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Abstract

A physical system is determined by a finite set of initial conditions and laws represented by equations. The system is computable if we can solve the equations in all instances using a "finite body of mathematical knowledge". In this case, if the laws of the system can be coded into a computer program, then given the system's initial conditions of the system, one can compute the system's evolution.

This scenario is tacitly taken for granted. But is this reasonable? The answer is negative, and a straightforward example is when the initial conditions or equations use irrational numbers, like Chaitin's Omega Number: no program can deal with such numbers because of their "infinity".

Are there incomputable physical systems? This question has been theoretically studied in the last 30–40 years. This article presents a class of quantum protocols producing quantum random bits. Theoretically, we prove that every infinite sequence generated by these quantum protocols is strongly incomputable – no algorithm computing any bit of such a sequence can be proved correct. This theoretical result is not only more robust than the ones in the literature: experimental results support and complement it.

1 Introduction

According to Einstein [23]

> Physics constitutes a logical system of thought which is in a state of evolution ... The justification (truth content) of the system rests in the proof of the usefulness of the resulting theorems on the basis of sense experiences, where the relations of the latter to the former can only be comprehended intuitively.

This is in agreement with Hertz's contemplation of the relationship between physical and formal entities [28]:

> We form for ourselves images or symbols of external objects, and the form which we give them is such that the necessary consequents of the images in thought are always the images of the necessary consequents in nature of the things pictured.

Both Einstein and Hertz do not perceive a 'strong' connection between physical entities and their corresponding categories of mind, but rather a homomorphism. This raises the question: how much trust can we place in the theoretical categories regarding their usefulness in physics? What criteria can be provided to ensure and certify their applicability relative to our assumptions?

A physical system is determined by a finite set of initial conditions and laws represented by equations. The system is computable if we can solve the equations in all instances using a "finite body of mathematical knowledge". If the laws of the system can be coded into a computer program, then given the system's initial conditions, one can compute the system's evolution.

One needs to differentiate between operational, empirically accessible, observables on the one hand, and, on the other hand, theoretical assumptions and conventions – such as the existence of the continuum – that are not operational ("almost all" elements are incomputable and even random), but provide a convenient formalism [10].

This scenario is tacitly taken for granted in discussing issues of incomputability. But is this reasonable? The answer is negative, and a straightforward example is when the initial conditions or equations include irrational numbers, like Chaitin's Omega Number: no program can deal with such numbers because of their *infinity*.

Incomputability in physics has been studied by many authors [53, 54, 56, 38, 21, 44, 17, 19, 63, 35, 20, 8, 41, 29, 33, 60, 41, 13, 34, 1, 3, 27, 7, 30]. The results in all these articles are mainly theoretical, so following Einstein's above citation, we can ask: *what is their justification?* The word "real" in the title means "a justification of incomputability claims based on usefulness".

For sufficiently complex systems (even reversible) determinism on a "one-by (to)-one" evolution basis does not imply predictability [52]. For example, take the n-body problem as example: the respective series solutions [59, 43, 51, 57, 58] may be "very slowly" convergent [22], or even encode the Halting Problem [55].

Quantum underdetermination can be expressed in terms of gaps in the physical description [25, 26, § III.12-14]: Due to the unitarity of quantum evolution, information cannot be created nor annihilated. Therefore, if a quantised system encodes a finite amount of information any query above this information must inevitably be indeterminate and value indefinite relative to the original quantum state. One could, of course, suppose that the entanglement can fulfil such a request with the measurement apparatus and thus the environment at large [36, 37], resulting in an unbounded nesting argument with an ever-increasing Heisenberg cut. Yet the fact remains that, due to the finite (possibly relational [62]) amount of information encoded in a quantised system, information in excess of this amount cannot reside or be encoded in any pre-selected state.

This article uses a located form of the Kochen-Specker Theorem to derive a class of quantum protocols producing quantum random bits. Theoretically, we prove that every infinite sequence generated with these quantum protocols is strongly incomputable – no algorithm computing any bit of such a sequence can be proved correct. Such a result is more robust than the ones in the literature and satisfies Einstein's requirement of justification: the experimental results presented here confirm and complement incomputability and, quite importantly, the choice of physical assumptions.

