left(q_g^r, Z^r\right) = \left(q_g^r - q_n^C\right)\pi_n^r\left(q_g^r, Z^r\right) - C_g\left(q_g^r\right).^6 \tag{11.12}$$

The optimal profit for generator g at node n in KKT regime r as a function of Z^r over the domain $q_g^C \leq q_g^r \leq q_g^U$ is given by:

$$\begin{aligned} \rho_g^{r+}\left(Z^r\right) = \max \qquad & \rho_g^r\left(q_g^r, Z^r\right) \\ \text{s.t.} \qquad & q_g^C \leq q_g^r \leq q_g^U. \end{aligned} \tag{11.13}$$

The optimal profit for generator g at node n in KKT regime r as a function of Z^r over the domain $q_g^U \leq q_g^r \leq q_g^C$ is given by:

$$\begin{aligned} \rho_g^{r-}\left(Z^r\right) = \max \qquad & \rho_g^r\left(q_g^r, Z^r\right) \\ \text{s.t.} \qquad & q_g^U \leq q_g^r \leq q_g^C. \end{aligned} \tag{11.14}$$

Theorem 11.6 and corollary 11.7, below, specify the necessary and sufficient conditions for a generator not to have any incentive to deviate from the unconstrained candidate equilibrium. These conditions will be in terms of the optimization problems defined above. We will subsequently recast these conditions in terms of the line capacities and fringe bounds.

Theorem 11.6. *Suppose the conditions* (11.3) *are satisfied, and that for generator g at node n, $q_g^U > q_g^C$. The necessary and sufficient conditions such that generator g has no incentive to deviate from its unconstrained candidate equilibrium quantity, q_g^U, are given by:*

$$\rho_g^{r+}\left(Z^r\right) \leq \rho_g^U, \quad \forall r \in \mathcal{R}_n^+.^7$$

[6] Here the nodal price is not only a function of the injections q, but also the line capacities and fringe bounds contained in Z^r.

[7] Recall that $\mathcal{R}_n^+$ is the set of KKT regimes pertaining to node n with arcs and fringes either at their upper bounds or unconstrained.

Proof. From lemmas 11.1 and 11.4 we know there is never incentive for generator g to inject more than q_g^U or less than q_g^C. Hence for generator g to attain a higher profit, it must withhold to some quantity $q_g \in \left(q_g^C, q_g^U\right)$. From lemma 5.6 we know that when withholding from an unconstrained point, lines will only congest toward node n and fringes will only approach their upper bounds. Therefore only constrained states corresponding to KKT regimes $r \in \mathcal{R}_n^+$ are possible.

As one KKT regime $r \in \mathcal{R}_n^+$ must correspond to the optimal dispatch, a sufficient condition ensuring that generator g has no incentive to deviate from the unconstrained candidate equilibrium is that the optimal profit from deviation within the range $q_g \in \left(q_g^C, q_g^U\right)$ be less than the candidate equilibrium profit for all possible KKT regimes. This condition is given below:

$$\rho_g^{r+}\left(Z^r\right) \leq \rho_g^U, \quad \forall r \in \mathcal{R}_n^+. \tag{11.15}$$

Furthermore, observe that if any inequality in (11.15) is violated there exists a KKT regime r giving:

$$\rho_g^{r+}\left(Z^r\right) = \left(q_g^{r*} - q_g^C\right) \pi_n^r\left(q_g^{r*}, Z^r\right) - C_g\left(q_g^{r*}\right) > \rho_g^U,$$

where q_g^{r*} yields the maximum profit for the problem given in (11.14).

Now by theorem 5.14, we know that for a given vector q for which the g^{th} element has been reduced from the unconstrained candidate equilibrium, the nodal price at node n from $\mathrm{P}(q)$ is given by

$$\pi_n^* = \max_{r \in \mathcal{R}_n^+} \pi_n^r\left(q_g^{r*}, Z^r\right).$$

So the profit actually made by generator g is

$$\rho_g^* = \left(q_g^{r*} - q_g^C\right) \pi_n^* - C_g\left(q_g^{r*}\right) \geq \rho_g^{r+}\left(Z^r\right) > \rho_g^U,$$

showing that generator g can in fact deviate profitably from q_g^U.

Therefore the conditions given in (11.15) are both necessary and sufficient to ensure that generator g cannot profitably deviate from the unconstrained equilibrium, q^U. □

Corollary 11.7. *Suppose the conditions* (11.3) *are satisfied, and that for generator g at node n, $q_g^U < q_g^C$. The necessary and sufficient conditions such that generator g has no incentive to deviate from its unconstrained candidate equilibrium quantity, q_g^U, are given by:*

$$\rho_g^{r-}\left(Z^r\right) \leq \rho_g^U, \quad \forall r \in \mathcal{R}_n^-.$$

Proof. The proof of this result is analogous to that proved in theorem 11.6 above. □

Now we will proceed to show that the conditions given in theorem 11.6 and corollary 11.7 can be equivalently represented by linear inequalities on Z^r. For this, we first need to prove some lemmas that characterise how the profit changes as a function of Z^r.

Lemma 11.8. *Suppose that for generator g at node n $q_g^U > q_g^C$, and for KKT regime $r \in \mathcal{R}_n^+$ generator g injects some $q_g^r > q_g^C$. The profit function of generator g for such an injection is decreasing in Z^r.*

Proof. By making the substitution for Z^r given in equation (11.11), equation (5.7) becomes

$$\sum_{i \in \mathcal{D}_r} S_i\left(\pi_n^r\right) = D^r\left(q\right) - q_g - Z^r, \tag{11.16}$$

where

$$D^r\left(q\right) = \sum_{i \in \mathcal{N}_r} \left(d_i - \sum_{h \in \mathcal{G}_i \setminus \{g\}} q_h \right)$$

is the total residual inelastic demand. Since $n \in \mathcal{N}_r$, it is clear that

$$\frac{\partial \pi_n^r}{\partial Z^r} = -\frac{\partial \pi_n^r}{\partial d_n},$$

therefore from equation (5.8), we have

$$\frac{\partial \pi_n^r}{\partial Z^r} = -\frac{1}{\sum\limits_{i \in \mathcal{D}_r} S_i{}'\left(\pi_n^r\right)}.$$

From equation (11.12), the profit function for generator g associated with this decomposition is

$$\rho_g^r = \left(q_g^r - q_g^C\right) \pi_n^r \left(q_g^r, Z^r\right) - C_g\left(q_g^r\right).$$

The derivative of this function with respect to Z^r is

$$\begin{aligned} \frac{\partial \rho_g^r}{\partial Z^r} &= \left(q_g^r - q_g^C\right) \frac{\partial \pi_n^r}{\partial Z^r} \\ &= -\frac{\left(q_g^r - q_g^C\right)}{\sum\limits_{i \in \mathcal{D}_r} S_i'\left(\pi_n^r\right)} \end{aligned}$$

and as $q_g^r > q_g^C$ and $S_i'(p) > 0, \forall i \in \mathcal{N}$, the profit is decreasing with Z^r. □

Corollary 11.9. *Suppose that for generator g at node n $q_g^U < q_g^C$, and for KKT regime $r \in \mathcal{R}_n^-$ generator g injects some $q_g^r < q_g^C$. The profit function of generator g for such an injection is increasing in Z^r.*

Proof. The proof of this result is analogous to that proved in lemma 11.8 above. □

We will now present a lemma and corollary that show how the *optimal* profit for generator g within KKT regime r varies with Z^r.

Lemma 11.10. *Consider the optimal value function $\rho_g^{r+}(Z^r)$ for some KKT regime $r \in \mathcal{R}_n^+$; this function is decreasing in Z^r so long as the optimal solution to* (11.14) *satisfies $q_g^r > q_g^C$.*

Proof. Suppose that generator g's optimal profit from the optimization problem in (11.14) for $Z^r = Z_1$ and $Z^r = Z_2$, with $Z_1 < Z_2$ are given by

$$\rho_g^{r*}(Z_1) = \rho_g^r(q_1, Z_1), \quad \rho_g^{r*}(Z_2) = \rho_g^r(q_2, Z_2).$$

Since q_1 and q_2 must not violate the bounds of (11.14) and because $\rho_g^r\left(q_g^r, Z_1\right)$ is maximized at $q_g^r = q_1$ the following inequality holds

$$\rho_g^r(q_2, Z_1) \leq \rho_g^r(q_1, Z_1).$$

By assumption $q_2 > q_n^C$, therefore by lemma 11.8 we have that

$$\rho_g^r(q_2, Z_2) < \rho_g^r(q_2, Z_1).$$

These inequalities give

$$\rho_g^{r+}(Z_2) = \rho_g^r(q_2, Z_2) < \rho_g^r(q_1, Z_1) = \rho_g^{r+}(Z_1), \tag{11.17}$$

as required. □

Corollary 11.11. *Consider the optimal value function $\rho_g^{r-}(Z^r)$ for some KKT regime $r \in \mathcal{R}_n^-$; this function is increasing in Z^r so long as the optimal solution to* (11.15) *satisfies $q_g^r < q_g^C$.*

Proof. The proof of this result is analogous to that proved in lemma 11.10 above. □

At this stage we know how the optimal profit of generator g, within some KKT regime r, varies with Z^r. In the next lemma, we show that there exists some critical value of Z^r above which there is no incentive for generator g to deviate from the unconstrained candidate equilibrium.

Lemma 11.12. *Consider the optimal value function, $\rho_n^{r+}(Z^r)$, for some KKT regime $r \in \mathcal{R}_n^+$; if $q_g^U > q_g^C$ there exists some Z_g^{r+} such that $\rho_g^U \geq \rho_g^{r+}(Z^r)$ if and only if $Z^r \in \left[Z_g^{r+}, \infty\right)$.*

Proof. From equation (11.16), we can find some Z_1 such that $\pi_n^r\left(q_g^C, Z_1\right) = 0$:

$$Z_1 = D^r(q) - q_g^C - \sum_{i \in \mathcal{D}_r} S_i(0).$$

For such a Z_1, we have

$$\begin{aligned} \rho_g^{r+}(Z_1) &= \rho_g^r\left(q_g^C, Z_1\right) \\ &= -C_n\left(q_g^C\right). \end{aligned}$$

Since generator g is at a global maximum in the unconstrained candidate equilibrium we have

$$\rho_g^U \geq -C_g\left(q_g^C\right) = \rho_g^{r+}\left(Z_1\right). \tag{11.18}$$

On the other hand, from equation (11.16), we can find some Z_2 such that $\pi_n^r\left(q_g^U, Z_2\right) = \pi_n^U\left(q_g^U\right)$, namely

$$Z_2 = D^r\left(q\right) - q_g^U - \sum_{i \in \mathcal{D}_r} S_i\left(\pi_n^U\left(q_g^U\right)\right),$$

giving a profit of

$$\begin{aligned} \rho_g^U &= \rho_g^r\left(q_g^U, Z_2\right) \\ &\leq \rho_g^{r+}\left(Z_2\right). \end{aligned} \tag{11.19}$$

From inequalities (11.18) and (11.19) above, we find that

$$\rho_g^{r+}\left(Z_1\right) \leq \rho_g^U \leq \rho_g^{r+}\left(Z_2\right). \tag{11.20}$$

Now we will consider two cases:

- If $\rho_g^U = \rho_g^{r+}\left(Z_2\right)$ then from lemma 11.10, we know for all $Z^r < Z_2$ that $\rho_g^U < \rho_g^{r+}\left(Z^r\right)$ and for all $Z^r \geq Z_2$ that $\rho_g^U \geq \rho_g^{r+}\left(Z^r\right)$. In this case we define $Z_g^{r+} = Z_2$.
- On the other hand, if $\rho_n^U < \rho_n^{r+}\left(Z_2\right)$ then from lemma 11.10 and the fact that $\rho_g^{r+}\left(Z_1\right) \leq \rho_g^U$, we are guaranteed that there must exist some Z_g^{r+} such that for all $Z^r \geq Z_g^{r+}$, $\rho_g^U \geq \rho_g^{r+}\left(Z^r\right)$. Conversely, for all for $Z^r < Z_g^{r+}$, $\rho_g^U < \rho_g^{r+}\left(Z^r\right)$.

In both of the cases above there exists some Z_g^{r+} such that if and only if $Z^r \in \left[Z_g^{r+}, \infty\right)$ then $\rho_g^U \geq \rho_g^{r*}\left(Z^r\right)$, as required. □

Corollary 11.13. *Consider the optimal value function, $\rho_g^{r-}\left(Z^r\right)$, for some KKT regime $r \in \mathcal{R}_n^+$; if $q_g^U < q_g^C$ there exists some Z_g^{r-} such that $\rho_n^U \geq \rho_g^{r-}\left(Z^r\right)$ if and only if $Z^r \in \left(-\infty, Z_g^{r-}\right]$.*

Proof. The proof of this result is analogous to that proved in lemma 11.12 above. □

So far we have shown that if $q_g^U > q_g^C$ then for any KKT regime $r \in \mathcal{R}_n^+$, there exists a critical value Z_g^{r+} such that for any $Z^r < Z_g^{r+}$ we have $\rho_g^{r+}\left(Z^r\right) > \rho_g^U$, and for any $Z^r \geq Z_g^{r+}$ we have $\rho_g^{r+}\left(Z^r\right) \leq \rho_g^U$.

We have also shown for $q_g^U < q_g^C$ that for any KKT regime $r \in \mathcal{R}_n^-$, there exists a critical value Z_g^{r-} such that for any $Z^r < Z_g^{r-}$ we have $\rho_g^{r-}\left(Z^r\right) < \rho_g^U$, and for any $Z^r \geq Z_g^{r-}$ we have $\rho_g^{r-}\left(Z^r\right) \geq \rho_g^U$.

In the following lemmas we will use the above results to give the necessary and sufficient conditions ensuring that there is no incentive for generator g to deviate from the unconstrained candidate equilibrium in terms of the line capacities and fringe bounds.

Lemma 11.14. *Suppose the conditions* (11.3) *are satisfied and* $q_g^U > q_g^C$ *at the unconstrained equilibrium. The necessary and sufficient conditions for which generator* g *at node* n *has no incentive to deviate from the unconstrained candidate equilibrium form a convex polyhedral set of the form*

$$\sum_{ij \in \mathcal{B}_r^+} K_{ij} + \sum_{i \in \mathcal{D}_r^+} x_i^+ \geq Z_g^{r+}, \quad \forall r \in \mathcal{R}_n^+.$$

Proof. From theorem 11.6, we know that if $q_g^U > q_g^C$ then the necessary and sufficient conditions such that generator g has no incentive to deviate from the unconstrained equilibrium are given by:

$$\rho_g^U \geq \rho_g^{r+}(Z^r), \quad \forall r \in \mathcal{R}_n^+.$$

Now, from lemma 11.12, the above conditions can be written as a set of linear constraints in terms of Z^r as follows

$$Z^r \geq Z_g^{r+}, \quad \forall r \in \mathcal{R}_n^+,$$

which from equation (11.11) can be written as:

$$\sum_{ij \in \mathcal{B}_r^+} K_{ij} + \sum_{i \in \mathcal{D}_r^+} x_i^+ \geq Z_g^{r+}, \quad \forall r \in \mathcal{R}_n^+. \tag{11.21}$$

This is a set of linear inequalities, forming a convex polyhedral set. □

Corollary 11.15. *Suppose conditions* (11.3) *are satisfied and* $q_g^U < q_g^C$ *at the unconstrained equilibrium. The necessary and sufficient conditions for which generator* n *has no incentive to deviate from the unconstrained candidate equilibrium form a convex polyhedral set of the form*

$$-\sum_{ij \in \mathcal{B}_r^-} K_{ij} + \sum_{i \in \mathcal{D}_r^-} x_i^- \leq Z_g^{r-}, \quad \forall r \in \mathcal{R}_n^-.$$

Finally, by considering the above conditions for all generators, we prove that the necessary and sufficient conditions guaranteeing the existence of the unconstrained equilibrium form a convex set.

Theorem 11.16. *The set of necessary and sufficient conditions on the line capacities and fringe bounds which ensure the existence of the unconstrained equilibrium form a convex polyhedral set, which we call the* competitive capacity set*.*

Proof. The unconstrained equilibrium exists if and only if

(i) the candidate equilibrium is consistent and

(ii) no generator has incentive to deviate.

From earlier in this section, the necessary and sufficient conditions for (i) are given by:

$$\begin{aligned} -K_{ij} \leq f_{ij}^U \leq K_{ij} \quad & \forall ij \in \mathcal{A}_n, \\ -x_i^- \leq x_i^U \leq x_i^+ \quad & \forall i \in \mathcal{N}. \end{aligned}$$

For (ii), if for generator g in the unconstrained equilibrium $q_g^U > q_g^C$, by lemma 11.14 the necessary and sufficient conditions are given by:

$$\sum_{ij \in \mathcal{B}_r^+} K_{ij} + \sum_{i \in \mathcal{D}_r^+} x_i^+ \geq Z_g^{r+}, \quad \forall r \in \mathcal{R}_n^+.$$

On the other hand, if $q_g^U < q_g^C$, by corollary 11.15 the necessary and sufficient conditions are given by:

$$-\sum_{ij \in \mathcal{B}_r^-} K_{ij} + \sum_{i \in \mathcal{D}_r^-} x_i^- \leq Z_g^{r-}, \quad \forall r \in \mathcal{R}_n^-.$$

Finally if $q_g^U = q_g^C$, we know there is never incentive to deviate. So for no generator to have incentive to deviate, we require that line capacities and fringe bounds must lie in the set formed by the intersection of the non-deviation sets for each generator.

As the necessary and sufficient conditions for (i) and (ii) are convex polyhedral sets, the competitive capacity set must also be a convex polyhedral set as it is the intersection of these sets. □

In this section we have shown, in the case where every generator is owned by a separate firm and all the contracted load is located at the same node as the generator, the set of line capacities and fringe bounds that guarantee the existence of an unconstrained equilibrium is a convex polyhedral set. In the next section we will construct this set for the case where the fringe supply functions are linear and each generator has a quadratic cost function.

11.2 Linear fringes and quadratic costs

In this section, we will derive the competitive capacity set, defined in the previous section, for the case where the fringe supply functions are linear and the generation costs are quadratic.

We set the fringe supply functions to be of the form:

$$S_i(p) = a_i p,$$

where $a_i > 0$, for all $i \in \mathcal{N}$.

We retain the capacity V_g on each strategic generator and define generator g's costs to be given by a convex quadratic function of the form:

$$C_g(q) = u_g q + v_g q^2,$$

where $u_g \geq 0$ and $v_g \geq 0$, for all $g \in \mathcal{G}$.

Note that the above functions satisfy the hypothesis of theorem 11.16, hence we are guaranteed that the competitive capacity set is convex. In what follows we derive the specific conditions that describe the competitive capacity set explicitly in this setting.

11.2.1 Competitive capacity set

For the case where the fringes have no bounds and the lines have no capacities, we know from section 9.3.1 that there always exists a unique unconstrained candidate equilibrium when the fringes are linear. We will now derive the conditions guaranteeing that the candidate equilibrium strategies, q^U, defined by the solution to the linear complementarity problem (11.2), still constitute a valid equilibrium for the game in the presence of line capacities and bounds on the fringes.

Our approach is to first derive the conditions such that the unconstrained candidate equilibrium is consistent, and then to find the additional conditions guaranteeing that no generator has incentive to deviate into another KKT regime by changing its strategy.

A necessary condition for the unconstrained candidate equilibrium to be a bona fide equilibrium, is that the candidate equilibrium is consistent. The conditions guaranteeing this are given by the inequalities (11.3). These conditions consist of linear inequalities on the line capacities and fringe bounds.

In what follows we will derive further conditions that are both necessary and sufficient to ensure that no generator has an incentive to deviate from the unconstrained candidate equilibrium. These conditions together with the candidate equilibrium being consistent ensure that the unconstrained equilibrium is valid.

No incentive for a particular generator to deviate

Consider generator g situated at node n. Note that from lemmas 11.1 and 11.4 we know that if $q_g^U \geq q_g^C$ there is no incentive for generator g to deviate to a quantity $q_g \notin \left(q_g^C, q_g^U\right)$. On the other hand, from the corresponding corollaries, we know that if $q_g^U \leq q_g^C$ there is no incentive for generator g to deviate to a quantity $q_g \notin \left(q_g^U, q_g^C\right)$. Therefore we will only consider deviating within these intervals.

As the candidate equilibrium consists of all generators simultaneously maximizing their profit in the unconstrained KKT regime, any change of generator g's injection quantity will not improve its profit unless some line becomes congested or some fringe hits a bound. We therefore investigate all possible constrained states of the network by considering all KKT regimes $r \in \mathcal{R}_n$. The profit function for generator g, for KKT regime r is therefore given by

$$\rho_g^r\left(q_g^r\right) = \left(q_g^r - q_g^C\right)\pi_n^r\left(q_g^r, Z^r\right) - u_g q_g^r - v_g\left(q_g^r\right)^2, \tag{11.22}$$

where $\pi_n^r\left(q_g^r, Z^r\right)$ is defined implicitly by equation (11.16). In the case of linear supply functions, we can derive an explicit form for π_n^r, namely

$$\pi_n^r\left(q_g^r, Z^r\right) = \frac{D^r\left(q^U\right) - q_g^r - Z^r}{\sum\limits_{i\in\mathcal{D}_r} a_i + \sum\limits_{i\in\mathcal{N}_r}\epsilon}.^8 \tag{11.23}$$

To simplify the upcoming expressions, we will use the following substitution for the rest of this section:

$$A^r = \sum_{i\in\mathcal{D}_r} a_i + \sum_{i\in\mathcal{N}_r}\epsilon. \tag{11.24}$$

For each KKT regime we determine the point, q_g^{r*}, that maximizes generator g's profit defined by (11.22). Note that at least one KKT regime will correspond to the state of the network and fringes when $q_g = q_g^{r*}$.

For $q_g^U > q_g^C$, from theorem 11.6 we know that the conditions

$$\rho_g^{r+} \leq \rho_g^U, \quad \forall r \in \mathcal{R}_n^+$$

are necessary and sufficient to guarantee that generator g has no incentive to deviate from the unconstrained candidate equilibrium.

To find these conditions in terms of Z^r, we consider the following profit maximization problem for generator g, for each KKT regime $r \in \mathcal{R}_n^+$:

$$\begin{aligned}\rho_g^{r+} := \max \quad & \left(q_g^r - q_g^C\right)\tfrac{D^r\left(q^U\right) - q_g^r - Z^r}{A^r} - u_g q_g^r - v_g\left(q_g^r\right)^2 \\ \text{s.t.} \quad & q_g^C \leq q_g^r \leq q_g^U.\end{aligned}$$

[8] ϵ is present here because the dispatch problem has been relaxed to ensure feasibility.

The above maximization problem has a strictly concave objective, so the following KKT conditions are necessary and sufficient to determine the unique solution:

$$\begin{aligned} \frac{1}{A^r}\left(2q_g^r - q_g^C - D^r\left(q^U\right) + Z^r\right) + u_g + 2v_g q_g^r - \mu_1 + \mu_2 &= 0 \\ 0 \le q_g^U - q_g^r \quad \perp \quad \mu_1 &\ge 0 \\ 0 \le q_g^r - q_g^C \quad \perp \quad \mu_2 &\ge 0. \end{aligned} \tag{11.25}$$

Since we are interested in a solution that yields a profit strictly higher than the unconstrained equilibrium, by lemmas 11.1 and 11.4, we only need to consider $q_g^{r*} \in \left(q_g^C, q_g^U\right)$. This simplifies system 11.25 to

$$\frac{1}{A^r}\left(2q_g^{r*} - q_g^C - D^r\left(q^U\right) + Z^r\right) + u_g + 2v_g q_g^{r*} = 0,$$

which gives the optimal injection quantity as

$$q_g^{r*} = \frac{D^r\left(q^U\right) + q_g^C - Z^r - u_g A^r}{2\left(1 + v_g A^r\right)}. \tag{11.26}$$

Substituting q_g^{r*} into equations (11.22) and (11.23) provides the optimal profit associated with KKT regime r:

$$\rho_g^{r*} = \left(\frac{1}{A^r} + v_g\right)\left(q_g^{r*} - q_g^C\right)^2 - u_g q_g^C - v_g\left(q_g^C\right)^2. \tag{11.27}$$

From above, the inequality $\rho_g^{r+} \le \rho_g^U$ is therefore implied by the union of the three following inequalities:

$$\rho_g^{r*} \le \rho_g^U, \quad \text{or} \quad q_g^{r*} \le q_g^C, \quad \text{or} \quad q_g^{r*} \ge q_g^U.$$

From equation (11.27), the first inequality above can be written as

$$\left(q_g^{r*} - q_g^C\right)^2 \le \frac{\hat{\rho}_g^U}{\frac{1}{A^r} + v_g},$$

where

$$\hat{\rho}_g^U = \rho_g^U + u_g q_g^C + v_g\left(q_g^C\right)^2.$$

Thus we have

$$q_g^C - \sqrt{\frac{\hat{\rho}_g^U}{\frac{1}{A^r} + v_g}} \le q_g^{r*} \le q_g^C + \sqrt{\frac{\hat{\rho}_g^U}{\frac{1}{A^r} + v_g}}.$$

The union of this condition with $q_g^{r*} \le q_g^C$ and $q_g^{r*} \ge q_g^U$ gives

$$q_g^{r*} \le q_g^C + \sqrt{\frac{\hat{\rho}_g^U}{\frac{1}{A^r} + v_g}} \quad \text{or} \quad q_g^{r*} \ge q_g^U. \tag{11.28}$$

When a generator is not at capacity in the unconstrained equilibrium, the conditions (11.28) can be simplified further.

Lemma 11.17. *Suppose at the unconstrained candidate equilibrium, generator g is not at capacity ($q_g^U < V_g$). If $q_g^{r*} \geq q_g^U$, for some KKT regime $r \in \mathcal{R}_n^+$, then $\rho_g^{r*} \leq \rho_g^U$. This reduces the condition guaranteeing generator g will not deviate from the candidate equilibrium to the following inequality*

$$q_g^{r*} \leq q_g^C + \sqrt{\frac{\hat{\rho}_g^U}{\frac{1}{A^r} + v_g}}. \tag{11.29}$$

Proof. From inequalities (11.3) and equation (11.11), for the candidate equilibrium to be consistent, we require that

$$\begin{aligned} Z^r &= \sum_{ij \in \mathcal{B}_r^+} K_{ij} + \sum_{i \in \mathcal{D}_r^+} x_i^+ \\ &\geq \sum_{i \in \mathcal{B}_r^+} \left| f_{ij}^U \right| + \sum_{i \in \mathcal{D}_r^+} x_i^U. \end{aligned} \tag{11.30}$$

This then implies from theorem 5.14 that $\pi_n^U(q_g) \geq \pi_n^r(q_g)$ for any $q_g \geq q_g^U$. If $q_g^{r*} \geq q_g^U$ then it is clear that $\rho_n^U \geq \rho_n^{r*}$, since q_n^U is globally optimal for the unconstrained regime. Hence, if $q_g^U < V_g$ then the condition that $q_g^{r*} \geq q_n^U$ implies that $\rho_g^{r*} \leq \rho_g^U$. This reduces the condition guaranteeing generator g will not deviate from the candidate equilibrium within KKT regime r to inequality (11.29). ☐

Next, let us explore the case when $q_g^U = V_g$.

Lemma 11.18. *Suppose, at the unconstrained candidate equilibrium, that generator g is at capacity ($q_g^U = V_g$). For some KKT regime $r \in \mathcal{R}_n^+$, if*

$$\frac{D^r\left(q^U\right) + q_g^C - \sum_{ij \in \mathcal{B}_r^+} \left| f_{ij}^U \right| - \sum_{i \in \mathcal{D}_r^+} x_i^U - u_g A^r}{2\left(1 + v_g A^r\right)} < q_g^U,$$

then $\rho_g^{r+} \leq \rho_g^U$ reduces to inequality (11.29). *Otherwise any Z^r satisfying inequality* (11.30) *yields $\rho_g^{r+} \leq \rho_g^U$.*

Proof. For the unconstrained candidate equilibrium to be consistent, we require that Z^r satisfies inequality (11.30). This, with the expression for q_g^{r*} in equation (11.26), gives the following upper bound on q_g^{r*}:

$$q_g^{r*} \leq \frac{D^r\left(q^U\right) + q_g^C - \sum_{ij \in \mathcal{B}_r^+} \left| f_{ij}^U \right| - \sum_{i \in \mathcal{D}_r^+} x_i^U - u_g A^r}{2\left(1 + v_g A^r\right)}. \tag{11.31}$$

Now consider the two following possibilities:

1. If

$$\frac{D^r\left(q^U\right) + q_g^C - \sum_{ij \in \mathcal{B}_r^+} \left| f_{ij}^U \right| - \sum_{i \in \mathcal{D}_r^+} x_i^U - u_g A^r}{2\left(1 + v_g A^r\right)} < q_g^U,$$

then by (11.31) we are guaranteed that $q_n^{r*} < q_n^U$; this reduces the condition under which generator g will not deviate within KKT regime to (11.29).

2. Alternatively, suppose

$$\frac{D^r\left(q^U\right) + q_g^C - \sum_{ij \in \mathcal{B}_r^+} \left|f_{ij}^U\right| - \sum_{i \in \mathcal{D}_r^+} x_i^U - u_g A^r}{2\left(1 + v_g A^r\right)} \geq q_g^U.$$

Since q_g^{r*} is a continuous decreasing linear function of Z^r, there must exist some Z^r, satisfying the inequality (11.30), which gives

$$q_g^{r*} = q_g^U.$$

From theorem 5.14, we know that the optimal KKT regime yields the highest price. Moreover, we know that at the unconstrained candidate equilibrium, q^U, no constraints are binding. Thus $\pi_n^U \geq \pi_n^{r*}$, which clearly yields $\rho_g^U \geq \rho_g^{r*}$.

