


# Attaining Fairness in Communication for Omniscience <sup>†</sup>

Ni Ding <sup>1,\*</sup> , Parastoo Sadeghi <sup>2</sup>, David Smith <sup>3</sup> and Thierry Rakotoarivelo <sup>3</sup><sup>1</sup> School of Computing and Information Systems, The University of Melbourne, Parkville, VIC 3010, Australia<sup>2</sup> School of Engineering and Information Technology, University of New South Wales, Northcott Dr, Campbell, ACT 2612, Australia; p.sadeghi@unsw.edu.au<sup>3</sup> Data61, CSIRO, Level 5/13 Garden St, Eveleigh, NSW 2015, Australia; david.smith@data61.csiro.au (D.S.); thierry.rakotoarivelo@data61.csiro.au (T.R.)

\* Correspondence: ni.ding@unimelb.edu.au

<sup>†</sup> This paper is an extended version of our paper published in the 2016 IEEE International Symposium on Information Theory (ISIT), Barcelona, Spain, 10–15 July 2016, the 2015 22nd International Conference on Telecommunications (ICT), Sydney, Australia, 27–29 April 2015, the 2016 Australian Communications Theory Workshop (AusCTW), Melbourne, Australia, 20–22 January 2016 and the 2018 IEEE International Symposium on Information Theory (ISIT), Vail, CO, USA, 17–22 June 2018.

**Abstract:** This paper studies how to attain fairness in communication for omniscience that models the multi-terminal compress sensing problem and the coded cooperative data exchange problem where a set of users exchange their observations of a discrete multiple random source to attain omniscience—the state that all users recover the entire source. The optimal rate region containing all source coding rate vectors that achieve omniscience with the minimum sum rate is shown to coincide with the core (the solution set) of a coalitional game. Two game-theoretic fairness solutions are studied: the Shapley value and the egalitarian solution. It is shown that the Shapley value assigns each user the source coding rate measured by their remaining information of the multiple source given the common randomness that is shared by all users, while the egalitarian solution simply distributes the rates as evenly as possible in the core. To avoid the exponentially growing complexity of obtaining the Shapley value, a polynomial-time approximation method is proposed which utilizes the fact that the Shapley value is the mean value over all extreme points in the core. In addition, a steepest descent algorithm is proposed that converges in polynomial time on the fractional egalitarian solution in the core, which can be implemented by network coding schemes. Finally, it is shown that the game can be decomposed into subgames so that both the Shapley value and the egalitarian solution can be obtained within each subgame in a distributed manner with reduced complexity.

**Keywords:** coalitional game; communication for omniscience; fairness; submodularity

**Citation:** Ding, N.; Sadeghi, P.; Smith, D.; Rakotoarivelo, T. Attaining Fairness in Communication for Omniscience. *Entropy* **2022**, *24*, 109. <https://doi.org/10.3390/e24010109>

Academic Editor: Syed A. Jafar

Received: 30 November 2021

Accepted: 9 January 2022

Published: 11 January 2022

**Publisher's Note:** MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



**Copyright:** © 2022 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

## 1. Introduction

The communication for omniscience (CO) problem is formulated in [1]. It is assumed that there are a finite number of users in a system that are indexed by the set  $V$ . Each user  $i \in V$  observes a distinct component  $Z_i$  of a discrete multiple random source  $Z_V = (Z_i : i \in V)$  in private. The users are allowed to exchange their observations over public authenticated broadcast channels so as to attain omniscience, the state where each user recovers the observation sequence of the entire source  $Z_V$ . Originally, the CO problem was studied in [1] due to its dual relationship with the multi-terminal secret capacity (Theorem 1 in [1]). The interactive data exchange process was also studied in other source coding scenarios, e.g., the interactive function computation problem [2–4] (Refs. [2–4] studied the two-terminal  $|V| = 2$  lossy source coding problem, where each user communicates with the other to compute a function. The aim is to characterize the sum-rate-distortion region. This paper considers lossless source coding, which, for  $|V| = 2$ , corresponds to zero-distortion case (see Theorem 2 in [3], Section IV-B in [4]), but the focus is to find how to distribute the sum-rate evenly among more than two users). More recently, CO was

also cast into the *coded cooperative data exchange (CCDE)* problem [5–7], in which the users are mobile clients broadcasting linear combinations of packets over noiseless peer-to-peer (P2P) wireless channels and the communication rates are restricted to being integral.

One main optimization problem that arises in CO is how to minimize the overall source coding rate to attain omniscience. We call it the *minimum sum-rate problem* and denote the value of the minimum sum-rate by  $R^*$ . By utilizing submodular function minimization (SFM) techniques, the value of  $R^*$ , along with an optimal rate vector, are determined in  $O(|V|^2 \cdot \text{SFM}(|V|))$  time in [8] for the asymptotic model, where the communication rates are real-valued (In an asymptotic model, the observation sequence is assumed to be infinitely long. The CCDE corresponds to the finite linear source model, an example of the non-asymptotic model. In the non-asymptotic model, each user only obtains a finite length of observations, and the broadcasts are integer numbers of linear combinations of observations (Section II in [8])), and in [9,10] for CCDE. Here,  $\text{SFM}(|V|)$  is the complexity of a SFM algorithm and is polynomial (Chapter VI in [11]). A more efficient algorithm can be found in [12] by simulating the communications based on the random linear network coding scheme [13,14]. The complexity was further reduced to  $O(|V| \cdot \text{SFM}(|V|))$  in [15].

While [8–10,12] only determined one optimal rate vector, it is shown in (Section III-B in [8]) that the optimal rate region is not a singleton in general. Thus, it is natural to consider how to choose an optimal rate vector that also attains fairness, particularly when the intention is to promote the mobile clients' cooperation in CCDE or even out the battery usage in a wireless sensor network (WSN). The problem of how to attain fairness has been previously considered in [16,17] for CCDE. In [17], a multi-layer acyclic graph is proposed, based on which, a constrained quadratic programming was formulated to determine the Jain's fairness solution [18]. The algorithm proposed in [16] is a greedy approach, where, in each iteration, a unit rate is assigned to the user that optimizes a fairness measure, so that the resulting solution converges on a fair and integer-valued optimal rate vector (The fair solutions in [16,17] coincide with the egalitarian solution [19] in coalitional game theory due to the equivalence between the submodular base polyhedron and the optimal rate region (Section III-B in [8]), both of which, as will be shown in Section 3 in this paper, coincide with the core of a coalitional game).

However, neither of them applies to systems where the communication rates are non-integral, e.g., the asymptotic model, or where packet splitting (and hence fractional transmission rates) is allowed in CCDE. The main purpose of this paper is to study how to attain fairness in the optimal rate region for the CO problem, where the broadcast rates are not constrained to be integer-valued. We start the study by showing the equivalence between the optimal rate region and the core (the solution set) of a coalitional game. We consider two fair solutions proposed in coalitional game theory: the Shapley value [20] and the egalitarian solution [19]. We propose a steepest descent algorithm (SDA) for searching a fractional egalitarian solution that can be implemented by packet splitting in CCDE. Finally, we show that the game can be decomposed by the fundamental partition  $\mathcal{P}^*$  into subgames (The fundamental partition  $\mathcal{P}^*$  is an optimizer that determines the minimum sum-rate  $R^*$  [8]. See also Section 2.1), each of which can attain fairness, either being the Shapley value or the egalitarian solution, on its own. This decomposition leads to a distributed computation method for fairness and reduces the complexity.

### 1.1. Summary of Main Results

Our main results are summarized as follows:

(1) We formulate the problem of attaining omniscience with the minimum sum-rate  $R^*$  by a coalitional game model, where the characteristic cost function, denoted by  $\hat{f}_{R^*}(X)$  for all user groups  $X \subseteq V$ , quantifies the remaining randomness in  $Z_X = (Z_i: i \in X)$  given the common randomness  $\Lambda = H(V) - R^*$  shared by all users in  $V$  (The game model is closely related to the dual relationship (Theorem 1 in [1,21]):  $R^* = H(V) - \Lambda$ , where  $H(V)$  is the entropy of  $Z_V$  and  $\Lambda$  is the common randomness that is shared by all the users in  $V$  [22,23]. The interpretation is that attaining omniscience by the minimum sum-rate

$R^*$  is equivalent to determining how to let the users encode the remaining randomness in  $Z_V$  given the common randomness  $\Lambda$ ). That is,  $\hat{f}_{R^*}$  specifies the source coding rate/cost upper bound to each user subset  $X \subseteq V$ . We show that (a) the core of the game coincides with the optimal rate region containing all the solutions to the minimum sum-rate problem, and (b) the game describes the users' cooperation when they jointly encode the remaining randomness in  $Z_V$  to reach omniscience with the sum-rate exactly equal to  $R^*$ .

(2) We show that the Shapley value [20] assigns each user the expected marginal cost  $\hat{f}_{R^*}(X \cup \{i\}) - \hat{f}_{R^*}(X)$  over all  $X \subseteq V \setminus \{i\}$ . This value is fair in that it penalizes each user according to the source coding rate he/she incurred in CO. While the complexity of obtaining the exact Shapley value is exponentially growing in the number of users  $|V|$ , we show that the Shapley value is the mean over all extreme points in the core. By randomly generating an extreme point set of a desired size, an approximation of the Shapley value can be obtained in polynomial time.

(3) The egalitarian solution [19] aims to equalize the rate/cost allocation in the optimal rate region regardless of the marginal costs. We show that this solution is more suitable for those systems with equally privileged users, e.g., CCDE and WSN. We propose a steepest descent algorithm (SDA) for searching a fractional egalitarian solution that can be implemented in CCDE by splitting each packet into  $|\mathcal{P}^*| - 1$  chunks, where  $|\mathcal{P}^*|$  denotes the number of user subsets in the fundamental partition  $\mathcal{P}^*$ . Based on an optimality criterion for the egalitarian solution stating that the local optimum implies the global optimum, we show that the estimation sequence generated by the SDA converges in the fractional egalitarian solution in  $O(|\mathcal{P}^*| \cdot L(V) \cdot |V| \cdot \text{SFM}(|V|))$  time, where  $L(V)$  is the maximum  $\ell_1$ -norm over all pairs of points in the optimal rate region. In addition, the steepest direction in each iteration of SDA can be computed in a distributed manner.

(4) The game is decomposable such that the users in each subset  $C \in \mathcal{P}^*$  form a subgame with the characteristic cost function  $\hat{f}_{R^*}(X)$  for all  $X \subseteq C$ . This is due to the mutual independence between  $Z_C$  and  $Z_{C'}$  for any two distinct subsets  $C, C' \in \mathcal{P}^*$  given the common randomness  $\Lambda$ . To attain fairness, it suffices to let the users within each subgame  $C \in \mathcal{P}^*$  decide how to allocate the source coding rates fairly, which not only reduces complexity, but also allows parallel computation.

## 1.2. Organization

The rest of the paper is organized as follows. The system model is described in Section 2, where we also review existing results on the minimum sum-rate problem. In Section 3, we formulate the coalitional game model and show that it can be decomposed by the fundamental partition  $\mathcal{P}^*$ . In Section 4, we show how to attain fairness in the optimal rate region by the Shapley value and discuss how to approximate it to avoid the exponentially growing complexity. In Section 5, we propose the SDA algorithm for searching the fractional egalitarian solution. In both Sections 4 and 5, we also present methods to obtain the Shapley value and egalitarian solution by the decomposition method.

## 2. Communication for Omniscience

Let  $V$  with  $|V| > 1$  be a finite set that indexes the terminals in a discrete memoryless multiple source  $Z_V = (Z_i : i \in V)$ . Each component  $Z_i$  is a discrete random variable that takes its values in the finite alphabet  $\mathcal{Z}_i$  according to the joint probability mass function  $P_{Z_V}$ . Let there be  $|V|$  users. Each user  $i \in V$  observes an i.i.d.  $n$ -sequence  $Z_i^n$  of the component  $Z_i$  in private. The users are allowed to exchange compressed versions of their observations over noiseless broadcast channels. The purpose is to attain *omniscience*, the state where all users recover the observation sequence  $Z_V^n$ . This problem is called *communication for omniscience (CO)* [1] (The CO problem was originally formulated in [1] based on a study on the secret capacity in a more general setting where a set of users  $A \subseteq V$  serve as helpers that assist the active users in generating the secret key. The CO problem considered in this paper is the case when  $A = V$ ).