The paper is organised as follows. In Section 2, we present the theoretical framework for the Localised Kochen-Specker Theorem, allowing the construction of strongly incomputable sequences via measurements of value-indefinite observables. In Section 3, we use a standard superconducting transmon system to implement logical states as qutrits and realise the theoretical quantum protocols in Section 2. In Section 4, we present a method to empirically show the incomputability of the outputs generated in Section 3. The last Section 5, we briefly discuss the results presented in this article and suggest further continuations.

2 3D-QRNG – Theory

In this section, we present the theoretical framework allowing the construction of value-indefinite observables, their tolerance to measurement errors and the certification of the degree of randomness of their outcomes.

2.1 Notation and definitions

The set of positive integers will be denoted by $\mathbb{N}$. Consider the alphabet $A_b = \{0, 1, \ldots, b-1\}$, where $b \geq 2$ is an integer; the elements of A_b are the digits used in natural positional representations of numbers in the interval $[0, 1)$ at base b. By A_b^* and A_b^ω we denote the sets of (finite) strings and (infinite) sequences over the alphabet A_b. Strings will be denoted by x, y, u, w; the length of the string $x = x_1 x_2 \ldots x_m$, $x_i \in A_b$, is denoted by $|x|_b = m$ (the subscript b will be omitted if it is clear from the context); A_b^m is the set of all strings of length m. Sequences will be denoted by $\mathbf{x} = x_1 x_2 \ldots$; the prefix of length m of $\mathbf{x}$ is the string $\mathbf{x}(m) = x_1 x_2 \ldots x_m$. Strings will be ordered quasi-lexicographically according to the natural order $0 < 1 < 2 < \cdots < b-1$ on the alphabet A_b. For example, for $b = 2$, we have $0 < 1 < 00 < 01 < 10 < 11 < 000 \ldots$. We assume knowledge of elementary computability theory over different size alphabets [11].

By $\mathbb{C}$, we denote the set of complex numbers. We then fix a positive integer $n \geq 2$ and let $O \subseteq \{P_\psi : |\psi\rangle \in \mathbb{C}^n\}$ be a non-empty set of one-dimensional projection observables on the Hilbert space $\mathbb{C}^n$.

A set $C \subset O$ is a *context* of O if C has n elements and for all $P_\psi, P_\phi \in C$ with $P_\psi \neq P_\phi, \langle\psi|\phi\rangle = O$. A *value assignment function* (on O) is a partial function $v : O \to \{0, 1\}$ assigning values to some (possibly all) observables in O. The partiality of the function v means that $v(P)$ can be $0, 1$ or indefinite. An observable $P \in O$ is *value definite* (under the assignment function v) if $v(P)$ is defined, i.e. it is 0 or 1; otherwise, it is *value indefinite* (under v). Similarly, we call O *value definite* (under v) if every observable $P \in O$ is value definite.

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2.2 Localised Kochen-Specker Theorem

We next present the main result used to construct a value indefinite observable. First, we assume the following premises:

- **Admissibility.** This assumption guarantees agreement with quantum mechanics predictions. Fix a set O of one-dimensional projection observables on $\mathbb{C}^n$ and the value assignment function $v : O \to \{0, 1\}$. Then v is *admissible* if for every context C of O, we have that $\sum_{P \in C} v(P) = 1$. Accordingly, only one projection observable in a context can be assigned the value 1.

- **Non-contextuality of definite values.** Every outcome obtained by measuring a value definite observable is *non-contextual*, i.e. it does not depend on other compatible observables which may be measured alongside it.

- **Eigenstate principle.** If a quantum system is prepared in the state $|\psi\rangle$, then the projection observable P_ψ is value definite.