Now if Z^r were reduced, q_g^{r*} would increase, hence the condition $q_g^{r*} > q_g^U$ would be satisfied (meaning that there would be no incentive to deviate). On the other hand if Z^r were increased, by lemma 11.10, the deviation profit must decrease. Hence the deviation profit would be strictly less than that of the unconstrained candidate equilibrium, ρ_g^U.

For case 1, above, there is incentive to deviate from the candidate equilibrium if and only if the inequality given by (11.29) is satisfied. Whereas for case 2, there is never incentive to deviate. □

We will proceed to show that the above conditions form a convex set in the line capacities and fringe bounds. With equation (11.26), the inequality (11.29) becomes

$$\frac{D^r\left(q^U\right) + q_g^C - Z^r - u_g A^r}{2\left(1 + v_g A^r\right)} \leq q_g^C + \sqrt{\frac{\hat{\rho}_g^U}{\frac{1}{A^r} + v_g}},$$

which yields the following inequality on Z^r

$$Z^r \geq D^r\left(q^U\right) - u_g A^r - \left(1 + 2v_g A^r\right) q_g^C - 2\left(1 + v_g A^r\right)\sqrt{\frac{\hat{\rho}_g^U}{\frac{1}{A^r} + v_g}}.$$

Recall from equation (11.24) that

$$A^r = \sum_{i \in \mathcal{D}_r} a_i + \sum_{i \in \mathcal{N}_r} \epsilon.$$

To reflect the high penalty for an infeasible solution, ϵ is a very small positive number. Hence we compute

$$\lim_{\varepsilon \to 0} A^r = \sum_{i \in \mathcal{D}_r} a_i.$$

Using the limit above we now substitute A^r out of the inequality on Z^r, yielding

$$Z^r \geq D^r\left(q^U\right) - u_g \sum_{i\in\mathcal{D}_r} a_i - q_g^C\left(1 + 2v_g \sum_{i\in\mathcal{D}_r} a_i\right) - 2\sqrt{\hat{\rho}_g^U\left(1 + v_g \sum_{i\in\mathcal{D}_r} a_i\right)\sum_{i\in\mathcal{D}_r} a_i}.$$

Finally, from equation (11.11), we can write the above inequality in terms of the line capacities, K_{ij}, and the fringe upper bounds, x_i^+, as follows

$$\begin{aligned}\sum_{ij\in\mathcal{B}_r^+} K_{ij} + \sum_{i\in\mathcal{D}_r^+} x_i^+ \quad \geq \quad & D^r\left(q^U\right) - u_g \sum_{i\in\mathcal{D}_r} a_i - q_g^C\left(1 + 2v_g \sum_{i\in\mathcal{D}_r} a_i\right) \\ & -2\sqrt{\hat{\rho}_g^U\left(1 + v_g \sum_{i\in\mathcal{D}_r} a_i\right)\sum_{i\in\mathcal{D}_r} a_i}. \end{aligned} \tag{11.32}$$

Alternatively for $q_g^U < q_g^C$, we consider the following profit maximization problem for generator g, for each KKT regime $r \in \mathcal{R}_n^-$:

$$\begin{aligned}\rho_g^{r-} := \max \quad & \left(q_g^r - q_g^C\right)\left(\frac{D^r\left(q^U\right) - q_n^r - Z^r}{A^r}\right) - u_g q_g^r - v_g\left(q_g^r\right)^2 \\ \text{s.t.} \quad & q_g^U \leq q_g^r \leq q_g^C. \end{aligned}$$

Using the same method as above, here we find that the necessary and sufficient conditions guaranteeing that generator g with $q_g^U > 0$ will not deviate within a KKT regime $r \in \mathcal{R}_n^-$ is

$$\begin{aligned}\sum_{i\in\mathcal{B}_r^-} K_{ij} - \sum_{i\in\mathcal{D}_r^-} x_i^- \quad \geq \quad & -D^r\left(q^U\right) + u_g \sum_{i\in\mathcal{D}_r} a_i + q_g^C\left(1 + 2v_g \sum_{i\in\mathcal{D}_r} a_i\right) \\ & +2\sqrt{\hat{\rho}_g^U\left(1 + v_g \sum_{i\in\mathcal{D}_r} a_i\right)\sum_{i\in\mathcal{D}_r} a_i}. \end{aligned} \tag{11.33}$$

However, if $q_g^U = 0$ then inequality (11.33) is no longer a necessary condition. If

$$\frac{D^r\left(q^U\right) + q_g^C + \sum_{ij\in\mathcal{B}_r^-}\left|f_{ij}^U\right| - \sum_{i\in\mathcal{D}_r^-} x_i^U - u_g \sum_{i\in\mathcal{D}_r} a_i}{2\left(1 + v_g \sum_{i\in\mathcal{D}_r} a_i\right)} > 0,$$

then so long as the candidate equilibrium is consistent, generator g will not have incentive to deviate. Otherwise, the necessary and sufficient condition for generator g to have no incentive to deviate within KKT regime $r \in \mathcal{R}_n^-$ is given by inequality (11.33) above.

Conditions for existence of unconstrained equilibrium

In the previous section we have derived conditions under which a particular generator has no incentive to deviate from the unconstrained candidate equilibrium. We will now use these to form the set of conditions on the line capacities and fringes bounds such that the unconstrained candidate equilibrium is in fact a bona fide equilibrium. We know that the unconstrained equilibrium is valid if and only if

(i) the candidate equilibrium is consistent, and

(ii) no generator has incentive to deviate.

The conditions for (i) are given by the inequalities in (11.3). These inequalities ensure that the unconstrained candidate equilibrium is consistent with the assumption that no line is constrained and no fringe is at a bound. We repeat these conditions below for convenience:

$$\begin{aligned} -K_{ij} \leq f_{ij}^U \leq K_{ij} \quad & \forall ij \in \mathcal{A}_n, \\ -x_i^- \leq x_i^U \leq x_i^+ \quad & \forall i \in \mathcal{N}. \end{aligned}$$

Now we will give the conditions equivalent to (ii). For the set of generators such that $q_g^U > q_g^C$, we have

$$\begin{aligned} \sum_{ij \in \mathcal{B}_r^+} K_{ij} + \sum_{i \in \mathcal{D}_r^+} x_i^+ \quad \geq \quad & D^r\left(q^U\right) - u_g \sum_{i \in \mathcal{D}_r} a_i - q_g^C \left(1 + 2v_g \sum_{i \in \mathcal{D}_r} a_i\right) \\ & -2\sqrt{\hat{\rho}_g^U \left(1 + v_g \sum_{i \in \mathcal{D}_r} a_i\right) \sum_{i \in \mathcal{D}_r} a_i}, \\ & \forall r \in \mathcal{R}_n^+, \quad \forall \left\{g \in \mathcal{G}_n \mid q_g^C < q_g^U\right\}, \quad \forall n \in \mathcal{N}. \end{aligned}$$

However, from lemma 11.18, if $q_g^U = V_g$, then the above condition is not required if

$$\frac{D^r\left(q^U\right) + q_g^C - \sum_{ij \in \mathcal{B}_r^+} \left|f_{ij}^U\right| - \sum_{i \in \mathcal{D}_r^+} x_i^U - u_g A^r}{2\left(1 + v_g A^r\right)} < V_n.$$

For the set of generators such that $q_g^U < q_g^C$, we have

$$\begin{aligned} \sum_{i \in \mathcal{B}_r^-} K_{ij} - \sum_{i \in \mathcal{D}_r^-} x_i^- \quad \geq \quad & -D^r\left(q^U\right) + u_g \sum_{i \in \mathcal{D}_r} a_i + q_g^C \left(1 + 2v_g \sum_{i \in \mathcal{D}_r} a_i\right) \\ & +2\sqrt{\hat{\rho}_g^U \left(1 + v_g \sum_{i \in \mathcal{D}_r} a_i\right) \sum_{i \in \mathcal{D}_r} a_i}, \\ & \forall r \in \mathcal{R}_n^+, \quad \forall \left\{g \in \mathcal{G}_n \mid q_g^U < q_g^C\right\}, \quad \forall n \in \mathcal{N}. \end{aligned}$$

However, if $q_g^U = 0$ then the above condition is not required if

$$D^r\left(q^U\right) + q_g^C + \sum_{ij \in \mathcal{B}_\delta^-} \left|f_{ij}^U\right| - \sum_{i \in \mathcal{D}_r^-} x_i^U - u_g \sum_{i \in \mathcal{D}_r} a_i > 0.$$

Imposing the above conditions simultaneously yields a convex polyhedral set in the line capacities and fringe bounds. This set ensures that the unconstrained equilibrium is consistent and that no generator has incentive to deviate, which means that for any vector of line capacities and fringe bounds that satisfy the above inequalities the candidate unconstrained equilibrium will be a bona fide equilibrium.

Simplifications

Here we will consider a situation where we simplify some of the details of the model. If the generators had no capacities, no contracting and constant marginal costs, the above expression simplify considerably. If we set $q_n^C = 0$, then we know that at the unconstrained equilibrium, we cannot have $q_n^U < q_n^C$ therefore the inequality (11.33) is redundant. For generator g at node n, inequality (11.32) simplifies to:

$$\sum_{i \in \mathcal{B}_r^+} K_{ij} + \sum_{i \in \mathcal{D}_r^+} x_i^+ \geq D^r\left(q^U\right) - u_g \sum_{i \in \mathcal{D}_r} a_i - 2\sqrt{\rho_g^U \sum_{i \in \mathcal{D}_r} a_i}.$$

Examples

To illustrate the competitive capacity set, we will now consider two examples.

Example 11.19. *This example relates to the three-node network shown in figure 11.1. Each node has a fringe supply function of $S_i(p) = p$, and each generator has a cost function of $C_i(q) = 0$. The demands at nodes 1, 2 and 3 are 180, 320 and 100 respectively. For the purposes of this example, there will be no contracting, and no capacities on the generators. To keep this example simple, we will assume that the fringes are infinite, and therefore it is not possible for them to hit a bound when a generator deviates.*

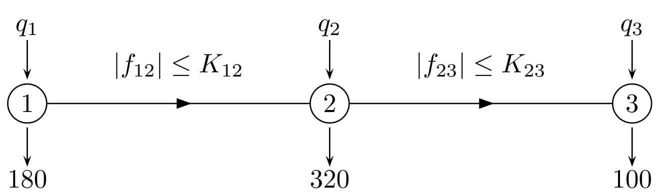

Figure 11.1: Three-node radial network.

In order to find the conditions guaranteeing the existence of an unconstrained equilibrium, we must first compute the unconstrained equilibrium for this example. This can be found from the solution to the LCP given in section 11.1.1, yielding:

$$\begin{array}{lll} q_1^U = q_2^U = q_3^U = 150, & \pi_1^U = \pi_2^U = \pi_3^U = 50, & f_{12}^U = 20, \\ x_1^U = x_2^U = x_3^U = 50, & \rho_1^U = \rho_2^U = \rho_3^U = 7500, & f_{23}^U = -100. \end{array}$$

For this to be a true equilibrium, we must first ensure that the candidate equilibrium is consistent. For this the conditions are

$$\begin{aligned} K_{12} &\geq 20, \\ K_{23} &\geq 100. \end{aligned}$$

Now we will consider possible deviations; generator 1 has two possible KKT regimes to deviate within:

- $\mathcal{N}_1 = \{1\}$, $\mathcal{B}_1^+ = \{12\}$, $\mathcal{D}_1 = \{1\}$:

$$\begin{aligned} K_{12} &\geq 180 - 2\sqrt{7500} \\ &\approx 6.795. \end{aligned}$$

- $\mathcal{N}_2 = \{1,2\}$, $\mathcal{B}_2^+ = \{23\}$, $\mathcal{D}_2 = \{1,2\}$:

$$\begin{aligned} K_{23} &\geq 180 + 320 - 150 - 2\sqrt{7500 \times 2} \\ &\approx 105.051. \end{aligned}$$

Generator 2 has three possible decompositions to deviate within:

- $\mathcal{N}_2 = \{1,2\}$, $\mathcal{B}_2^+ = \{23\}$, $\mathcal{D}_2 = \{1,2\}$:

$$\begin{aligned} K_{23} &\geq 180 + 320 - 150 - 2\sqrt{7500 \times 2} \\ &\approx 105.051. \end{aligned}$$

- $\mathcal{N}_3 = \{2,3\}$, $\mathcal{B}_3^+ = \{12\}$, $\mathcal{D}_3 = \{2,3\}$:

$$\begin{aligned} K_{12} &\geq 320 + 100 - 150 - 2\sqrt{7500 \times 2} \\ &\approx 25.051. \end{aligned}$$

- $\mathcal{N}_4 = \{2\}$, $\mathcal{B}_4^+ = \{12, 23\}$, $\mathcal{D}_4 = \{2\}$:

$$\begin{aligned} K_{12} + K_{23} &\geq 320 - 2\sqrt{7500} \\ &\approx 146.795. \end{aligned}$$

Lastly, generator 3 has two possible decompositions to deviate within:

- $\mathcal{N}_3 = \{2,3\}$, $\mathcal{B}_3^+ = \{12\}$, $\mathcal{D}_3 = \{2,3\}$:

$$\begin{aligned} K_{12} &\geq 320 + 100 - 150 - 2\sqrt{7500 \times 2} \\ &\approx 25.051. \end{aligned}$$

- $\mathcal{N}_5 = \{3\}$, $\mathcal{B}_5^+ = \{23\}$, $\mathcal{D}_5 = \{3\}$:

$$\begin{aligned} K_{23} &\geq 100 - 2\sqrt{7500} \\ &\approx -73.205. \end{aligned}$$

Together these conditions yield the following set of capacities,

$$\begin{aligned} K_{12} &\geq 25.05, \\ K_{23} &\geq 105.05, \\ K_{12} + K_{23} &\geq 146.80, \end{aligned}$$

which ensure the validity of the unconstrained candidate equilibrium; this competitive capacity set is given by the unshaded region in figure 11.2.

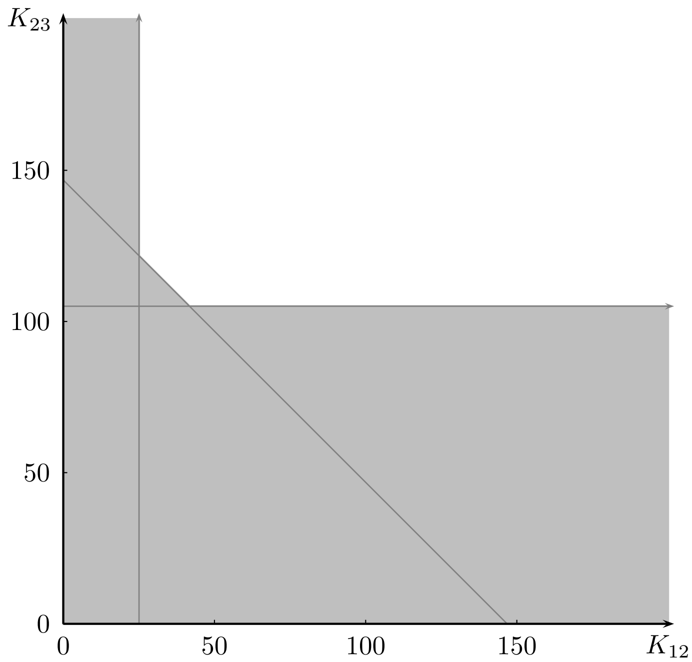

Figure 11.2: Example 11.19 – competitive capacity set (unshaded).

Example 11.20. *Now we will consider a two-node example with constant marginal cost and contracting; this network is shown in figure 11.3. The properties of the demand and generators are given below.*

Generator 1 *has a cost function is given by* $C_1(q) = 10q$ *and a capacity,* $V_1 = 500$. *Generator* 1 *has a contract for differences for* $q_1^C = 200$ *at node* 1. *Generator* 2 *has a cost function is given by* $C_2(q) = 30q$ *and a capacity,* $V_2 = 160$. *The demand at node* 1 *is* $d_1 = 400$, *and the demand at node* 2 *is* $d_2 = 500$. *The fringe supply functions at the nodes are* $S_1(p) = S_2(p) = p$.

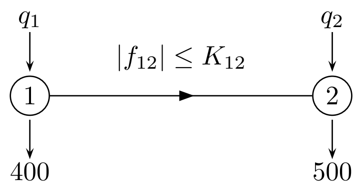

Figure 11.3: Two-node linear network.

We must first compute the unconstrained equilibrium for this example. This can be found to be:

$$\begin{aligned}
&q_1^U = 460, \quad \pi_1^U = \pi_2^U = 140, \quad x_1^U = x_2^U = 140, \quad f_{12}^U = 100, \\
&q_2^U = 160, \qquad \rho_1^U = 31800, \qquad\quad \rho_2^U = 17600.
\end{aligned}$$

We will now proceed to construct the competitive capacity set. We first must ensure that the candidate equilibrium is consistent; this gives the following set of constraints:

$$\begin{aligned}
K_{12} &\geq 100, \\
x_1^+ &\geq 140, \\
x_2^+ &\geq 140, \\
x_1^- &\leq 140, \\
x_2^- &\leq 140.
\end{aligned}$$

Now we will find the conditions preventing generator 1 from deviating; to do this we must consider the conditions associated with generator 1 deviating within all KKT regimes in $\mathcal{R}_1^+$:

- $\mathcal{N}_1 = \{1\}$, $\mathcal{B}_1^+ = \{12\}$, $\mathcal{D}_1^+ = \emptyset$:

$$\begin{aligned}
K_{12} &\geq 500 - 10 - 200 - 2\sqrt{31800 + 10 \times 200} \\
&\approx -77.696.
\end{aligned}$$

- $\mathcal{N}_2 = \{1\}$, $\mathcal{B}_2^+ = \{12\}$, $\mathcal{D}_2^+ = \{1\}$:

$$\begin{aligned} K_{12} + x_1^H &\geq 500 - 200 \\ &= 300. \end{aligned}$$

- $\mathcal{N}_3 = \{1, 2\}$, $\mathcal{B}_3^+ = \emptyset$, $\mathcal{D}_3^+ = \{1\}$:

$$\begin{aligned} x_1^H &\geq 900 - 160 - 10 - 200 - 2\sqrt{31800 + 10 \times 200} \\ &\approx 162.304. \end{aligned}$$

- $\mathcal{N}_4 = \{1, 2\}$, $\mathcal{B}_4^+ = \emptyset$, $\mathcal{D}_4^+ = \{2\}$:

$$\begin{aligned} x_2^H &\geq 900 - 160 - 10 - 200 - 2\sqrt{31800 + 10 \times 200} \\ &\approx 162.304. \end{aligned}$$

- $\mathcal{N}_5 = \{1, 2\}$, $\mathcal{B}_5^+ = \emptyset$, $\mathcal{D}_5^+ = \{1, 2\}$:

$$\begin{aligned} x_1^H + x_2^H &\geq 900 - 160 - 200 \\ &= 540. \end{aligned}$$

and for generator 2 these are

- $\mathcal{N}_6 = \{2\}$, $\mathcal{B}_6^+ = \{12\}$, $\mathcal{D}_6^+ = \emptyset$:

$$\begin{aligned} K_{12} &\geq 400 - 30 - 2\sqrt{17600} \\ &\approx 104.670. \end{aligned}$$

- $\mathcal{N}_7 = \{2\}$, $\mathcal{B}_7^+ = \{12\}$, $\mathcal{D}_7^+ = \{2\}$:

$$K_{12} + x_2^+ \geq 400.$$

- $\mathcal{N}_8 = \{1, 2\}$, $\mathcal{B}_8^+ = \emptyset$, $\mathcal{D}_8^+ = \{1\}$:

$$\begin{aligned} x_1^+ &\geq 900 - 460 - 30 - 2\sqrt{17600} \\ &\approx 144.670. \end{aligned}$$

- $\mathcal{N}_9 = \{1, 2\}$, $\mathcal{B}_9^+ = \emptyset$, $\mathcal{D}_9^+ = \{2\}$:

$$\begin{aligned} x_2^+ &\geq 900 - 460 - 30 - 2\sqrt{17600} \\ &\approx 144.670. \end{aligned}$$

- $\mathcal{N}_{10} = \{1, 2\}$, $\mathcal{B}_{10}^{+} = \emptyset$, $\mathcal{D}_{10}^{+} = \{1, 2\}$:

$$\begin{aligned} x_1^{+} + x_2^{+} &\geq 900 - 460 \\ &= 440. \end{aligned}$$

To plot the region representing the competitive capacity set, it is convenient to set $x_2^{+} = \infty$ and examine the two-dimensional region formed from x_1^{+} and K_{12}. The competitive capacity set is shown by the unshaded region in figure 11.4.

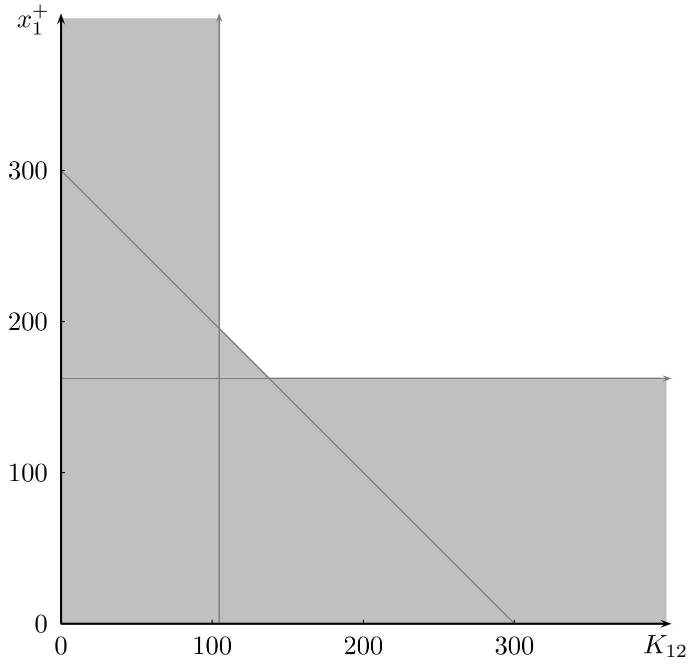

Figure 11.4: Example 11.20 – competitive capacity set (unshaded).

Note that some of these conditions prevent a generator from becoming net pivotal. For example, the constraint: $K_{12} + x_1^{H} \geq 300$ is preventing generator 1 from being in a position whereby it could withhold to some quantity above its contracted load and receive an infinite price. This is equivalent to being able to unilaterally set the price due to a lack of competition.

Chapter 12

Unconstrained equilibrium: extensions

In this chapter we consider the effects of relaxing some of the assumptions of the previous chapter. First, we will discuss how the competitive capacity set may change if generators can have CFDs at multiple nodes, and present an example. We will then examine how the nature of the competitive capacity set is affected once the assumption that each generator is owned by a separate firm is relaxed.

12.1 Contracts for differences at multiple nodes

We will now consider the implications of having contracted load at multiple nodes. Clearly, the location of the contracted load has no effect on the unconstrained candidate equilibrium, since the prices, across all nodes, are the same. However, the conditions ensuring that the unconstrained equilibrium is valid may be affected. In this section, we will first prove that so long as $q_g^U \geq q_g^C$, $\forall g \in \mathcal{G}$, the conditions derived in section 11.2.1 are still necessary for the validity of the unconstrained candidate equilibrium, q^U. Following this we will present an example demonstrating that those conditions are indeed necessary, but are not sufficient.

The following theorem states that when generators generate more than their contracted quantities at equilibrium, the conditions from section 11.2.1 are necessary to ensure that no generator has incentive to deviate.

Theorem 12.1. *Suppose that at an unconstrained candidate equilibrium,* $q_g^U \geq \sum_{i \in \mathcal{N}} q_{gi}^C$, $\forall g \in \mathcal{G}$. *Now consider the conditions given by* (11.32)*; these conditions are necessary for* q_g^U *to be a bona fide equilibrium.*

Proof. If all of the contracted load for each of the generators were located at its own node then, from theorem 11.16, we know that the conditions given by (11.32) are both necessary and sufficient for the unconstrained candidate equilibrium to be valid. This means that not only is the candidate equilibrium consistent, but there is also no incentive for any generator to deviate.

Now suppose that generators have contracts for differences at multiple nodes (the total contract level is fixed). First note that the location of the CFDs does not affect the candidate equilibrium, since there is no congestion and the prices at all nodes are the same; hence the conditions ensuring that the candidate equilibrium is consistent are unchanged.

In what follows we will prove that if there is incentive for a generator to deviate when its CFDs are aggregated at that generator's own node, then there will still be incentive to deviate when the CFDs are based at other nodes.

Suppose that, when all contracted load is at node n, there is incentive for generator g at node n to deviate to some KKT regime r by injecting q_g^r. This can be written as:

$$\rho_g^r(q_g^r) = q_g^r \pi_n^r - C_g(q_g^r) - \sum_{i \in \mathcal{N}} q_{gi}^C \pi_n^r > \rho_g^U.$$

We know from lemmas 11.1 and 11.4 that $q_g^r \in \left(q_g^C, q_g^U\right)$; moreover, from lemma 7.1 and the proof of lemma 5.8, we know that as a generator at node n withholds, the price at all nodes are non-decreasing and

$$\pi_n \geq \pi_i, \ \forall i \in \mathcal{N}.$$

Therefore with CFDs at multiple nodes, the profit made by generator g, when injecting q_g^r, is at least $\rho_g^r(q_g^r)$ and potentially exceeds it. Hence there is incentive to deviate from the equilibrium. □

We will now demonstrate the result of the above theorem by way of an example over a two-node linear network.

Example 12.2. *Consider a two-node linear network with a strategic generator and competitive fringe at each node. We set* $q_{11}^C = q_1^C \theta_1$ *and* $q_{12}^C = q_1^C(1-\theta_1)$, $q_{21}^C = q_2^C \theta_2$ *and* $q_{22}^C = q_2^C(1-\theta_2)$, *where* θ_1 *and* θ_2 *are between* 0 *and* 1. *Each strategic generator can have contracted load at both nodes, satisfying the condition that,*

$$q_1^C + q_2^C \leq d_1 + d_2. \tag{12.1}$$

Unconstrained equilibrium

First we can solve for the unconstrained candidate equilibrium (corresponding to the KKT regime with no congestion). Here the location of the contracted load is not important because the prices are the same at both nodes. Assuming no costs, the profit functions for the generators are

$$\begin{aligned}\rho_1^U &= \left(q_1 - q_1^C\right)\left(\frac{d_1 + d_2 - q_1 - q_2}{a_1 + a_2}\right),\\ \rho_2^U &= \left(q_2 - q_2^C\right)\left(\frac{d_1 + d_2 - q_1 - q_2}{a_1 + a_2}\right).\end{aligned}$$

From these profit functions we can explicitly calculate the candidate equilibrium generation quantities to be:

$$\begin{aligned}q_1^U &= \frac{1}{3}\left(d_1 + d_2 + 2q_1^C - q_2^C\right),\\ q_2^U &= \frac{1}{3}\left(d_1 + d_2 - q_1^C + 2q_2^C\right).\end{aligned}$$

From equation (12.1) we can see that

$$\begin{aligned}q_1^U &\geq q_1^C,\\ q_2^U &\geq q_2^C.\end{aligned}$$

This equilibrium has a price of

$$\pi^U = \frac{d_1 + d_2 - q_1^C - q_2^C}{3\left(a_1 + a_2\right)},$$

which yields profits of

$$\begin{aligned}\rho_1^U &= \frac{\left(d_1 + d_2 - q_1^C - q_2^C\right)^2}{9\left(a_1 + a_2\right)},\\ \rho_2^U &= \frac{\left(d_1 + d_2 - q_1^C - q_2^C\right)^2}{9\left(a_1 + a_2\right)}.\end{aligned}$$

If we were to assume that all contracted load for each generator were located at that generator's node ($\theta_1 = \theta_2 = 1$) then the conditions given in section 11.2.1 are necessary and sufficient for the candidate equilibrium to be a bona fide equilibrium. For this problem these conditions simplify to:

$$\begin{aligned}K_{12} &\geq d_1 - q_1^C - 2\sqrt{\rho_1^U a_1},\\ K_{12} &\geq d_2 - q_2^C - 2\sqrt{\rho_2^U a_2}.\end{aligned}$$

By substituting the above equilibrium profits, these become:

$$\begin{aligned}K_{12} &\geq d_1 - q_1^C - \frac{2}{3}\left(d_1 + d_2 - q_1^C - q_2^C\right)\sqrt{\frac{a_1}{a_1 + a_2}},\\ K_{12} &\geq d_2 - q_2^C - \frac{2}{3}\left(d_1 + d_2 - q_1^C - q_2^C\right)\sqrt{\frac{a_2}{a_1 + a_2}}.\end{aligned}$$

An interesting question is how does the total contracted load affect the minimum line capacity supporting the candidate equilibrium. It is clear from the above inequalities that the more highly contracted a generator is the smaller the minimum line capacity to prevent that generator deviating becomes (that is they have less incentive to withhold).[1] However, the capacity required to ensure that the other generator has no incentive to deviate increases. We now examine a specific example with $d_1 = 2$, $d_2 = 1$, $a_1 = 2$, $a_2 = 1$ and $q_2^C = 0$; here the constraints become:

$$K_{12} \geq 2 - q_1^C - \frac{2}{3}\left(3 - q_1^C\right)\sqrt{\frac{2}{3}},$$
$$K_{12} \geq 1 - \frac{2}{3}\left(3 - q_1^C\right)\sqrt{\frac{1}{3}}.$$

Of course, to ensure the equilibrium is consistent we also require: $K_{12} \geq \left|f_{12}^U\right|$. These conditions are plotted at functions of q_1^C in figure 12.1 below.