### 2.1. Preliminaries

We review the existing results on minimum sum-rate and optimal rate region as follows. For  $X \subseteq V$ , let  $H(X)$  be the amount of randomness in  $Z_X$  measured by Shannon entropy [24]. For a (source coding) rate vector  $\mathbf{r}_V = (r_i : i \in V)$ , each dimension  $r_i$  denotes the code rate at which user  $i$  encodes their observation  $Z_i^n$ . Let  $r : 2^V \mapsto \mathbb{R}_+$  be the sum-rate function associated with  $\mathbf{r}_V$  such that  $r(X) = \sum_{i \in X} r_i, \forall X \subseteq V$  with the convention  $r(\emptyset) = 0$ . Here,  $r(X)$  denotes the rates at which the users in  $X$  jointly encode  $Z_X^n$ . A source coding rate vector  $\mathbf{r}_V$  at which omniscience is attainable satisfies the Slepian-Wolf (SW) constraints  $r(X) \geq H(X|V \setminus X), \forall X \subsetneq V$  [1]. The achievable rate region is

$$\mathcal{R}(V) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} : r(X) \geq H(X|V \setminus X), \forall X \subsetneq V\}. \tag{1}$$

The fundamental problem concerning the efficiency in CO is to minimize the sum-rate for attaining omniscience

$$R^* = \min\{r(V) : \mathbf{r}_V \in \mathcal{R}(V)\}. \tag{2}$$

This minimum sum-rate problem has been studied and solved efficiently in [8,25] without dealing with the exponentially large number of constraints in the linear programming (2). We review some results in [8] as follows. They will be used in Section 3 to formulate the game model.

For sum-rate  $\alpha \in \mathbb{R}_+$ , define  $f_\alpha(X) = \alpha - H(V \setminus X|X)$  for  $X \neq \emptyset$  and  $f_\alpha(X) = 0$  for  $X = \emptyset$ . Let  $\Pi(V)$  be the set containing all partitions of  $V$ . The Dilworth truncation of  $f_\alpha$  is  $\hat{f}_\alpha(X) = \min_{\mathcal{P} \in \Pi(X)} \sum_{C \in \mathcal{P}} f_\alpha(C)$  for all  $X \subseteq V$  [26]. It is shown in (Theorem 4 and Corollary 46 in [8]) that

$$R^* = \min\{\alpha : f_\alpha(V) = \hat{f}_\alpha(V)\}. \tag{3}$$

The optimal rate region  $\mathcal{R}^*(V)$  that contains all achievable rate vectors  $\mathbf{r}_V$  with sum-rate  $r(V) = R^*$  coincides with  $B(\hat{f}_{R^*})$ , the base polyhedron of  $\hat{f}_{R^*}$  (Section 2.3 in [11] and Definition 9.7.1 in [27]):

$$\begin{aligned} \mathcal{R}^*(V) &= \{\mathbf{r}_V \in \mathcal{R}(V) : r(V) = R^*\} \\ &= \{\mathbf{r}_V \in P(\hat{f}_{R^*}) : r(V) = \hat{f}_{R^*}(V) = R^*\} \\ &= B(\hat{f}_{R^*}), \end{aligned} \tag{4}$$

where  $P(\hat{f}_{R^*}) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} : r(X) \leq \hat{f}_{R^*}(X), \forall X \subseteq V\}$  is the polyhedron of  $\hat{f}_{R^*}$ , which coincides with  $P(f_{R^*}) = \{\mathbf{r}_V \in \mathbb{R}^{|V|} : r(X) \leq f_{R^*}(X), \forall X \subseteq V\}$  (Theorems 2.5(i) and 2.6(i) in [11]). Here, the polyhedron  $P(f_{R^*})$  is induced by the SW constraints: the inequality  $r(X) \geq H(X|V \setminus X)$  in (1) is converted to  $r(V \setminus X) \leq R^* - H(X|V \setminus X)$  under the constraint  $r(V) = R^*$  in  $B(\hat{f}_{R^*})$ .

Problem (3) can be solved in  $O(|V|^2 \cdot \text{SFM}(|V|))$  time by the modified decomposition algorithm (MDA) proposed in (Section V-A in [8]) (The efficiency of the MDA algorithm relies on the submodularity of the entropy function  $H$ .  $\text{SFM}(|V|)$  denotes the complexity of solving a submodular function. See Appendix A for the definition of the submodularity and a brief note on  $\text{SFM}(|V|)$ ), which also returns an optimal rate vector in  $\mathcal{R}^*(V)$ . Let  $\mathcal{P}^*$  be the finest minimizer that determines the Dilworth truncation:

$$\hat{f}_{R^*}(V) = \min_{\mathcal{P} \in \Pi(V)} \sum_{C \in \mathcal{P}} f_{R^*}(C). \tag{5}$$

We call  $\mathcal{P}^*$  the fundamental partition, which is also returned by the MDA algorithm.

### 2.2. Fairness

While the optimal rate region  $\mathcal{R}^*(V)$  is not necessarily a singleton, the MDA algorithm, as well as (Algorithm 3 in [9] and Appendix F in [10]) for solving the minimum sum-rate

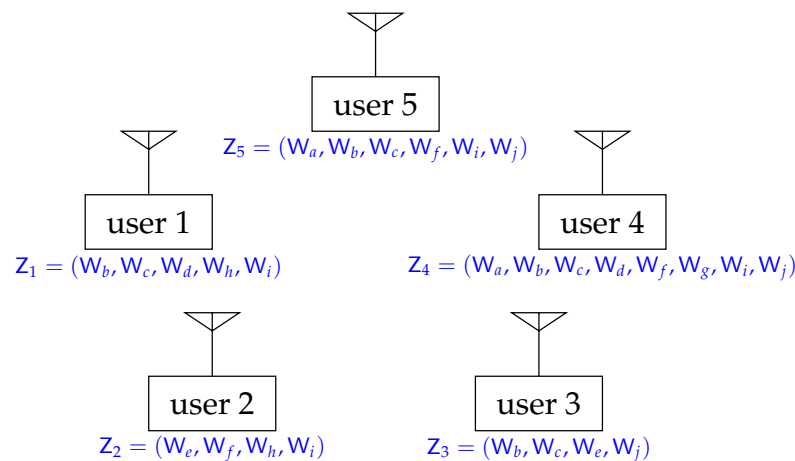
problem in CCDE determine an extreme point (a vertex) in  $\mathcal{R}^*(V)$ , as illustrated in the following example.

**Example 1.** There are five users  $V = \{1, \dots, 5\}$  in Figure 1 observing, respectively,

$$\begin{aligned} Z_1 &= (W_b, W_c, W_d, W_h, W_i), \\ Z_2 &= (W_e, W_f, W_h, W_i), \\ Z_3 &= (W_b, W_c, W_e, W_j), \\ Z_4 &= (W_a, W_b, W_c, W_d, W_f, W_g, W_i, W_j), \\ Z_5 &= (W_a, W_b, W_c, W_f, W_i, W_j), \end{aligned}$$

with  $W_j$  for all  $j \in \{a, \dots, e\}$  being an independent uniformly distributed random bit. In CCDE, each  $W_j$  represents a packet and omniscience refers to the recovery of all packets in  $Z_V$  by users' broadcasting linear combinations of  $Z_i$ 's over P2P channels [5].

By applying the MDA algorithm (Algorithm 1 in [8]), we determine the minimum sum-rate  $R^* = \frac{13}{2}$  and an optimal rate vector  $(1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0)$ , which is an extreme point in  $\mathcal{R}^*(V)$  (Corollary 10 in [8]), and also the fundamental partition  $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$ , which is the finest minimizer of (5). It is not difficult to see that we can improve the fairness of the returned optimal rate vector in  $\mathcal{R}^*(V)$ . For example,  $(1, \frac{1}{2}, \frac{1}{2}, 4, \frac{1}{2}) \in \mathcal{R}^*(V)$  is fairer in that user 5 also takes part in the CO instead of being a free rider.



**Figure 1.** The 5-user system with  $V = \{1, \dots, 5\}$  in Example 1. The users encode and broadcast  $Z_i$ 's so as to attain omniscience of the source  $Z_V$ . In the corresponding CCDE problem, each  $W_j$  denotes a packet that belongs to a field  $\mathbb{F}_q$ , and each user  $i \in V$  broadcasts linear combinations of  $Z_i$  to help others recover all packets in  $Z_V$ .

The fairness considered in Example 1 corresponds to the egalitarian solution [19,28], which tries to make the users have an equal share of the coding rates. The purpose is to motivate them to take part in the CO. In a system where the users' contribution is unequal, fairness could mean that each user should be penalized proportionally by the coding rates he/she incurs in the CO. In Example 1, user 4 should transmit more since he/she incurs the most coding rates for attaining omniscience, even if the overall coding rates can be distributed to the users more evenly (see Section 5). This is another fairness metric called the Shapley value in coalitional game theory. These two fairness metrics are both studied in this paper.

For a fractional rate vector  $\mathbf{r}_V$ , if  $K \in \mathbb{Z}_+$  is the least common multiple (LCM) of all denominators of  $r_i$ , i.e.,  $K\mathbf{r}_V = (Kr_i: i \in V) \in \mathbb{Z}_+^{|V|}$ , this rate vector can be implemented by  $K$ -packet-splitting in CCDE [9,10,17,29]: dividing each packet into  $K$  chunks and letting the users broadcast linear combinations of packet chunks at rate  $K\mathbf{r}_V$ . In Example 1, both

$(1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0)$  and  $(1, \frac{1}{2}, \frac{1}{2}, 4, \frac{1}{2})$  can be achieved by 2-packet-splitting. Therefore, in CCDE, we are also interested in determining a fair fractional optimal rate vector.

### 3. Decomposable Coalitional Game

We formulate a coalitional game model (The coalitional game was first formulated to propose the Shapley value as a fair rate allocation for CO. This paper introduces the decomposition property and focuses on the mutual dependence, cooperation among the users and distributed computation of fair solutions) in this section and show the equivalence of the optimal rate region  $\mathcal{R}^*(V)$  and the core of this game. The purpose is to introduce two game-theoretic solutions, the Shapley value and egalitarian solution in Sections 4 and 5, respectively, for attaining fairness in  $\mathcal{R}^*(V)$ . We also show the decomposition of this game model, a property that will be utilized in Sections 4 and 5 to propose a decomposition method for obtaining the Shapley value and egalitarian solution, respectively.

#### 3.1. Coalition Game Model

Let the users in  $V$  be self-autonomous decision makers that take part in the CO, and assume that, instead of being selfish, they may cooperate with others to form groups. We call  $X \subseteq V$  a *coalition* and  $V$  the *grand coalition*. Consider the function  $f_{R^*}(X) = H(X) + R^* - H(V)$ . Here,  $R^* - H(V)$  equals the common randomness  $\Lambda$  in  $Z_V$  that is shared by all users in  $V$  due to the dual relationship (Theorem 1 in [1]) [21]

$$R^* = H(V) - \Lambda. \tag{6}$$

Here,  $\Lambda$  is called the multivariate mutual information in [23], or shared information in [22]. Assume that  $\Lambda$  is obtained by a random variable  $U$ , which does not need to be broadcast over the public channels. Then, the problem is how to encode the remaining randomness in  $Z_X$  given  $U$  for all  $X \subseteq V$  that is measured by the Dilworth truncation [23]

$$H(X|U) = \hat{f}_{R^*}(X) = \min_{\mathcal{P} \in \Pi(X)} \sum_{C \in \mathcal{P}} f_{R^*}(C). \tag{7}$$

We call  $\hat{f}_{R^*}$  the *characteristic cost function* in that  $\hat{f}_{R^*}(X)$  specifies the upper bound on the (source) coding cost when the users in  $X$  form a coalition so as to jointly encode the randomness in  $Z_X$  given  $U$ . The *coalitional game model* is characterized by the user set  $V$  and the characteristic cost function  $\hat{f}_{R^*}$ . We denote it by  $\Omega(V, \hat{f}_{R^*})$ . In this sense, the game  $\Omega(V, \hat{f}_{R^*})$  formulates a multi-terminal data compression problem where the users jointly encode the remaining randomness in  $Z_V$  that is specified by the set function  $\hat{f}_{R^*}$ .

**Example 2.** For the 5-user system in Example 1, the common randomness  $\Lambda = H(V) - R^* = 10 - \frac{13}{2} = \frac{7}{2}$  is obtained by the random variable  $U$ . For users 1 and 2, we have

$$\begin{aligned} H(\{1,2\}|U) &= \hat{f}_{13/2}(\{1,2\}) \\ &= \min \{f_{13/2}(\{1\}) + f_{13/2}(\{2\}), f_{13/2}(\{1,2\})\} \\ &= \min \{H(\{1\}) + H(\{2\}) - 2H(U), H(\{1,2\}) - H(U)\} \\ &= H(\{1\}) + H(\{2\}) - 2H(U) = 2, \end{aligned}$$

being the remaining randomness in  $Z_{\{1,2\}}$  given  $U$ . The interpretation is that, in order to attain omniscience with sum-rate  $R^*$ , the rate for users 1 and 2 to jointly encode their observations is no more than 2 bits. Alternatively, the maximum cost incurred by users 1 and 2 cooperating with each other is 2 bits of coding rate. One can show that (7) holds for all  $X \subseteq V$  (An explanation of (7) can be found in (Section IV-B in [23])).