The last assumption is motivated by Einstein, Podolsky and Rosen definition of *physical reality* [24, p. 777]: *If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists a* definite value *prior to observation corresponding to this physical quantity.* A criterion for value-definiteness results: *if a quantum system is prepared in an arbitrary state $|\psi\rangle \in \mathbb{C}^n$, then the measurement of the observable P_ψ should yield the outcome 1, hence, if $P_\psi \in O$, then $v(P_\psi) = 1$.*

We can now state the main result:

Theorem 1 (Localised Kochen-Specker Theorem [4, 5, 34, 7]) *Assume a quantum system prepared in the state $|\psi\rangle$ in a dimension $n \geq 3$ Hilbert space $\mathbf{C}^n$, and let $|\phi\rangle$ be any quantum state such that $0 < |\langle\psi|\phi\rangle| < 1$. If the following three conditions are satisfied: i) admissibility, ii) non-contextuality and iii) eigenstate principle, then the projection observable P_ψ is value indefinite.*

Theorem 1 states that, under the given assumptions, any quantum state $|\phi\rangle$ that is neither orthogonal nor parallel to $|\psi\rangle$ is *value indefinite*. This result has two major consequences:

1. it shows how to construct a value indefinite observable effectively,
2. it guarantees that the status of "value-indefiniteness" is invariant under minor errors in measurements: this is a significant property as no measurement is exact.

We note that Theorem 1, as the original Kochen-Specker Theorem [32], is not valid in $\mathbf{C}^2$, hence the requirement to work in $\mathbf{C}^3$.

How "good" is such a 3D-QRNG, i.e. what randomness properties can be *certified* for their outcomes? For example, can we prove that the outcomes of the 3D-QRNG are "better" than the outcomes produced by *any* pseudo-random number generator (PRNG)?

For certification, we use the following assumption, which is motivated by the fact that a computable sequence is the strongest form of "deterministic hidden variable":

- **epr principle**: If a repetition of measurements of an observable generates a computable sequence, then these observables are value definite.

Based on the Eigenstate and epr principles, we can prove that the answer to the last question is affirmative: *Any infinite repetition of the experiment measuring a quantum value indefinite observable generates an incomputable infinite sequence $x_1x_2\ldots$: no PRNG has this randomness property.*

A stronger result is true. Informally, a sequence $\mathbf{x}$ is bi-immune if no algorithm can generate infinitely many correct values of its elements (pairs, (i, x_i)). Formally, a. sequence $\mathbf{x} \in A_b^\omega$ ($b \geq 2$) is *bi-immune* if there is no partially computable function φ from $\mathbb{N}$ to A_b having an infinite domain $\mathrm{dom}(\varphi)$ with the property that $\varphi(i) = x_i$ for all $i \in \mathrm{dom}(\varphi)$ [9]).

Theorem 2 ([1, 7]) *Assume the Eigenstate and epr principles. An infinite repetition of the experiment measuring a quantum value indefinite observable in $\mathbb{C}^b$ always generates a b-bi-immune sequence $\mathbf{x} \in A_2^\omega$, for every $b \geq 2$.*

In particular, every sequence generated by the 3D-QRNG is 3-bi-immune.

Theorem 3 ([7]) *Assume the epr and Eigenstate principles. Let $\mathbf{x}$ be an infinite sequence obtained by measuring a quantum value indefinite observable in $\mathbb{C}^b$ in an infinite repetition of the experiment E. Then, no single bit x_i can be predicted.*

In particular, no single digit of every sequence $\mathbf{x} \in A_3^\omega$ generated by the 3D-QRNG can be algorithmically predicted.

The following simple morphism $\varphi\colon A_3 \to A_2$ transforms a ternary sequence into a binary sequence:

$$\varphi(a) = \begin{cases} 0, & \text{if } a = 0, \\ 1, & \text{if } a = 1, \\ 0 & \text{if } a = 2, \end{cases} \tag{1}$$

, which can be extended sequentially for strings, $\mathbf{y}(n) = \varphi(\mathbf{x}(n)) = \varphi(x_1)\varphi(x_2) \ldots \varphi(x_n)$ and sequences $\mathbf{y} = \varphi(\mathbf{x}) = \varphi(x_1)\varphi(x_2)\ldots\varphi(x_n)\ldots$. This transformation preserves 2-bimmunity:

Theorem 4 ([7]) *Assume the epr and Eigenstate principles. Let $\mathbf{y} = \varphi(\mathbf{x})$, where $\mathbf{x} \in A_3^\omega$ is a ternary sequence generated by the* 3D-QRNG *and φ is the alphabetic morphism defined in (1). Then, no single bit of $\mathbf{y} \in A_2^\omega$ can be predicted.*