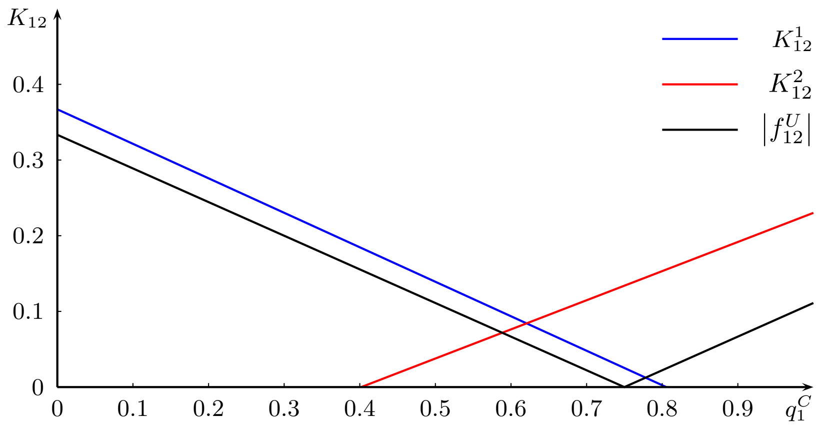

Figure 12.1: Effect of contracting level on line capacity.

Note that if θ_1 or θ_2 were less than 1 then these conditions would be necessary, but not sufficient. This means that the capacity on the line must at least satisfy these conditions for the unconstrained equilibrium to be supported.

We can, however, find the necessary conditions for this example, by considering the price at nodes outside $\mathcal{N}_r$; this is straightforward for a two-node example. The profit function for generator 1, deviating by congesting the line towards node 1, which we will refer to as regime r_1 is given by

$$\begin{aligned}\rho_1^{r_1} &= q_1\pi_1^{r_1} - q_{11}^C\pi_1^{r_1} - q_{12}^C\pi_2^{r_1} \\ &= \frac{\left(q_1 - q_{11}^C\right)\left(d_1 - K_{12} - q_1\right)}{a_1} - \frac{q_{12}^C\left(d_2 + K_{12} - q_2^U\right)}{a_2}\end{aligned}$$

[1] Allaz and Vila in [3] demonstrate a related concept whereby Cournot agents with forward contracts increase their output at equilibrium.

We can solve for the maximum profit from the first order condition:

$$\begin{aligned} \left.\frac{\partial \rho_1^{r_1}}{\partial q_1^{r_1}}\right|_{q_1^{r_1}=q_1^{r_1*}} &= 0 \\ \Rightarrow \quad \frac{d_1 - K_{12} - 2q_1^{r_1*} + q_{11}^C}{a_1} &= 0 \\ \Rightarrow \quad q_1^{r_1*} &= \frac{1}{2}\left(d_1 - K_{12} + q_{11}^C\right), \end{aligned}$$

giving

$$\rho_1^{r_1*} = \frac{\left(d_1 - K_{12} - q_{11}^C\right)^2}{4a_1} - \frac{q_{12}^C\left(d_2 + K_{12} - q_2^U\right)}{a_2}.$$

The condition on the line capacity to ensure that generator 1 has no incentive to deviate becomes

$$K_{12} \geq \left(d_1 - q_{11}^C\right) + 2\frac{a_1}{a_2}q_{12}^C - 2\sqrt{\frac{{a_1}^2}{{a_2}^2}{q_{12}^C}^2 + \frac{a_1}{a_2}q_{12}^C\left(d_1 + d_2 - q_{11}^C - q_2^U\right) + a_1\rho_1^U},$$

and similarly for generator 2:

$$K_{12} \geq \left(d_2 - q_{22}^C\right) + 2\frac{a_2}{a_1}q_{21}^C - 2\sqrt{\frac{{a_2}^2}{{a_1}^2}{q_{21}^C}^2 + \frac{a_2}{a_1}q_{21}^C\left(d_1 + d_2 - q_{22}^C - q_1^U\right) + a_2\rho_2^U}.$$

These inequalities are sufficient to ensure that no player has incentive to deviate from the candidate equilibrium, given above. From the derivation of these conditions it is clear that if load is contracted at a node at which the generator is not located, then they have more incentive to congest lines towards nodes where they have generation (by withholding). This is because the price they must pay to serve the load at the other node is less than the price they are paid for the electricity that they produce.

We can illustrate this effect by repeating the plot in figure 12.1 for the extreme values of θ_1.

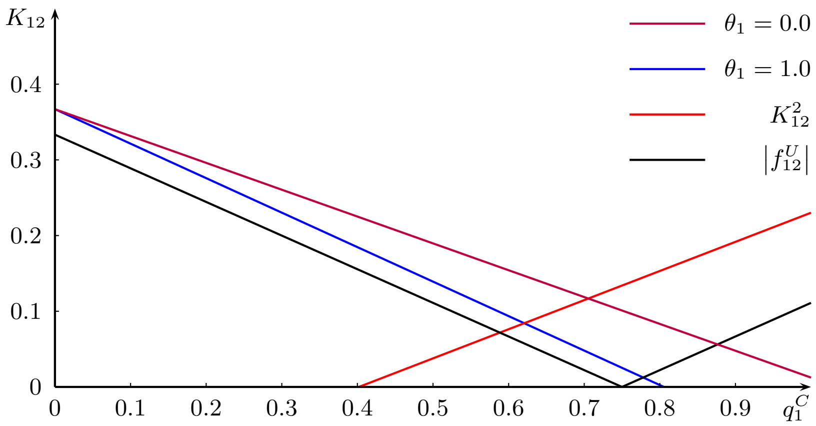

Figure 12.2: Effect of contracting location on line capacity.

Figure 12.2 demonstrates that by assuming that all contracted load is located at the node of the generator we underestimate the required size of the line. Under this assumption, for a contracted load of $q_1^C = 0.5$, a line capacity of $K_{12} = 0.1392$ would be sufficient to ensure that no player would deviate from the single-node equilibrium. However, if in fact $\theta_1 = 0$, all the contracted load would be located at node 2, and the true minimum line capacity required is $K_{12} = 0.1897$. As the flow on the line at the equilibrium is $f_{12} = -0.1111$, the *fat* needing to be built into the line to prevent a generator begin able to exploit his position in the grid varies from 0.028 to 0.079. This increase amounts to approximately 2.6% of the total load, and it is solely attributed to the location of the contracted load of generator 1.

12.2 Ownership of multiple generation units

So far we have assumed that all generators are individual profit maximizers, acting solely in their own interests. We will now examine the effect of allowing firms to own a number of generators located at different nodes around the grid. Here we are particularly interested in how this affects the nature of the set of capacities ensuring that the unconstrained candidate equilibrium is valid.

We will prove a theorem that states that, in certain circumstances, the conditions given in (11.32) are necessary (although not sufficient) for an unconstrained equilibrium to exist, even when firms own multiple generators.

Theorem 12.3. *Suppose we have a consistent candidate equilibrium corresponding to the unconstrained KKT regime, U. If $q_g^U \geq q_{gi}^C$, $\forall g \in \mathcal{G}_i$, $\forall i \in \mathcal{N}$ then the conditions of* (11.32) *are necessary to ensure there is no incentive to deviate from the candidate equilibrium, even if firms own multiple generators.*

Proof. Consider an unconstrained candidate equilibrium satisfying the condition of the theorem:

$$q_g^U \geq q_{gi}^C, \quad \forall g \in \mathcal{G}_i, \forall i \in \mathcal{N}.$$

Note that each of the inequalities in (11.32) corresponds to one generator withholding to increase their payoff. Moreover, from lemmas 5.6 and 7.1 we know that, for a radial network, as a generator withholds, the price at all nodes must be non-decreasing.

Now suppose that the line capacities or fringe bounds were such that some constraint in (11.32) were not satisfied. This would imply there is incentive for one generator to withhold to increase its payoff. Consider a situation where a firm owns that generator and also some other generators; from above it is clear that by leaving the generation level of its other generators fixed it would achieve at least as high a profit from deviating, meaning that the incentive for it deviate is still valid. Hence, when firms own multiple generation units, the incentive conditions given by equation (11.32) are still necessary (although not sufficient) for the unconstrained equilibrium to exist. □

In this section we will compute the necessary and sufficient conditions ensuring that an unconstrained equilibrium exists when firms own multiple plants. However, we will first provide a simple example demonstrating how to compute the optimal profit for a firm owning generators at multiple nodes.

Example 12.4. *Here we examine the incentives of a single strategic firm, who owns a plant at each node in a two-node network, to congest a transmission line. For simplicity we set the marginal costs of both plants to 0. The injections of the firm are q_{11} and q_{12} for the plants at nodes* 1 *and* 2 *respectively; this network is shown in figure 12.3 below.*

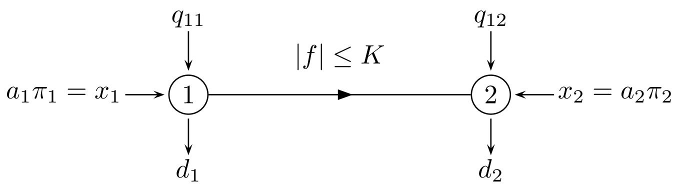

Figure 12.3: Monopolist firm owning plants at both nodes.

The dispatch problem for this situation is

$$\begin{aligned}
\min \quad & \tfrac{1}{2a_1}{x_1}^2 + \tfrac{1}{2a_2}{x_2}^2 \\[2ex]
\text{s.t.} \quad & x_1 - f = d_1 - q_{11} && [\pi_1] \\
& x_2 + f = d_2 - q_{12} && [\pi_2] \\
& |f| \leq K && \left[\eta^1, \eta^2\right].
\end{aligned}$$

This problem is convex, and hence it is equivalent to the following KKT system:

$$\begin{aligned}
x_1 - f &= d_1 - q_{11} \\
x_2 + f &= d_2 - q_{12} \\
x_1 - a_1\pi_1 &= 0 \\
x_2 - a_2\pi_2 &= 0 \\
\pi_1 - \pi_2 + \eta^1 - \eta^2 &= 0 \\
0 \leq K - f &\perp \eta^1 \geq 0 \\
0 \leq K + f &\perp \eta^2 \geq 0.
\end{aligned}$$

Since we only have one firm, we just need to consider a single profit maximization problem

$$\begin{aligned} \max \quad & q_{11}\pi_1 + q_{12}\pi_2 \\ \\ \text{s.t.} \quad & a_1\pi_1 - f = d_1 - q_{11} \\ & a_2\pi_2 + f = d_2 - q_{12} \\ & \pi_1 - \pi_2 + \eta^1 - \eta^2 = 0 \\ & 0 \leq K - f \perp \eta^1 \geq 0 \\ & 0 \leq K + f \perp \eta^2 \geq 0 \end{aligned}$$

This maximization problem has a non-convex feasible region, so we need to solve it parametrically. As with the examples in chapter 10, to do this we will enumerate the different line congestion cases, giving three possible congestion regimes:

(a) line not congested,

(b) line congested from node 1 to node 2,

(c) line congested from node 2 to node 1.

Each of these regimes can be written as its own maximization problem. For regime (a) we have

$$\begin{aligned} \max \quad & q_{11}\pi_1 + q_{12}\pi_2 \\ \\ \text{s.t.} \quad & a_1\pi_1 - f = d_1 - q_{11} \\ & a_2\pi_2 + f = d_2 - q_{12} \\ & \pi_1 - \pi_2 = 0. \end{aligned}$$

For this to be consistent, we require $|f| \leq K$. Solving this problem gives

$$\begin{aligned} q_{11} + q_{12} &= \frac{1}{2}(d_1 + d_2), \\ \rho_1 &= \frac{(d_1 + d_2)^2}{4(a_1 + a_2)}, \\ f &= q_{11} - d_1 + \frac{a_1(d_1 + d_2)}{2(a_1 + a_2)}, \\ K &\geq \left| q_{11} - d_1 + \frac{a_1(d_1 + d_2)}{2(a_1 + a_2)} \right|. \end{aligned}$$

For regime (b) we have

$$\max \quad q_{11}\pi_1 + q_{12}\pi_2$$

$$\begin{aligned} \text{s.t.} \quad & a_1\pi_1 - K = d_1 - q_{11} \\ & a_2\pi_2 + K = d_2 - q_{12} \\ & \pi_1 - \pi_2 + \eta^1 = 0. \end{aligned}$$

For this to be consistent we require $\eta^1 \geq 0$[2]. Solving this problem gives

$$\begin{aligned} q_{11} &= \frac{1}{2}(d_1 + K), \\ q_{12} &= \frac{1}{2}(d_2 - K), \\ \rho_1 &= \frac{(d_1 + K)^2}{4a_1} + \frac{(d_2 - K)^2}{4a_2}, \\ \pi_1 &= \frac{d_1 + K}{2a_1}, \\ \pi_2 &= \frac{d_2 - K}{2a_2}, \\ K &\leq \frac{a_1 d_2 - a_2 d_1}{a_1 + a_2}. \end{aligned} \tag{12.2}$$

For regime (c) we have

$$\max \quad q_{11}\pi_1 + q_{12}\pi_2$$

$$\begin{aligned} \text{s.t.} \quad & a_1\pi_1 + K = d_1 - q_{11} \\ & a_2\pi_2 - K = d_2 - q_{12} \\ & \pi_1 - \pi_2 - \eta^2 = 0. \end{aligned}$$

For this to be consistent, we require $\eta^2 \geq 0$. Solving this problem gives

$$\begin{aligned} q_{11} &= \frac{1}{2}(d_1 - K), \\ q_{12} &= \frac{1}{2}(d_2 + K), \\ \rho_1 &= \frac{(d_1 - K)^2}{4a_1} + \frac{(d_2 + K)^2}{4a_2}, \\ \pi_1 &= \frac{d_1 - K}{2a_1}, \\ \pi_2 &= \frac{d_2 + K}{2a_2}, \\ K &\leq \frac{a_2 d_1 - a_1 d_2}{a_1 + a_2}. \end{aligned}$$

Note that under all regimes the total generation of the monopolist firm is the same:

$$q_{11} + q_{12} = \frac{1}{2}(d_1 + d_2);$$

[2] That is, $\pi_2 \geq \pi_1$, which ensures that the optimally conditions of the dispatch problem are satisfied.

all that changes is the proportion that each plant generates. Now examine the profit of regime (b) as a function of K:

$$\begin{aligned}\rho_1 &= \frac{(d_1+K)^2}{4a_1} + \frac{(d_2-K)^2}{4a_2}\\ &= \frac{{d_1}^2+2d_1K+K^2}{4a_1} + \frac{{d_2}^2-2d_2K+K^2}{4a_2}\\ &= \left(\frac{1}{a_1}+\frac{1}{a_2}\right)K^2 + 2\left(\frac{d_1}{a_1}-\frac{d_2}{a_2}\right)K + \left(\frac{{d_1}^2}{a_1}+\frac{{d_2}^2}{a_2}\right) \qquad (12.3)\end{aligned}$$

We can clearly see that equation (12.3) is a convex quadratic function of K. This function is minimized at

$$K = \frac{a_1 d_2 - a_2 d_1}{a_1 + a_2},$$

which gives a profit of

$$\rho_1 = \frac{(d_1+d_2)^2}{4(a_1+a_2)}.$$

Note that the profit from regime (b) is minimized at the boundary of its feasible region given by inequality (12.2), and at this point the profit is equal to that of regime (a). Hence, it follows from the convexity of the profit function of regime (b) that so long as it is possible to constrain the line by playing a local monopolist strategy at each node, doing so will yield a higher profit than playing a strategy that does not constrain the line.[3] The same is true for regime (c) by symmetry. A plot of the profit functions of regime (a) and (b) are shown in figure 12.4, for the case where $d_1 = 1$, $d_2 = 2$ and $a_1 = a_2 = 1$.

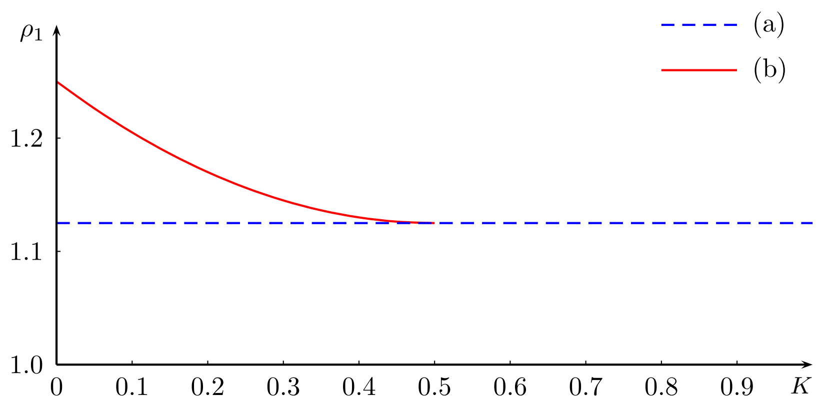

Figure 12.4: Monopoly profit as a function of K.

In this next example we will consider this result in the context of a Cournot game.

Example 12.5. *Once again consider a two node network, this time with two strategic firms; firm*

[3] In the presence of retail loads at either node or the ownership of an FTR, this result may change.

1 owns a plant at both node 1 and node 2, while firm 2 owns a single plant, located at node 2. The fringe slopes at the nodes are both equal to 1. This situation is shown in figure 12.5, below.

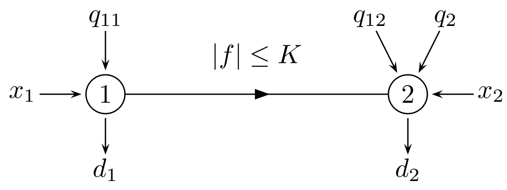

Figure 12.5: Two-node network, firm 1 owning 2 plants.

For the purposes of this example, the three plants have the following marginal costs:

$$c_{11} = 0, \quad c_{12} = \epsilon, \quad c_2 = 0,$$

where ϵ is some small positive value.

This yields the unconstrained candidate equilibrium:

$$q_{11}^U = \frac{1}{3}(d_1 + d_2), \quad q_{12}^U = 0, \quad q_2^U = \frac{1}{3}(d_1 + d_2),$$

with prices:

$$\pi_1^U = \pi_2^U = \frac{1}{6}(d_1 + d_2).$$

The unconstrained equilibrium therefore gives a profit for firm 1 of

$$\rho_1^U = \frac{1}{18}(d_1 + d_2)^2.$$

For this unconstrained candidate equilibrium to be consistent, we require that capacity of the line is large enough to support the equilibrium flow, this gives the inequality

$$K \geq \left| \frac{1}{2}(d_1 - d_2) \right|.$$

Let us initially assume that firm 1 had no plant at node 2. From inequality (11.32), we could calculate that in order for firm 1 to have no incentive to deviate from the equilibrium, we would require that

$$\begin{aligned} K &\geq d_1 - 2\sqrt{\rho_1^U} \\ &= d_1 - \frac{\sqrt{2}}{3}(d_1 + d_2). \end{aligned}$$

However, since firm 1 has a generator at node 2 (and there are no CFDs) we know from theorem 12.3 that the line capacity above is necessary but not sufficient to ensure there is no incentive for firm 1 to deviate. In the previous example, we examined the conditions under which a firm would have incentive to congest a line; the same results apply to this example, except that the demand at node 2 must be adjusted by firm 2's unconstrained equilibrium quantity. This gives the following inequality

$$\begin{aligned} K &\geq \frac{1}{2}\left(d_1 - \left(d_2 - q_2^U\right)\right) \\ &= \frac{1}{3}\left(2d_1 - d_2\right). \end{aligned}$$

If we compare these two conditions, we can see that when the firm has a second plant the line size ensuring that firm 1 has no incentive to deviate from the unconstrained equilibrium increases by

$$\frac{\sqrt{2}-1}{3}\left(d_1 + d_2\right).$$

This is because, when the line is congested (from node 2 to node 1), firm 1's plant at node 2 is able to increase its generation level without affecting the price at node 1, thus increasing the firm's profit.

This example is quite important as it reinforces the result of theorem 12.3. If a firm has multiple generators located at different nodes in the grid they will likely require larger lines to ensure they have no incentive to cause congestion in the network.

In the final example of this section, we examine the multiple ownership paradigm over a three-node network and derive the corresponding deviation set for a particular firm.

Example 12.6. *Suppose that we have a three-node linear network, with three generators; firm 1 has a plant at nodes 1 and 3, whereas firms 2 and 3 each have one generator located at nodes 2 and 3 respectively, as shown in figure 12.6. All the plants have zero marginal costs except firm 1's plant at node 3, which has a positive marginal cost, ϵ. Here we are interested in the set of line capacities such that firm 1 has no incentive to deviate from the unconstrained candidate equilibrium.*

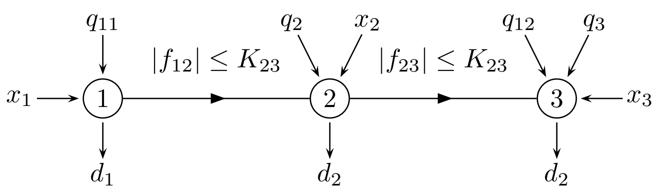

Figure 12.6: Three-node network, firm 1 owning 2 plants.

First, we will calculate the unconstrained equilibrium for this problem. Since the marginal cost of firm 1's plant at node 1 is slightly less than the marginal cost of its plant at node 3, the injection quantities at the candidate equilibrium can be found to be

$$q_{11}^U = q_2^U = q_3^U = \frac{d_1 + d_2 + d_3}{4}, \quad q_{12}^U = 0,$$

and the price at all nodes is given by

$$\pi^U = \frac{d_1 + d_2 + d_3}{4(a_1 + a_2 + a_3)}.$$

This gives profits to all players of

$$\rho_1^U = \rho_2^U = \rho_3^U = \frac{(d_1 + d_2 + d_3)^2}{16(a_1 + a_2 + a_3)}.$$

Now let us consider a specific example with the following parameters: $a_1 = a_2 = a_3 = 1$ and $d_1 = d_2 = d_3 = 5$. The unconstrained equilibrium from this example is $q_{11}^U = q_2^U = q_3^U = 3.75$ and $q_{12}^U = 0$, giving $\pi^U = 1.25$. The profits of each of the generators can be found to be 4.6875. We can also calculate the flow on the lines at the equilibrium: $f_{12}^U = f_{23}^U = 0$. For this example, we are interested in determining the set of $\{K_{12}, K_{23}\}$ such that firm 1 has no incentive to deviate from the single-node equilibrium. To do this, we must consider all the ways that firm 1 can deviate from the equilibrium, these are defined by the KKT regimes given in chapter 4. For each regime we can compute the set of capacities such that there is feasible incentive to deviate to that regime.

(1) $\mathcal{B}_1^+ = \emptyset$, $\mathcal{B}_1^- = \{12\}$,

(2) $\mathcal{B}_2^+ = \emptyset$, $\mathcal{B}_2^- = \{12, 23\}$,

(3) $\mathcal{B}_3^+ = \emptyset$, $\mathcal{B}_3^- = \{23\}$,

(4) $\mathcal{B}_4^+ = \{12\}$, $\mathcal{B}_4^- = \{23\}$.

These regimes are depicted in figure 12.7.

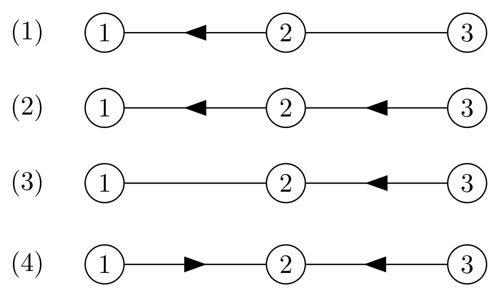

Figure 12.7: KKT regimes.

For each of these possible deviation regimes, we must compute the incentive and compatibility conditions, the complement of the union of these sets gives the non-deviation set for firm 1.

If we set $K_{12} = 2$, we can find the maximum profit firm 1 can make as a function of K_{23}. This profit is shown in figure 12.8. When K_{23} is less than 1.25, there is incentive for firm 1 to withhold at node 1 and inject at node 3 to deviate to KKT regime (3) as shown by the blue curve in figure 12.8. For K_{23} between 1.25 and 1.932, there is no incentive to deviate from the equilibrium, since K_{23} is too large for the firm's node 1 generator to profit by withholding, and yet too small for its node 3 generator to recover the forgone profits by injecting more. For K_{23} between 1.932 and 2.125, firm 1 has incentive to deviate to the boundary of KKT regimes (1) and (2) and finally for any K_{23} greater than 2.125, the optimal deviation strategy is to withhold at node 1 and inject more power at node 3 causing KKT regime (1).

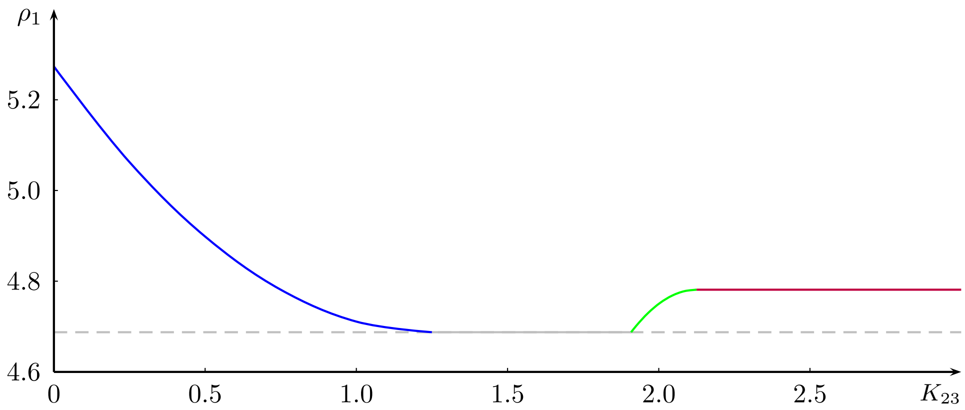

Figure 12.8: Maximum profit from deviating when $K_{12} = 2$ as a function of K_{23}.

In figure 12.9 below, for each KKT regime, we draw the regions of capacities for which firm 1 would have incentive to deviate. For any pair $\{K_{12}, K_{23}\}$ in one of these regions, there is (feasible) incentive for firm 1 to deviate from the unconstrained candidate equilibrium. In figure 12.10 we take the union of these sets to give the deviation set for firm 1. Recall that in the single ownership situation we are guaranteed that the competitive capacity set is convex. However, with multiple ownership, this convexity is no longer assured; in fact, it is possible that increasing the size of a line may create an incentive to congest a line that was previously unconstrained at equilibrium, even in a linear network.

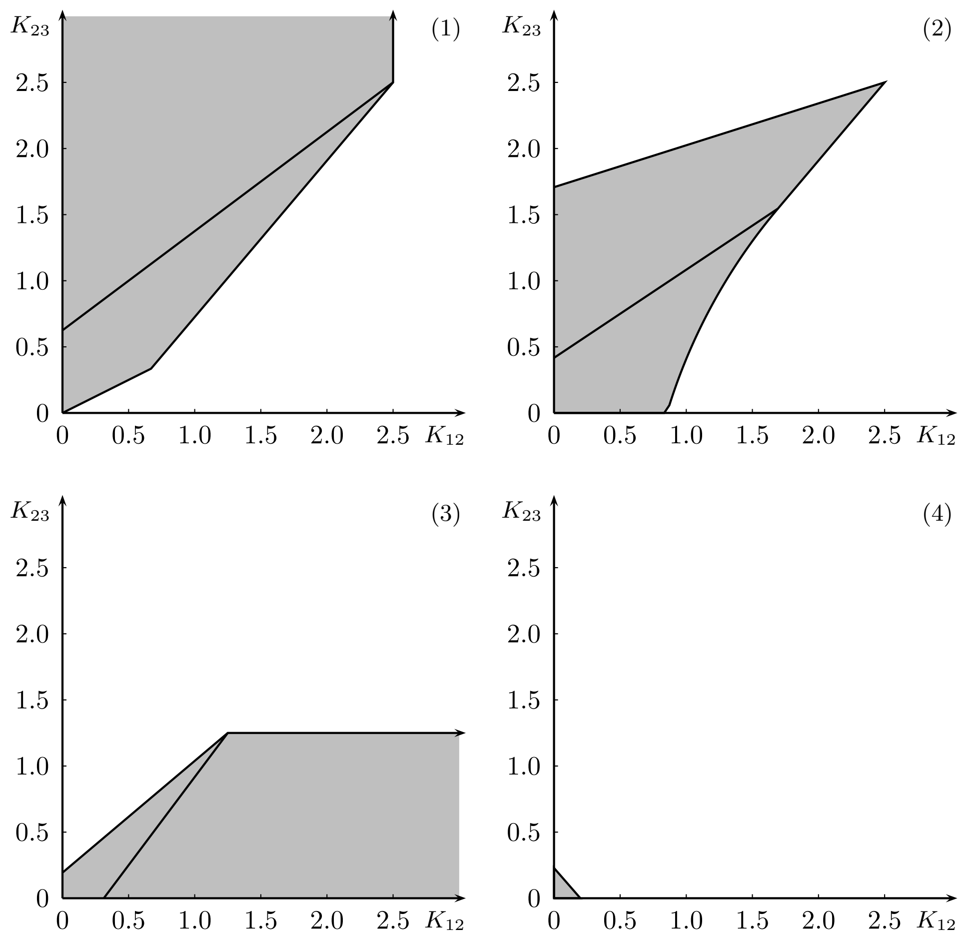

Figure 12.9: Deviation sets for firm 1 for various of KKT regimes.