#### 3.2. Core

While  $\hat{f}_{R^*}$  quantifies the maximum coding cost in each coalition, each  $\mathbf{r}_V$  denotes a cost allocation method, with each  $r_i$  being the source coding rate assigned to user  $i \in V$ .

The solution set of the game  $\Omega(V, f_{R^*})$  is called the *core* [30,31] which contains all  $\mathbf{r}_V$ s distributing exactly the total cost  $R^*$  to individual users such that  $r(X) \leq \hat{f}_{R^*}(X)$  holds for all coalitions  $X \subseteq V$ . It is not difficult to see from (4) that the core coincides with the optimal rate region  $\mathcal{R}^*(V)$ , which is nonempty (Theorem 4 in [8]) (The nonemptiness of the core  $\mathcal{R}^*(V)$  can also be explained by the submodularity of  $\hat{f}_{R^*}$ . See Appendix B). In the rest of the paper, we will refer to  $\mathcal{R}^*(V)$  as the core or the optimal rate region interchangeably.

The inequality  $r(X) \leq \hat{f}_{R^*}(X)$  in the core  $\mathcal{R}^*(V)$  also has an interpretation in coalitional game theory. If a cost allocation method  $\mathbf{r}_V$  results in  $r(X) > \hat{f}_{R^*}(X)$  for some  $X$ , the users in  $X$  may break the coalition  $X$  and seek another  $\mathbf{r}_V$  such that  $r(X) \leq \hat{f}_{R^*}(X)$ . This means the coalition  $X$  is not stable (This can also be explained by the definition of stability (Section 4.3 in [32]) and the fact that the core is a stable set in (Theorem 8 in [32])). On the other hand, if  $r(X) \leq \hat{f}_{R^*}(X)$  holds for all  $X \subseteq V$ , then no user has the incentive to break the coalition  $V$  and form a smaller one, i.e., the grand coalition  $V$  forms. In this sense, the core contains all cost allocation methods  $\mathbf{r}_V$  that exactly distribute the sum-cost  $r(V) = R^*$  to all users in a way such that all of them would like to cooperate with others for the purpose of attaining omniscience (Chapter 12 in [30]).

### 3.3. Decomposition

For any  $X, Y \subseteq V$  such that  $X \cap Y = \emptyset$ , let  $\sqcup$  denote the *disjoint union* and  $\mathbf{r}_X \oplus \mathbf{r}_Y = \mathbf{r}_{X \sqcup Y}$  be the *direct sum* of  $\mathbf{r}_X$  and  $\mathbf{r}_Y$ . For example, for  $\mathbf{r}_{\{1,3\}} = (r_1, r_3) = (3, 7)$  and  $\mathbf{r}_{\{2,5\}} = (r_2, r_5) = (2, 4)$ ,  $\mathbf{r}_{\{1,3\}} \oplus \mathbf{r}_{\{2,5\}} = \mathbf{r}_{\{1,2,3,5\}} = (3, 2, 7, 4)$ . For  $X \subseteq V$ , let  $\chi_X = (r_i : i \in V)$  be the *characteristic vector* of the subset  $X$  such that  $r_i = 1$  if  $i \in X$  and  $r_i = 0$  if  $i \notin X$ .

For the fundamental partition  $\mathcal{P}^*$ , each  $C \in \mathcal{P}^*$  defines a subgame  $\Omega(C, \hat{f}_{R^*})$  with the characteristic cost function  $\hat{f}_{R^*}(X)$  for all  $X \subseteq C$ . The core of the subgame  $\Omega(C, \hat{f}_{R^*})$  is

$$\mathcal{R}^*(C) = \{\mathbf{r}_C \in P_C(\hat{f}_{R^*}) : r(C) = \hat{f}_{R^*}(C)\},$$

where the polyhedron  $P_C(\hat{f}_{R^*}) = \{\mathbf{r}_C \in \mathbb{R}^{|C|} : r(X) \leq \hat{f}_{R^*}(X), \forall X \subseteq C\}$  is a *reduction/projection* of  $P(\hat{f}_{R^*})$  on to  $C$ . The following lemma shows the decomposition property of the game  $\Omega(V, \hat{f}_{R^*})$ .

**Lemma 1** (Theorem 38 and Lemma 39 in [8]). *The game  $\Omega(V, \hat{f}_\alpha)$  can be decomposed by the fundamental partition  $\mathcal{P}^*$  so that*

(a) *the dimension of  $\mathcal{R}^*(V)$  is  $|V| - |\mathcal{P}^*|$  and*

$$\begin{aligned} \mathcal{R}^*(V) &= \bigoplus_{C \in \mathcal{P}^*} \mathcal{R}^*(C) \\ &= \left\{ \bigoplus_{C \in \mathcal{P}^*} \mathbf{r}_C : \mathbf{r}_C \in \mathcal{R}^*(C), C \in \mathcal{P}^* \right\}. \end{aligned}$$

(b) *The following holds for any  $\mathbf{r}_V \in \mathcal{R}^*(V)$ :*

- (i) *For any  $C, C' \in \mathcal{P}^*$  such that  $C \neq C'$ ,  $\mathbf{r}_V + \epsilon(\chi_i - \chi_j) \notin \mathcal{R}^*(V)$ , for all  $\epsilon > 0$ ,  $i \in C$  and  $j \in C'$ ;*
- (ii) *For all  $C \in \mathcal{P}^*$ ,  $\mathbf{r}_V + \epsilon(\chi_i - \chi_j) \in \mathcal{R}^*(V)$  for some  $\epsilon > 0$  and  $i, j \in C$ .*

The decomposition of the core  $\mathcal{R}^*(V)$  in Lemma 1(a) interprets the decomposition of the solution set of  $\Omega(V, \hat{f}_{R^*})$  and the fact that it makes no difference for the users to cooperate in the grand coalition  $V$  or in subgames  $\Omega(C, \hat{f}_{R^*}), \forall C \in \mathcal{P}^*$  (This fact can be seen more clearly via the definition of the decomposable game in Appendix B). Lemma 1(b) states that the costs, or source coding rates, can be exchanged within a subgame, but not between subgames, which can be explained by the dependence relationship in the remaining randomness as follows.

Interpretation

Recall that  $\hat{f}_{R^*}(X) = H(X|U)$ . Due to the fact that  $\mathcal{P}^*$  is the finest minimizer of (5), we have ((8a) holds because  $\mathcal{P}^*$  is the minimizer of the Dilworth truncation; the strict inequality (8a) holds because otherwise  $\mathcal{P}^*$  is not the finest minimizer)

$$I(C;C'|U) = \hat{f}_{R^*}(C) + \hat{f}_{R^*}(C') - \hat{f}_{R^*}(C \sqcup C') = 0, \quad \forall C, C' \in \mathcal{P}^*: C \neq C'; \quad (8a)$$

$$I(X;C \setminus X|U) = \hat{f}_{R^*}(X) + \hat{f}_{R^*}(C \setminus X) - \hat{f}_{R^*}(C) > 0, \quad \forall X \subsetneq C. \quad (8b)$$

Here, (8a) means that given the common randomness, any two distinct coalitions  $C$  and  $C'$  in  $\mathcal{P}^*$  have  $Z_C$  and  $Z_{C'}$  mutually independent. That is, to attain omniscience with the minimum sum-rate  $R^*$ , the users in  $C$  and  $C'$  must encode the exact randomness  $H(C|U)$  and  $H(C'|U)$ , respectively. In other words, the costs or the source coding rates cannot transfer between any two users  $i \in C$  and  $j \in C'$ . This is the interpretation of Lemma 1(b)-(i) and we call it *zero exchange rate* between  $i$  and  $j$ . On the other hand, (8b) states that, given the common randomness, any two users  $i$  and  $j$  in the same coalition  $C$  are mutually dependent. In this case, the information amount  $I(X;C \setminus X|U)$  that is mutual to  $X$  and  $C \setminus X$  can be encoded by either  $i \in X$  or  $j \in C \setminus X$ , i.e., the costs or source coding rates can be transferred between users  $i$  and  $j$ : they have *nonzero exchange rate*.

**Example 3.** For the 5-user system in Example 1, we have the fundamental partition  $\mathcal{P}^* = \{\{1,4,5\}, \{2\}, \{3\}\}$ . The core  $\mathcal{R}^*(V)$  has the dimension of  $|V| - |\mathcal{P}^*| = 5 - 3 = 2$  and is decomposed as

$$\mathcal{R}^*(V) = \mathcal{R}^*({1,4,5}) \oplus \mathcal{R}^*({2}) \oplus \mathcal{R}^*({3})$$

where  $\mathcal{R}^*({1,4,5})$ , as shown in Figure 2, is a 2-dimensional plane and  $\mathcal{R}^*({2})$  and  $\mathcal{R}^*({3})$  are singletons containing single points  $r_2 = \frac{1}{2}$  and  $r_3 = \frac{1}{2}$ , respectively.

Given the common randomness  $\Lambda = H(V) - R^* = \frac{7}{2}$  that is obtained by  $U$ , any two distinct  $C, C' \in \mathcal{P}^*$  are independent, e.g.,

$$I(\{1,4,5\};\{2\}|U) = \hat{f}_{R^*}(\{1,4,5\}) + \hat{f}_{R^*}(\{2\}) - \hat{f}_{R^*}(\{1,2,4,5\}) = 0;$$

for any  $C \in \mathcal{P}^*$ , any two disjoint  $X, Y \subseteq C$  such that  $X \sqcup Y = C$  are mutually dependent; e.g.,

$$I(\{1,4\};\{5\}|U) = \hat{f}_{R^*}(\{1,4\}) + \hat{f}_{R^*}(\{5\}) - \hat{f}_{R^*}(\{1,4,5\}) = \frac{5}{2};$$

i.e., in the fundamental partition  $\mathcal{P}^*$ , we have zero exchange rate between coalitions and nonzero exchange rate within a coalition.

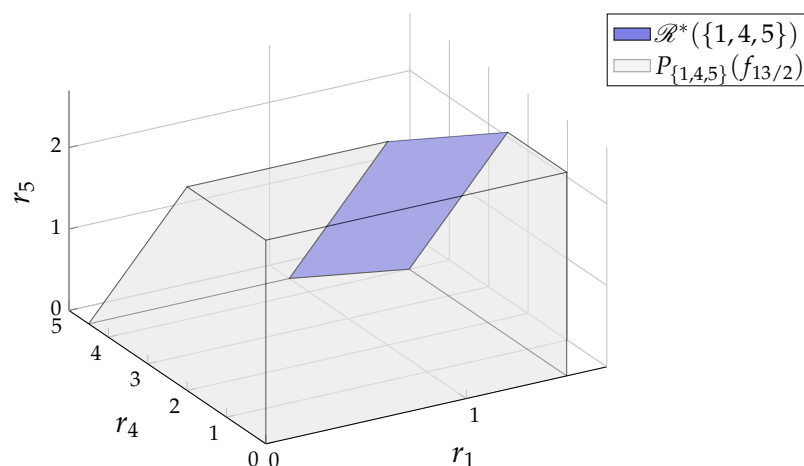


Figure 2. The core  $\mathcal{R}^*({1,4,5})$  of the subgame  $\Omega(\{1,4,5\}, \hat{f}_{R^*})$  of the 5-user system in Figure 1.



The decomposition property in Lemma 1 is useful when considering the fairness. Since there is no freedom for the users who belong to distinct coalitions in  $\mathcal{P}^*$  to negotiate how to allocate coding costs fairly, it suffices to just study how to attain fairness within each  $C \in \mathcal{P}^*$ . This will be further summarized in Theorem 1 in Section 4 and Theorem 3 in Section 5 that allow distributed computation for attaining the two fair solutions, the Shapley value and egalitarian solution, in the optimal rate region  $\mathcal{R}^*(V)$ .

#### 4. Shapley Value

For an omniscience-achievable rate vector  $\mathbf{r}_V$ , it is worth discussing how fairly it can distribute the source coding rates. In the game model  $\Omega(V, \hat{f}_{R^*})$ , fairness is also an important performance metric of a cost allocation method  $\mathbf{r}_V$  in that it promotes the users incentives to cooperate with each other. In this section, we discuss how to attain fairness by searching the Shapley value in the optimal rate region  $\mathcal{R}^*(V)$ .

The Shapley value  $\hat{\mathbf{r}}$  is defined in (Theorem 7 in [20]) as a unique solution in the core  $\mathcal{R}^*(V)$ , with each dimension being

$$\hat{r}_i = \sum_{X \subseteq V \setminus \{i\}} \frac{|X|!(|V| - |X| - 1)!}{|V|!} (\hat{f}_{R^*}(X \sqcup \{i\}) - \hat{f}_{R^*}(X)). \tag{9}$$

Here,  $\hat{f}_{R^*}(X \sqcup \{i\}) - \hat{f}_{R^*}(X) = H(X \sqcup \{i\}|U) - H(X|U) = H(\{i\}|X \cup U)$  is the remaining uniqueness in  $Z_i$  given the  $Z_X$  and the common randomness in  $U$ . The interpretation is that, to attain the omniscience by the minimum sum-rate  $R^*$ , if the users in  $X$  encode at the rate  $H(X|U)$  first, user  $i$  needs to encode at the rate  $H(\{i\}|X \cup U)$ .