As noted in [7], Theorem 1 shows that given a system prepared in state $|\psi\rangle$, a one-dimensional projection observable can only be value definite if it is an eigenstate of that observable. Consequently, for any diagonalisable observable O with spectral decomposition $O = \sum_{i=1}^{n} \lambda_i P_{\lambda_i}$, where λ_i denotes each distinct eigenvalue with corresponding eigenstate $|\lambda_i\rangle$, O has a predetermined measurement outcome if and only if each projector in its spectral decomposition has a predetermined measurement outcome. Thus, the previous result holds true to the outcome of the measurement of any observable with non-degenerate spectra. Such generalisation is particularly useful in the case when we use the value assignment function to represent a value definite observable. These results have been used to design the following quantum operators of the 3D-QRNG. These 3D-QRNGs operate in a succession of events of the form "preparation, measurement, reset", iterated indefinitely many times in an algorithmic fashion [1]. The first 3D-QRNG was designed in [1], realized in [34] and analysed in [2]. While the analysis failed to observe a strong advantage of the quantum random sequences due to incomputability, it has motivated the improvement in [7], in which the problematic probability zero branch $S_x = 0$ in Figure 1.

The next 3D-QRNG is presented in Figure 2. The unitary matrix U_x corresponding to the spin state operator S_x is

$$U_x = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}.$$

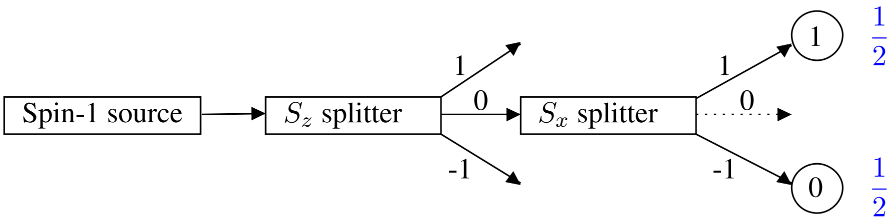

Figure 1: QRNG setup proposed in [1]; the values $\frac{1}{2}, \frac{1}{2}$ (in blue) correspond to the outcome probabilities

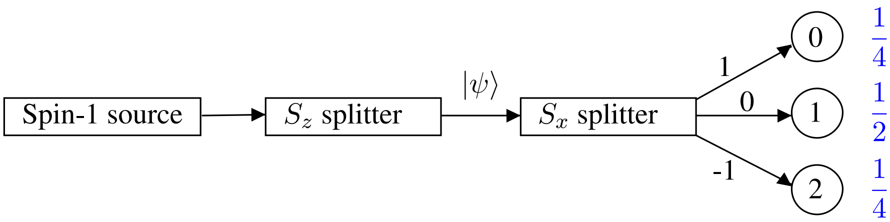

Figure 2: Blueprint for a new QRNG; the values $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ (in blue) correspond to the outcome probabilities of setups prepared in the state $|\psi\rangle = |\pm 1\rangle$

As U_x can be decomposed into two-dimensional transformations [16]

$$U_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ i\sqrt{\frac{2}{3}} & -\frac{i}{\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{i}{2} \\ 0 & 1 & 0 \\ \frac{i}{2} & 0 & -\frac{\sqrt{3}}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ 0 & i\sqrt{\frac{2}{3}} & -\frac{i}{\sqrt{3}} \end{pmatrix} .$$

a physical realisation of the unitary operator by a lossless beam splitter [**?**, 61] was obtained; the new outcome probabilities are 1/4,1/2,/1/4.

3 3D-QRNG – Physical Realisation

To realise the protocols shown in Figs. 1,2 we used a standard superconducting transmon system [34]. The transmon has a weakly anharmonic multi-level structure [31], and its three lowest energy eigenstates $|0\rangle, |1\rangle$ and $|2\rangle$ can be used as the logical states of a qutrit.