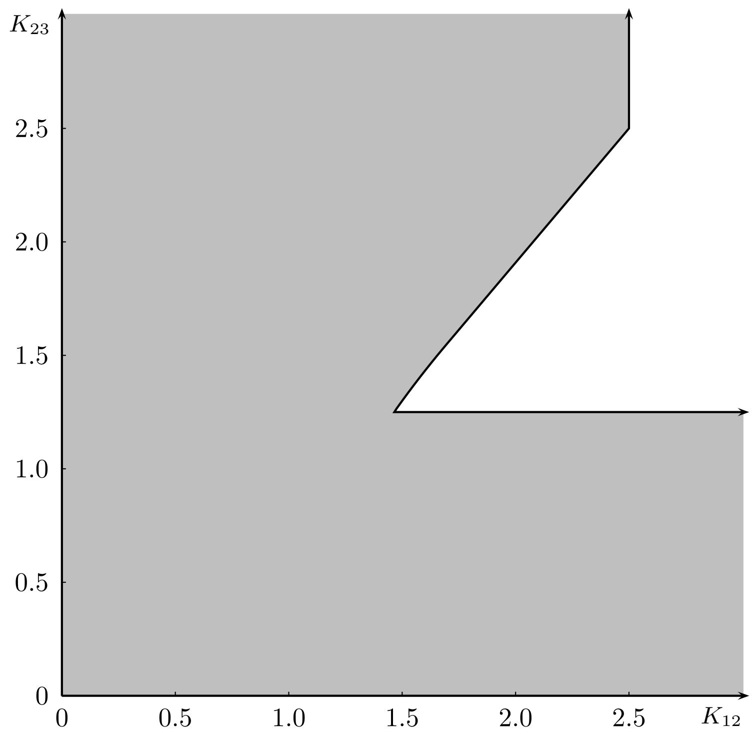

Figure 12.10: Deviation set for firm 1.

Chapter 13

Mixed strategy equilibria

In the previous chapters we have derived conditions on the line capacities of a network and fringe bounds guaranteeing the existence of various pure strategy equilibria. This has shown that for certain line capacities there may not exist any pure strategy equilibria. In this chapter we will discuss the potential for existence of mixed strategy equilibria. A mixed strategy can consist of a set of pure strategies each with some corresponding probability (these could be discrete strategies or a continuum of strategies). In a mixed strategy equilibrium each player is indifferent between all strategies over which it mixes, and these strategies must be globally optimal, as discussed by Stoft in [90].

Here we investigate the existence of these equilibria over a network consisting of two nodes joined by a single line. We will first examine the existence of mixed strategy equilibria in an asymmetric context, and then show how the nature of the equilibrium is different when the situation is symmetric.

13.1 Asymmetric mixed strategy equilibria

We will first consider mixed strategy equilibria in an asymmetric two-node setting; this network is shown in figure 13.1.

We assume that at each node there is an unconstrained fringe offering supply functions of the form $S_i(p) = a_i p$. We will analyze a family of mixed strategy equilibria in this setting and derive the conditions for which it is a valid equilibrium to the game. The mixed strategy equilibria that we will consider here consist of one generator playing a pure strategy, and the other other mixing

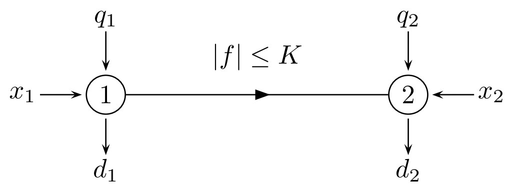

Figure 13.1: Two-node network.

over two strategies. In most discrete games, a mixed strategy equilibrium of this type is unlikely, as it relies on one player having multiple (non-unique) maximizers in response to another player's decision. Here, however, as we are dealing with continuous decision spaces, this situation can occur.

We will consider mixed strategy equilibria of the following types:

(A) Generator 2 chooses a quantity so that generator 1 is indifferent between withholding to congest the line and not congesting the line.

(B) Generator 1 chooses a quantity so that generator 2 is indifferent between withholding to congest the line and not congesting the line.

Without loss of generality, we will analyse equilibrium type (A). The condition such that generator 1 is indifferent between congesting the line and not, is found from the following profit maximization problem:

$$\begin{aligned}
\max \quad & q_1 \pi_1 \\
\\
\text{s.t.} \quad & q_1 + a_1 \pi_1 - f = d_1 \\
& q_2 + a_2 \pi_2 + f = d_2 \\
& \pi_1 - \pi_2 + \eta_1 - \eta_2 = 0 \\
& 0 \leq K - f \perp \eta_1 \geq 0 \\
& 0 \leq K + f \perp \eta_2 \geq 0.
\end{aligned} \tag{13.1}$$

First assuming the line does not congest, we can solve the above problem. This gives a profit of

$$\rho_1^A = \frac{1}{a_1 + a_2} \left(\frac{d_1 + d_2 - q_2}{2} \right)^2,$$

and then assuming that the line congests towards node 1 gives an optimal profit of

$$\rho_1^B = \frac{1}{a_1} \left(\frac{d_1 - K}{2} \right)^2.$$

We now solve for the value of q_2, namely $\hat{q}_2$, which makes these actions equally profitable

$$\rho_1^A\big|_{q_2=\hat{q}_2} = \rho_1^B.$$
$$\Rightarrow \quad \hat{q}_2 = d_1 + d_2 - (d_1 - K)\sqrt{\frac{a_1 + a_2}{a_1}}; \tag{13.2}$$

this is the pure strategy of generator 2. From this, we can compute the strategies of generator 1, for which it is indifferent; these are

$$q_1^A = \frac{1}{2}(d_1 - K)\sqrt{\frac{a_1 + a_2}{a_1}}, \tag{13.3}$$
$$q_1^B = \frac{1}{2}(d_1 - K). \tag{13.4}$$

Now presume that generator 1 injects q_1^A with probability θ and q_1^B with probability $1 - \theta$. The problem of maximizing generator 2's *expected* profit can be formulated as the following MPEC:

$$\begin{aligned}
\max \quad & q_2\left(\theta\pi_2^A + (1-\theta)\pi_2^B\right) \\
\\
\text{s.t.} \quad & q_1^A + a_1\pi_1^A - f^A = d_1 \\
& q_2 + a_2\pi_2^A + f^A = d_2 \\
& q_1^B + a_1\pi_1^B - f^B = d_1 \\
& q_2 + a_2\pi_2^B + f^B = d_2 \\
& \pi_1^A - \pi_2^A + \eta_1^A - \eta_2^A = 0 \\
& \pi_1^B - \pi_2^B + \eta_1^B - \eta_2^B = 0 \\
& 0 \le K - f^A \perp \eta_1^A \ge 0 \\
& 0 \le K + f^A \perp \eta_2^A \ge 0 \\
& 0 \le K - f^B \perp \eta_1^B \ge 0 \\
& 0 \le K + f^B \perp \eta_2^B \ge 0.
\end{aligned} \tag{13.5}$$

For the equilibrium of type (A) to exist it is necessary that the optimal solution to the above problem yields $\pi_1^A = \pi_2^A$, $\pi_1^B \ge \pi_2^B$, $f^B = -K$ and $\left|f^A\right| \le K$. To derive the conditions on the parameters yielding this result, we must first determine the value of the mixing generator, θ. So long as the line congests appropriately, we can compute the expected profit of generator 2 to be

$$\rho_2(q_2) = q_2 \times \left(\frac{d_1 + d_2 - q_1^A - q_2}{a_1 + a_2} \times \theta + \frac{d_2 + K - q_2}{a_2} \times (1-\theta)\right),$$

the unique maximizer of ρ_2 can be found from the first order condition, as follows

$$\left.\frac{\partial \rho_2}{\partial q_2}\right|_{q_2 = q_2^*} = 0$$
$$\Rightarrow \quad q_2^* = \frac{2a_1(d_2 + K)(1-\theta) + 2a_2(d_2 + K) + a_2(d_1 - K)\left(2 - \sqrt{\frac{a_1+a_2}{a_1}}\right)\theta}{4(a_1(1-\theta) + a_2)}.$$

Now we must compute the value of θ which yields $q_2^* = \hat{q}_2$, in other words, we find the mixing probability which ensures that generator 2 injects a quantity such that generator 1 is indifferent between congesting the line towards node 1 and not congesting the line. The mixing probability can be found as follows:

$$\begin{aligned} & q_2^*|_{\theta=\theta^*} = \hat{q}_2 \\ \Rightarrow \quad & \theta^* = \frac{2\left(a_1+a_2\right)\left(a_1\left(2d_1+d_2-K\right)-2\left(d_1-K\right)\sqrt{a_1\left(a_1+a_2\right)}\right)}{a_1\left(a_2\left(2-\sqrt{\frac{a_1+a_2}{a_1}}\right)\left(d_1-K\right)-4\left(d_1-K\right)\sqrt{a_1\left(a_1+a_2\right)}+2a_1\left(2d_1+d_2-K\right)\right)}. \end{aligned}$$

Since θ is a probability, we have that

$$0 \leq \theta \leq 1.$$

This condition is necessary for the mixed strategy to be feasible. Now we must determine the conditions which ensure that:

(i) playing the strategies, q_1^A, q_1^B and q_2^* congest the line as assumed, and

(ii) the strategies for both players are globally optimal.

The conditions described in (i) ensure that when generator 1 injects q_1^A and generator 2 injects q_2^* the line is unconstrained; this condition is given by

$$K \geq \left|\frac{\left(a_1+a_2\right)d_1-\left(a_1+\frac{1}{2}a_2\right)\left(d_1-K\right)\sqrt{\frac{a_1+a_2}{a_1}}}{a_1+a_2}\right|. \tag{13.6}$$

The conditions (i) also ensure that when generator 1 injects q_1^B and generator 2 injects q_2^* the line is constrained towards node 1; from (13.1) this can be shown to be

$$K \leq d_1. \tag{13.7}$$

The conditions described in (ii) are needed to ensure that generator 1's injections are globally optimal. Suppose the conditions (i) are satisfied then generator 1 is q_1^A and q_1^B are both locally optimal strategies (with the same profit). These must also be globally optimal strategies, since congesting the line away would require generator to inject more than q_1^A; this will not yield a better profit since (from section 5.3 and lemma 7.1) we know that the residual demand curve is concave for $q_1 > q_1^A$; hence no additional conditions are required to ensure this.

Now we must consider generator 2. Generator 2 is solving the MPEC in (13.5), given q_1^A, q_2^B and the mixing probability. To determine the conditions such that q_2^* is the globally optimal solution to this problem we need to consider different line congestion regime. It can be shown that we only need to consider the following four cases:

D1: $f^A = K$, $f^B = -K$, $\pi_1^A \leq \pi_2^A$, $\pi_1^B \geq \pi_2^B$,

D2: $f^A = K$, $f^B = -K$, $\pi_1^A \leq \pi_2^A$, $\pi_1^B = \pi_2^B$,

D3: $f^A = K$, $\left|f^B\right| \leq K$, $\pi_1^A \leq \pi_2^A$, $\pi_1^B = \pi_2^B$,

D4: $f^A = K$, $f^B = K$, $\pi_1^A \leq \pi_2^A$, $\pi_1^B \leq \pi_2^B$.

Assuming the conditions (i) are satisfied, for each of these cases we can determine the conditions ensuring that the maximum profit is lower than the profit given by q_2^*.

As an illustration of this procedure, consider case D1. In this case the profit function becomes

$$\rho_2^{D1} = q_2 \times \left(\frac{d_2 - K - q_2}{a_2} \times \theta + \frac{d_2 + K - q_2}{a_2} \times (1 - \theta) \right),$$

the unique maximizer of ρ_2^{D1} is given by

$$\begin{aligned} \left. \frac{\partial \rho_2}{\partial q_2} \right|_{q_2 = q_2^{D1}} &= 0 \\ \Rightarrow \quad q_2^{D1} &= \frac{1}{2} \left(d_2 + K - 2K\theta \right). \end{aligned}$$

If order for generator 2 to have incentive to deviate to this congestion regime we require that the profit from deviation is greater than the profit from not deviating (incentive condition), and that the deviation leads to lines congesting as specified. The incentive condition is given by

$$\rho_2^{D1}\left(q_2^{D1}\right) < \rho_2\left(q_2^*\right),$$

and the feasibility conditions are

$$\begin{aligned} \eta_1^A &= \frac{d_2 - K - q_2^{D1}}{a_2} - \frac{d_1 + K - q_1^A}{a_1} \geq 0, \\ \eta_2^B &= \frac{d_1 - K - q_1^B}{a_1} - \frac{d_2 + K - q_2^{D1}}{a_1} \geq 0. \end{aligned}$$

This methodology can be used for each possible congestion regime to determine the set of conditions where generator 2 has no incentive to deviate from the equilibrium.

For the case where $a_1 = a_2$, we will show the conditions which ensure the existence of the mixed strategy equilibrium (A) diagrammatically. To make this diagram general, we will make the following substitutions:

$$\alpha = \frac{d_1}{d_1 + d_2}, \quad \beta = \frac{K}{d_1 + d_2}.$$

The conditions for θ being between 0 and 1 and condition (i) are given by the unshaded regions of figure 13.2. Furthermore, conditions ensuring there is no incentive to deviate to each congestion regime (D1 – D4) are given by the unshaded regions in figure 13.3.

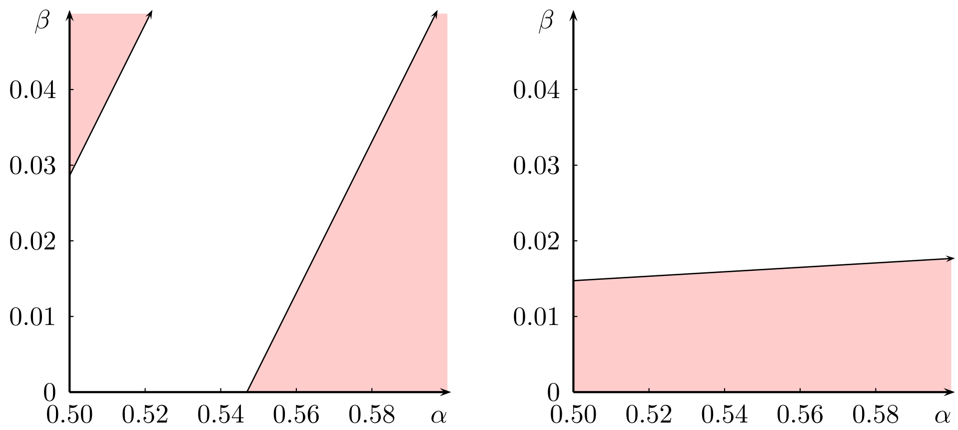

Figure 13.2: Conditions on θ (left) and condition (i) (right).

Putting all these conditions together gives the set of pairs (α, β) (which are proxies for the line capacity and relative amounts of demand at each node) that support the mixed strategy equilibrium of type (A), defined at the beginning of the chapter; this set is given by the unshaded region in figure 13.4.

Now we will plot the regions supporting various pure- and mixed strategy equilibria. The different types of equilibria shown in figure 13.5 are the unconstrained equilibrium (blue), an equilibrium with the line congested from node 1 to node 2 (red, $\alpha < 0.5$), an equilibrium with the line congested from node 2 to node 1 (red, $\alpha > 0.5$), the mixed strategy equilibrium outlined above (green, $\alpha > 0.5$) and its reverse with generator 2 indifferent between congesting the line and not (green, $\alpha < 0.5$).

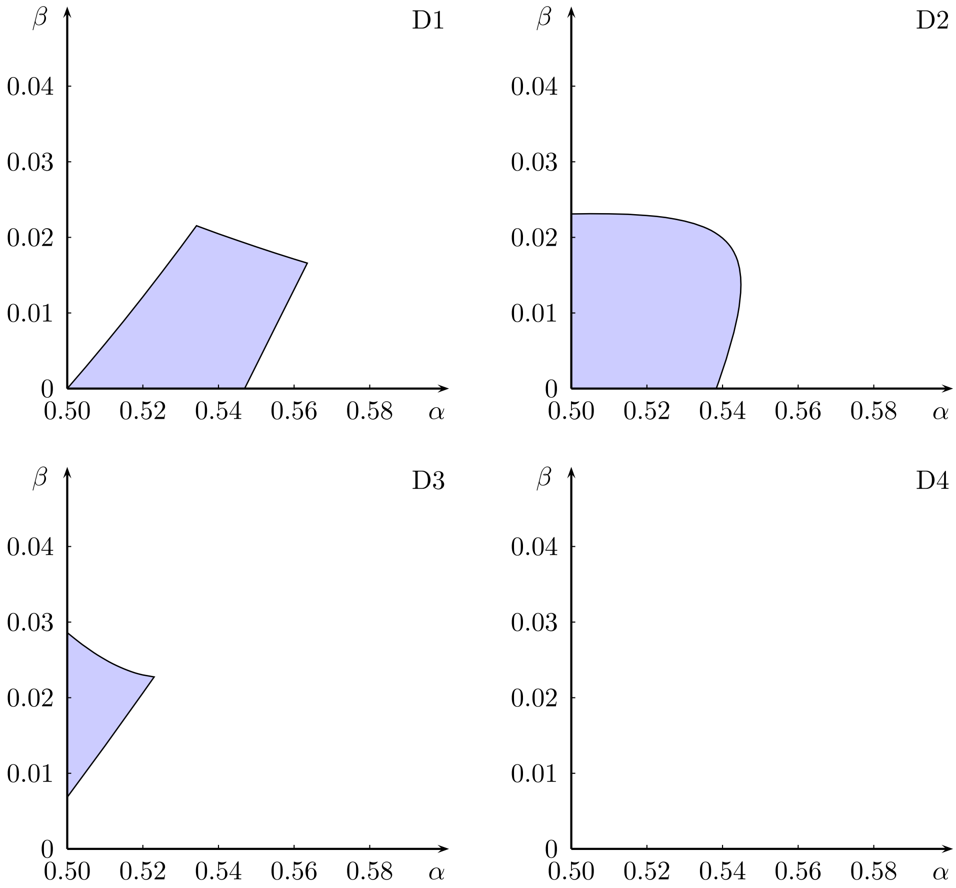

Figure 13.3: No incentive to deviate (unshaded regions).

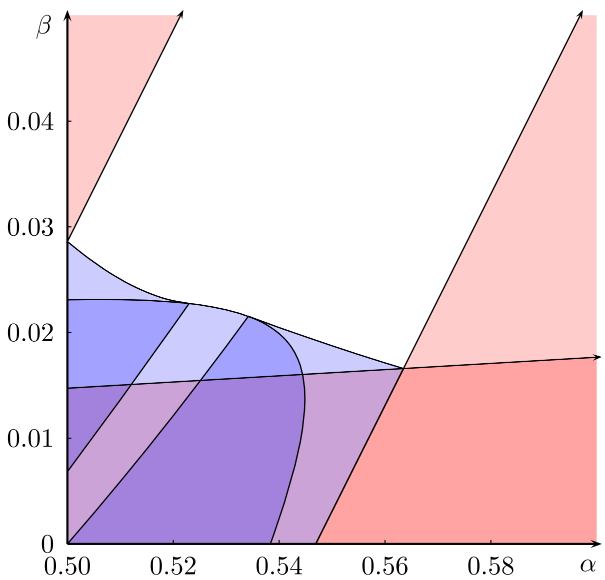

Figure 13.4: Existence of equilibrium of type (A) (given by the unshaded region).

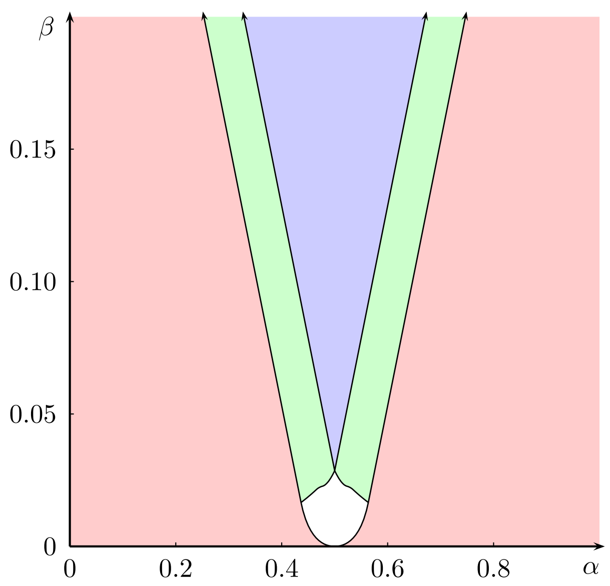

Figure 13.5: Various equilibria.

Example 13.1. *Now consider an example of this type of mixed strategy equilibrium. Here we will set $d_1 = 2$, $d_2 = 1$, $a_1 = 1$ and $a_2 = 1$. We will consider this situation over a range of K.*

First we must find the range of K for which this equilibrium is valid. We can see that the mixing ratio, shown in figure 13.6 (i), is only valid (between 0 and 1) over the following range of K:

$$0.359246 \leq K \leq 0.585786, . \tag{13.8}$$

We also require that conditions (i) and (ii) be satisfied. From inequalities (13.6) and (13.7), condition (i) can be shown to be satisfied over following range of K:

$$34 - 24\sqrt{2} \leq K \leq 2.$$

Furthermore, from figure 13.5 with an α value of $\frac{2}{3}$ we see that condition (ii) is always satisfied so long as the mixing ratio is between 0 and 1.

Since we require all conditions to be satisfied, the range of K such that this equilibrium is valid is found from the intersection of these ranges. Hence the range of K such that this equilibrium is valid given by (13.8).

Note that here, at the boundary of the supported values of K, both players are playing a pure strategy, as the value of θ is either 0 or 1. As K leaves the feasible range of the mixed strategy equilibrium, a pure strategy equilibrium persists. This is shown in figure 13.6.

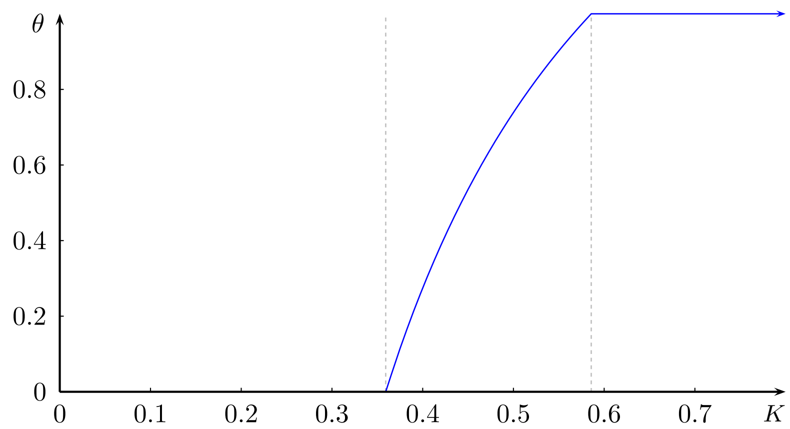

Figure 13.6: Mixing probability, θ, as a function of line capacity, K.

Now we compute the injection quantities q_1^A, q_1^B and q_2^* as a function of K. From equations (13.2), (13.3) and (13.4) these are:

$$
\begin{aligned}
q_1^A &= \frac{\sqrt{2}}{2}(2-K)\,, \\
q_1^B &= \frac{1}{2}(2-K)\,, \\
q_2^* &= 2-\sqrt{2}\,(1-K)\,,
\end{aligned}
$$

these injections as functions of K are shown in figure 13.7.

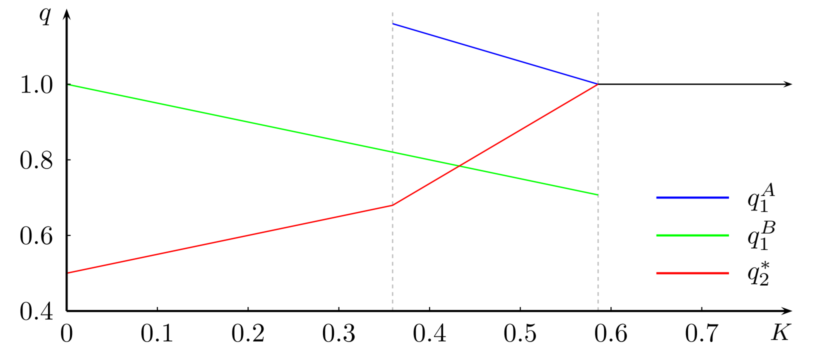

Figure 13.7: Injection quantities as a function of line capacity, K.

The expected nodal prices as a function of K, for the pure and mixed strategy equilibria are shown in figure 13.8. In this section, we have shown that when the demand and fringes are sufficiently

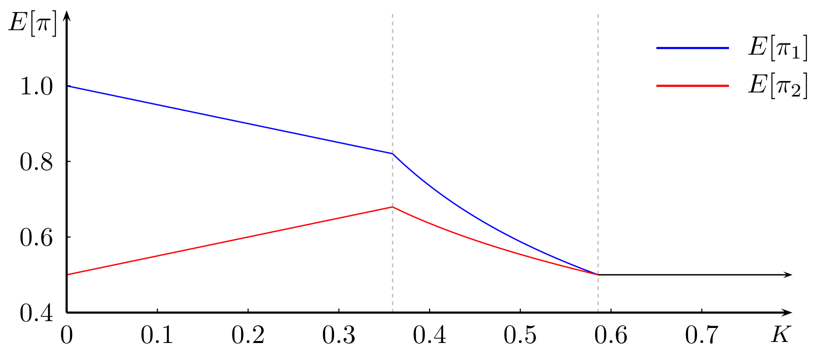

Figure 13.8: Expected prices as a function of line capacity, K.

asymmetric, that there exist mixed strategy equilibria where one generator has a pure strategy and another is mixing over two discrete strategies; in chapter 16 we derive mixed strategy equilibria of this type in the context of carbon charges. In the next section we will analyse a symmetric scenario.

13.2 Symmetric Case

From figure 13.4 in the previous section, we can see that when the demand and fringes are identical at both nodes, there is no mixed strategy equilibrium where one generator chooses a pure strategy and the other mixes over two strategies. Instead, in this situation, we will seek a symmetric mixed strategy equilibrium, as analysed by Stoft in [90].

13.2.1 Computing the equilibria

The simplest symmetric mixed strategy equilibrium has each player mixing over two strategies. We first find the conditions on line capacities which supports an equilibrium of this type.

In the symmetric case the dispatch problem P(q), given in chapter 5, gives the following price at node 1 as a function of the injection at each node:

$$\pi_1 (q_1, q_2) = \begin{cases} d - K - q_1, & q_1 - q_2 \leq -2K, \\ \frac{1}{2}(2d - q_1 - q_2), & -2K < q_1 - q_2 < 2K, \\ d + K - q_1, & q_1 - q_2 \geq 2K. \end{cases}$$

In order for a pair of mixed strategies to be an equilibrium, each player must be indifferent between the strategies, and each generator's strategy must maximize its profit function. So for a pair of strategies, q^A, q^B for each generator, with $q^A > q^B$, we know that when both players play the same strategy the flow on the line is 0; furthermore, if $q_1 = q^A$ and $q_2 = q^B$ the line must congest toward node 1, otherwise the line will not congest for any pair of injections. This would mean that the profit function is strictly concave, precluding the existence of a mixed strategy equilibrium. From the dispatch problem the flow on the line (when unconstrained) is given by $f = \frac{1}{2}(q_1 - q_2)$. Therefore the above conditions can be written in terms of the line capacity K as follows:

$$0 \leq K \leq \frac{1}{2}\left(q^A - q^B\right).$$

Now suppose generator 2 injects q_2^A with a probability $1 - \theta$ and q_2^B with a probability θ. The profit function for generator 1 becomes:

$$\rho_1 (q_1) = (1 - \theta)\, \rho_1 \left(q_1, q_2^A\right) + \theta \rho_1 \left(q_1, q_2^B\right).$$

So if generator 1 injects q_1^A the profit is

$$\rho_1 \left(q_1^A\right) = \frac{1 - \theta}{2}\left(q_1^A \left(2d - q_1^A - q_2^A\right)\right) + \theta \left(q_1^A \left(d + K - q_1^A\right)\right), \tag{13.9}$$

and if generator 1 injects q_1^B the profit is

$$\rho_1\left(q_1^B\right) = (1-\theta)\left(q_1^B\left(d-K-q_1^B+\frac{\theta}{2}\left(q_1^B\left(2d-q_1^B-q_2^B\right)\right)\right)\right) \tag{13.10}$$

To find the mixed strategy equilibrium, we solve a system of equation which require that both injection quantities are local maxima, that the profit is identical for both injections, and that the injections for generator 1 and 2 are the same, therefore we have the following system of equations:

$$\begin{aligned}
\left.\frac{d\rho_1}{dq}\right|_{q=q_1^A} &= 0 \\
\left.\frac{d\rho_1}{dq}\right|_{q=q_1^B} &= 0 \\
\rho_1\left(q_1^A\right) &= \rho_1\left(q_1^B\right) \\
q_1^A &= q_2^A \\
q_1^B &= q_2^B.
\end{aligned}$$

The above system of equations can be solved in terms of θ to give

$$\begin{aligned}
q_1^A = q_2^A &= \frac{-2d\left((-2+\theta)\left(-3+2\theta+\theta^2\right)^2+(-4+\theta)\,\theta\left(-3+2\theta+\theta^2\right)\sqrt{(2+\theta-\theta^2)}\right)}{54-135\theta+24\theta^2+55\theta^3+19\theta^4-18\theta^5-\theta^6+2\theta^7} \\
q_1^B = q_2^B &= \frac{2d\left(\theta\left(-4-3\theta+\theta^2\right)+\left(-3+2\theta+\theta^2\right)\sqrt{(2+\theta-\theta^2)}\right)}{-18+33\theta+8\theta^2-2\theta^3-5\theta^4+2\theta^5} \\
K &= \frac{d\left((-4+\theta)\left(-3+2\theta+\theta^2\right)\sqrt{(2+\theta-\theta^2)}+18-17\theta-5\theta^3+6\theta^4-2\theta^5\right)}{18-51\theta+25\theta^2+10\theta^3+3\theta^4-7\theta^5+2\theta^6}.
\end{aligned} \tag{13.11}$$

For θ between 0 and 1, solution given above satisfies the conditions that each generator is indifferent between the two strategies and that both strategies are local maxima, however it is possible that the strategies are not global maxima. Consider a strategy for generator 1, q_1^C which congests the line towards node 1 for both of generator 2's strategies; the profit for such a strategy would be:

$$\begin{aligned}
\rho_1\left(q_1^C\right) &= (1-\theta)\left(q_1^C\left(d-K-q_1^C\right)\right)+\theta\left(q_1^C\left(d-K-q_1^C\right)\right) \\
&= q_1^C\left(d-K-q_1^C\right).
\end{aligned} \tag{13.12}$$

This function is maximized at

$$q_1^{C*} = \frac{d-K}{2},$$

yielding a profit of

$$\rho_1\left(q_1^{C*}\right) = \frac{1}{4}\left(d-K\right)^2.$$

So for the mixed strategy equilibrium, q^A, q^B, θ to be valid we require:

$$\rho_1\left(q_1^{C*}\right) \leq \rho_1\left(q^A\right) = \rho_1\left(q^B\right).$$

This yields the following constraint on K

$$K \geq 0.0135123d,$$

at which point the equilibrium is $q^A = 0.598091$, $q^B = 0.544738$, $\theta = 0.360250$.