In the game model  $\Omega(V, \hat{f}_{R^*})$ ,  $\hat{f}_{R^*}(X \sqcup \{i\}) - \hat{f}_{R^*}(X)$  is the marginal coding cost incurred by user  $i$  when he/she joins the coalition  $X$ . Let  $\Phi = (\phi_1, \dots, \phi_{|V|})$  such that  $\phi_i \in V$  and  $\phi_i \neq \phi_j$  for all  $i \neq j$  be a permutation of  $V$ . Here, each  $\Phi$  denotes the order that the users join the grand coalition  $V$ , for which, the total cost  $R^*$  can be assigned to individual users by the Edmond greedy algorithm [33]: For  $i$  increasing from 1 to  $|V|$ , we assign each user the marginal cost

$$r_i := \hat{f}_{R^*}(V_i) - \hat{f}_{R^*}(V_{i-1}),$$

where  $V_0 = \emptyset$  and  $V_i = \{\phi_1, \dots, \phi_i\}$  for all  $i \in \{1, \dots, |V|\}$ . The resulting  $\mathbf{r}_V$  satisfies  $\mathbf{r}_V \in \mathcal{R}^*(V)$ . The Shapley value  $\hat{\mathbf{r}}_V$  is based on the assumption that all the permutations are equiprobable. For each  $X \subseteq V \setminus \{i\}$ , user  $i$  will be assigned the marginal coding cost  $\hat{f}_{R^*}(X \sqcup \{i\}) - \hat{f}_{R^*}(X)$  for  $|X|!(|V| - |X| - 1)!$  out of  $|V|!$  times. Then,  $\hat{\mathbf{r}}_V$  assigns each user the expected marginal coding cost he/she incurs over all permutations.

##### 4.1. Decomposition

The fairness of  $\hat{\mathbf{r}}_V$  can also be explained by its relationship with the extreme points in the core  $\mathcal{R}^*(V)$ . Let  $\text{EX}(V)$  be the extreme point set containing all vertices of the core  $\mathcal{R}^*(V)$ . For a particular permutation  $\Phi$ , the optimal rate vector returned by the Edmond greedy algorithm is an extreme point of  $\mathcal{R}^*(V)$  and  $\text{EX}(V)$  can be constructed by applying the Edmond greedy algorithm for all  $|V|!$  permutations of  $V$  (Section 3.2 in [11]). Based on the definition (9), the Shapley value is the mean value of  $\text{EX}(V)$  [20] (In this sense, the Shapley value is the gravity center of  $\mathcal{R}^*(V)$  [20]):

$$\hat{\mathbf{r}}_V = \frac{\sum_{\mathbf{r}_V \in \text{EX}(V)} \mathbf{r}_V}{|\text{EX}(V)|}. \tag{10}$$

Since the core  $\mathcal{R}^*(V)$  is decomposed by the fundamental partition  $\mathcal{P}^*$  (Lemma 1(a)), we have the extreme point set also decomposed as  $\text{EX}(V) = \bigoplus_{C \in \mathcal{P}^*} \text{EX}(C)$ , which leads to the decomposition of the Shapley value in Theorem 1 below (Theorem 1 is a special case of Theorem 1 when the minimum sum-rate  $R^* = H(V)$ ).

**Theorem 1.** For the Shapley value  $\hat{\mathbf{r}}_V$  in the core  $\mathcal{R}^*(V)$ , we have  $\hat{\mathbf{r}}_V = \bigoplus_{C \in \mathcal{P}^*} \hat{\mathbf{r}}_C$ , where  $\hat{\mathbf{r}}_C = \frac{\sum_{\mathbf{r}_C \in \text{EX}(C)} \mathbf{r}_C}{|\text{EX}(C)|}$  is the Shapley value in the core  $\mathcal{R}^*(C)$  of the subgame  $\Omega(C, \hat{f}_{R^*})$ .

**Proof.** For the fundamental partition  $\mathcal{P}^*$ , since  $\text{EX}(V) = \bigoplus_{C \in \mathcal{P}^*} \text{EX}(C)$ , we have

$$\begin{aligned} \hat{\mathbf{r}}_V &= \frac{\sum_{\mathbf{r}_V \in \text{EX}(V)} \mathbf{r}_V}{|\text{EX}(V)|} \\ &= \frac{\sum_{\mathbf{r}_V \in \bigoplus_{C \in \mathcal{P}^*} \text{EX}(C)} \mathbf{r}_V}{|\bigoplus_{C \in \mathcal{P}^*} \text{EX}(C)|} \\ &= \frac{\bigoplus_{C \in \mathcal{P}^*} \left( \prod_{C' \in \mathcal{P}^*: C' \neq C} |\text{EX}(C')| \sum_{\mathbf{r}_C \in \text{EX}(C)} \mathbf{r}_C \right)}{\prod_{C \in \mathcal{P}^*} |\text{EX}(C)|} \\ &= \bigoplus_{C \in \mathcal{P}^*} \frac{\sum_{\mathbf{r}_C \in \text{EX}(C)} \mathbf{r}_C}{|\text{EX}(C)|} \\ &= \bigoplus_{C \in \mathcal{P}^*} \hat{\mathbf{r}}_C. \end{aligned}$$

Theorem holds.  $\square$

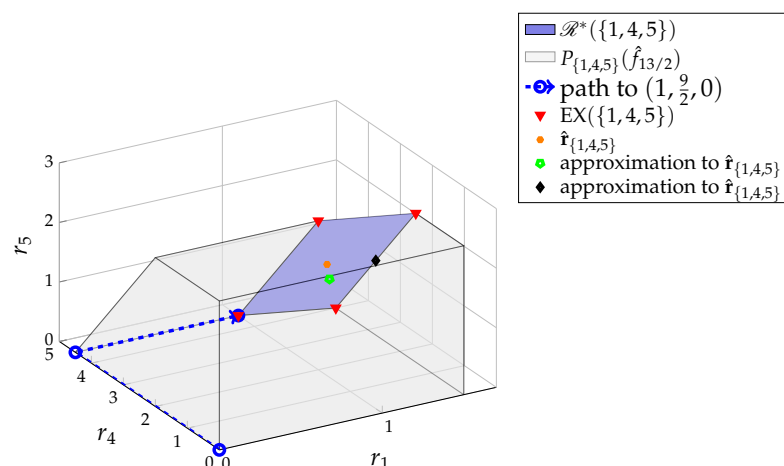
**Example 4.** In the core  $\mathcal{R}^*(V)$  of the 5-user system in Example 1, the Shapley value by the definition (9) is  $\hat{\mathbf{r}}_V = (\frac{5}{4}, \frac{1}{2}, \frac{1}{2}, 3, \frac{5}{4})$ . We have four extreme points in

$$\text{EX}(V) = \left\{ \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 4, 0\right), \left(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right), \left(1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0\right), \left(1, \frac{1}{2}, \frac{1}{2}, 2, \frac{5}{2}\right) \right\}$$

such that  $\hat{\mathbf{r}}_V = \frac{\sum_{\mathbf{r}_V \in \text{EX}(V)} \mathbf{r}_V}{4}$ . Recall that we have the fundamental partition  $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$  that decomposes the game  $\Omega(V, \hat{f}_{R^*})$  as in Example 3. According to Theorem 1, we have

$$\hat{\mathbf{r}}_V = \hat{\mathbf{r}}_{\{1,4,5\}} \oplus \hat{r}_2 \oplus \hat{r}_3,$$

where  $\hat{\mathbf{r}}_{\{1,4,5\}} = (\frac{5}{4}, 3, \frac{5}{4}) = \frac{\sum_{\mathbf{r}_{\{1,4,5\}} \in \text{EX}(\{1,4,5\})} \mathbf{r}_{\{1,4,5\}}}{4}$  is the Shapley value of the subgame  $\Omega(\{1, 4, 5\}, \hat{f}_{R^*})$  as shown in Figure 3,  $\hat{r}_2 = \frac{1}{2}$  and  $\hat{r}_3 = \frac{1}{2}$ .



**Figure 3.** For the core  $\mathcal{R}^*(\{1, 4, 5\})$  of the subgame  $\Omega(\{1, 4, 5\}, \hat{f}_{R^*})$ , the extreme point set is  $\text{EX}(\{1, 4, 5\}) = \{(\frac{3}{2}, 4, 0), (\frac{3}{2}, \frac{3}{2}, \frac{5}{2}), (1, \frac{9}{2}, 0), (1, 2, \frac{5}{2})\}$ , the mean value of which is the Shapley value  $\hat{\mathbf{r}}_{\{1,4,5\}} = (\frac{5}{4}, 3, \frac{5}{4})$ . We apply the random permutation method twice as in Example 5. We randomly generate 3 permutations of 1, 4 and 5 each time and get the two approximations of  $\hat{\mathbf{r}}_{\{1,4,5\}}$ . In this figure, the path to  $(1, \frac{9}{2}, 0)$  shows an example of how the Edmond algorithm (Algorithm 3 in [8]) finds the vertex  $(1, \frac{9}{2}, 0)$  corresponding to the permutation  $(4, 5, 1)$ .

#### 4.2. Complexity and Approximation

The complexity of computing the Shapley value is exponentially large in the problem size  $|V|$ , since the values of  $f_{R^*}(X)$  for all  $X \subseteq V$  are required to be calculated to get  $\hat{r}_V$  in (9). What makes the situation worse is that determining the value of the Dilworth truncation  $\hat{f}_{R^*}(X)$  for a given  $X$  requires calling SFM algorithms and their complexity is  $O(|X| \cdot \text{SFM}(|X|))$ . Therefore, it is impractical to obtain the exact value of  $\hat{r}_V$  in large systems.

One alternative approach is to utilize the decomposition property in Theorem 1 to allow distributed and parallel computation. For each coalition  $C$  in the fundamental partition  $\mathcal{P}^*$ , let the users in  $C$  obtain the Shapley value  $\hat{r}_C$  in the subgame  $\Omega(C, \hat{f}_{R^*})$  by themselves; All  $\hat{r}_C$  are combined to form the Shapley value  $\hat{r}_V$  of the entire game  $\Omega(V, \hat{f}_{R^*})$ . By doing so, the complexity is determined by the subgame of maximum size  $\hat{C} = \text{argmax}\{|C| : C \in \mathcal{P}^*\}$ . However, the complexity to obtain the Shapley value  $\hat{r}_{\hat{C}}$  in the subgame  $\Omega(\hat{C}, \hat{f}_{R^*})$  is again exponentially growing in  $|\hat{C}|$ .

While the high computational complexity is an intrinsic problem of the Shapley value, there are various approximation algorithms proposed in the literature to alleviate this complexity problem. For example, the random permutation method in [34] utilizes the fact that the Shapley value is the mean value over the extreme point set in (10). The idea is to randomly generate a set of permutations of  $V$  of a desired size, e.g.,  $|V|$  or  $|V|^2$  permutations, and apply the Edmond greedy algorithm to determine the corresponding extreme points, the mean of which is an approximation of the Shapley value  $\hat{r}_V$ . This approximation method can also be used in combination with the decomposition method in Theorem 1.

**Example 5.** For the 5-user system in Example 1, we first decompose the game into subgames  $\Omega(\{1, 4, 5\}, \hat{f}_{R^*})$ ,  $\Omega(\{2\}, \hat{f}_{R^*})$  and  $\Omega(\{3\}, \hat{f}_{R^*})$ . For the subgame  $\Omega(\{1, 4, 5\}, \hat{f}_{R^*})$ , we randomly select  $|\{1, 4, 5\}| = 3$  permutations. For example, for  $\Phi = (1, 4, 5)$ ,  $(1, 5, 4)$  and  $(4, 1, 5)$ , we can generate three extreme points:

$$\left\{ \left( \frac{3}{2}, 4, 0 \right), \left( \frac{3}{2}, \frac{3}{2}, \frac{5}{2} \right), \left( 1, \frac{9}{2}, 0 \right) \right\} \subseteq EX(\{1, 4, 5\}),$$

respectively, so that the mean value  $(\frac{4}{3}, \frac{10}{3}, \frac{6}{5})$  is an approximation of the Shapley value  $\hat{r}_{\{1,4,5\}}$  in  $\mathcal{R}^*(\{1, 4, 5\})$ . Note, different permutations might result in different approximations. For example, if we choose three permutations  $\Phi = (1, 4, 5)$ ,  $(1, 5, 4)$  and  $(5, 1, 4)$ , we would have the approximation  $(\frac{3}{2}, \frac{11}{4}, \frac{5}{4})$ . See the two approximations in Figure 3.