To implement the protocol shown in Fig. 1 we followed the recipe from [34] where the eigenstates of the S_z operator were mapped to the states of the qutrit as follows

$$\{|z,-1\rangle, |z,0\rangle, |z,+1\rangle\} \to \{|2\rangle, |0\rangle, |1\rangle\}. \tag{2}$$

This mapping provided an advantage of preparing $|z,0\rangle$ state by cooling down the transmon to the base temperature of a dilution refrigerator ($\sim 20\,$mK).

To perform an arbitrary rotation of the qutrit quantum state $R_{\hat{n}}^{i,i+1}(\phi)$ we applied microwave pulses resonant to the $|0\rangle \leftrightarrow |1\rangle$ or $|1\rangle \leftrightarrow |2\rangle$ transition frequencies, respectively. Two rotations $R_y^{12}(\pi) \cdot R_y^{01}(\pi/2)$ of the state before the dispersive measurement were used to engineer a measurement in the eigenbasis of S_x. The resulting measurement outcomes of the transmon energy eigenstates were mapped to the following outcomes of the measurement of S_x operator: $\{|0\rangle, |1\rangle, |2\rangle\} \rightarrow \{|x,+1\rangle, |x,-1\rangle, |x,0\rangle\}$.

To implement the protocol shown in Fig. 2 we used a slightly different encoding:

$$\{|z,-1\rangle, |z,0\rangle, |z,+1\rangle\} \rightarrow \{|1\rangle, |2\rangle, |0\rangle\}. \tag{3}$$

In this case, state $|z,+1\rangle$ was prepared by cooling the transmon. The following measurement in the eigenbasis of S_x was engineered by applying the same rotations $R_y^{01}(\pi/2) \cdot R_y^{12}(\pi/2)$ before the dispersive measurements. The measurement outcomes of the transmon were then mapped to the following outcomes of the measurement of S_x operator: $\{|0\rangle, |1\rangle, |2\rangle\} \rightarrow \{|x,0\rangle, |x,-1\rangle, |x,+1\rangle\}$.

To measure the transmon, we used the standard dispersive readout scheme where the transmon is capacitively coupled to a co-planar waveguide resonator. The difference between the frequency of the resonator ($f_r = 7.63$ GHz) and the $|0\rangle \leftrightarrow |1\rangle$ ($f_{01} = 5.49$ GHz) and $|1\rangle \leftrightarrow |2\rangle$ ($f_{12} = 5.16$ GHz) transitions of the transmon was designed to be much larger than the qubit-resonator coupling to ensure that the system is in the dispersive regime. In this regime, the frequency of the resonator depended on the states of the transmon and underwent shifts of -8.5 MHz or -15.5 MHz when the transmon was excited in $|1\rangle$ or $|2\rangle$ states, relative to f_r when the transmon was prepared in its ground state $|0\langle$ [31]. We used a Josephson parametric amplifier to distinguish between three different transmon states with high fidelity. In addition, we set the readout pulse frequency close to the cavity frequency corresponding to the $|1\rangle$ state of the qutrit, which allowed the three possible qutrit states to be well separated on the I-Q plane for the time-integrated signal measured with the heterodyne detection scheme. The readout frequency was then fine-tuned to maximise the three-level readout fidelity. The measurement response was classified using a convolutional neural network (CNN) to increase the readout fidelity further, as described in [42].

The procedure used to generate the random numbers required an initial calibration procedure typical for circuit quantum electrodynamics setups. This involved calibration of f_r, f_{01} and the $R_y^{01}(\pi)$ and $R_y^{01}(\pi/2)$ pulses. Two $R_y^{01}(\pi/2)$ pulses were used to fine-tune f_{01} using a Ramsey measurement. The $R_y^{01}(\pi)$ and $R_y^{01}(\pi/2)$ pulses were then fine-tuned with repeated pulses. A similar procedure was followed to calibrate for f_{12} and the $R_y^{12}(\pi)$ and $R_y^{12}(\pi/2)$ pulses.

After initial calibrations, we optimised the readout frequency of a single-shot readout using the Josephson parametric amplifier. The CNN is then trained for 50 cycles using 1024 measurements of the readout resonator after preparing each of the three states, $|0\rangle$,$|1\rangle$ and $|2\rangle$ as described in [42].