For $K < 0.0135123d$ the equilibrium given by (13.11) is not valid; there is in fact a mixed strategy equilibrium where each generator mixes over three strategies. To compute an equilibrium in this case, a similar technique to that above can be used, however each generator would have three (or more) strategies. The mixing probabilities, injections and expected price as a function of K are shown in figures 13.9, 13.10 and 13.11 respectively.

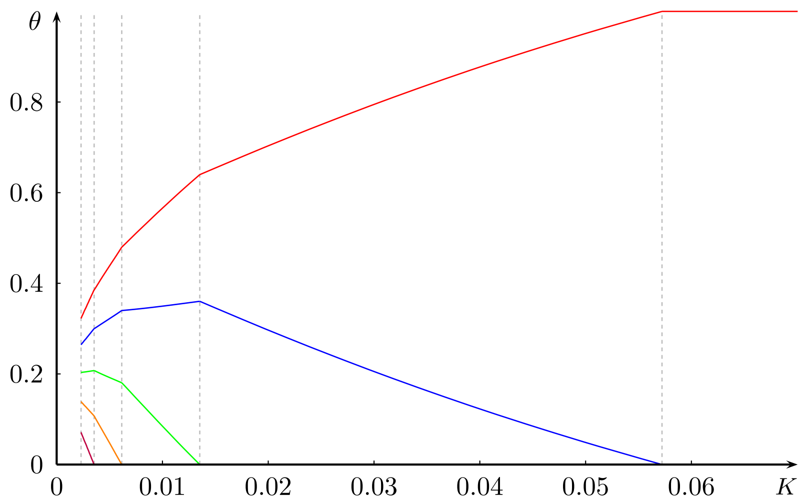

Figure 13.9: Mixing probabilities as functions of line capacity, K.

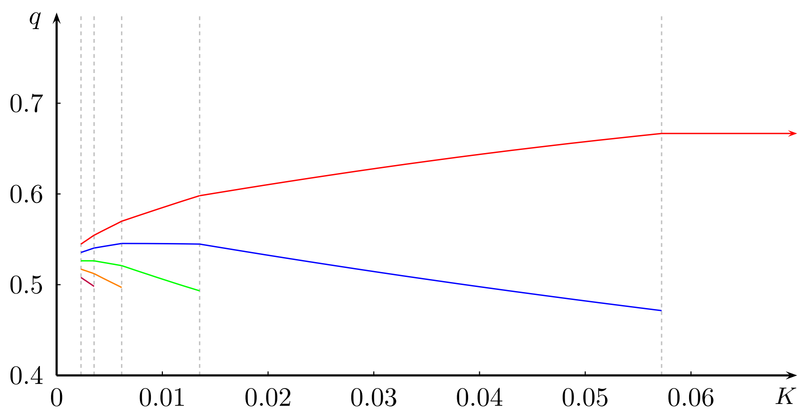

Figure 13.10: Equilibrium injection quantities as functions of line capacity, K.

Notice that for small values of K the above graphs are not completed. This because as the number of strategies being mixed over becomes large the problem of finding the equilibrium becomes extremely

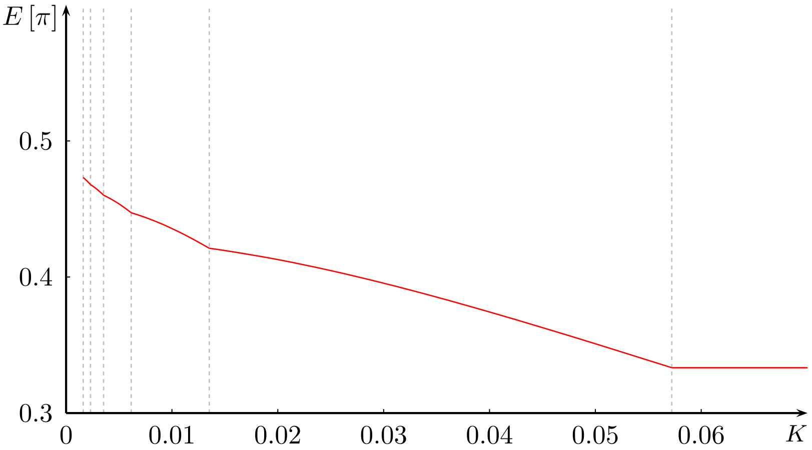

Figure 13.11: Expected price as function of line capacity, K.

difficult to solve analytically. Stoft in [90] and Borenstein *et al.* in [12], discuss numerical methods for solving for these mixed strategy equilibria. Moreover, they state that as $K \to 0$ the number of strategies each generator must mix over approaches infinity. We will not discuss the numerical methods to compute these mixed strategy equilibria as it is outside the scope of this work.

Chapter 14

Losses in a Cournot setting

In this chapter, we examine the effect that quadratic losses on lines may have on the outcome of Cournot games. This has been examined previously by e.g. Borenstein *et al.* in [12] and Leite da Silva in [22]. In chapter 6 we showed that, in general, the presence of quadratic losses mean that the dispatch problem has a non-convex feasible region. This makes the computation of a Cournot equilibrium difficult, as the optimal dispatch must be embedded in the optimization problem of each generator.

Due to this potential non-convexity, we will focus on radial networks; here the dispatch problem can be replaced with a convex equivalent as discussed earlier in chapter 6 and by Palma-Benhke *et al.* in [76] (so long as prices are positive). In this chapter, we examine a two-node network, with linear fringe generation at both nodes for two different approximations of quadratic losses. For each approximation we investigate the shape of a generator's revenue function and find that it is not guaranteed to be *quasi-concave*.

The lack of quasi-concavity of a generator's revenue function can cause significant problems, as this would mean that the KKT conditions of a generators' profit maximization problem would not be sufficient to guarantee that a particular strategy is globally optimal [57]. Specifically, we will show how the solution to the KKT conditions may find only local maxima or even local minima.

In this two-node setting, we also consider how the outcome of Cournot games can be affected by the presence of line losses, and derive necessary conditions on the parameters of the game which ensure that there exists a unique pure-strategy Nash equilibrium.

14.1 Revenue functions

Here we will restrict our attention to a network with only two nodes, connected by a lossy line, with loss coefficient $r \geq 0$. We assume that at each node, i, there is a competitive fringe, offering a linear supply function $S_i(p) = a_i p$, where $a_i > 0$. Initially, we will investigate the revenue function of a single strategic generator who injects q_1 at node 1, thus we fix the injections of all other strategic generators and subtract their injections from the demand at the appropriate node to give a net demand at node i of d_i.

In this section, we will consider the two methods of modelling quadratic losses that were introduced in chapter 6. First we examine the revenue function of a generator assuming that the losses are proportional to the square of the average flow on the line. We will compare this to the revenue function associated with losses being proportional to the sent power.

For each approximation, we are particularly interested in deriving conditions that guarantee that a generator's revenue function is quasi-concave (i.e., a function with convex level sets; see for example [65]). Quasi-concavity is important as it guarantees that any local maximum is the global maximum, which is a crucial property when considering equilibria.

14.1.1 Loss as function of average flow

We first examine the loss approximation where the loss is computed based on the average flow on the line. In this situation the dispatch problem is:

$$\begin{aligned}
\mathrm{P}(q_1): \min \quad & \tfrac{1}{2a_1}{x_1}^2 + \tfrac{1}{2a_2}{x_2}^2 \\
\text{s.t.} \quad & x_1 - f - \tfrac{1}{2}rf^2 = d_1 - q_1 \quad [\pi_1] \\
& x_2 + f - \tfrac{1}{2}rf^2 = d_2 \qquad\quad [\pi_2] \\
& x_1,\, x_2 \geq 0.
\end{aligned}$$

As discussed in chapter 6, this is a non-convex problem because we have equality constrained node balance constraints with quadratic terms. However, if we allow for free disposal of electricity by formulating the node balance constraints with inequalities, then we are able to regain convexity. From theorem 6.1 of chapter 6, we know that the amount of surplus on the constraint will be zero when nodal prices are positive. Since the strategic generator is a profit maximizer, a negative price at node 1 is not a possible outcome. Moreover, so long as the flow on the line is between $-\frac{1}{r}$ and $\frac{1}{r}$ (which are the implicit bounds; see figure 6.1), the price at node 2 must also be positive. Therefore we will consider the following convex relaxation of the dispatch problem

$$
\begin{aligned}
\mathrm{RP}(q_1): \min \quad & \tfrac{1}{2a_1}{x_1}^2 + \tfrac{1}{2a_2}{x_2}^2 \\
\text{s.t.} \quad & x_1 - f - \tfrac{1}{2} r f^2 \geq d_1 - q_1 \quad [\pi_1] \\
& x_2 + f - \tfrac{1}{2} r f^2 \geq d_2 \qquad\quad [\pi_2] \\
& x_1,\, x_2 \geq 0.
\end{aligned}
$$

Because we now have a convex program, we can replace it by its equivalent KKT formulation

$$
\begin{aligned}
0 \leq q_1 + x_1 - f - \frac{1}{2} r f^2 - d_1 \quad & \perp \quad \pi_1 \geq 0, \\
0 \leq x_2 + f - \frac{1}{2} r f^2 - d_2 \quad & \perp \quad \pi_2 \geq 0, \\
x_1 - a_1 \pi_1 \quad & \perp \quad x_1 \geq 0, \\
x_2 - a_2 \pi_2 \quad & \perp \quad x_2 \geq 0, \\
-(1 + r f)\, \pi_1 + (1 - r f)\, \pi_2 \quad & = \quad 0,
\end{aligned}
$$

and when prices are positive, this system simplifies to

$$
\begin{aligned}
q_1 + a_1 \pi_1 - f - \frac{1}{2} r f^2 - d_1 \quad & = \quad 0, \\
a_2 \pi_2 + f - \frac{1}{2} r f^2 - d_2 \quad & = \quad 0, \\
-(1 + r f)\, \pi_1 + (1 - r f)\, \pi_2 \quad & = \quad 0, \\
\pi_1, \pi_2 \quad & > \quad 0.
\end{aligned}
$$

In the Cournot game, the generators will each maximize their profits simultaneously, with their prices being determined by the dispatch problem. To compute the equilibrium to this game, we will embed the optimality conditions of the dispatch problem into each generator's revenue maximization problem. The revenue maximization problem for generator 1 is shown below

$$
\begin{aligned}
\max \quad & q_1 \pi_1 \\
\text{s.t.} \quad & q_1 + a_1 \pi_1 - f - \tfrac{1}{2} r f^2 - d_1 = 0 \\
& a_2 \pi_2 + f - \tfrac{1}{2} r f^2 - d_2 = 0 \\
& -(1 + r f)\, \pi_1 + (1 - r f)\, \pi_2 = 0 \\
& \pi_1, \pi_2 \geq 0.
\end{aligned}
$$

The above optimization problem for generator 1 is non-convex. Note that, unlike for the dispatch problem, here it is not possible to relax the problem using inequalities, as the objective function is maximizing with respect to q_1 and π_1 (meaning that if the constraints were relaxed the problem would become unbounded). Moreover, the third constraint is bilinear and cannot be relaxed. Therefore the solutions to the KKT conditions of this problem are not guaranteed to be globally optimal. However, we will proceed to derive the conditions under which there exists only one stationary point in generator 1's revenue function when prices are positive. These conditions will ensure that the revenue function is quasi-concave over the feasible region; and hence guarantee that a solution to its KKT conditions will be globally optimal.[1]

Quasi-concave revenue function

Here we will derive the conditions which ensure that a generator injecting at node 1 has a quasi-concave revenue function. To do this, we first solve RP(q_1) analytically for generator 1's injection quantity, q_1. For any q_1, we can then express q_1 and the nodal prices, π_1 and π_2, as functions of the optimal flow, f, on the line giving

$$\pi_1 = \frac{1-rf}{1+rf} \times \frac{d_2 - f + \frac{1}{2}rf^2}{a_2}, \tag{14.1}$$

$$\pi_2 = \frac{d_2 - f + \frac{1}{2}rf^2}{a_2}, \quad \text{and} \tag{14.2}$$

$$q_1 = d_1 + f + \frac{1}{2}rf^2 - a_1\pi_1. \tag{14.3}$$

To simplify the notation, for much of this section we will make the following substitutions

$$\begin{aligned} g &= rf, \\ \alpha &= \frac{a_2}{a_1}, \\ e_1 &= rd_1, \quad \text{and} \\ e_2 &= rd_2. \end{aligned}$$

With these substitutions, the above expressions become

$$\pi_1 = \frac{1-g}{1+g} \times \frac{e_2 - g + \frac{1}{2}g^2}{a_1\alpha r}, \tag{14.4}$$

$$\pi_2 = \frac{e_2 - g + \frac{1}{2}g^2}{a_1\alpha r}, \quad \text{and} \tag{14.5}$$

$$q_1 = \frac{e_1 + g + \frac{1}{2}g^2}{r} - a_1\pi_1. \tag{14.6}$$

[1]Note that although we are deriving conditions for the generator's revenue function being quasi-concave, if the generator has some convex cost function, these same conditions will be sufficient to ensure the profit function is quasi-concave.

Hence, we can compute generator 1's revenue as a function of g:

$$\begin{aligned} R &= \pi_1 q_1 \\ &= \pi_1 \left(\frac{e_1 + g + \frac{1}{2}g^2}{r} - a_1 \pi_1 \right). \end{aligned} \tag{14.7}$$

$$\tag{14.8}$$

Since the dispatch problem has constraints which ensure that the nodal prices are positive, these must continue to be enforced. In the following lemma we define the domain over which both nodal prices are positive.

Lemma 14.1. *Consider the expressions for π_1 and π_2 given by equations* (14.4) *and* (14.5)*. We will denote the set over which both π_1 and π_2 are positive by $\mathcal{S}$. Where $\mathcal{S}$ is defined as the following Cartesian product:*

$$S \equiv g \in (-1, 1) \times e_2 \in \left(g - \tfrac{1}{2}g^2, \infty\right).$$

Proof. From equations (14.4) and (14.5) we have

$$\pi_1 = \frac{1-g}{1+g} \times \pi_2,$$

hence we know that in order for π_1 and π_2 to have the same sign

$$-1 < g < 1.$$

The condition for π_2, given in equation (14.5), being positive is given by

$$\pi_2 = \frac{e_2 - g + \frac{1}{2}g^2}{a_1 \alpha r} > 0.$$

Since $a_1 > 0$, $\alpha > 0$ and $r > 0$, the conditions over which both π_1 and π_2 are positive are

$$\begin{aligned} e_2 &> g - \tfrac{1}{2}g^2, \\ g &< 1, \quad \text{and} \\ g &> -1, \end{aligned}$$

as required. □

Next, we seek conditions ensuring that there is at most one turning point in generator 1's revenue function with respect to its injection quantity, so long as prices are positive. Typically a sufficient condition for this is concavity, which is equivalent to a negative second derivative. Unfortunately, in this case, the revenue function is not concave in general; however, will find conditions ensuring that the revenue is quasi-concave. To find these conditions, we define the following function:

$$\Phi\left(g, e_2\right) = \frac{d}{dg}\left(\frac{dR}{dg}\Big/\frac{d\pi_1}{dg}\right).^{2} \tag{14.9}$$

The following lemmas and theorem 14.4 state that so long as Φ is positive, we are guaranteed that generator 1's revenue function will be quasi-concave.

In the first lemma, below, we prove that π_1 is a strictly decreasing function of g.

Lemma 14.2. *Consider π_1, given in equation* (14.4)*, for some arbitrary but fixed e_2. For all g such that $\{g, e_2\} \in \mathcal{S}$, π_1 is strictly decreasing in g.*

Proof. From (14.4) is we know that

$$\pi_1 = \frac{1-g}{1+g} \times \frac{e_2 - g + \frac{1}{2}g^2}{a_1 \alpha r}.$$

We show that the derivative of π_1 with respect to g, given below, is negative.

$$\frac{d\pi_1}{dg} = -\frac{g^3 - 3g + 2e_2 + 1}{a_1 \alpha r\left(1+g\right)^2}. \tag{14.10}$$

As the denominator of equation (14.10) is strictly positive over the set $\mathcal{S}$, the zeros of the numerator of this expression are the turning points of π_1 with respect to g, we will now show there exist no such turning points of π_1 over the set $\mathcal{S}$ given by lemma 14.1.

The numerator of equation (14.10) is a cubic polynomial in g and linear in e_2, we will denote this function $\chi(g, e_2)$:

$$\chi(g, e_2) = -g^3 + 3g - 2e_2 - 1,$$

As $\mathcal{S}$ defines a lower bound on e_2 and $\frac{\partial \chi}{\partial e_2} < 0$, we have

$$\begin{aligned} \sup_{e_2|\{g,e_2\}\in\mathcal{S}} \chi(g, e_2) &= -g^3 + 3g - 2\left(g - \tfrac{1}{2}g^2\right) - 1 \\ &= -(1+g)(1-g)^2. \end{aligned} \tag{14.11}$$

It is clear that equation (14.11) is strictly negative over the the domain $g \in (-1, 1)$, hence for all $\{g, e_2\} \in \mathcal{S}$, π_1 is a decreasing function of g, as required. □

We will now prove a lemma which states that q_1 is an increasing function of g.

Lemma 14.3. *Consider q_1, given in equation* (14.6)*. For all g such that $\{g, e_2\} \in \mathcal{S}$, q_1 is strictly increasing in g.*

[2] Since e_2 does not depend on g, total derivatives are used in this expression.

Proof. From equation (14.6) we have

$$q_1 = \frac{e_1 + g + \frac{1}{2}g^2}{r} - a_1\pi_1.$$

Taking the derivative with respect to g gives

$$\frac{dq_1}{dg} = \frac{1+g}{r} - a_1\frac{d\pi_1}{dg}.$$

From lemma 14.2 we know that $\frac{d\pi_1}{dg} < 0$, and since $1+g$ is positive we therefore have that

$$\frac{dq_1}{dg} > 0, \quad \forall \{g, e_2\} \in \mathcal{S},$$

as required. □

Finally, we prove a theorem which states that $\Phi > 0$ implies that generator 1's revenue function is quasi-concave.

Theorem 14.4. *Suppose that Φ, defined in equation* (14.9)*, is strictly positive for some arbitrary but fixed e_2 for all g such that $\{g, e_2\} \in \mathcal{S}$ then there is at most one turning point in generator 1's revenue function, defined in equation* (14.7)*, with respect to q_1 when prices are positive.*

Proof. Consider Φ for some e_2; if $\Phi > 0$ for all g such that $\{g, e_2\} \in \mathcal{S}$, equation (14.9) gives

$$\frac{d}{dg}\left(\frac{dR}{dg}\bigg/\frac{d\pi_1}{dg}\right) > 0, \quad \forall g \mid \{g, e_2\} \in \mathcal{S}.$$

This implies that $\frac{dR}{dg}\Big/\frac{d\pi_1}{dg}$ is strictly increasing with g, and hence has at most one zero within $\mathcal{S}$ for a given e_2. Now, from lemma 14.2, we know that $\frac{d\pi_1}{dg} < 0$, therefore we have that the zeros of $\frac{dR}{dg}$ are the same as those of $\frac{dR}{dg}\Big/\frac{d\pi_1}{dg}$, hence there can be at most one zero for $\frac{dR}{dg}$ with $\mathcal{S}$ (for a fixed e_2). Using the chain rule we can write $\frac{dR}{dg}$ as

$$\frac{dR}{dg} = \frac{dR}{dq_1}\frac{dq_1}{dg}.$$

Finally from lemma 14.3, we know that $\frac{dq_1}{dg} > 0$, thus $\frac{dR}{dq_1}$ has at most one zero when prices are positive, which means that there is at most one turning point in generator 1's revenue function. □

The above theorem has shown that $\Phi > 0$ implies the quasi-concavity of R. In what follows we will derive the conditions ensuring that $\Phi > 0$ for all $\{g, e_2\} \in \mathcal{S}$. Φ can be written as

$$\Phi = \frac{a_1\phi}{2r\alpha\,(1+g)^2\,(1+2e_2-3g+g^3)^2}, \tag{14.12}$$

where,

$$\phi = z_0 + z_1 e_2 + z_2 {e_2}^2 + z_3 {e_2}^3, \tag{14.13}$$

and

$$
\begin{aligned}
z_0 &= 4\left(1-3g+g^3\right)^3+\alpha\left(1+g\right)^3\left(4-12g+6g^2+26g^3-21g^4-3g^5+4g^6\right), \\
z_1 &= 24\left(1-3g+g^3\right)^2+4\alpha\left(1+g\right)^3\left(1-3g-6g^2+4g^3\right), \\
z_2 &= 48\left(1-3g+g^3\right)+4\alpha\left(1+g\right)^3\left(1+3g\right), \\
z_3 &= 32.
\end{aligned}
$$

From the proof of lemma 14.2, it is clear that the denominator of (14.12) is positive for $\{g, e_2\} \in \mathcal{S}$. So Φ and ϕ have the same sign, hence we want to find the conditions on α such that ϕ is greater than zero for all $\{g, e_2\} \in \mathcal{S}$. We will first prove that ϕ is positive for all $g \in \left[-\frac{1}{3}, 1\right)$.

Lemma 14.5. *Consider the function ϕ for some e_2. ϕ is strictly positive for all $g \in \left[-\frac{1}{3}, 1\right)$ such that $\{g, e_2\} \in \mathcal{S}$.*

Proof. To show this result, we make the following substitutions for e_2 and g,

$$
\begin{aligned}
e_2 &= g-\tfrac{1}{2}g^2+\tilde{e}_2 \\
g &= -\tfrac{1}{3}+\tfrac{4}{3}\tilde{g}
\end{aligned}
$$

The conditions on $\tilde{e}_2$ and $\tilde{g}$ such that $g \in \left[-\frac{1}{3}, 1\right)$ and $\{g, e_2\} \in \mathcal{S}$ are:

$$\tilde{e}_2 \in (0, \infty) \times \tilde{g} \in [0, 1). \tag{14.14}$$

With these substitutions, ϕ becomes,

$$
\begin{aligned}
\phi &= \frac{32a_1}{19638}\left[\left(27\tilde{e}_2+16(1-\tilde{g})^2\left(1+2\tilde{g}\right)\right)^3\right. \\
&+ \left.4(1+2\tilde{g})^3\left(729\tilde{g}\tilde{e}_2+216\tilde{e}_2(1-\tilde{g})^2\left(1+2\tilde{g}\right)+256(1-\tilde{g})^4(1+2\tilde{g})^2\right)\alpha\right],
\end{aligned}
$$

it is clear that this is strictly positive over the domain of interest, as required. □

Now note that if we treat g as a parameter, ϕ is a cubic polynomial in e_2. The following lemma gives three necessary conditions which ensure that ϕ is positive for all e_2 given that $g \in \left(-1, -\frac{1}{3}\right)$.

Lemma 14.6. *Consider the function ϕ for some $g \in \left(-1, -\frac{1}{3}\right)$. ϕ is strictly positive for all e_2 such that $\{g, e_2\} \in \mathcal{S}$, if and only if one or more of the following three conditions are satisfied:*

1. *ϕ has only one real root (and two imaginary roots),*
2. *$\hat{e}_2$ is not real, or*
3. *$\hat{e}_2 \leq g - \frac{1}{2}g^2$,*

where $\hat{e}_2$ is the location of the convex turning point of ϕ with respect to e_2.[3]

Proof. We first note that the feasible domain of e_2 for a given g is:

$$e_2 \in \left(g - \tfrac{1}{2}g^2, \infty\right).$$

We can determine the limits of the cubic at the boundary of the feasible region to be

$$\begin{aligned}\lim_{e_2 \to \infty} \phi &= \infty, \\ \lim_{e_2 \to g - \frac{1}{2}g^2} \phi &= 4\left(1-g\right)^4\left(1+g\right)^3\left(\left(1-g\right)^2 + \left(1+g\right)^2\alpha\right).\end{aligned}$$

As the limits of the cubic are both positive, for ϕ to be strictly positive for all e_2 within this interval, it is clear that any of the following conditions are sufficient:

1. ϕ has only one real root, e_2^*, and it is not a repeated root. If ϕ has only one (unique) real root, it is clear that $\{g, e_2^*\} \notin \mathcal{S}$, hence ϕ could not have been less than or equal to zero within the range, since it would have needed to have had at least a repeated root.

2. $\hat{e}_2$ is not real. As the turning points of a cubic are complex conjugates, if $\hat{e}_2$ is not real then both turning points are complex. Therefore the function must be monotonic, so it can not have any zero over the relevant domain.

3. $\hat{e}_2 \leq g - \frac{1}{2}g^2$. Since $z_3 > 0$, the convex turning point of ϕ occurs at a larger value of e_2 than the concave turning point. Therefore if the location of the convex turning point satisfies this condition, then the function is monotonic within the domain, and thus cannot contain any zero.

Conversely, if none of the conditions are met, i.e. the turning points of ϕ with respect to e_2 are real, the convex turning point, $\hat{e}_2$, occurs inside $\mathcal{S}$, and there is not one real zero (and two imaginary ones), it is clear that ϕ cannot be strictly positive for all e_2 such that $\{g, e_2\} \in \mathcal{S}$.

□

We will now proceed to compute the conditions on the parameters of the problem that correspond to each of the conditions given in lemma 14.6. We examine condition 1 in lemma 14.7 below.

Lemma 14.7. *Consider the function ϕ for some $g \in \left(-1, -\frac{1}{3}\right)$; so long as α is positive, condition (1) from lemma 14.6 (that ϕ only has one real root) is equivalent to:*

$$\alpha < -\frac{18\left(1-g\right)^2\left(3g + 12g^2 + g^3 + \sqrt{\left(-1-4g+g^2\right)^3}\right)}{\left(1+4g\right)\left(1+3g\right)^2\left(1+g\right)^2}.$$

[3] Note that $\hat{e}_2$ is not required to be in $\mathcal{S}$.

Proof. From [51], we know that the sign of the discriminant of a cubic defines the number of distinct real zeros the function has. A cubic function has one real zero if and only if its discriminant is strictly positive. For ϕ, the discriminant is given by

$$\begin{aligned} D_\phi &= \frac{-{z_1}^2{z_2}^2+4z_0{z_2}^3-18z_0z_1z_2z_3+4{z_1}^3z_3+27{z_0}^2{z_3}^2}{{z_3}^4} \\ &= \frac{3}{4096}\left(1-g\right)^4\left(1+g\right)^{12}\alpha^2\Big(324\left(1-g\right)^4+36\left(1-g\right)^2g\left(3+12g+g^2\right)\alpha \\ &\quad+\left(1+g\right)^2\left(1+3g\right)^2\left(1+4g\right)\alpha^2\Big). \end{aligned}$$

Here we will find the conditions in terms of α and g such that ϕ has one real zero (and two imaginary ones). We find that $D_\phi > 0$, for

$$\alpha < -\frac{18\left(1-g\right)^2\left(3g+12g^2+g^3+\sqrt{\left(-1-4g+g^2\right)^3}\right)}{\left(1+4g\right)\left(1+3g\right)^2\left(1+g\right)^2}$$

and

$$\alpha > -\frac{18\left(1-g\right)^2\left(3g+12g^2+g^3-\sqrt{\left(-1-4g+g^2\right)^3}\right)}{\left(1+4g\right)\left(1+3g\right)^2\left(1+g\right)^2}.$$

It it easy to verify, for $g \in \left(-1,-\frac{1}{3}\right)$, that the right-hand side of these inequalities are both real. Furthermore, it can be shown that the right-hand side of the second inequality is strictly negative for g in the range. It follows that so long as $\alpha > 0$ the condition simplifies to

$$\alpha < -\frac{18\left(1-g\right)^2\left(3g+12g^2+g^3+\sqrt{\left(-1-4g+g^2\right)^3}\right)}{\left(1+4g\right)\left(1+3g\right)^2\left(1+g\right)^2},$$

as required. □

In the following lemma we find the necessary and sufficient condition on α which ensures that the convex turning point of ϕ is not real.

Lemma 14.8. *Consider ϕ for some $g \in \left(-1,-\frac{1}{3}\right)$. When $\alpha > 0$, the convex turning point of ϕ with respect to g is not real if and only if*

$$\alpha < -\frac{72\left(1-g\right)^2 g}{\left(1+g\right)^2\left(1+3g\right)^2}. \tag{14.15}$$

Proof. The expression for the convex turning point $\hat{e}_2$, can be found from the first order condition of ϕ to be

$$\begin{aligned} \hat{e}_2 &= \frac{1}{24}\Big(-12\left(1-3g+g^3\right)-\left(1+g\right)^3\left(1+3g\right)\alpha \\ &\quad+\left(1+g\right)^2\sqrt{\alpha\left(72\left(1-g\right)^2g+\left(1+g\right)^2\left(1+3g\right)^2\alpha\right)}\Big). \end{aligned} \tag{14.16}$$

This expression contains a square-root of a polynomial. For $\hat{e}_2$ to not be real it is necessary and sufficient for the polynomial, under the square-root, to be negative. For $\alpha > 0$, this can be written as

$$72\left(1-g\right)^2 g + \left(1+g\right)^2\left(1+3g\right)^2\alpha < 0.$$

For $g \in \left(-1, -\frac{1}{3}\right)$, this gives

$$\alpha < -\frac{72\left(1-g\right)^2 g}{\left(1+g\right)^2\left(1+3g\right)^2}.$$

□

The following lemma shows that the third condition of lemma 14.6 is never satisfied.