By combining the approximation of  $\hat{r}_{\{1,4,5\}}$  with the ones obtained in other subgames, we have the approximation of the Shapley value  $\hat{r}_V$  of the game  $\Omega(V, \hat{f}_{R^*})$ . For example, the above two approximations generate  $(\frac{4}{3}, \frac{1}{2}, \frac{1}{2}, \frac{10}{3}, \frac{6}{5})$  and  $(\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{11}{4}, \frac{5}{4})$ , which are the two approximations to  $\hat{r}_V$ .

In Example 5, we chose no more than  $|C|$  permutations for each subgame  $C \in \mathcal{P}^*$ , where the extreme point corresponding to each permutation can be determined by Algorithm 3 in [8] (The algorithm (Algorithm 3 in [8]) can be considered as a modified Edmond greedy algorithm. See Appendix B in [8] for the explanation. In Figure 3, the path towards the extreme point  $(1, \frac{9}{2}, 0)$  is generated by Algorithm 3 in [8] for the permutation  $\Phi = (4, 5, 1)$ ) in  $O(|C| \cdot \text{SFM}(|C|))$  time. Therefore, the overall complexity for approximating the Shapley value  $\hat{r}_V$  is determined by the subgame  $\Omega(\hat{C}, \hat{f}_{R^*})$  of maximum size as polynomial time  $O(|\hat{C}|^2 \cdot \text{SFM}(|\hat{C}|))$ . Accordingly, if we choose  $|C|^2$  permutations for each subgame  $C \in \mathcal{P}^*$ , the complexity would be  $O(|\hat{C}|^3 \cdot \text{SFM}(|\hat{C}|))$ . We also remark that the approximation algorithm is not unique. In fact, there are many other existing methods, e.g., [34–36], that can be implemented to approximate the Shapley value  $\hat{r}_V$ .

#### 5. Egalitarian Solution

The Shapely value  $\hat{r}_V$  is fair in that it penalizes each user based on the expected marginal cost he/she incurs in game  $\Omega(V, \hat{f}_{R^*})$ . For example, in the 5-user system in

Figure 1, user 4 incurs the most expected marginal cost  $\hat{f}_{R^*}(\{4\} \sqcup X) - \hat{f}_{R^*}(X)$  over all  $X \subseteq V \setminus \{4\}$ , the Shapley value  $\hat{\mathbf{r}}_V = (\frac{5}{4}, \frac{1}{2}, \frac{1}{2}, 3, \frac{5}{4})$  in Example 4 assigns him/her the most coding cost.

However, this fairness suggested by the Shapley value might not be the desired one in some practical systems. For example, in CCDE where mobile clients are considered as equally privileged peers, it is desirable to find a  $\mathbf{r}_V \in \mathcal{R}^*(V)$  that allocates the source coding rate as evenly as possible without considering users' prior knowledge of the source  $Z_V$ . Another example is a WSN containing a large number of battery-powered sensors with equal initial energy budget, where the even allocation of the source coding rate prolongs the overall lifetime of the WSN. In these cases, it might be more suitable to consider the egalitarian solution  $\mathbf{r}_V^*$  [19,28], the minimizer of  $\min\{\sum_{i \in V} r_i^2 : \mathbf{r}_V \in \mathcal{R}^*(V)\}$ . In this section, we consider a more general quadratic programming [37,38]:

$$\min \left\{ g(\mathbf{r}_V) : \mathbf{r}_V \in \mathcal{R}^*(V) \right\}, \tag{11}$$

where  $g(\mathbf{r}_V) = \sum_{i \in V} \frac{r_i^2}{w_i}$  and  $\mathbf{w}_V \in \mathbb{R}_{++}^{|V|}$  is a positive weight vector which could have some interpretations in practical scenarios. For example,  $w_i$  could denote the quality of the wireless transmission of user  $i$  in CCDE or the remaining battery energy of sensor node  $i$  in a WSN.

It is shown [39,40] that, if the function value of  $\hat{f}_{R^*}$  can be obtained directly, problem (11) can be solved in  $O(|V| \cdot \text{SFM}(|V|))$  time (In the case when  $\mathbf{w}_V = \mathbf{1} = (1, \dots, 1)$ , the minimizer of (11) is also called the minimum-norm point in  $\mathcal{R}^*(V)$ , which can be searched by the algorithm in [41] by polynomial time calls of the Dilworth truncation  $\hat{f}_{R^*}$ ). However, determining the Dilworth truncation  $\hat{f}_{R^*}(X)$  for a given  $X$  has the complexity  $O(|X| \cdot \text{SFM}(|X|))$ . In addition, the minimizer of (11) may not be fractional or, if it is fractional, may require splitting each packet into more than  $|\mathcal{P}^*| - 1$  chunks in CCDE. Since  $|\mathcal{P}^*| \leq |V|$  and it is shown in (Corollary 28 in [8]) that there exists an optimal rate vector in  $\mathcal{R}^*(V)$  with an LCM  $|\mathcal{P}^*| - 1$ , it would be of interest to see if we can find a fair optimal rate vector in  $\mathcal{R}^*(V)$  still with LCM  $|\mathcal{P}^*| - 1$ .

**Example 6.** Consider the minimizer  $\mathbf{r}_V^*$  of (11) for the 5-user system in Example 1, we have  $\mathbf{r}_V^* = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2)$  for  $\mathbf{w}_V = \mathbf{1} = (1, \dots, 1)$  and  $\mathbf{r}_V^* = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{12}{5}, \frac{8}{5})$  for  $\mathbf{w}_V = (6, 1, 1, 3, 2)$ . While the former can be implemented by 2-packet-splitting, the latter requires dividing each packets into 10 chunks.

In fact, not only the minimizer of (11), but also the Shapley value have the problem of incurring more than  $(|\mathcal{P}^*| - 1)$ -packet-splitting. For example, the Shapley value  $(\frac{5}{4}, \frac{1}{2}, \frac{1}{2}, 3, \frac{5}{4})$  in Example 4 requires 4-packet-splitting, where  $4 > |\mathcal{P}^*| - 1 = 2$ , and its approximation  $(\frac{4}{3}, \frac{1}{2}, \frac{1}{2}, \frac{10}{3}, \frac{6}{5})$  in Example 5 even requires 30-packet-splitting. Such dividing and reconstructing of packets could be cumbersome or even very impractical. In the next subsection, we consider how to search for an egalitarian solution in  $\mathcal{R}^*(V)$  that can be implemented by  $(|\mathcal{P}^*| - 1)$ -packet-splitting.

### 5.1. Steepest Descent Algorithm

For  $K = |\mathcal{P}^*| - 1$ , let  $\mathbb{Q}_K = \frac{\mathbb{Z}}{K}$  be the set containing all rational numbers that are divisible by  $K$ . Consider the problem

$$\min \left\{ g(\mathbf{r}_V) : \mathbf{r}_V \in \mathcal{R}^*(V) \cap \mathbb{Q}_K^{|V|} \right\}. \tag{12}$$

The purpose is to search for a fractional egalitarian solution  $\mathbf{r}_V^*$  with an LCM  $|\mathcal{P}^*| - 1$ . The objective function in (12) is a separable convex function, for which local optimality w.r.t. the elementary exchange  $\chi_i - \chi_j$  implies the global optimality. See Lemma 2 below. Here,  $\chi_i - \chi_j$  denotes the cost/rate exchange between users  $i$  and  $j$  in the game  $\Omega(V, \hat{f}_{R^*})$  (The optimization criterion in Lemma 2 is related to the discrete convexity: The problem in (11)

exhibits  $M$ -convexity on the real number set (Section 1.4.2 in [42]), which also leads to the  $M$ -convexity on the fractional number set of (12). This is essentially due to the  $M$ -convexity of a submodular base polyhedron (Theorem 4.12 and Proposition 4.13 in [42]). See also Appendix A for the definition of the submodular base polyhedron).

**Lemma 2.** In CCDE,  $\mathbf{r}_V^*$  is the minimizer of (12) if and only if, for all  $i, j \in V$  and positive integer  $\zeta \in \mathbb{Z}_{++}$  such that  $\mathbf{r}_V^* + \frac{\zeta}{K}(\chi_i - \chi_j) \in \mathcal{R}^*(V)$ ,

$$g(\mathbf{r}_V^*) \leq g(\mathbf{r}_V^* + \frac{\zeta}{K}(\chi_i - \chi_j)),$$

where  $K = |\mathcal{P}^*| - 1$ .

**Proof.** The proof is based on a necessary and sufficient condition for the minimizer of (11) for any convex function  $g$  in (Theorem 20.3 in [11]):  $\mathbf{r}_V^*$  is the minimizer of (11) if and only if, for all  $i, j \in V$  and positive integer  $\epsilon > 0$  such that  $\mathbf{r}_V^* + \epsilon(\chi_i - \chi_j) \in \mathcal{R}^*(V)$ ,  $g(\mathbf{r}_V^*) \leq g(\mathbf{r}_V^* + \epsilon(\chi_i - \chi_j))$ . In CCDE, the entropy function  $H$  is integer-valued and  $\mathcal{R}^*(V)$  is fractional with denominator  $K = |\mathcal{P}^*| - 1$  so that the value of  $\hat{f}_{\mathcal{R}^*}(X)$  has the denominator  $K = |\mathcal{P}^*| - 1$  for all  $X \subseteq V$ . Furthermore, all extreme points in  $\text{EX}(V)$  have the LCM  $K = |\mathcal{P}^*| - 1$  (Corollary 10 in [8]). Therefore, for any  $\mathbf{r}_V \in \mathcal{R}^*(V) \cap \mathbb{Q}_K^{|V|}$ , if  $\mathbf{r}_V + \epsilon(\chi_i - \chi_j) \in \mathcal{R}^*(V)$ , then  $\mathbf{r}_V + \frac{1}{K}(\chi_i - \chi_j) \in \mathcal{R}^*(V) \cap \mathbb{Q}_K^{|V|}$ . So, Lemma 2 is the result of Theorem 20.3 in [11] on the set  $\mathcal{R}^*(V) \cap \mathbb{Q}_K^{|V|}$ .  $\square$

---

**Algorithm 1:** Steepest descent algorithm (SDA)

---

```

input : a positive integer  $K = |\mathcal{P}^*| - 1$  and an initial point  $\mathbf{r}_V^{(0)} \in \mathcal{R}^*(V) \cap \mathbb{Q}_{|\mathcal{P}^*|-1}^{|V|}$ 
output:  $\mathbf{r}_V^{(n)}$ , the minimizer of (12)
1 begin
2    $n \leftarrow 0$ ;
3   repeat
4     forall  $i \in V$  do
5        $\text{dep}(\mathbf{r}_V^{(n)}, i) \leftarrow$  the minimal minimizer of
                                      $\min\{f_{\mathcal{R}^*}(X) - r^{(n)}(X) : i \in X \subseteq V\};$ 
                                     (13)
6     end
7      $(i^*, j^*) \leftarrow \text{argmin}\{g(\mathbf{r}_V^{(n)} + \frac{1}{K}(\chi_i - \chi_j)) : i, j \in V, j \in \text{dep}(\mathbf{r}_V^{(n)}, i) \setminus \{i\}\};$ 
8     if  $g(\mathbf{r}_V^{(n)} + \frac{1}{K}(\chi_{i^*} - \chi_{j^*})) < g(\mathbf{r}_V^{(n)})$  then
9        $\mathbf{r}_V^{(n+1)} \leftarrow \mathbf{r}_V^{(n)} + \frac{1}{K}(\chi_{i^*} - \chi_{j^*});$ 
10       $n \leftarrow n + 1$ ;
11    else
12       $\mathbf{r}_V^{(n+1)} \leftarrow \mathbf{r}_V^{(n)}$ ;
13    endif
14  until  $\mathbf{r}_V^{(n+1)} = \mathbf{r}_V^{(n)}$ ;
15  return  $\mathbf{r}_V^{(n)}$ ;
16 end

```

---

Lemma 2 directly suggests the steepest descent algorithm (SDA) in Algorithm 1 (The SDA algorithm is also based on a discrete convex minimization algorithm in (Section 10.1.1 in [42]), which has been adopted in (Algorithm 1) for determining an integer-valued egalitarian solution for CCDE. The difference is that we use a dependence function  $\text{dep}$  to search the steepest descent direction, which is more efficient than the brute-force search in (Algorithm 1). Furthermore, note that (Algorithm 1) only determines a real-valued

egalitarian solution, which may be unable to be implemented in some practical systems, e.g., CCDE).

Furthermore, note that, as an input to the SDA, the initial point  $\mathbf{r}_V^{(0)} \in \mathcal{R}^*(V) \cap \mathbb{Q}_{|\mathcal{P}^*|-1}^{|V|}$  can be searched by the MDA algorithm at the same time when the minimum sum-rate problem is solved (Corollary 28(a) in [8]). The optimality of the SDA algorithm is stated below.

**Theorem 2.** For CCDE, the SDA algorithm generates an estimation sequence  $\{\mathbf{r}_V^{(n)}\}$  that converges on the minimizer  $\mathbf{r}_V^*$  of (12).

**Proof.** Consider the recursive process

$$\mathbf{r}_V^{(n+1)} = \mathbf{r}_V^{(n)} + \frac{1}{K}(\chi_{i^*} - \chi_{j^*}),$$

where  $(i^*, j^*) = \operatorname{argmin}\{f(\mathbf{r}_V^{(n)} + \frac{1}{K}(\chi_i - \chi_j)) : \mathbf{r}_V^{(n)} + \frac{1}{K}(\chi_i - \chi_j) \in \mathcal{R}^*(V), i, j \in V\}$ . This is a steepest descent approach: in each iteration  $n$ , we move from the current estimation  $\mathbf{r}_V^{(n)}$  in the steepest elementary exchange  $\chi_{i^*} - \chi_{j^*}$  by a constant step size  $\frac{1}{K}$ . Based on Lemma 2, starting with any initial  $\mathbf{r}_V^{(0)} \in \mathcal{R}^*(V) \cap \mathbb{Q}_K^{|V|}$ , the minimum of (12) is reached when this recursion converges, i.e., when  $\mathbf{r}_V^{(n+1)} = \mathbf{r}_V^{(n)}$ .

For  $\mathbf{r}_V \in \mathcal{R}^*(V) \cap \mathbb{Q}_K^{|V|}$ , consider the dependence function (Sections 2.2 and 2.3, Equations (2.14), (2.15), (2.18) and (2.19) in [11])

$$\begin{aligned} \operatorname{dep}(\mathbf{r}_V, i) &= \{j \in V : \max\{\epsilon : \mathbf{r}_V + \epsilon(\chi_i - \chi_j) \in \mathcal{R}^*(V)\} > 0\} \\ &= \bigcap \operatorname{argmin}\{f_{R^*}(X) - r(X) : i \in X \subseteq V\}. \end{aligned} \tag{14}$$

The last equality (14) states that  $\operatorname{dep}(\mathbf{r}_V, i)$  is the minimal minimizer of  $\min\{f_{R^*}(X) - r(X) : i \in X \subseteq V\}$  (The last equality (14) is shown in (Equations (2.14) and (2.15) in [11]) due to the min-max theorem (Corollary 3.4 in [11])). The minimizers of  $\min\{f_{R^*}(X) - r(X) : i \in X \subseteq V\}$  form a set lattice and the smallest/minimal is the intersection of all minimizers. See Sections 2.2 and 2.3 in [11] for details). A trivial case is that  $i \in \operatorname{dep}(\mathbf{r}_V, i)$ . Based on (14), we have  $\mathbf{r}_V + \frac{1}{K}(\chi_i - \chi_j) \notin \mathcal{R}^*(V) \cap \mathbb{Q}_K^{|V|}$  for all  $i, j \in V : j \notin \operatorname{dep}(\mathbf{r}_V, i) \setminus \{i\}$ . So, for all iterations  $n$  of the recursion above,  $\mathbf{r}_V^{(n)} \in \mathcal{R}^*(V) \cap \mathbb{Q}_K^{|V|}$  and

$$(i^*, j^*) = \operatorname{argmin}\{f(\mathbf{r}_V^{(n)} + \frac{1}{K}(\chi_i - \chi_j)) : i, j \in V, j \in \operatorname{dep}(\mathbf{r}_V, i) \setminus \{i\}\}.$$

Therefore, theorem holds.  $\square$

**Remark 1.** According to the proofs of Lemma 2 and Theorem 2, if  $K \neq |\mathcal{P}^*| - 1$ , we could have  $\mathbf{r}_V^{(n)} \notin \mathcal{R}^*(V)$  for some iteration  $n$  in the SDA algorithm, or the estimation sequence converges on, but may not reach exactly, the minimizer of (12), i.e., the output vector  $\mathbf{r}_V^{(n)}$  can be a suboptimal solution of (12).

**Example 7.** For the 5-user system in Example 1, we first apply the MDA algorithm in [8] and get the minimum sum-rate  $R^* = \frac{13}{2}$ , the fundamental partition  $\mathcal{P}^* = \{\{1, 4, 5\}, \{2\}, \{3\}\}$  and an extreme point  $(1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0) \in \operatorname{EX}(V)$  in the core  $\mathcal{R}^*(V)$ . By setting  $K = |\mathcal{P}^*| - 1 = 2$  and  $\mathbf{w}_V = \mathbf{1}$ , we start the SDA algorithm with the initial point  $\mathbf{r}_V^{(0)} = (1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0)$ .

At the first iteration  $n = 1$ , we have

$$\begin{aligned} \text{dep}(\mathbf{r}_V^{(0)}, 1) &= \{1, 4\}, \text{dep}(\mathbf{r}_V^{(0)}, 2) = \{2\}, \\ \text{dep}(\mathbf{r}_V^{(0)}, 3) &= \{3\}, \text{dep}(\mathbf{r}_V^{(0)}, 4) = \{4\}, \\ \text{dep}(\mathbf{r}_V^{(0)}, 5) &= \{4, 5\}. \end{aligned} \tag{15}$$

Then,  $\{(i, j) : j \in \text{dep}(\mathbf{r}_V^{(0)}, i) \setminus \{i\}\} = \{(1, 4), (4, 5)\}$ . For  $\mathbf{r}_V^{(0)} + \frac{1}{2}(\chi_1 - \chi_4) = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 4, 0)$  and  $\mathbf{r}_V^{(0)} + \frac{1}{2}(\chi_4 - \chi_5) = (1, \frac{1}{2}, \frac{1}{2}, 4, \frac{1}{2})$ , we have  $g(\mathbf{r}_V^{(0)} + \frac{1}{2}(\chi_4 - \chi_5)) < g(\mathbf{r}_V^{(0)} + \frac{1}{2}(\chi_1 - \chi_4))$  and, therefore,  $(i^*, j^*) = (4, 5)$ . Since  $g(\mathbf{r}_V^{(0)} + \frac{1}{2}(\chi_4 - \chi_5)) < g(\mathbf{r}_V^{(0)})$ , we assign  $\mathbf{r}_V^{(1)} = \mathbf{r}_V^{(0)} + \frac{1}{2}(\chi_4 - \chi_5) = (1, \frac{1}{2}, \frac{1}{2}, 4, \frac{1}{2})$  and continue the iteration.

By repeating the same procedure in each iteration, we get the estimation sequence  $\{\mathbf{r}_V^{(n)}\}$  that results in the update path

$$\begin{aligned} (1, \frac{1}{2}, \frac{1}{2}, \frac{9}{2}, 0) &\rightarrow (1, \frac{1}{2}, \frac{1}{2}, 4, \frac{1}{2}) \rightarrow (1, \frac{1}{2}, \frac{1}{2}, \frac{7}{2}, 1) \\ &\rightarrow (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 3, 1) \rightarrow (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{5}{2}, \frac{3}{2}) \rightarrow (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2). \end{aligned}$$

The recursion converges at  $n = 6$ , where we have  $\mathbf{r}_V^{(6)} = \mathbf{r}_V^{(5)} = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2)$ , which is the minimizer  $\mathbf{r}_V^* = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2)$  of (12) for  $|\mathcal{P}^*| - 1 = 2$  and  $\mathbf{w}_V = \mathbf{1}$ . Here,  $\mathbf{r}_V^* = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2)$  is a fractional egalitarian solution, a fair optimal rate vector in  $\mathcal{R}^*(V)$ , that can be implemented by 2-packet-splitting in CCDE.

### 5.2. Dependence Function

Based on (14), Lemma 1(b) and the discussion in Section 3.3, it is not difficult to see that, for all  $\mathbf{r}_V \in \mathcal{R}^*(V)$ , if  $j \in \text{dep}(\mathbf{r}_V, i)$  for any  $i, j \in V$ , then  $Z_i$  and  $Z_j$  are mutually dependent given the common randomness  $\Lambda = H(V) - R^*$  in  $U$ , i.e.,  $I(\{i\}; \{j\} | U) \neq 0$ , hence the name dependence function. Moreover, due to the fact that  $j \in \text{dep}(\mathbf{r}_V, i)$ , we can transfer arbitrarily small, but nonzero, coding cost from user  $j$  to user  $i$  for encoding the mutually shared information between users  $i$  and  $j$ , which is consistent with the nonzero exchange rate in Section 3.3.

In addition, we must have  $\text{dep}(\mathbf{r}_V, i) \subseteq C$  for the coalition  $C \in \mathcal{P}^*$  such that  $i \in C$ , e.g., (15). This is because  $I(\{i\}; \{j\} | U) = 0$  for all  $i \in C, j \in C'$  such that  $C \neq C'$  and  $I(\{i\}; \{j\} | U) \neq 0$  for all  $i, j \in C$ , i.e., given the common randomness in  $U$ , any  $Z_i$  is only mutually dependent on any other  $Z_j$  in the same coalition  $C \in \mathcal{P}^*$ . This will be formally stated as the decomposition of  $\mathbf{r}_V^*$  in Theorem 3.

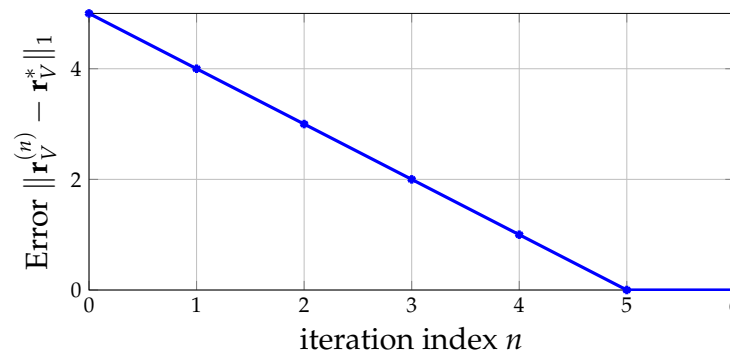
### 5.3. Complexity and Distributed Implementation

The SDA algorithm requires oracle calls of  $f_{R^*}$ , instead of  $\hat{f}_{R^*}$ , which is equivalent to the entry of the entropy function  $H$  and avoids the complexity of calculating the Dilworth truncation. We derive the worst-case complexity of SDA as follows. For any initial point  $\mathbf{r}_V^{(0)}$ , the total number of iterations of the SDA algorithm is  $\frac{K \cdot \|\mathbf{r}_V^{(0)} - \mathbf{r}_V^*\|_1}{2}$ . Let

$$L(V) = \max \left\{ \|\mathbf{r}_V - \mathbf{r}'_V\|_1 : \mathbf{r}_V, \mathbf{r}'_V \in \mathcal{R}^*(V) \cap \mathbb{Q}_K^{|V|} \right\}$$

denote the  $\ell_1$ -size of the core  $\mathcal{R}^*(V)$ . The maximum number of iterations of the SDA algorithm is  $\frac{K \cdot L(V)}{2}$ . The minimization problem (13) in step 5 in the SDA algorithm is a SFM due to the intersecting submodularity of  $f_{R^*}$  (Lemma 3 in [8]). Thus, each iteration of the SDA algorithm completes in  $O(|V| \cdot \text{SFM}(|V|))$  time and the overall complexity is  $O(K \cdot L(V) \cdot |V| \cdot \text{SFM}(|V|))$  (The reason that the  $\ell_1$ -size determines the upper bound on the number of iterations is explained in detail in (Section 10.1.1 in [42])).

**Example 8.** For the estimation sequence  $\{\mathbf{r}_V^{(n)}\}$  generated in Example 7 by the SDA algorithm, we show the error of the estimation  $\mathbf{r}_V^{(n)}$  in terms of the  $\ell_1$ -norm  $\|\mathbf{r}_V^{(n)} - \mathbf{r}_V^*\|_1$  in Figure 4. Since in each iteration of the SDA algorithm, the estimation  $\mathbf{r}_V^{(n)}$  is updated along the steepest elementary exchange  $\chi_{i^*} - \chi_{j^*}$  by step size  $\frac{1}{K} = \frac{1}{2}$  toward the optimizer  $\mathbf{r}_V^*$ , we necessarily have  $\|\mathbf{r}_V^{(n)} - \mathbf{r}_V^*\|_1$  decreased by  $\frac{2}{K} = 1$  each time. As in Figure 4, we have the error  $\|\mathbf{r}_V^{(n)} - \mathbf{r}_V^*\|_1$ , a linearly decreasing curve. In this case, there are  $\frac{K \cdot \|\mathbf{r}_V^{(0)} - \mathbf{r}_V^*\|_1}{2} = 5$  iterations in the SDA algorithm so that we incur  $5 \cdot |V|$  calls of  $O(\text{SFM}(|V|))$ . In general, since the  $\ell_1$ -size of  $\mathcal{R}^*(V)$  is  $L(V) = 6$ , the worst-case complexity of the SDA algorithm when applied to the 5-user system in Figure 1, is  $6 \cdot |V|$  calls of  $O(\text{SFM}(|V|))$ .