Lemma 14.9. *The condition that $\hat{e}_2 \leq g - \frac{1}{2}g^2$ is never satisfied for $g \in \left(-1, -\frac{1}{3}\right)$.*

Proof. From equation (14.16), we can rewrite $\hat{e}_2 \leq g - \frac{1}{2}g^2$ as

$$\frac{1+g}{24}\left(12\left(1-g\right)^2 + \left(1+g\right)^2\left(1+3g\right)\alpha - \left(1+g\right)\sqrt{\alpha\left(72\left(1-g\right)^2 g + \left(1+g\right)^2\left(1+3g\right)^2\alpha\right)}\right) \geq 0.$$

For $g \in \left(-1, -\frac{1}{3}\right)$, for the condition to be satisfied, it is necessary (but not sufficient) that

$$\begin{aligned}
&\frac{1+g}{24}\left(12\left(1-g\right)^2 + \left(1+g\right)^2\left(1+3g\right)\alpha\right) \geq 0 \\
\Leftrightarrow\quad & 12\left(1-g\right)^2 + \left(1+g\right)^2\left(1+3g\right)\alpha \geq 0 \\
\Leftrightarrow\quad & \alpha \leq -\frac{12\left(1-g\right)^2}{\left(1+g\right)^2\left(1+3g\right)}, \quad \forall g \in \left(-1, -\tfrac{1}{3}\right).
\end{aligned} \tag{14.17}$$

An additional necessary requirement is that the convex turning point is real. From lemma 14.8, we write this condition as

$$\alpha \geq -\frac{6g}{1+3g} \times \frac{12\left(1-g\right)^2}{\left(1+g\right)^2\left(1+3g\right)}.$$

Therefore, over the domain $g \in \left(-1, -\frac{1}{3}\right)$, to ensure that condition 3 from lemma 14.6 is satisfied, we require that both of the above conditions are satisfied. We will now show that there is no positive α that simultaneously satisfies both conditions. Inequality (14.17) can be written as

$$\alpha \leq -1 \times \frac{12\left(1-g\right)^2}{\left(1+g\right)^2\left(1+3g\right)}.$$

It is easy to verify that

$$\frac{6g}{1+3g} > 1, \quad \forall g \in \left(-1, -\tfrac{1}{3}\right),$$

hence we know that there does not exist a positive α for which condition 3 is satisfied for $g \in \left(-1, -\frac{1}{3}\right)$. □

Finally, we derive a condition on α ensuring that ϕ is non-negative for all $\{g, e_2\} \in \mathcal{S}$.

Lemma 14.10. *So long as* α *is less than* $744+304\sqrt{6}$*, we are guaranteed that* ϕ *will be non-negative over the set* $\mathcal{S}$.

Proof. Lemma 14.5 shows that, for any $g \in \left[-\frac{1}{3}, 1\right)$, ϕ must be strictly positive for all e_2 such that $\{g, e_2\} \in \mathcal{S}$. Lemma 14.6 gives the conditions such that for any $g \in \left(-1, -\frac{1}{3}\right)$, ϕ is strictly positive for all e_2 such that $\{g, e_2\} \in \mathcal{S}$. Finally, lemma 14.9 shows that for $g \in \left(-1, -\frac{1}{3}\right)$ condition 3 is never satisfied.

From the lemmas outlined above, we need only consider conditions 1 and 2 for $g \in \left(-1, -\frac{1}{3}\right)$. Condition 1 is equivalent to the following inequality

$$\alpha < -\frac{18\left(1-g\right)^2\left(3g+12g^2+g^3+\sqrt{\left(-1-4g+g^2\right)^3}\right)}{\left(1+4g\right)\left(1+3g\right)^2\left(1+g\right)^2},$$

and condition 2 is equivalent to

$$\alpha < -\frac{72\left(1-g\right)^2 g}{\left(1+g\right)^2\left(1+3g\right)^2}.$$

It is easy to verify that for all $g \in \left(-1, -\frac{1}{3}\right)$

$$-\frac{72\left(1-g\right)^2 g}{\left(1+g\right)^2\left(1+3g\right)^2} < -\frac{18\left(1-g\right)^2\left(3g+12g^2+g^3+\sqrt{\left(-1-4g+g^2\right)^3}\right)}{\left(1+4g\right)\left(1+3g\right)^2\left(1+g\right)^2}.$$

This means that the conditions of lemma 14.6 simplify to

$$\alpha < -\frac{18\left(1-g\right)^2\left(3g+12g^2+g^3+\sqrt{\left(-1-4g+g^2\right)^3}\right)}{\left(1+4g\right)\left(1+3g\right)^2\left(1+g\right)^2}.$$

Note that the above upper bound on α is a function of g. We now find the smallest upper bound on α satisfying these conditions[4] at g^*, given by

$$\frac{d}{dg}\left(-\frac{18\left(1-g\right)^2\left(3g+12g^2+g^3+\sqrt{\left(-1-4g+g^2\right)^3}\right)}{\left(1+4g\right)\left(1+3g\right)^2\left(1+g\right)^2}\right)\Bigg|_{g=g^*} = 0$$

$$\Rightarrow \quad g^* = \frac{1}{23}\left(55-28\sqrt{6}\right).$$

The upper bound on α for $g = g^*$ is

$$-\frac{18\left(1-g\right)^2\left(3g+12g^2+g^3+\sqrt{\left(-1-4g+g^2\right)^3}\right)}{\left(1+4g\right)\left(1+3g\right)^2\left(1+g\right)^2}\Bigg|_{g=g^*} = 8\left(93+38\sqrt{6}\right).$$

[4] This upper bound on α will be sufficient to ensure that ϕ is positive regardless of g.

For any α satisfying the condition

$$\alpha < 8\left(93 + 38\sqrt{6}\right) \approx 1488.6$$

we are guaranteed that ϕ is positive for all $\{g, e_2\} \in \mathcal{S}$. □

Now suppose that $\alpha \leq 1488.6$. From lemma 14.10 we know that $\phi > 0$ and hence from theorem 14.4 we have that generator 1's revenue function has only one turning point when prices are positive. Conversely, for any value of α greater than or equal to this value, there exists some g with $\{g, \hat{e}_2\} \in \mathcal{S}$ for which ϕ is negative.

Recall that α is equal to a_2/a_1. Therefore we can write the condition which guarantees that generator 1's revenue function is quasi-concave as

$$a_2 \leq 1488.6 a_1.$$ [5]

So long as the above condition is satisfied, generator 1's revenue function is quasi-concave over the range of q_1 that give positive prices at both nodes, irrespective of the loss coefficient or the demand at either node.

In the next section, we give the corresponding condition for the case where losses are approximated to be proportional to the square of sent power.

14.1.2 Loss proportional to square of sent power

We will now examine the nature of the revenue function of generator 1, for the case where the losses are approximated to be proportional to the square of the power leaving a node.

Quasi-concave profit function

For $f \leq 0$, we solve for generator 1's quantity, q_1, and nodal prices, π_1 and π_2, as functions of the flow on the line, f, giving

$$\pi_1 = \frac{1}{1 + 2rf} \times \frac{d_2 - f}{a_2}, \tag{14.18}$$

$$\pi_2 = \frac{d_2 - f}{a_2}, \quad \text{and} \tag{14.19}$$

$$q_1 = d_1 + f + rf^2 - a_1\pi_1. \tag{14.20}$$

[5] We present an example of a revenue function that is not quasi-concave (with $\alpha = 5000$) in figure 14.7, later in this chapter.

For much of this section we will make the these substitutions:

$$\begin{aligned} g &= rf, \\ \alpha &= \frac{a_2}{a_1}, \\ e_1 &= rd_1, \quad \text{and} \\ e_2 &= rd_2. \end{aligned}$$

The above expressions simplify to give

$$\begin{aligned} \pi_1^- &= \frac{1}{1+2g} \times \frac{e_2 - g}{a_1 \alpha r}, \\ \pi_2^- &= \frac{e_2 - g}{a_1 \alpha r}, \quad \text{and} \\ q_1^- &= \frac{e_1 + g + g^2}{r} - a_1 \pi_1. \end{aligned}$$

For $f \geq 0$, we find that the expressions for π_1, π_2 and q_1 are

$$\begin{aligned} \pi_1^+ &= (1 - 2g) \times \frac{e_2 - g + g^2}{a_1 \alpha r}, \\ \pi_2^+ &= \frac{e_2 - g + g^2}{a_1 \alpha r}, \quad \text{and} \\ q_1^+ &= \frac{e_1 + g}{r} - a_1 \pi_1. \end{aligned}$$

Combining these to form piecewise functions of g gives

$$\pi_1 = \begin{cases} \pi_1^- & g \leq 0, \\ \pi_1^+, & \text{otherwise}, \end{cases} \tag{14.21}$$

$$\pi_2 = \begin{cases} \pi_2^-, & g \leq 0, \\ \pi_2^+, & \text{otherwise}, \end{cases} \tag{14.22}$$

$$q_1 = \begin{cases} q_1^-, & g \leq 0, \\ q_1^+, & \text{otherwise}. \end{cases} \tag{14.23}$$

From above, we can compute generator 1's revenue as a piecewise function of g.

As in the previous section, over the domain where π_1 and π_2 are both positive, we will find the conditions such that there is only one turning point in generator 1's revenue function with respect to its injection quantity, q_1.

It can be shown that there is a maximum value of α, below which we are guaranteed that there is at most one turning point in generator 1's revenue function. In this case the maximum value of α is 24 (compared to $\alpha < 1488$ in the previous section, where losses were proportional to the square of the average flow). The following is an example where the revenue function is not quasi-concave.

Example 14.11. *Consider a network with these parameters: $a_1 = 1$, $a_2 = 40$, $d_1 = 50$, $d_2 = 210$ and $r = 0.01$. For this situation, the residual demand curve faced by generator 1 and its revenue are shown in figures 14.1 and 14.2, respectively.*

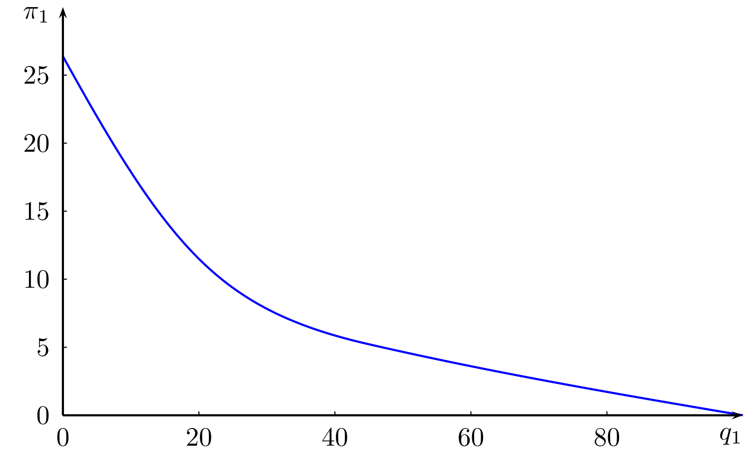

Figure 14.1: Price as function of injection.

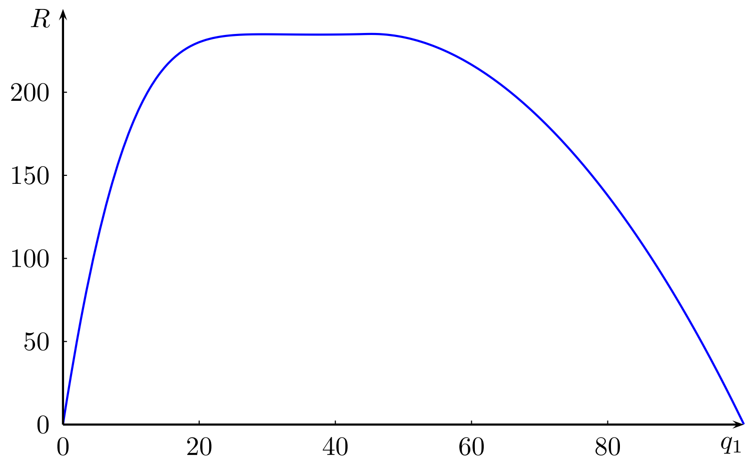

Figure 14.2: Revenue as function of injection.

Note that the revenue function has three turning points, these occur at, $q_1 = 29.42$, $q_1 = 37.30$ and $q_1 = 45.31$, these injections cause a flow on the line of -14.79, -6.88 and 0.49 respectively. Solving this problem using its first order conditions may yield any of these stationary points. Note here that the flow on the line for each stationary point is much less than the implicit limit of $\frac{1}{2r} = 50$, so one might believe that using this loss approximation is not materially affecting the results. However, we know that since the ratio of the fringe slopes is less than 1488.6, if the previous loss approximation is used, then the corresponding revenue function will be quasi-concave. Therefore one should be careful, when deciding how to model losses, not to cause revenue functions that are not quasi-concave solely due to the approximation of losses.

14.2 Cournot competition with transmission losses

In the previous section we have derived conditions on the ratio of the fringe slopes ensuring that a generator has a quasi-concave revenue function. We will now consider these revenue functions in the context of a Cournot game and derive conditions for existence and uniqueness of equilibria.

We will first apply an existence theorem of Rosen [84] to guarantee that there exists an pure-strategy equilibrium to a Cournot game over a two-node transmission network with losses. Theorem 2.1 in [84] states that there exists an equilibrium to a game so long as it satisfies the following conditions: each player's payoff function must be quasi-concave in its own action and continuous in the actions of others; and the joint decision space of all players is non-empty, closed and convex.

Consider a game with generators $g \in \mathcal{G}$ each with capacity V_g and a profit function given by

$$\rho_g \left(q_g \right) = q_g \pi_g.$$

By theorem 14.4, so long as $a_2 < 1488a_1$ and (by symmetry) $a_1 < 1488a_2$, we are guaranteed that each generator's profit function is quasi-concave. As the generators each have a capacity (and a lower bound of 0), the joint decision space is clearly closed and bounded. Finally, π_1 and π_2 can be shown to be continuous functions of the injections at either node. This game therefore satisfies the conditions of theorem 2.1 in [84] and hence we are guaranteed that there exists a pure-strategy equilibrium.

In what follows, we will demonstrate that there always exists an equilibrium in a symmetric two-node situation with linear fringe supply functions, and then derive conditions for uniqueness of equilibria when the fringes are asymmetric.

14.2.1 Symmetric fringes

We will first examine a symmetric situation with two nodes connected by a lossy line. There is one generator at each node, as shown in figure 14.3, a demand d at each node and a fringe at each node offering a linear supply curve. We will set the slopes of the supply curves to be the same (i.e. $a_1 = a_2$).

This game was discussed by Borenstein *et al.* in [12]; here Borenstein *et al.* made the assertion that due to the implicit capacity on a lossy line, large loss coefficients can lead to situations where

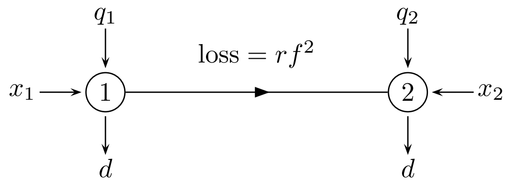

Figure 14.3: Two-node network with losses.

there is no pure-strategy equilibrium. They state, for a symmetric two-node game, that:

> "If losses are sufficiently large, however, they can eliminate the existence of a Cournot equilibrium."

The above results show that this assertion is incorrect; moreover, these results are consistent with the observations of Downward *et al.* in [26]. Since $a_1 = a_2 = a$, from theorem 14.4 and lemma 14.10, we know that each generator has a quasi-concave revenue function, regardless of which of the two quadratic loss approximations we use. Therefore by theorem 2.1 in [84] a pure-strategy equilibrium exists.

It is true, as shown in chapter 8 that small line capacities may lead to a situation where no pure-strategy equilibria exist, however, the implicit capacity imposed by a lossy line cannot be treated as merely a capacity. A lossy line not only affects the maximum flow on the line, it also affects the prices paid to the generators. The effect of this is that as the loss coefficient is increased, the equilibrium injection quantities vary. We will see that the equilibrium outcome becomes more profitable for generators as the loss coefficient is increased, which offsets any incentives a generator might have to reduce its output and cause the price at its node to increase. In fact, as the revenue functions are quasi-concave we are guaranteed that there is no incentive to deviate from the equilibrium.

Furthermore, due to the quasi-concave revenue functions, the KKT conditions of the generators' revenue maximization problem yield a global optimal, and any solution to the all-KKT system is a bona-fide equilibrium [83]. In fact, for this example, we can compute the equilibrium analytically from the KKT conditions. The equilibrium as a function of the loss coefficient is:

$$\begin{aligned} q_1 = q_2 = q^C &= \frac{3 + 6r - \sqrt{9 + 4r + 4r^2}}{8r}, \\ \pi_1 = \pi_2 = \pi^C &= \frac{2r - 3 + \sqrt{9 + 4r + 4r^2}}{8r}, \\ f &= 0, \end{aligned}$$

and the revenue for each generator is

$$\begin{aligned}\bar{R} &= \pi^C q^C \\ &= \frac{4r^2 - 8r - 9 + (3+2r)\sqrt{9+4r+4r^2}}{32r^2}\end{aligned}$$

Looking at the candidate equilibrium as a function of the loss coefficient, we find that when taking the limit as $r \to 0$, we end up at the duopoly equilibrium point ($q^D = \frac{2d}{3}, \pi^D = \frac{d}{3a}$), and when taking the limit as $r \to \infty$ the equilibrium is that of two local monopolists ($q^M = \frac{d}{2}, \pi^M = \frac{d}{2a}$), as though there were no line connecting the nodes. The results at the limits are consistent with what one might expect. In figure 14.4, below, we plot how the equilibrium quantities and prices change as a function of the loss coefficient, when $a = 1$.

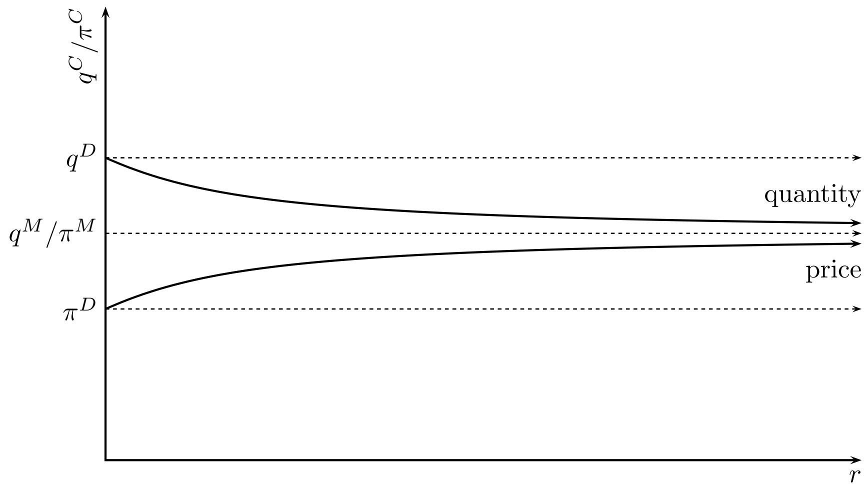

Figure 14.4: Two-node equilibrium as function of loss coefficient.

14.2.2 Asymmetric fringes

In the symmetric case above, we were able to find analytic expressions for the equilibrium injections and prices as functions of the loss coefficient; this is not possible in general. In what follows we consider a game over a two-node network, this time with non-identical fringes ($\alpha \neq 1$) and derive the conditions guaranteeing the existence of a unique equilibrium.

Consider a network with two nodes connected by a lossy line. Let there be a competitive fringe offering a linear supply function at each node with $a_2 < 1488a_1$. There are two strategic generators located at node 1 and no strategic generators located at node 2, as depicted in figure 14.5. In order to compute an equilibrium to this game, we could solve a joint KKT system by simultaneously optimizing each generator's revenue function, assuming the other generator's injection was fixed

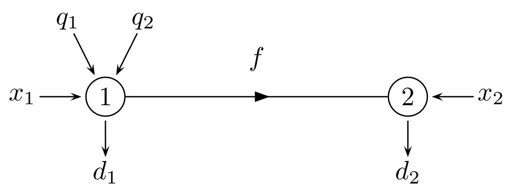

Figure 14.5: Cournot game with both strategic generators at node 1.

(as discussed earlier in chapter 9). This, however, cannot be done analytically, so the problem would need to be solved numerically. At this stage we have no proof that any pure-strategy Nash-equilibrium will be unique, so we can not rely on such a method to find all equilibria.

An alternate method for finding Nash-equilibria is to construct the best response curves for the generators. This technique allows us to gain a deeper understanding of the game. Using this method, we are also able to find all of the pure-strategy Nash-equilibria. In what follows, we will define the best response curves of the generators parametrically, and then find the conditions guaranteeing that there is a unique pure-strategy Nash-Equilibrium.

From equations (14.4) and (14.6) the revenue for generator 1 can be written as

$$R = \pi_1 \left(\frac{e_1 + g + \frac{1}{2}g^2}{r} - a_1\pi_1 \right).$$

Because we have generator 2 also situated at node 1, e_1 will be a function of both the demand at the node and the injection quantity of generator 2. To derive the best response curve, we need to separate these quantities out; to do this, we will make the following substitution

$$e_1 = \bar{e}_1 - rq_2.$$

Generator 1's revenue therefore becomes

$$\begin{aligned} R &= \pi_1 \left(\frac{\bar{e}_1 - rq_2 + g + \frac{1}{2}g^2}{r} - a_1\pi_1 \right) \\ &= -\pi_1 q_2 + \pi_1 \left(\frac{\bar{e}_1 + g + \frac{1}{2}g^2}{r} - a_1\pi_1 \right). \end{aligned}$$

Since $a_1 < 1488a_2$, there is only one turning point in each generator's revenue function. Moreover, if we assume that the capacities of the generators are sufficiently large, then we can solve the first-order condition, $\frac{dR}{dg} = 0$, to find the maximum revenue in terms of q_2. This can be rewritten as

$$-\frac{d\pi_1}{dg} q_2^* + \frac{d\pi_1}{dg} \left(\frac{\bar{e}_1 + g + \frac{1}{2}g^2}{r} - a_1\pi_1 \right) + \pi_1 \left(\frac{1+g}{r} - a_1 \frac{d\pi_1}{dg} \right) = 0$$

which gives

$$q_2^*(g) = \left(\frac{\bar{e}_1 + g + \frac{1}{2}g^2}{r} - a_1\pi_1 \right) + \frac{\pi_1}{\frac{d\pi_1}{dg}} \left(\frac{1+g}{r} - a_1 \frac{d\pi_1}{dg} \right). \tag{14.24}$$

From equation (14.6), for a given g, this value of $q_2^*(g)$ uniquely determines a value of $q_1^*(g)$:

$$q_1^*(g) = \left(\frac{\bar{e}_1 + g + \frac{1}{2}g^2}{r} - a_1\pi_1 \right) - q_2^*(g). \tag{14.25}$$

We can now find the best response curve of generator 1 by parametrically plotting (q_1^*, q_2^*) for all $g \in (-1, 1)$. By symmetry, a similar best response curve can be constructed for generator 2.

In the following lemma we give the conditions on the best response curves to ensure that multiple equilibria do not exist.

Lemma 14.12. *In a symmetric two player game, sufficient conditions ensuring that equilibrium is unique, if it exists, are that the best response curves are continuous, smooth and their slopes are strictly less than* 1 *and strictly greater than* -1.[6]

Proof. Consider a smooth continuous best response curve for the players, $f(x)$ such that

$$-1 < \frac{df}{dx} < 1.$$

An equilibrium is any point (a, b) that satisfies the following equations simultaneously

$$\begin{aligned} a &= f(b) \\ b &= f(a). \end{aligned}$$

We will proceed to show that there is at most one solution to equations above. We will first show that there can be at most one symmetric solution (i.e. a solution such that $a = b$). To achieve this, we show that $f(x)$ has at most one *fixed point* (a point that is mapped to itself by the function) [27]. Therefore the fixed points of $f(x)$ are the roots of the following equation:

$$f(x) - x = 0.$$

As $\frac{df}{dx} < 1$, the above function is strictly decreasing with x, therefore it can have at most one root.

On the other hand, suppose we have an asymmetric solution ($a < b$). Therefore, from above, the following expression must be satisfied:

$$\frac{f(b) - f(a)}{b - a} = -1,$$

by assumption $\frac{df}{dx} > -1$, therefore we have a contradiction by the mean value theorem [58]. Hence there is at most one equilibrium and it is symmetric if it exists. □

[6] See Vives [94] for more general results relating to existence and uniqueness of equilibria.

Lemma 14.12 states that if $-1 < \frac{dq_1^*}{dq_2^*} < 1$ in a symmetric game, then there can be at most one equilibrium. We will now find the conditions on α such that we are guaranteed that this condition is satisfied. The gradient of the best response curve is given by

$$\frac{dq_1^*}{dq_2^*} = \frac{dq_1^*}{dg} \Big/ \frac{dq_2^*}{dg}.$$

Initially we can examine how q_2^* moves as a function of g; this can be shown to give

$$\frac{dq_2^*}{dg} = \Phi$$

where Φ is given by equation (14.9). Also, from equation (14.25) above, we have that

$$\frac{dq_1^*}{dg} = \left(\frac{1+g}{r} - a_1 \frac{d\pi_1}{dg}\right) - \Phi.$$

Now from lemma 14.12, we are guaranteed that there will be a unique equilibrium if the following condition is satisfied

$$-1 < \frac{\left(\frac{1+g}{r} - a_1 \frac{d\pi_1}{dg}\right) - \Phi}{\Phi} < 1.$$

As we know that when prices are positive and $0 < \alpha < 1488$, Φ is strictly positive; we therefore have

$$\frac{1+g}{r} - a_1 \frac{d\pi_1}{dg} > 0, \quad \text{and} \tag{14.26}$$

$$2\Phi - \left(\frac{1+g}{r} - a_1 \frac{d\pi_1}{dg}\right) > 0 \tag{14.27}$$

We will now find the conditions on the parameters of the game for which the above conditions are satisfied. First we will present a lemma which states that condition (14.26) is satisfied so long as nodal prices are positive.

Lemma 14.13. *Consider the condition given by inequality* (14.26). *So long as nodal prices are positive, this condition will be satisfied.*

Proof. From lemma 14.1, we have that if both nodal prices are positive, then g must satisfy: $-1 < g < 1$. This implies that

$$\frac{1+g}{r} > 0.$$

Furthermore, from lemma 14.2 we know that $\frac{d\pi_1}{dg} < 0$. Finally, from above, and as $a_1 > 0$, condition (14.26) must be satisfied. □

Next we define Θ to equal the left hand side of condition (14.27). By substituting in for Φ we have the following function

$$\Theta = \frac{a_1 \theta}{r\alpha (1+g)^2 (1 + 2e_2 - 3g + g^3)^2},$$

where

$$\theta = y_0 + y_1 e_2 + y_2 {e_2}^2 + y_3 {e_2}^3,$$

and

$$\begin{aligned}
y_0 &= 3\left(1-3g+g^3\right)^3 + 3\left(1+g\right)^3\left(1-\left(1-g\right)\left(2+3g-5g^2+g^4\right)g\right), \\
y_1 &= 18\left(1-3g+g^3\right)^2 - 12\alpha\left(2-g\right)\left(1+g\right)^3 g^2, \\
y_2 &= 36\left(1-3g+g^3\right) + 12\alpha\left(1+g\right)^3 g, \\
y_3 &= 24.
\end{aligned}$$

We will now derive a condition on α such that Θ satisfies condition (14.27). Note that so long as prices are positive, from the proof of lemma 14.1, the denominator of Θ is strictly positive, hence the sign of θ is the same as the sign of Θ.

To simplify the upcoming expressions, we will make the following substitution

$$e_2 = g + \tfrac{1}{2}g^2 + \tilde{e}_2,$$

this means the set $\mathcal{S}$ (within which prices are positive), is equivalent to the following Cartesian product

$$g \in (-1,1) \times \tilde{e}_2 \in (0,\infty)\,. \tag{14.28}$$

With the substitution, we have

$$\theta = 3\left(\left(2\tilde{e}_2 + (1-g)^2(1+g)\right)^3 + (1+g)^3\left((1-g)^4(1+g)^2 + 4\tilde{e_2}^2 g\right)\alpha\right). \tag{14.29}$$

We are interested in finding the largest α, below which we are guaranteed that θ is strictly positive. In the following lemma we find a condition on α guaranteeing that θ is positive.

Lemma 14.14. *Consider the function θ for some $\{g, \tilde{e}_2\} \in \mathcal{S}$ and $\alpha > 0$. θ is always positive so long as*

$$(1-g)^4(1+g)^2 + 4\tilde{e_2}^2 g \geq 0,$$

otherwise the following condition is sufficient to ensure that θ is strictly positive

$$\alpha < -\frac{\left(2\tilde{e}_2 + (1-g)^2(1+g)\right)^3}{(1+g)^3\left(4\tilde{e_2}^2 g + (1-g)^4(1+g)^2\right)}. \tag{14.30}$$

Proof. From equation (14.29) we have

$$\theta = 3\left(\left(2\tilde{e}_2 + (1-g)^2(1+g)\right)^3 + (1+g)^3\left((1-g)^4(1+g)^2 + 4\tilde{e_2}^2 g\right)\alpha\right).$$

From the definition of $\mathcal{S}$ given by (14.28). It is clear that

$$2\tilde{e}_2 + (1-g)^2 (1+g) > 0.$$

Furthermore, as $1+g>0$ and $\alpha > 0$, if

$$(1-g)^4 (1+g)^2 + 4\tilde{e_2}^2 g \geq 0,$$

then we are guaranteed that $\theta > 0$. Otherwise if

$$(1-g)^4 (1+g)^2 + 4\tilde{e_2}^2 g < 0,$$

then the condition that $\theta > 0$ is given as

$$\alpha < -\frac{\left(2\tilde{e}_2 + (1-g)^2 (1+g)\right)^3}{(1+g)^3 \left(4\tilde{e_2}^2 g + (1-g)^4 (1+g)^2\right)}.$$

□

Lemma 14.14 provides a sufficient condition (dependent on g and $\tilde{e}_2$) such that θ is positive. However, we are interested in finding a sufficient condition on α, independent of g and $\tilde{e}_2$; the following lemma gives such a condition.

Lemma 14.15. *For any $\{g, \tilde{e}_2\} \in \mathcal{S}$, $\alpha < 56 + 40\sqrt{2}$ satisfies the conditions, ensuring that θ is positive, given in lemma 14.14.*

Proof. To find the condition that $\theta > 0$, we consider the turning points and limits of the right hand side of (14.30). It can be shown that when approaching either of the limits with respect to $\tilde{e}_2$, the right hand side of (14.30) tends to positive infinity. Therefore we are only interested in the turning points; these are found from the first order condition

$$\frac{\partial}{\partial \tilde{e}_2} \left(-\frac{\left(2\tilde{e}_2 + (1-g)^2 (1+g)\right)^3}{(1+g)^3 \left(4\tilde{e_2}^2 g + (1-g)^4 (1+g)^2\right)} \right) \Bigg|_{\tilde{e}_2 = \tilde{e}_2^*} = 0.$$

This yields one solution satisfying the condition $\tilde{e}_2^* \geq 0$

$$\tilde{e}_2^* = \frac{(1-g)^2 (1+g)\, g - \sqrt{-(3-g)(1-g)^4 (1+g)^2 g}}{2g}.$$

By setting $\tilde{e}_2 = \tilde{e}_2^*$, inequality (14.30) becomes

$$\alpha < \frac{(3-g)(1-g)^2 \sqrt{-(3-g)\, g} - (1-g)^2 (9+g)\, g}{2(1+g)^2 g^2}.$$

Now we will find minimum value of this expression with respect to g. The limits on g for the inequality in (14.30) to hold are $-1 < g < 0$, clearly at these limits the expression tend to infinity. Hence we will examine the turning points with respect to g,

$$\frac{d}{dg}\left(\frac{(3-g)(1-g)^2\sqrt{-(3-g)g}-(1-g)^2(9+g)g}{2(1+g)^2 g^2}\right)\Bigg|_{g=g^*}=0$$

this gives a single turning point in the relevant domain:

$$g^* = \tfrac{3}{7}\left(5-4\sqrt{2}\right).$$

Setting $g = g^*$, inequality (14.30) becomes

$$\alpha < \frac{(3-g)(1-g)^2\sqrt{-(3-g)g}-(1-g)^2(9+g)g}{2(1+g)^2 g^2}\Bigg|_{g=g^*} = 8\left(7+5\sqrt{2}\right).$$

Therefore $\theta > 0$, for any α satisfying the condition

$$\alpha < 56 + 40\sqrt{2} \approx 112.6, \tag{14.31}$$

for any $\{g, \tilde{e}_2\} \in \mathcal{S}$. □

Finally we will present a theorem stating a condition on the fringe slopes ensuring that there exists a unique pure-strategy equilibrium to the game discussed at the beginning of this section.

Theorem 14.16. *There exists a unique equilibrium to the game represented by figure 14.5, so long as $a_2 < 112.6a_1$.*

Proof. If $a_2 < 112.6a_1$, then we have that $\alpha < 112.6$. Clearly this satisfies $\alpha < 1488$, hence from theorem 14.4, we have a quasi-concave revenue function for each generator. This implies that there exists a pure-strategy equilibrium to the game.

Furthermore, from lemma 14.15, we have that $\theta > 0$, for all $\{g, \tilde{e}_2\} \in \mathcal{S}$. This therefore implies that condition (14.27) is satisfied. As, from lemma 14.13, (14.26) is always satisfied, we therefore have from lemma 14.12 that any equilibrium is unique.

From above, there exists a unique equilibrium to the game. □

14.2.3 Examples

Now we will consider some examples. We will set $d_1 = 90$, $d_2 = 500$, $a_1 = 1$ and $r = 0.01$. For these parameters, we draw the best response curves for each of the two strategic generators as functions of the other generator. In figure 14.6 we draw these curves for $a_2 = 400$ and $a_2 = 100$.

When $a_2 = 400$ there are three equilibria, shown by black dots; this is possible because the ratio of the fringe slopes exceeds the bound of 112.6 given in condition (14.31).[7] On the other hand, when $a_1 = 100$ there is only one equilibrium; this could have been predicted since the ratio of the fringe slope now satisfies condition (14.31), which guarantees at most one equilibrium.

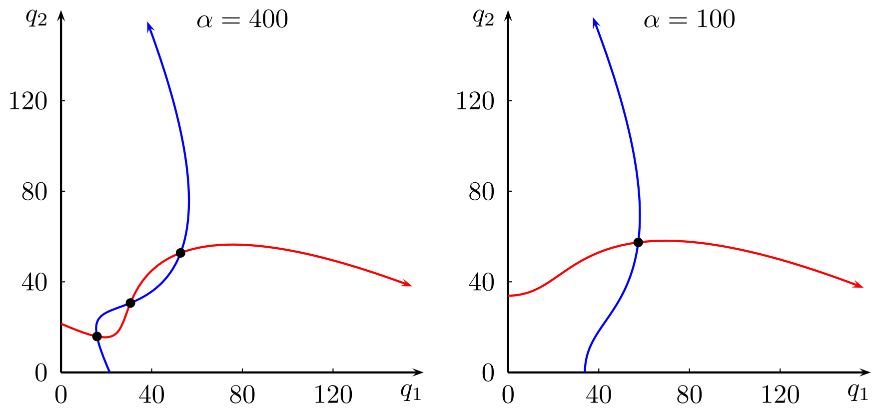

Figure 14.6: Best response curves (both generators at node 1).

It is also possible to have just two pure strategy equilibria. Consider the situation where $d_1 = 80$, $d_2 = 500$, $a_1 = 1$, $a_2 = 5000$ and $r = 0.01$. In this case, the ratio of the fringe slopes exceeds 1488.6, so by lemma 14.10 we are not guaranteed that each generator's profit function is unimodal. Figure 14.7 gives generator 1's profit as a function of its injection quantity q_1 for $q_2 = 24.72$; it demonstrates that using the stationary points for each generator does not yield a best reply curve when a generator's revenue is not quasi-concave; it gives the three turning points, two of which are local maxima and the other is a local minimum. By solving the joint KKT system, one would find three equilibria: $q_1 = q_2 = 10.05$, $q_1 = q_2 = 24.72$ and $q_1 = q_2 = 54.94$. However, the 2$^{\text{nd}}$ stationary point is not actually a global maximum for both players. For generator 1, $q_1 = 24.72$ is a local maximum, with the global maximum actually found at $q_1 = 3.31$.[8] The best reply curves in figure 14.8 show the two valid equilibria, with the invalid turning points in each generator's best reply curve represented by dotted lines. For the equilibrium with both generators injecting 10.05, the line is flowing towards node 1 and is incurring large losses; this leads to a high price at node 1. Neither generator is willing to inject more because the price at node 1 reduces rapidly as a function of its injection. On the other hand, for the equilibrium where both generators inject 54.94, there is a flow on the line from node 1 to node 2. Here each firm, given that the other firm

[7] Note that the second of the three equilbria is not stable. If there is any perturbation from this point, players will have incentive to move towards one of the other two equilibria.

[8] By symmetry, the same is true for generator 2.

is injecting a large amount of power, has no incentive to withhold because the price does not rise steeply enough. Thus we have two equilibria: one where firms have incentive to withhold so as to achieve a high price; and the other where firms inject large quantities, but have lower prices.

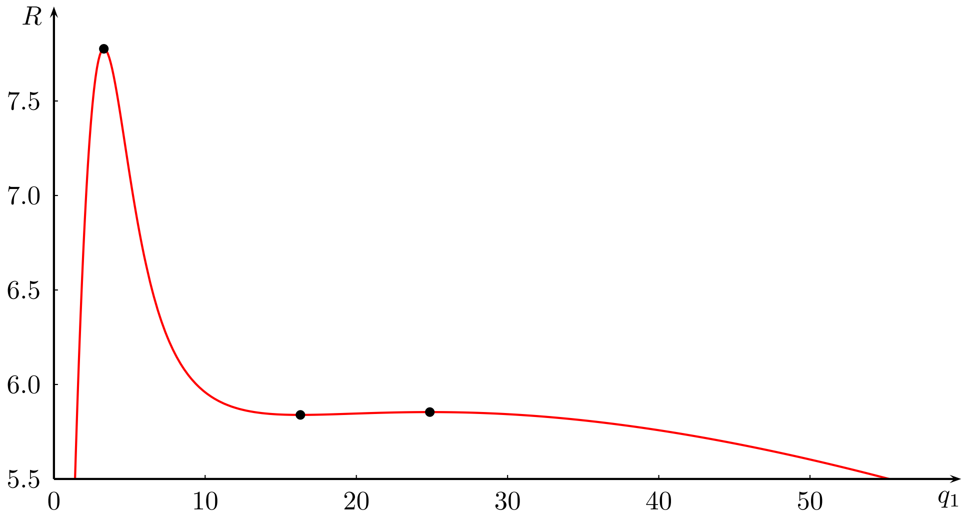

Figure 14.7: Generator 1's revenue function with multiple turning points.

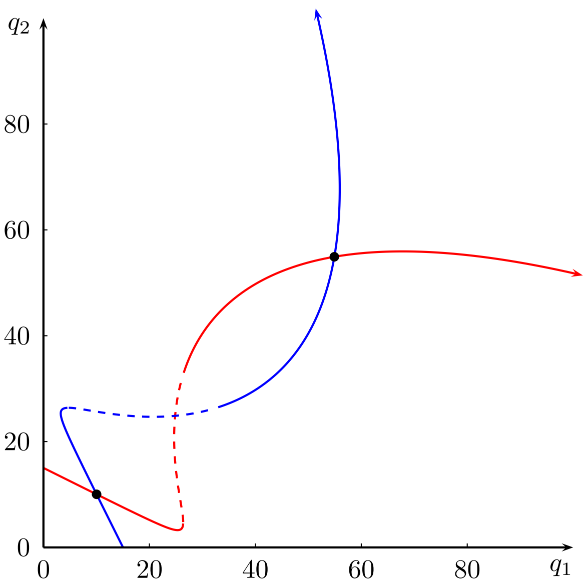

Figure 14.8: Jump in best response curve.

Part IV

Applications

Chapter 15

Capacity expansion

15.1 Capacity expansion planning

In this section, we formulate a mixed-integer stochastic program (see [61]) that seeks to plan transmission investment such that unconstrained Cournot competition between firms located in different regions is facilitated at least cost.

Decisions about transmission investment must be made several years before construction starts; this is because there is often a long lead-time in designing and building transmission lines, due land-use rights issue [100]. Moreover, in New Zealand any investment proposed by Transpower must satisfy the *grid investment test* (GIT), which is overseen by the Electricity Commission [28]. The GIT specifies many criteria that must be taken into account so as to ensure that the best transmission decisions are made. These include a comprehensive cost/benefit analysis and quantifiable competition benefits.

This work is particularly focused on the last point. The mixed-integer problem, below, aims to minimize the cost of the planned expansion, while facilitating unconstrained competition amongst the firms (ensuring existence of an unconstrained equilibrium). In general, due to the non-convexity of the set of line capacities ensuring that the unconstrained equilibrium exists, this problem is difficult to solve. We will construct a mathematical model of this problem, however for simplicity, we give an example of the model over a radial network with linear fringes. This simplification allows us to use the results of chapter 11, which ensure that the constraints yielding the unconstrained equilibrium form a convex polyhedral set.

We will first formulate the mixed-integer stochastic program.

Model formulation

Sets: $\mathcal{N}$ is the set the nodes in the network,

$\mathcal{A}$ is the set of arcs in the network.

Parameters: α_{ij} is the fixed cost of choosing to expand the capacity of arc ij,

β_{ij} is the marginal cost of expanding the capacity of arc ij,

L_{ij} is the lower-bound for expanding arc ij,

U_{ij} is the upper-bound for expanding arc ij,

K^0_{ij} is the initial capacity of arc ij,

d^0 is the initial vector of demands,

δ is the discount factor,

$\gamma(\omega)$ is the (uncertain) demand growth factor.

Variables: y^t_{ij} is the amount in MW by which arc ij is expanded,

z^t_{ij} is a boolean variable equal to 1 if arc ij is upgraded and 0 otherwise,

K^t_{ij} is the capacity of arc ij in period t ($t \neq 0$).

$$V^t\left(K^{t-1}, d\right) := \min \quad \sum_{ij\in\mathcal{A}} \left(\alpha_{ij} z^t_{ij} + \beta_{ij} y^t_{ij}\right) + E_\omega\left[\delta \times V\left(K^t, \gamma(\omega)\, d\right)\right]$$

$$\begin{aligned} \text{s.t.} \quad & y^t_{ij} - L_{ij} z^t_{ij} \geq 0 && \forall ij \in \mathcal{A} \\ & U_{ij} z^t_{ij} - y^t_{ij} \geq 0 && \forall ij \in \mathcal{A} \\ & K^t_{ij} - K^{t-1}_{ij} - y^t_{ij} = 0 && \forall ij \in \mathcal{A} \\ & z^t_{ij} \in \{0,1\} && \forall ij \in \mathcal{A} \\ & K^t \in \mathcal{S}^t(d), \end{aligned}$$

where $\mathcal{S}^t(d)$ is the set of line capacities ensuring that, at time period t, the unconstrained equilibrium exists for the vector of demands, d.

The above stochastic program, $V^1\left(K^0, d^0\right)$, aims to plan the timing of investments in the electricity grid at minimum cost, while ensuring that in all time periods the line capacities are sufficient to guarantee that no firm has incentive to deviate from the unconstrained equilibrium. For simplicity we have assumed that the random demand growth is independent between stages, however it is possible to incorporate a more sophisticated growth model (for example a Markov chain; see [55]).

The objective function of this problem is to minimize the expected cost of the planned expansion. The first two constraints ensure that if a line is being expanded then the expansion lies between some lower and upper bound. The third constraint updates the total capacity of lines joining a pair of nodes. The binary requirement on z ensures that a line is either upgraded (incurring a fixed cost) or not (incurring no cost). Finally, the last constraint ensures that the line capacities, K^t, for each time period, t, are sufficient to ensure that the unconstrained equilibrium exists.

We will now present a deterministic example of this model over a three-node radial network. This model will simply serve as an illustration of how the model may work.

Example 15.1. *Consider the three-node network, shown in figure 15.1; note that initially both lines have capacity* 40*. We have one strategic generator at each node with no costs and there is a fringe at each node, each submitting a linear supply function of the form:* $S(p) = p$ *(independent of the time period). Finally, the demand forecast for period* 1 *at nodes* 1 *and* 3 *is* 100*, and for node* 2 *is* 200*.*

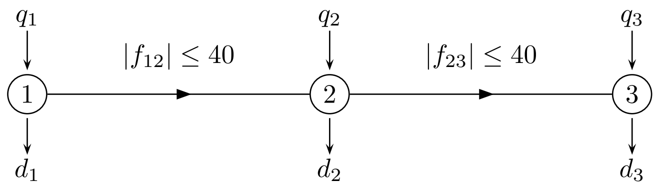

Figure 15.1: Expansion of line capacities in three node radial network.

We will first find the (symmetric) unconstrained equilibrium for a given period:

$$q^t = 100\gamma^t,$$

with a nodal price of:

$$\pi^t = \frac{100\gamma^t}{3}.$$

Hence the profit of each generator is:

$$\rho^t = \frac{10000\gamma^{2t}}{3}.$$

Therefore, from the inequalities given by the inequalities (11.32), we can compute the conditions ensuring that the unconstrained equilibrium exists for a given period; these are:

$$\begin{aligned}
K_{12}^t &\geq 100\gamma^t - 2\sqrt{\rho^t}, \\
K_{12}^t &\geq 300\gamma^t - q^t - 2\sqrt{2\rho^t}, \\
K_{23}^t &\geq 100\gamma^t - 2\sqrt{\rho^t}, \\
K_{23}^t &\geq 300\gamma^t - q^t - 2\sqrt{2\rho^t}, \\
K_{12}^t + K_{23}^t &\geq 200\gamma^t - 2\sqrt{\rho^t}.
\end{aligned}$$

In table 15.1 below, we give the minimum required line capacities for each period, from the above conditions.

Period	K_{12}	K_{23}	$K_{12} + K_{23}$
1	36.7	36.7	84.5
2	40.4	40.4	93.0
3	44.4	44.4	102.3
4	48.8	48.8	112.5
5	53.7	53.7	123.8
6	59.1	59.1	136.1
7	65.0	65.0	149.8

Table 15.1: Minimum required line capacities for unconstrained equilibrium in each period.

We solve the mixed-integer program, $V^1(K^0, d^0)$ over the seven period time horizon above for three discount rates. The results of this are shown in table 15.2 below.

	$\delta = 1.0$		$\delta = 0.9$		$\delta = 0.8$	
Period	K_{12}	K_{23}	K_{12}	K_{23}	K_{12}	K_{23}
0	40.0	40.0	40.0	40.0	40.0	40.0
1	65.0	84.7	44.5	40.0	44.5	40.0
2	65.0	84.7	44.5	57.8	44.5	48.5
3	65.0	84.7	66.0	57.8	53.8	48.5
4	65.0	84.7	66.0	57.8	53.8	58.7
5	65.0	84.7	66.0	57.8	65.1	58.7
6	65.0	84.7	66.0	83.7	65.1	71.1
7	65.0	84.7	66.0	83.7	78.7	71.1

Table 15.2: Capacity expansion plans for three discount rates.

From the results, we can see that the assumed discount rate greatly affects the investment plan. Specifically, for discount rates near 1.0 there is no incentive to delay investments, the results of this that the lines are upgraded one time, in period 1. For lower discount rates, we see that investments are smaller and more frequent. In fact, for $\delta = 0.8$ investments are made alternatively in each line for each period. Furthermore, note that the example is symmetric, however the resulting line expansion plans are not. This occurs, because the presence of fixed costs associated with a upgrading lines favours solutions with large investments in single lines, as opposed to smaller investments in multiple lines. Observe that an asymmetric solution to a symmetric problem means that there are, in fact, multiple solutions. That is, the planned upgrades on the lines could be swapped and yield the same cost. For larger networks there could be a set of optimal investment plans that all have the same costs. To decide on an specific plan, either a subjective judgement of the best plan could be made, or the costs of upgrading lines can be adjusted so that there are not multiple solutions.

This example illustrates how the concept of competitive capacity sets can be incorporated into multi-stage integer programming problems to minimize the cost of expanding the transmission network while ensuring that firms' strategic incentive to exercise market power is mitigated.

15.2 Example within NZ framework

In this section, will will consider the idea of a competitive capacity set in a New Zealand context. In order to be able to compute the minimum transmission capacities, an analytically tractable model of the New Zealand electricity market is required. A number of simplifications need to be made to achieve this; the grid must be simplified and assumptions must be made about the behaviour of the generators.

We construct a Cournot model over a greatly simplified transmission network to allow us to investigate the competition effects of changing the capacities of lines within an electricity grid. Specifically, we compute the conditions that ensure that the unconstrained Cournot equilibrium exists.

In this section, we first outline the modelling assumptions and describe the participants in the market. We then calculate the unconstrained equilibrium and characterize the conditions on the regional transfer capacities of the transmission network which ensure that this equilibrium is valid.

15.2.1 Modelling assumptions

Generation

For the electricity market model that we are using, we need to categorize generators as being either strategic, part of a competitive fringe, or assumed to offer in however much they can produce at $0 (e.g. wind / geothermal); the specific assumptions are outlined below.

Strategic Generators: We determined that there are six major strategic generators in the New Zealand market; the ownership and capacities of these generators are shown in table 15.3 below.

		Capacity (MW)		
Generator	Firm	2010	2015	2020
Huntly	Genesis	1413	1413	1413
P40	Genesis	50	50	50
E3P	Genesis	385	385	385
Waikato Hydro	Mighty River	776	776	776
Otahuhu B	Contact	404	404	404
Taranaki Combined Cycle	Contact	377	377	377
Clutha Hydro	Contact	700	790	790
Waitaki Hydro	Meridian	2718	3008	3183

Table 15.3: Expected generation capacity.

The short-run marginal costs (excluding any possible carbon charge) for gas and coal plants, were computed based on the heat rates and fuel costs from [29]. These are shown in tables 15.4 and 15.5 below.

Fuel Type	2010	2015	2020
Gas	6.5	8.0	10.0
Coal	4.0	4.0	4.0

Table 15.4: Expected fuel costs ($ / GJ).

For all the hydro generators, we assume that at any given time of year there is a constant water value; this represents the opportunity cost of water.

Wind and geothermal: Wind farms and geothermal plants are assumed to have zero short-run marginal costs and have no ability/incentive to act strategically. We therefore subtract their production directly from the demand. In the case of wind farms we assume that on average they are running at 40% of capacity and that geothermals are running at 100% of capacity. Note that if a different utilization of wind power were used, the analysis would need to be repeated.

Generator	Fuel Type	Heat Rate
Huntly	Coal	10.50
P40	Gas	9.50
E3P	Gas	7.08
Otahuhu B	Gas	7.05
Taranaki Combined Cycle	Gas	7.30

Table 15.5: Thermal generator heat rates (GJ / MWh).

Competitive fringe: We assume that all remaining generators form part of a competitive fringe that creates the price elasticity at each node. We wish to explore how the incentives of the strategy firms change as the relative slopes of the competitive fringe offers change. To do this we define the offer curve of the competitive fringe in the north and south to be $S_N(p) = 15\alpha p$ and $S_S(p) = 15\,(1-\alpha)\,p$, respectively. As we vary α between 0 and 1, we change the relative price elasticity at the two nodes, but keep the combined (uncongested) offer curve unchanged.

Transmission grid

In this work, we simplify the entire New Zealand network to a two-node grid. We are specifically interested in the size of the Otahuhu–Whakamaru line necessary to ensure that an unconstrained equilibrium exists. However, note that in order for this simplification of the network to be valid, we must assume that all other lines are large enough that they would not constrain regardless of the actions of the generators.

Due to the simplification of the network, we must reassign each of the generators to one of the two nodes; these assignments have simply been determined based on the physical locations of the generators. These are shown in table 15.6 below.

Finally, we aggregate demand forecasts and expected wind and geothermal generation levels for each of the two nodes; these give peak demand levels and are shown in tables 15.7 and 15.8, respectively.

For a contract level of 40% of plant capacity, table 15.9 gives the unconstrained equilibria for the three years, for varying opportunity costs of water.

From this table we can see that the equilibrium price increases with water value, and with time. In 2010 we have an equilibrium peak price of \$82.57 with a \$0 opportunity cost for water; this

Generator	Node
Huntly	OTA
P40	OTA
E3P	OTA
Waikato Hydro	WKM
Otahuhu B	OTA
Taranaki Combined Cycle	WKM
Clutha Hydro	WKM
Waitaki Hydro	WKM

Table 15.6: Generator locations.

Node	2010	2015	2020
OTA	2288	2631	2987
WKM	5005	5447	5830

Table 15.7: Aggregated demand forecasts (MW).

increases to \$133.41 in 2020 with an opportunity cost for water of \$80. These results are expected, because the fuel costs are increasing over time and so is the demand. Also note that the generation at, for example, Otahuhu does not strictly increase with time. This occurs because, although the net demand is increasing over time, the fuel costs are also increasing.

In this work, we are particularly interested in the incentives of firms to deviate from these equilibria and how these incentives change over time or due to changes in water value or contract levels. We will first examine the incentives of firms to deviate from the unconstrained equilibrium in 2010 with contract levels fixed at 40% of the plants' capacities and a \$40 water value. The minimum line capacity, K^*, to ensure that there is no incentive to deviate is shown, for each firm, in figure 15.2 below. In each of the graphs in this figure we also plot (the absolute value of) the equilibrium flow on the line to give perspective on the extra capacity that must be provided to ensure that there exists an unconstrained equilibrium.

From the figure, above, we can see that Genesis has the most incentive to congest the line for $\alpha < 0.45$, whereas for $\alpha > 0.45$ Meridian sets the minimum line capacity. Note that due to Contact owning plants of both sides of the line, there is incentive to deviate by congesting the line towards Otahuhu for small values of α and conversely congesting the line towards Whakamaru for

Node	2010	2015	2020
OTA	10	70	70
WKM	701	1081	1081

Table 15.8: Aggregated expected wind and geothermal generation (MW).

Water Value	\$0 / MWh			\$40 / MWh			\$80 / MWh		
Generator	2010	2015	2020	2010	2015	2020	2010	2015	2020
Huntly	1000.0	1000.0	1000.0	1000.0	1000.0	1000.0	1000.0	1000.0	1000.0
P40	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
E3P	122.3	84.65	148.9	272.3	242.15	304.8	385.0	385.0	385.0
Waikato Hydro	776.0	776.0	776.0	776.0	776.0	776.0	776.0	776.0	776.0
Otahuhu B	404.0	352.6	404.0	404.0	404.0	404.0	404.0	404.0	404.0
Taranaki CC	15.3	0.0	0.0	165.3	76.15	132.2	377.0	377.0	377.0
Clutha Hydro	700.0	790.0	790.0	700.0	790.0	790.0	325.2	425.5	648.5
Waitaki Hydro	2325.8	2563.5	2910.1	1875.8	2120.9	2466.0	1601	1781.3	2074.3
Price	82.57	90.68	109.13	92.57	101.18	119.52	114.25	118.54	133.41

Table 15.9: Equilibrium outcomes.

large values of α.

These results show that the size of the line needed depends heavily on the relative fringe slopes. Of particular importance is that the generator with incentive to deviate changes depending on the price elasticity at its node. Again we have set the contract level to be 40% of generation capacity and the water value to \$40 per megawatt hour; in figure 15.3 we see that for $\alpha < 0.4$ the required size of the line to prevent Huntly having incentive to deviate increases substantially between 2010 and 2020. Whereas for α between 0.6 and 0.9, the required size of the line to prevent Meridian from withhold actually decreases with time. These results reflect future changes in relative generation capacity and demand in both islands, as well as fuel costs. For a given value of α the equilibrium flow on the line is increasing from Whakamaru to Otahuhu (or decreasing from Otahuhu to Whakamaru) from 2010 to 2020; this is because although there is significant demand growth in Otahuhu, most of the increase in generation is located South of Whakamaru. This means (for low values of α) the opportunity to exercise market power is increasing. In early 2010, Transpower began construction on a project to upgrade the transmission grid connecting Otahuhu and Whakmaru, building a new

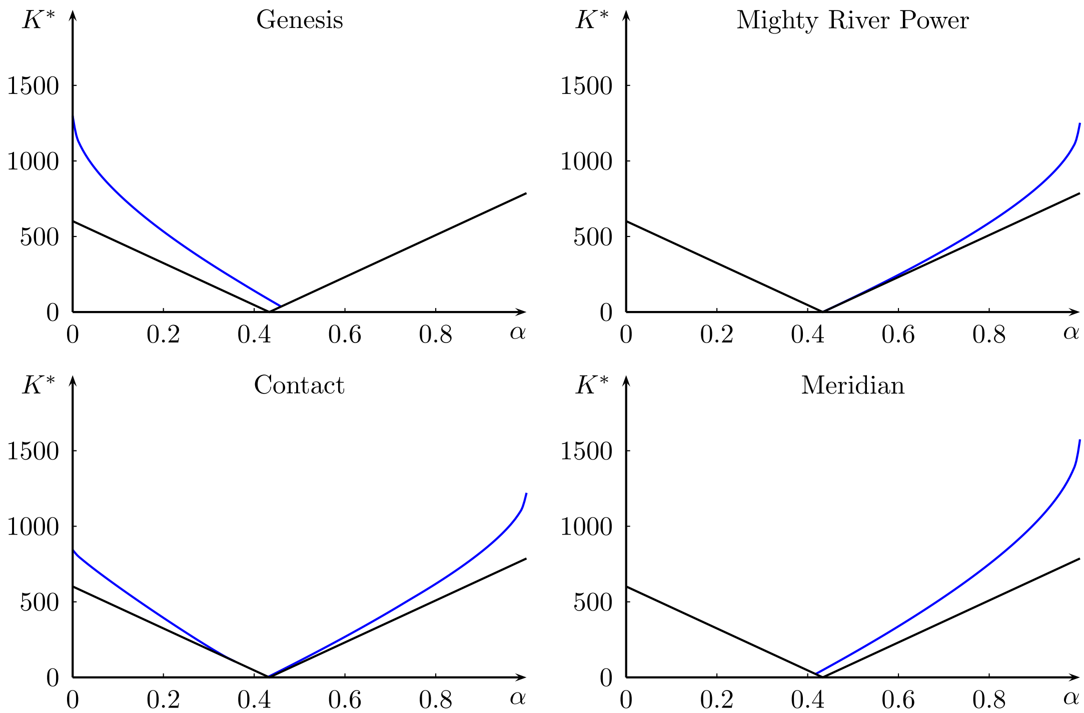

Figure 15.2: Incentive to deviate for each firm.

220kV line which could later be upgraded to 400kV [92]. This project aims to ensure the security of supply to the upper North Island, by enabling more power generated South of Auckland to be delivered [91]. The competition benefits of this project may be substantial, particularly for low values of α (relatively expensive fringe generation at Auckland), since it will reduce the incentive and ability of firms to withhold capacity and congest the line.