**Figure 4.** The error measured by the  $\ell_1$ -norm  $\|\mathbf{r}_V^{(n)} - \mathbf{r}_V^*\|_1$  of the estimation sequence  $\{\mathbf{r}_V^{(n)}\}$  generated by the SDA algorithm in Example 7 to determine the fractional egalitarian solution in  $\mathcal{R}^*(V)$ , the minimizer of  $\min \{ \sum_{i \in V} r_i^2 : \mathbf{r}_V \in \mathcal{R}^*(V) \cap \mathbb{Q}_{|P|_*-1}^{|V|} \}$ . The error linearly decreases to zero with gradient  $-1$ ; i.e., the  $\ell_1$ -norm  $\|\mathbf{r}_V^{(n)} - \mathbf{r}_V^*\|_1$  is reduced by  $\frac{2}{|P^*|-1} = 1$  in each iteration.

The SDA algorithm can also be implemented in a decentralized manner: let each user  $i$  obtain the dependence function  $\text{dep}(\mathbf{r}_V^{(n)}, i)$ , a set of mutually dependent users given the common randomness in  $U$ , by him/herself in steps 4 to 6; the steps 7 to 13 can be completed by users' communications over the broadcast channels. By doing so, the computational complexity incurred at each user is  $O(K \cdot L(V) \cdot \text{SFM}(|V|))$ .

#### 5.4. Decomposition

Similar to the decomposition of the Shapley value in Theorem 1, we also have the decomposition property of the egalitarian solution in Theorem 3. We omit the proof since it is a direct result of Corollary 42 in [8], Lemma 1 and Lemma 2(b).

**Theorem 3.** With  $\mathbf{r}_V^*$  as the egalitarian solution, the minimizer of (11), or the fractional egalitarian solution, the minimizer of (12),  $\mathbf{r}_V^* = \bigoplus_{C \in \mathcal{P}^*} \mathbf{r}_C^*$ , where  $\mathbf{r}_C^*$  is the egalitarian solution or fractional egalitarian solution, respectively, in the core  $\mathcal{R}^*(C)$  of the subgame  $\Omega(C, \hat{f}_{R^*})$ .

Theorem 3 states that the egalitarian solution  $\mathbf{r}_V^*$  can be determined by allowing the subgames  $\Omega(C, \hat{f}_{R^*})$  for all  $C \in \mathcal{P}^*$  to obtain their own  $\mathbf{r}_C^*$ . This decomposition method can be used in combination with the SDA algorithm so that the complexity is reduced to  $O(K \cdot L(\hat{C}) \cdot |\hat{C}| \cdot \text{SFM}(|\hat{C}|))$ , where  $L(\hat{C})$  is the  $\ell_1$ -size of the core  $\mathcal{R}^*(\hat{C})$  of the subgame  $\Omega(\hat{C}, \hat{f}_{R^*})$  of maximum size. In addition, the users in each subgame can run the SDA algorithm in a distributed manner as discussed in Section 5.3 and therefore the complexity incurred at each user is  $O(K \cdot L(\hat{C}) \cdot \text{SFM}(|\hat{C}|))$ .

**Remark 2.** Theorems 1 and 3 justify the exchange rate resulted from the mutual dependence in Section 3.3 when the game  $\Omega(V, \hat{f}_{R^*})$  is decomposed by the fundamental partition  $\mathcal{P}^*$  into the subgames  $\Omega(C, \hat{f}_{R^*})$  for all  $C \in \mathcal{P}^*$ : since the exchange rate, or mutual dependence, is only nonzero



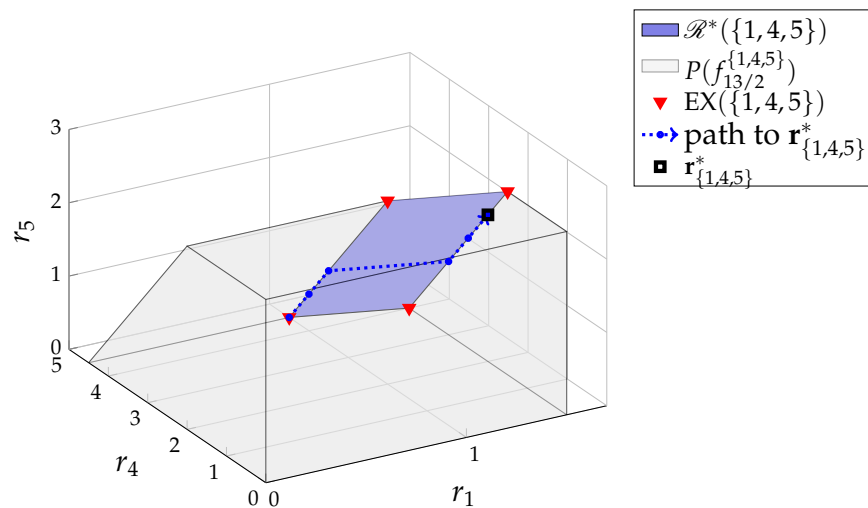
inside each subgame  $\Omega(C, \hat{f}_{R^*})$ , we just need to let the users cooperating in the same  $\Omega(C, \hat{f}_{R^*})$  decide how to attain fairness.

**Example 9.** For the 5-user system in Example 1, consider searching the fractional egalitarian solution w.r.t.  $\mathbf{w}_V = \mathbf{1}$  in Example 7 by the decomposition method in Theorem 3. We first decompose  $\Omega(V, \hat{f}_{R^*})$  into subgames  $\Omega(\{1, 4, 5\}, \hat{f}_{R^*})$ ,  $\Omega(\{2\}, \hat{f}_{R^*})$  and  $\Omega(\{3\}, \hat{f}_{R^*})$ . For the subgames  $\Omega(\{2\}, \hat{f}_{R^*})$  and  $\Omega(\{3\}, \hat{f}_{R^*})$ , we can directly assign  $r_2^* = \frac{1}{2}$  and  $r_3^* = \frac{1}{2}$ , respectively. For the subgame  $\Omega(\{1, 4, 5\}, \hat{f}_{R^*})$ , we apply the SDA algorithm and get the following update path to the fractional egalitarian solution  $\mathbf{r}_{\{1,4,5\}}^* = (\frac{3}{2}, 2, 2)$ :

$$(1, \frac{9}{2}, 0) \rightarrow (1, 4, \frac{1}{2}) \rightarrow (1, \frac{7}{2}, 1) \rightarrow (\frac{3}{2}, 3, 1) \rightarrow (\frac{3}{2}, \frac{5}{2}, \frac{3}{2}) \rightarrow (\frac{3}{2}, 2, 2).$$

See Figure 5. Then, we get  $\mathbf{r}_V^* = r_2^* \oplus r_3^* \oplus \mathbf{r}_{\{1,4,5\}}^* = (\frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 2, 2)$ , the fractional egalitarian solution w.r.t.  $\mathbf{w}_V = \mathbf{1}$  in  $\mathcal{R}^*(V) \cap \mathbb{Q}_2^5$ .

In this case, we still have 5 iterations in the SDA algorithm and the convergence performance is exactly the same as in Figure 4. However, the complexity reduces to  $5 \cdot |\{1, 4, 5\}|$  calls of  $O(\text{SFM}(|\{1, 4, 5\}|))$ . In general, since  $L(\{1, 4, 5\}) = 6$ , the complexity of the SDA algorithm when applied to the subgame  $\Omega(\{1, 4, 5\}, \hat{f}_{R^*})$ , is  $6 \cdot |\{1, 4, 5\}|$  calls of  $O(\text{SFM}(|\{1, 4, 5\}|))$ .



**Figure 5.** By applying the SDA algorithm to the subgame  $\Omega(\{1, 4, 5\}, \hat{f}_{R^*})$  of the 5-user system in Example 1 with the initial point  $\mathbf{r}_{\{1,4,5\}}^{(0)} = (1, \frac{9}{2}, 0)$ , we get the estimation sequence  $\{\mathbf{r}_{\{1,4,5\}}^{(n)}\}$  resulting an update path toward the fractional egalitarian solution  $\mathbf{r}_{\{1,4,5\}}^*$ , the minimizer of  $\min \{ \sum_{i \in \{1,4,5\}} r_i^2 : \mathbf{r}_{\{1,4,5\}} \in \mathcal{R}^*(\{1, 4, 5\}) \cap \mathbb{Q}_{|\mathcal{P}^*|-1}^3 \}$ .

### 6. Conclusions

We established the equivalence between the optimal rate region of CO and the core of a coalitional game with the characteristic cost function being the Dilworth truncation  $\hat{f}_{R^*}$  measuring the remaining information  $H(X|U)$  in  $Z_X$  for all subsets  $X \subseteq V$  given the common randomness in  $U$ . For attaining fairness in the optimal rate region, we considered the Shapley value and the egalitarian solution. The Shapley value differs from the egalitarian solution in that the fairness is attained if each user  $i$  is penalized by the expected marginal cost or source coding rate  $H(X \sqcup \{i\}|U) - H(X|U)$  he/she incurs if in coalition  $X$ . By utilizing the fact that the Shapley value is the average over all extreme points in the core, we showed that an approximation, instead of the exact Shapley value, can be obtained by taking the mean over a desired number of randomly generated extreme points. We also proposed the SDA algorithm for obtaining the egalitarian solution in the core that can be implemented in CCDE by  $(|\mathcal{P}^*| - 1)$ -packet-splitting. We showed that the

game is itself decomposable by the fundamental partition  $\mathcal{P}^*$  so that, given the common randomness,  $Z_C$  and  $Z_{C'}$  for any two distinct  $C, C' \in \mathcal{P}^*$  are mutually independent, while  $Z_i$  and  $Z_j$  for all  $i, j \in C$  are mutually dependent. This dependence relationship leads to a decomposition method for obtaining the fair solutions: the Shapley value and the egalitarian solution can be obtained independently within each subgame.

The methods for searching the Shapley value and the egalitarian solution in this paper require the solutions to the minimum sum-rate problem, the value of  $R^*$  and  $\mathcal{P}^*$  and also an optimal rate vector in  $\mathcal{R}^*(V)$  to initiate the SDA. To further improve the efficiency of attaining fairness in CO, it is worth studying whether we can directly attain the fairness in the optimal rate region without solving the minimum sum-rate problem first. On the other hand, apart from the fact that the egalitarian solution is more suitable to CCDE and WSN, it is worth understanding to which scenarios the fairness suggested by the Shapley value applies. Finally, the fractional egalitarian solution only determines a fair rate assigned to each user in CCDE. We still need a complete network coding scheme that also specifies the coefficients in the linear combination of chunks in each transmission.

**Author Contributions:** Conceptualization, methodology, validation, formal analysis and investigation, N.D.; writing—original draft preparation, N.D.; writing—review and editing, N.D., P.S., D.S. and T.R. All authors have read and agreed to the published version of the manuscript.

**Funding:** This research received no external funding.

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Not applicable.

**Acknowledgments:** The authors would like to thank Rodney Kennedy for his comments on the early version of the SDA algorithm. They are also very grateful to Chung Chan, Qiaoqiao Zhou and Tie Liu for their early contributions to the coalitional game model of CO and the decomposition property of the egalitarian solution, as well as the insightful discussions about the interpretation of the Shapley value. The authors would like to thank Ali Al-Bashabsheh for his comments on the egalitarian solution in Section 5 of this paper.

**Conflicts of Interest:** The authors declare no conflict of interest.

## Appendix A. Preliminaries

A set function  $f: 2^V \mapsto \mathbb{R}$  is *submodular* if

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y) \quad (\text{A1})$$

holds for all  $X, Y \subseteq V$  (Section 2.3 in [11]). A set function  $f$  is *intersecting submodular* if the submodular inequality (A1) holds for all  $X, Y \subseteq V$  such that  $X \cap Y \neq \emptyset$  (Section 2.3 in [11]).  $B(f) = \{\mathbf{r}_V \in P(f): \mathbf{r}_V = f(V)\}$  is a *submodular base polyhedron* if  $f$  is submodular. For a submodular function  $f: 2^V \mapsto \mathbb{R}$ ,

$$\min\{f(X): X \subseteq V\} \quad (\text{A2})$$

is called *submodular function minimization (SFM)* problem. We assume that the value of  $f(X)$  for any  $X \subseteq V$  can be obtained by an oracle call and  $\delta$  refers to the upper bound on the computation time of this oracle call. It is shown in [41,43–47] that an SFM problem can be solved in time polynomial in  $\delta$ . The SFM algorithms proposed in [34–39] vary in computational complexity. The exact completion time of an SFM depends on the size of the ground set  $V$ . For example, the SFM algorithm proposed in [48] completes in  $O(|V|^5 \cdot \delta + |V|^6)$  time. With  $O(\text{SFM}(|V|))$ , we denote the computational complexity of solving the SFM problem (A2).

A set function  $f$  is a *polymatroid rank function* if it is (a) normalized:  $f(\emptyset) = 0$ ; (b) monotonic:  $f(X) \geq f(Y)$  for all  $X, Y \subseteq V$  such that  $Y \subseteq X$ ; and (c) submodular

(Section 2.2 in [11]). It is shown in (Section 4.2 in [49]) that the entropy function  $H$  is a polymatroid rank function.

### Appendix B. Background on Coalitional Game

Due to the submodularity of  $\hat{f}_{R^*}$ , the game  $\Omega(V, \hat{f}_{R^*})$  is a convex game, for which the core is always nonempty (Section 2 in [32]). This also explains the nonemptiness of the core, or the optimal rate region,  $\mathcal{R}^*(V)$ . The decomposition property is originally defined for the convex games in [32], which is consistent with the definition of disconnected submodular system in [11,50].

**Definition A1** (Decomposable Convex Game (Theorems 3.32 and 3.38, Lemma 3.37 in [11]) (This definition is based on the concept of the separator of a disconnected submodular system in [11,50])). *A convex game  $\Omega(V, f)$  with the characteristic cost function  $f$  is decomposable if*

$$f(X) = \sum_{C \in \mathcal{P}} f(X \cap C), \quad X \subseteq V, \quad (\text{A3})$$

*for some decomposer  $\mathcal{P} \in \Pi(V)$  such that  $\mathcal{P} \neq \{V\}$ ; Otherwise,  $\Omega(V, f)$  is indecomposable. For a decomposable convex game  $\Omega(V, f)$ , the subgame  $\Omega(C, f)$  is convex for each  $C \in \mathcal{P}$ .*

Since (A3) always holds for  $\mathcal{P} = \{V\}$ , an indecomposable game can be considered as convex game with the only decomposer being  $\{V\}$  so that the core  $\mathcal{R}^*(V)$  has the full dimension  $|V| - 1$  (Theorem 6(a) in [32]). If a game is decomposable, it must have at least one decomposer other than  $\{V\}$  and all decomposers form a partition lattice, where the finest and coarsest partitions uniquely exist [32,51]. It is shown in (Theorem 38 in [8]) that the fundamental partition  $\mathcal{P}^*$  is the finest decomposer of the game  $\Omega(V, \hat{f}_{R^*})$ .

### References

1. Csiszár, I.; Narayan, P. Secrecy capacities for multiple terminals. *IEEE Trans. Inf. Theory* **2004**, *50*, 3047–3061. [\[CrossRef\]](#)
2. Ma, N.; Ishwar, P. The Infinite-Message Limit of Two-Terminal Interactive Source Coding. *IEEE Trans. Inf. Theory* **2013**, *59*, 4071–4094. [\[CrossRef\]](#)
3. Orlitsky, A.; Roche, J. Coding for computing. *IEEE Trans. Inf. Theory* **2001**, *47*, 903–917. [\[CrossRef\]](#)
4. Zaidi, A. On two terminal interactive source coding for function computation with remote sources. In Proceedings of the 2015 IEEE Information Theory Workshop-Fall (ITW), Jeju Island, Korea, 11–15 October 2015; pp. 302–306.
5. El Rouayheb, S.; Sprintson, A.; Sadeghi, P. On coding for cooperative data exchange. In Proceedings of the 2010 IEEE Information Theory Workshop on Information Theory (ITW 2010, Cairo), Cairo, Egypt, 6–8 January 2010; pp. 1–5.
6. Courtade, T.; Wesel, R. Efficient universal recovery in broadcast networks. In Proceedings of the 2010 48th Annual Allerton Conference on Communication, Control, and Computing (Allerton), Monticello, IL, USA, 29 September–1 October 2010; pp. 1542–1549.
7. Courtade, T.; Xie, B.; Wesel, R. Optimal exchange of packets for universal recovery in broadcast networks. In Proceedings of the 2010-Milcom 2010 Military Communications Conference, San Jose, CA, USA, 31 October–3 November 2010; pp. 2250–2255.
8. Ding, N.; Chan, C.; Zhou, Q.; Kennedy, R.A.; Sadeghi, P. Determining Optimal Rates for Communication for Omniscience. *IEEE Trans. Inf. Theory* **2018**, *64*, 1919–1944. [\[CrossRef\]](#)
9. Milosavljevic, N.; Pawar, S.; Rouayheb, S.E.; Gastpar, M.; Ramchandran, K. Efficient Algorithms for the Data Exchange Problem. *IEEE Trans. Inf. Theory* **2016**, *62*, 1878–1896. [\[CrossRef\]](#)
10. Courtade, T.; Wesel, R. Coded Cooperative Data Exchange in Multihop Networks. *IEEE Trans. Inf. Theory* **2014**, *60*, 1136–1158. [\[CrossRef\]](#)
11. Fujishige, S. *Submodular Functions and Optimization*, 2nd ed.; Elsevier: Amsterdam, The Netherlands, 2005.
12. Sprintson, A.; Sadeghi, P.; Booker, G.; El Rouayheb, S. A randomized algorithm and performance bounds for coded cooperative data exchange. In Proceedings of the 2010 IEEE International Symposium on Information Theory, Austin, TX, USA, 18–12 June 2010; pp. 1888–1892.
13. Ahlswede, R.; Cai, N.; Li, S.R.; Yeung, R.W. Network information flow. *IEEE Trans. Inf. Theory* **2000**, *46*, 1204–1216. [\[CrossRef\]](#)
14. Ho, T.; Medard, M.; Koetter, R.; Karger, D.R.; Effros, M.; Shi, J.; Leong, B. A Random Linear Network Coding Approach to Multicast. *IEEE Trans. Inf. Theory* **2006**, *52*, 4413–4430. [\[CrossRef\]](#)
15. Ding, N.; Sadeghi, P.; Rakotoarivelo, T. Improving Computational Efficiency of Communication for Omniscience and Successive Omniscience. *IEEE Trans. Inf. Theory* **2021**, *67*, 4728–4746. [\[CrossRef\]](#)

16. Milosavljevic, N.; Pawar, S.; Gastpar, M.; Ramchandran, K. Efficient algorithms for the data exchange problem under fairness constraints. In Proceedings of the 2012 50th Annual Allerton Conference on Communication, Control, and Computing (Allerton), Monticello, IL, USA, 1–5 October 2012; pp. 502–508.
17. Tajbakhsh, S.; Sadeghi, P.; Shams, R. A Generalized Model for Cost and Fairness Analysis in Coded Cooperative Data Exchange. In Proceedings of the 2011 International Symposium on Networking Coding, Beijing, China, 25–27 July 2011; pp. 1–6.
18. Jain, R.; Chiu, D.M.; Hawe, W.R. *A Quantitative Measure of Fairness and Discrimination for Resource Allocation in Shared Computer System*; Eastern Research Laboratory, Digital Equipment Corporation: Hudson, MA, USA, 1984.
19. Dutta, B.; Ray, D. A concept of egalitarianism under participation constraints. *Econom. J. Econom. Soc.* **1989**, *57*, 615–635. [[CrossRef](#)]
20. Shapley, L.S. A value for  $n$ -person games. *Contrib. Theory Games* **1953**, *2*, 307–317.
21. Chan, C. On tightness of mutual dependence upperbound for secret-key capacity of multiple terminals. *arXiv* **2008**, arXiv:0805.3200.
22. Narayan, P.; Tyagi, H. Multiterminal Secrecy by Public Discussion. *Found. Trends Commun. Inf. Theory* **2016**, *13*, 129–275. [[CrossRef](#)]
23. Chan, C.; Al-Bashabsheh, A.; Ebrahimi, J.; Kaced, T.; Liu, T. Multivariate Mutual Information Inspired by Secret-Key Agreement. *Proc. IEEE* **2015**, *103*, 1883–1913. [[CrossRef](#)]
24. Cover, T.M.; Thomas, J.A. *Elements of Information Theory*; John Wiley & Sons: Hoboken, NJ, USA, 2012.
25. Ding, N.; Chan, C.; Zhou, Q.; Kennedy, R.A.; Sadeghi, P. A Faster Algorithm for Asymptotic Communication for Omniscience. In Proceedings of the 2016 IEEE Globecom Workshops (GC Wkshps), Washington, DC, USA, 4–8 December 2016; pp. 1–6.
26. Dilworth, R.P. Dependence relations in a semi-modular lattice. *Duke Math. J.* **1944**, *11*, 575–587. [[CrossRef](#)]
27. Narayanan, H. *Submodular Functions and Electrical Networks*; Elsevier: Amsterdam, The Netherlands, 1997.
28. Dutta, B. The egalitarian solution and reduced game properties in convex games. *Int. J. Game Theory* **1990**, *19*, 153–169. [[CrossRef](#)]
29. Courtade, T.; Wesel, R. Weighted universal recovery, practical secrecy, and an efficient algorithm for solving both. In Proceedings of the 2011 49th Annual Allerton Conference on Communication, Control, and Computing (Allerton), Monticello, IL, USA, 28–30 September 2011; pp. 1349–1357.
30. Shoham, Y.; Leyton-Brown, K. *Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations*; Cambridge University Press: New York, NY, USA, 2008.
31. Shapley, L.S.; Shubik, M. On market games. *J. Econ. Theory* **1969**, *1*, 9–25. [[CrossRef](#)]
32. Shapley, L.S. Cores of convex games. *Int. J. Game Theory* **1971**, *1*, 11–26. [[CrossRef](#)]
33. Edmonds, J. Submodular functions, matroids, and certain polyhedra. In *Combinatorial Optimization—Eureka, You Shrink!* Springer: Berlin/Heidelberg, Germany, 2003; pp. 11–26.
34. Liben-Nowell, D.; Sharp, A.; Wexler, T.; Woods, K. Computing Shapley Value in Supermodular Coalitional Games. In Proceedings of the 18th Annual International Conference (COCOON 2012), Sydney, Australia, 20–22 August 2012; pp. 568–579.
35. Conitzer, V.; Sandholm, T. Computing Shapley Values, Manipulating Value Division Schemes, and Checking Core Membership in Multi-issue Domains. In Proceedings of the 19th National Conference on Artificial Intelligence, San Jose, CA, USA, 25–29 July 2004; pp. 219–225.
36. Fatima, S.S.; Wooldridge, M.; Jennings, N.R. A Linear Approximation Method for the Shapley Value. *Artif. Intell.* **2008**, *172*, 1673–1699. [[CrossRef](#)]
37. Hokari, T. Monotone-path Dutta-Ray solutions on convex games. *Soc. Choice Welf.* **2002**, *19*, 825–844. [[CrossRef](#)]
38. Hokari, T.; van Gellekom, A. Population monotonicity and consistency in convex games: Some logical relations. *Int. J. Game Theory* **2003**, *31*, 593–607. [[CrossRef](#)]
39. Nagano, K.; Aihara, K. Equivalence of convex minimization problems over base polytopes. *Jpn. J. Ind. Appl. Math.* **2012**, *29*, 519–534. [[CrossRef](#)]
40. Nagano, K.; Kawahara, Y. Structured convex optimization under submodular constraints. *arXiv* **2013**, arXiv:1309.6850.
41. Fujishige, S.; Isotani, S. A submodular function minimization algorithm based on the minimum-norm base. *Pac. J. Optim.* **2011**, *7*, 3–17.
42. Murota, K. *Discrete Convex Analysis*; SIAM: Philadelphia, PA, USA, 2003.
43. Khachiyan, L. Polynomial algorithms in linear programming. *USSR Comput. Math. Math. Phys.* **1980**, *20*, 53–72. [[CrossRef](#)]
44. Grötschel, M.; Lovász, L.; Schrijver, A. *Geometric Algorithms and Combinatorial Optimization*; Springer: Berlin/Heidelberg, Germany, 2012.
45. Grötschel, M.; Lovász, L.; Schrijver, A. The ellipsoid method and its consequences in combinatorial optimization. *Combinatorica* **1981**, *1*, 169–197. [[CrossRef](#)]
46. Iwata, S. Submodular function minimization. *Math. Program.* **2007**, *112*, 45–64. [[CrossRef](#)]
47. Iwata, S.; Fleischer, L.; Fujishige, S. A Combinatorial Strongly Polynomial Algorithm for Minimizing Submodular Functions. *J. ACM* **2001**, *48*, 761–777. [[CrossRef](#)]
48. Orlin, J.B. A faster strongly polynomial time algorithm for submodular function minimization. *Math. Program.* **2009**, *118*, 237–251. [[CrossRef](#)]
49. Fujishige, S. Polymatroidal dependence structure of a set of random variables. *Inf. Control* **1978**, *39*, 55–72. [[CrossRef](#)]

- 
50. Bixby, R.E.; Cunningham, W.H.; Topkis, D.M. The Partial Order of a Polymatroid Extreme Point. *Math. Oper. Res.* **1985**, *10*, 367–378. [[CrossRef](#)]
  51. Narayanan, H. The principal lattice of partitions of a submodular function. *Linear Algebra Appl.* **1991**, *144*, 179–216. [[CrossRef](#)]