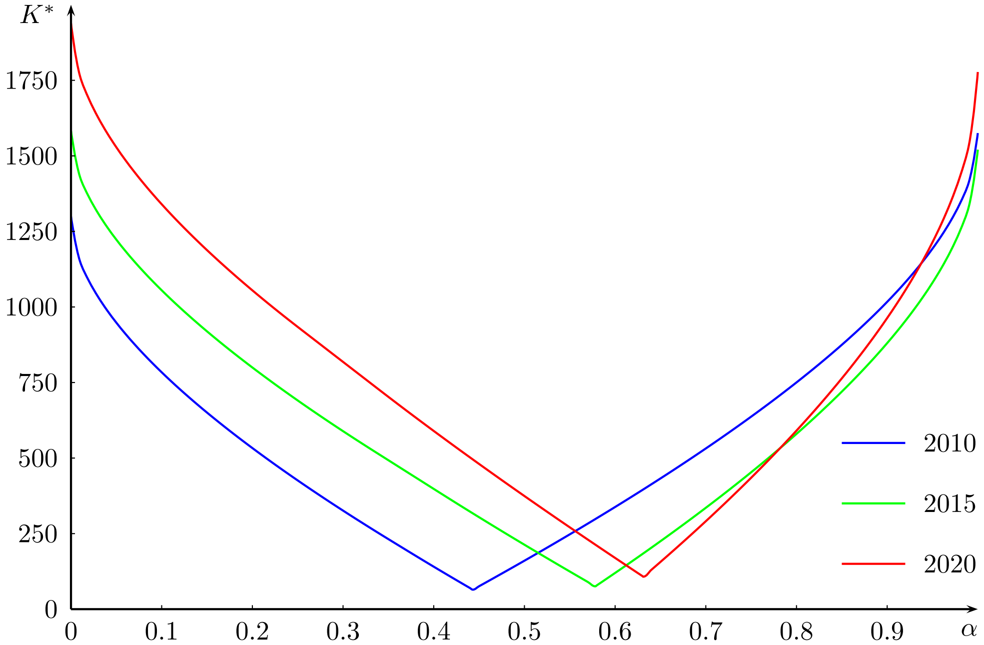

Figure 15.3: Minimum capacity supporting unconstrained equilibrium (2010-2020).

Chapter 16

Policy

16.1 Carbon emissions

Over the past few years, the climate change debate has moved away from whether global warming has been caused by anthropomorphic CO_2 emissions, and is now focusing on the design of policy instruments to reduce these emissions. The favoured options in most countries are to either introduce a tax on activities that generate emissions, or to form an emissions trading (cap-and-trade) scheme, in which emitters buy credits from those who offset their emissions, as described by Green in [40]. In a cap-and-trade scheme, one is guaranteed (or at least hopes for) a reduction of emissions to the cap. Whereas with a tax, one expects emissions to be reduced, as they incur higher costs, but the actual reductions are not specified, as discussed by Metcalf in [67].

In this chapter, we examine the effect a carbon charge may have on an electricity market under various assumptions about the market. Specifically, using a perfectly competitive model with convex generator costs, we prove that CO_2 charges (either from a tax or from the need to buy permits from outside the system) will always reduce emissions. We then consider the impact of a carbon charge in the Cournot framework. Here we show that if one confines attention to the electricity system in isolation, then carbon charges may lead to an *increase* in CO_2 emissions by the electricity generators.[1] This result highlights the need for care to be taken when predicting outcomes of such policy instruments when they are applied to electricity systems.

We will first give a brief description of the measures that are being implemented to reduce emissions.

[1]This example has appeared in [24]. To see the potential impact of carbon charges in a New Zealand context see Philpott and Downward [79].

16.1.1 Carbon taxes or emissions trading schemes

There are two policy instruments that are currently being implemented to try to reduce emissions of CO_2 and other greenhouse gasses. These are a carbon tax or an emissions trading scheme, as discussed by Hepburn in [45].

Carbon taxes

Under a carbon tax, the price of carbon emissions is set by the policy makers. This type of policy is referred to as a price instrument, as the price of emitting a tonne of carbon is defined outside the market. Therefore the various sectors receive a direct price incentive to emit less carbon dioxide. This type of measure is not guaranteed to reduce the emissions, but it disincentivises carbon-intensive activity and encourages alternatives.

Cap and trade

Conversely, a cap and trade or emissions trading scheme is a quantity instrument. Here the policy makers issue credits for a set amount of emissions. These permits are then traded amongst the affected companies, and a price of carbon is determined by the market. An emissions trading scheme will reduce the total emissions, however if the emissions target is too low the price of carbon may become extremely high.

16.1.2 Modelling

We will now consider the impact that a carbon charge may have on electricity prices and emissions. In this work we consider the electricity market in isolation and take the carbon charge to be some fixed price, whether it be from a tax on emissions or the need to purchase credits.

To model this, we denote the emissions (tonnes of CO_2) per MWh for generator g: γ_g. Therefore the total emissions is given by

$$E = \sum_{g \in \mathcal{G}} \gamma_g q_g, \tag{16.1}$$

where $\mathcal{G}$ is the set of all generators and q_g is the generation level of generator g.

We define the carbon charge (in \$ per tonne of CO_2) to be α, which means that the marginal cost of generator g increases by $\alpha\gamma_g$. Finally we will assume that demand is given by an invertible decreasing function of price: $D(p)$, which gives an inverse demand curve of $p = D^{-1}(d)$.

We will first examine this in the context of a perfectly competitive model and then in a Cournot model with generators behaving strategically.

Perfect competition

Here we examine the effect that the imposition of a carbon tax may have on emissions and prices under the assumption of perfect competition at a single node. In what follows we will prove that, in this framework, any positive charge on emissions will decrease the total emissions.

To simplify this proof, we will approximate the generators' capacities in their cost functions instead of placing explicit bounds on their production. We will denote the continuously differentiable increasing convex cost functions for each generator, g, by $C_g(q_g)$.

The dispatch problem is therefore given as:

$$\begin{array}{lll} \max & F(q,\alpha) = \int_0^d D^{-1}(\delta)\,.d\delta - \sum\limits_{g\in\mathcal{G}} \left(C(q_g) + \alpha\gamma_g q_g\right) & \\ & & \\ \text{s.t.} & d - \sum\limits_{g\in\mathcal{G}} q_g = 0 & [\pi], \end{array}$$

here the objective is to maximize total welfare, while meeting the demand. First we prove a theorem that guarantees that emissions will not increase due to an increase in carbon charge under the perfect competition assumption.

Theorem 16.1. *In a perfectly competitive market over a lossless network, the total emissions is non-increasing as the carbon charge, α, is increased.*

Proof. Suppose we have an optimal solution to the dispatch problem $(\bar{q},\bar{d})$ for $\alpha = \bar{\alpha}$. If α is then increased to $\hat{\alpha}$, then since α is not present in the constraints of the dispatch problem, $(\bar{q},\bar{d})$ is feasible for the dispatch with a higher carbon charge.

If the optimal solution to the dispatch problem for $\alpha = \hat{\alpha}$ is $(\hat{q},\hat{d})$ then we have the following inequalities:

$$F(\bar{q},\bar{\alpha}) \geq F(\hat{q},\bar{\alpha}), \tag{16.2}$$

$$F(\hat{q},\hat{\alpha}) \geq F(\bar{q},\hat{\alpha}). \tag{16.3}$$

Let us define the total emissions as a function of q to be:

$$E(q) = \sum_{g\in\mathcal{G}} \gamma_g q_g.$$

Note that for any vectors (q, d), we have

$$F(q, \hat{\alpha}) = F(q, \bar{\alpha}) - (\hat{\alpha} - \bar{\alpha})E(q).$$

Hence inequality (16.3) can be written as

$$F(\hat{q}, \bar{\alpha}) - (\hat{\alpha} - \bar{\alpha})E(\hat{q}) \geq F(\bar{q}, \bar{\alpha}) - (\hat{\alpha} - \bar{\alpha})E(\bar{q}).$$

Subtracting the above inequality from inequality (16.2) gives

$$(\hat{\alpha} - \bar{\alpha})E(\hat{q}) \leq (\hat{\alpha} - \bar{\alpha})E(\bar{q}),$$

Finally, since $\hat{\alpha} > \bar{\alpha}$ we have

$$E(\hat{q}) \leq E(\bar{q}),$$

where the left-hand side of the above equation is equal to the emissions with a tax of $\hat{\alpha}$ and the right-hand side is equal to the emissions with a tax of $\bar{\alpha}$; therefore the emissions are non-increasing with an increased tax. □

In the next section, we will consider the effect of a carbon charge on emissions when generators are acting strategically.

Cournot model

Now we will consider the implications of a carbon tax in a Cournot paradigm; where generators aim to maximize their profits. In this section, we will prove that, in the case of constant marginal costs and a linear demand curve at a single node, an increased charge on emissions will lead to an emissions reduction. However, if generators are are located at different nodes in a network, it is possible that the presence of transmission congestion may mean that an increase in a charge on carbon can lead to an increase in emissions. First we will consider the single node situation.

Single Node We will define the demand curve at the node to be

$$D(p) = d - ap.$$

Furthermore the cost for each generator, g is given by

$$C_g(q_g) = u_g q_g.$$

From chapter 9, we know that the Cournot equilibrium at a single node is unique and is given by the solution to the following LCP

$$0 \leq q_g + \sum_{h \in \mathcal{G}} q_h - d + a\left(u_g + \alpha\gamma_g\right) + \mu_g \quad \perp \quad q_g \geq 0, \quad \forall g \in \mathcal{G},$$
$$0 \leq V_g - q_g \quad \perp \quad \mu_g \geq 0, \quad \forall g \in \mathcal{G}.$$

At the solution to the above LCP, we define: $g \in \mathcal{G}^-$ if and only if $q_g = 0$; $g \in \mathcal{G}^+$ if and only if $q_g = V_g$; and $g \in \mathcal{G}^U$ if and only if $0 < q_g < V_g$. Therefore the solution to the following system of equations gives the quantities for the generators not at bound

$$q_g + \sum_{h \in \mathcal{G}} q_h - d + a\left(u_g + \alpha\gamma_g\right) = 0, \quad \forall g \in \mathcal{G}^U.$$

If we define N to equal the number of generators in $\mathcal{G}^U$, then this yields the following solution:

$$q_g = \frac{d - \sum\limits_{h \in \mathcal{G}^+} V_h + a \sum\limits_{h \in \mathcal{G}^U} \left(u_h + \alpha\gamma_h\right)}{N+1} - a\left(u_g + \alpha\gamma_g\right), \quad \forall g \in \mathcal{G}^U.$$

We can therefore compute the total emissions at equilibrium to be

$$\begin{aligned} E &= \sum_{g \in \mathcal{G}} \gamma_g q_g \\ &= \sum_{g \in \mathcal{G}^U} \gamma_g \left[\frac{d - \sum\limits_{h \in \mathcal{G}^+} V_h + a \sum\limits_{h \in \mathcal{G}^U} \left(u_h + \alpha\gamma_h\right)}{N+1} - a\left(c_g + \alpha\gamma_g\right)\right] + \sum_{g \in \mathcal{G}^+} \gamma_g V_g. \end{aligned} \tag{16.4}$$

The following theorem states that the total emissions, E, are non-increasing as the carbon charge, α, is increased.

Theorem 16.2. *The total emissions, E is a non-increasing function of the carbon charge, α.*

Proof. Consider the expression for total emissions (for an assumed set of generators not at bound) given in equation (16.4). The derivative of this with respect to α is given by:

$$\frac{dE}{d\alpha} = \frac{a \sum\limits_{g \in \mathcal{G}^U} \sum\limits_{h \in \mathcal{G}^U} \gamma_h \gamma_g}{N+1} - a \sum_{g \in \mathcal{G}^U} {\gamma_g}^2.$$

This can be rewritten as:

$$\begin{aligned} \frac{dE}{d\alpha} &= \frac{a}{N+1}\left[\sum_{g \in \mathcal{G}^U} \sum_{h \in \mathcal{G}^U} \gamma_g \gamma_h - (N+1) \sum_{g \in \mathcal{G}^U} {\gamma_g}^2\right] \\ &= \frac{a}{N+1}\left[\left(\sum_{g \in \mathcal{G}^U} \gamma_g\right)^2 - (N+1) \sum_{g \in \mathcal{G}^U} {\gamma_g}^2\right]. \end{aligned}$$

Suppose that $\sum\limits_{g \in \mathcal{G}^U} \gamma_g = T$. It can be shown that the minimum value of $\sum\limits_{g \in \mathcal{G}^U} {\gamma_g}^2$ is attained at $\gamma_g = \frac{T}{N}$. Therefore the maximum value of $\frac{dE}{d\alpha}$ is given by:

$$\frac{dE}{d\alpha} \leq \frac{a}{N+1}\left[T^2 - (N+1) \sum_{g \in \mathcal{G}^U} \left(\frac{T}{N}\right)^2\right] \tag{16.5}$$

$$= \frac{a}{N+1}\left[T^2 - \frac{(N+1)}{N} T^2\right] \tag{16.6}$$

$$= -\frac{aT^2}{N(N+1)} \leq 0. \tag{16.7}$$

Hence, for any particular equilibrium, the emissions are decreasing (not increasing) with α. As the solution to this LCP is equivalent to a strictly convex quadratic program, we know that the solution is continuous in the parameter α (see [19]). This therefore implies that the total emissions at equilibrium are a non-increasing function of α. □

We will now provide a counter-example to this result in the context of a two-node network with transmission constraints.

Two-node example with carbon tax It has been demonstrated that the reduction of market power can lead to an increase in negative social or environmental outcomes (see for example [60, 39, 33]). Moreover, Buchanan in [15] shows, in a market with a monopolist firm, that in the absence of pollution, increasing the output of the firm increases social welfare; however, when pollution is a concern, it may be beneficial to reduce output. Thus it is unclear a priori whether a tax on pollution will increase or decrease welfare. The counter-intuitive result that we discuss in this paper occurs because, rather than solely discouraging fossil-fuel use, the imposition of a carbon charge can also reduce market power. In fact, as we will demonstrate, the reduction of market power may be the dominant effect, leading to an increase in emissions.

A diagram of the electricity system which we consider is shown in Figure 16.1. Here, there are two nodes connected by a line of capacity K. At each node, i, there is a generator, with a constant marginal fuel cost, c_i, injecting a quantity of electricity, q_i. The nodal prices are calculated from inverse demand curves given by $y_i = d_i - a_i p_i$, where d_i and a_i are positive constants and y_i is the satisfied demand at node i. We shall assume that generator 1 is a coal-fired power plant and generator 2 is a gas-turbine plant, and their emissions per megawatt are γ_1 and γ_2 tonnes of CO_2 respectively. The tax charged to generators is $\$\alpha$ per tonne of CO_2 emitted. This means that the tax which generator i would pay is $\$\alpha\gamma_i$ per megawatt. We model the system as a one-shot Cournot game in which the generators are assumed to act with full rationality, maximizing their profits. Cournot models of electricity markets have some potential drawbacks, as shown by [12], however they are still widely used in modelling competition in electricity markets, (e.g. [68, 70]). In our model the optimal dispatch of electricity is determined by a system operator (ISO) who determines the prices and power flow on the line, with the objective to maximize total welfare.

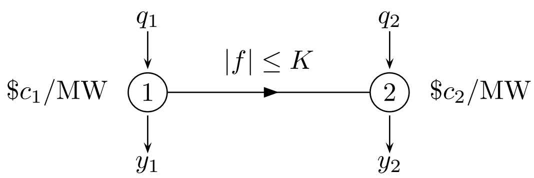

Figure 16.1: Two node electricity network.

The formulation for this problem is shown below.

$$\max \quad \tfrac{1}{a_1}\left(d_1 y_1 - \tfrac{1}{2}{y_1}^2\right) + \tfrac{1}{a_2}\left(d_2 y_2 - \tfrac{1}{2}{y_2}^2\right)$$

$$\begin{aligned} \text{s.t.} \quad & y_1 + f = q_1 \\ & y_2 - f = q_2 \\ & f \leq K \\ & -f \leq K \end{aligned}$$

Now consider an example in which $K = 125$MW and the inverse demand curves at the nodes are, $y_1 = 400 - 3.2p_1$ and $y_2 = 500 - 2p_2$, respectively. Let $\gamma_1 = 1.0$ and $\gamma_2 = 0.4$; this corresponds to a higher CO_2 charge per megawatt on the coal plant. We will compare the Cournot equilibrium solutions for two situations; one without a carbon tax ($\alpha = 0$), and one with a carbon tax ($\alpha = 26$).

We will first consider the case where there is no CO_2 charge. Here, as the marginal cost of gas is higher than the cost of coal, the gas plant has an incentive to withhold electricity, rather than try to compete with the less expensive coal plant. This leads to an equilibrium where both generators act as local monopolists, with the line congested towards node 2. The equilibrium result is shown in Figure 16.2. The nodal prices are $p_1 = \$102.03$ and $p_2 = \$118.75$, these prices differ due to the line congestion.

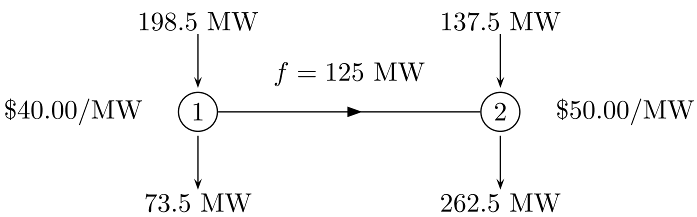

Figure 16.2: No carbon charge.

However, if the generators are subject to a CO_2 charge, the effective marginal cost of generation for the gas plant is now lower than that of the coal plant. This means that gas plant behaves in a more competitive manner, leading to an equilibrium in which the line is no longer congested;

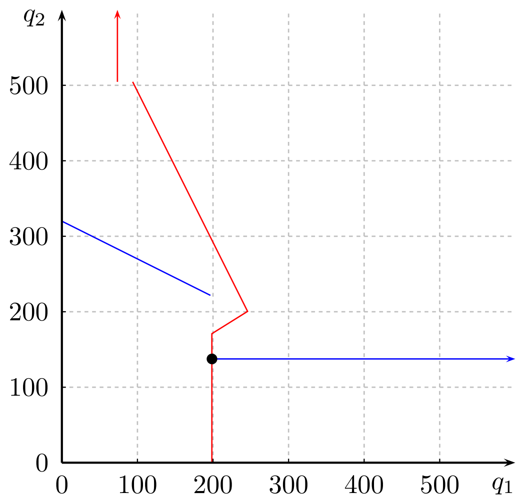

Figure 16.3: Best response correspondences: $\alpha = 0$.

this is shown in Figure 16.4. As there is no congestion, the prices at both nodes are the same, $p_1 = p_2 = \$99.83$; this equilibrium is equivalent to a single node duopoly equilibrium.

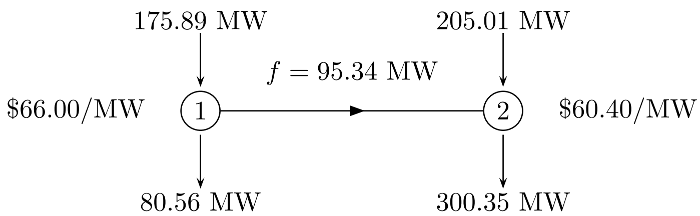

Figure 16.4: With carbon charge.

Looking at the welfare for these two situations, we see in Table 16.1 that the producer welfare drops considerably when the carbon tax is applied. This is because the tax has increased their marginal costs. The consumer welfare, however, has increased; this is due to the generators changing from acting as monopolists to competing in a duopoly, leading to lower electricity prices. The congestion rent has dropped to zero, because the line joining the nodes is no longer congested, after the tax is applied.

	$\alpha = 0$	$\alpha = 26$
Producer Welfare	\$21,766	\$14,032
Consumer Welfare	\$18,071	\$23,566
Congestion Rents	\$2,090	\$0

Table 16.1: Comparison of welfare.

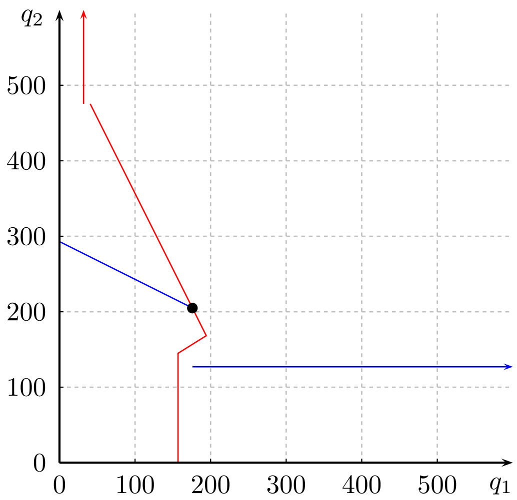

Figure 16.5: Best response correspondences: $\alpha = 26$.

Table 16.2 shows that the total emission of CO_2 increases when the carbon charge is applied. The CO_2 emissions increase 1.74% at equilibrium when the tax is introduced. The total amount of tax collected in the equilibrium is \$6,705.

	$\alpha = 0$	$\alpha = 26$
CO_2 Emissions	253.5 t	257.9 t
Tax Collected	\$0	\$6,705

Table 16.2: Comparison of emissions and tax revenue.

As the carbon tax is just a transfer of wealth from producers to the government, we will include the collected tax in the total welfare. Table 16.3 shows that the total welfare has in fact increased after the tax has been introduced.

	$\alpha = 0$	$\alpha = 26$
Total Welfare	\$41,927	\$44,304

Table 16.3: Total welfare.

To summarize this result, the application of a tax on emissions of carbon dioxide has increased generation costs, which predictably decreases the profits of generators. However, it has unexpectedly lowered electricity prices, increasing the welfare accruing to consumers, and it has eliminated congestion in the electricity grid. The most interesting result is that the total emissions from the generators have increased.

The cause of this apparent paradox is the combination of the strategic behaviour of the generators and the transmission constraint. Together these conspire to produce two different Nash equilibria, with the more competitive one arising from a regime of carbon charges. In the absence of strategic behaviour, or without a transmission constraint, this paradox would not occur. The example highlights the need to account for these effects when designing CO_2 policy instruments for constrained electricity transmission systems.

However, one must be careful when interpreting this result. If we consider the welfare associated with the emissions, assuming that the carbon charge is set at the marginal social damages, we can calculate that the increase in emissions of 4.4 tonnes per hour reduces welfare by only \$114.4 per hour; this reduction in welfare is less than the increase in welfare from the alleviation of congestion in the network, leading to an overall increase in welfare.

Mixed strategy equilibria So far we have computed equilibria for two carbon charge levels: first without a carbon charge, and then with a carbon charge of \$26 / tonne. With $\alpha = 0$, we have the line congested at equilibrium, whereas with $\alpha = 26$ there is no congestion on the line at equilibrium. It can be shown that for values of α between 2.38 and 25.93 no pure-strategy equilibrium exists. However, over this range of α it is possible to compute a mixed-strategy equilibrium of the type discussed earlier in section 13.1. Here the gas plant mixes over two strategies, while the coal plant has a single strategy. The expected injections of each plant are shown in figure 16.7, for α between 0 and 40. Then in figure 16.8 we show the expected total emissions as a function of the carbon charge. From this figure, we can see that as α increases from 0, initially the total emissions decrease; this is because the congested pure-strategy equilibrium is still valid for small α, and the charge has the effect of reducing output of both generators. However, as the carbon charge continues to increase the congested equilibrium is no longer valid, and due to the high carbon content of coal, the gas plant (at equilibrium) chooses a mixed-strategy that, in expectation, rises with α; this increase in generation from the gas plant offsets the reduction in coal usage, leading to an increase in expected emissions. Finally for α greater than 25.93 we once again have a pure-strategy equilibrium (this time unconstrained) and increasing the carbon charge further leads to reduced emission levels. Note that for carbon charges between \$14.7 and \$27.8 / tonne, the total expected emissions from the gas and coal plants at equilibrium in fact exceed the emissions without a charge on emissions.

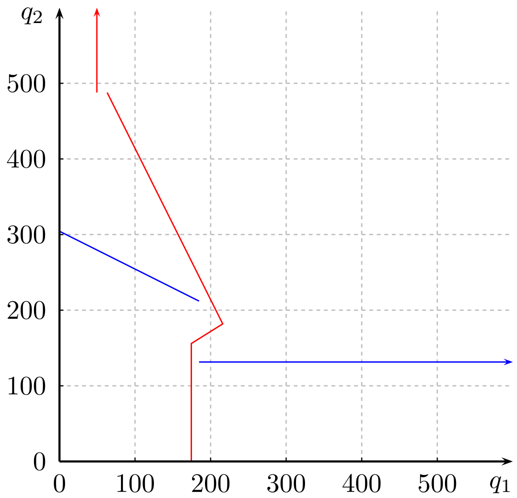

Figure 16.6: Best response curves: $\alpha = 15$.

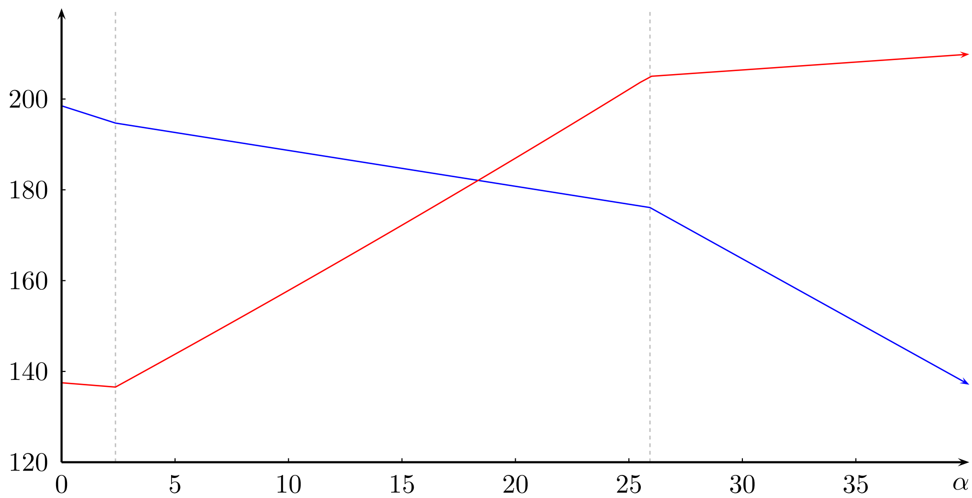

Figure 16.7: Injection quantities at equilibrium as a function of carbon charge, α.

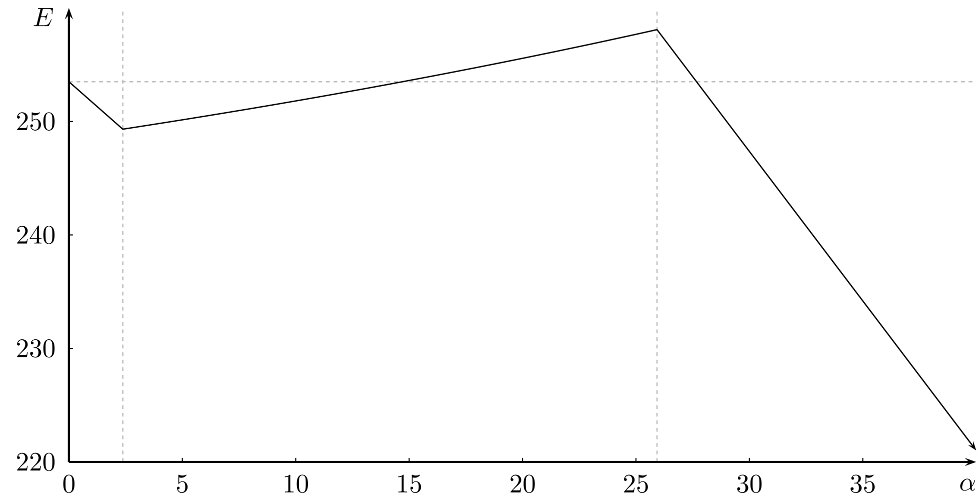

Figure 16.8: Carbon emissions at equilibrium as a function of carbon charge, α.

Chapter 17

Future Research Directions

In this thesis, we have examined how the presence of a transmission network can alter the incentives of firms competing in the electricity wholesale market. We have explored the effects of loops, losses and line capacities, firstly on the economic dispatch problem and then in the context of Cournot games. In this chapter we will discuss some research questions which have not been explored.

An interesting question that has been left unanswered by this thesis is how firms may behave in a constrained network with both loops and losses. There are, however, significant challenges to analyzing this, due to the non-convex nature of the dispatch problem. Chapter 6 mentions a few of the counter-intuitive incentives firms may face when prices become negative.

In this work, for the purposes of the analysis, we have clearly delineated *radial* and *looped* networks. However, networks may be mostly radial with a couple of loops. It would be interesting to explore whether the results relating to properties of the dispatch problem for radial networks, can be relaxed to allow for the presence of loops.[1] If such a relaxation were possible, it may cut down on the number of KKT regimes that must be enumerated to compute non-deviation sets associated with general networks.

One potential weakness of this work is that it focuses solely on Cournot competition. Cournot competition was chosen, mainly due its comparative tractability over supply function models. However, note that the results of chapters 4–6 do not assume a Cournot paradigm, and apply equally in a supply function setting (albeit without demand uncertainty). An interesting extension could be apply the results from chapter 5 to games where firms are offering supply functions, rather than quantities.

[1] For example, a simple relaxation would be to allow for loops without line capacities.

Finally, the capacity expansion planning stochastic program could be improved upon in a number of ways. First, it could be extended to allow for new lines to be built (rather than solely upgrading old lines); this would require that line impedance data be incorporated as part of the decision (due to new loops potentially being created). Another possible extension would be to allow for the equilibrium to be congested and incorporate the social welfare associated with any equilibrium into the objective, instead of *gold-plating* the network by ensuring the existence of the uncongested equilibrium. This extension is difficult, as there may not always exist pure strategy equilibria, and mixed strategy equilibria (and their corresponding welfare) are extremely difficult to compute.

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