The procedure so far involved repeated measurements where the transmon was reset to $|0\rangle$ state by waiting $35\ \mu$s to reach thermal equilibrium (at a decay rate of 250 kHz). We used an active reset protocol described in [39] to increase the experiment cycle time. This involved a *reset pulse* to transfer the $|2\rangle$ state population to the readout resonator and let it decay much faster (at a decay rate of 4 MHz). An $R_y^{12}(\pi)$ pulse is then used to transfer the unwanted $|1\rangle$ state population to the $|2\rangle$ state, and the reset pulse was used again to transfer $|2\rangle$ state population to the readout resonator. The $R_y^{12}(\pi)$ (40 ns), reset pulse (370 ns), and a wait time (50 ns) for the readout resonator to decay were used four times in series to ensure the transmon is in the ground state, taking 1.84 us in total. The reset protocol was tested using standard acquisition methods and the CNN to ensure the CNN was performing as intended. The reset time, the preparation pulses for the protocol and the measurement pulse time amounted to 3.2 us, corresponding to a rate of 312.5 kHz.

To ensure robust generation of 100 Gbit of random numbers, the procedure outlined in Algorithm 2 was followed, involving intermittent checks of the CNN without reset, retraining the CNN if necessary and re-calibrating the transmon as shown in Algorithm 1 if that fails.

Algorithm 1 Calibration

procedure CALIBRATE ▷ Calibrates the transmon preparation and readout
    $T_{\text{rep}} \leftarrow 40\ \mu$s
    set measurement frequency to f_r
    set previously calibrated settings
    Ramsey frequency calibration for f_{01}
    Calibrate $R_y^{01}(\pi)$ and $R_y^{01}(\pi/2)$ pulses
    Ramsey frequency calibration for f_{12}
    Calibrate $R_y^{12}(\pi)$ and $R_y^{12}(\pi/2)$ pulses
    Calibrate reset pulse frequency
    set measurement frequency to $f_r - 9$ MHz
    Create convolutional neural network (CNN)
    Train CNN for 50 training cycles
end procedure

The errors of our protocol consisted of the initialisation errors, the errors of the control pulses, and the measurement errors. As the initialisation and control errors were calibrated to be kept within $< 1\%$, the measurement error was the dominant error of our protocol due to the relaxation of the higher excited states of the qutrit to the lower energy states during the readout time. The typical assignment fidelities were 95%, 88%, and 78% for the ground, first and second excited states, respectively. All the fidelities were continuously monitored during random number generation, and a drop in the value of the average assignment fidelity was used to trigger the re-calibration of the protocol (see Algorithm 2).

Algorithm 2 Generation

procedure RUNINDEX
    if files exist **then**
        $r \leftarrow 1+$ last *random_xxx.rbf* file number
    else
        return $r \leftarrow 0$
    end if
    return r
end procedure
$T_{\text{rep}} \leftarrow 40\ \mu\text{s}$
Prepare $|0\rangle$,$|1\rangle$ and $|2\rangle$ ▷ Cyclically for each repetition
Create convolutional neural network (CNN)
Train CNN for 50 training cycles
$f \leftarrow$ measurement accuracy
▷ Assignment fidelity as defined in [42]
$c \leftarrow 0$ ▷ Calibration counter used to terminate
$l \leftarrow 0$ ▷ Low f counter used to calibrate
$r \leftarrow$RUNINDEX
while r < 750 **do**
    while $f < 0.86$ **do**
        if $l > 20$ **then**
            if $c > 5$ **then**
                ERROR ▷ Calibrated 5 times already. Failed
            end if
            CALIBRATE
            $c \leftarrow c + 1$
            $l \leftarrow 0$
        end if
        $l \leftarrow l + 1$
        Train CNN for 20 more training cycles
        $f \leftarrow$ measurement accuracy
    end while
    $T_{\text{rep}} \leftarrow 3.2\ \mu\text{s}$
    Program protocol pulses
    Measure 2^{26} repetitions
    Store measurements in *random_r.rbf*
    $T_{\text{rep}} \leftarrow 40\ \mu\text{s}$
end while
On Error Log error and restart

4 Testing

In this section, we present an empirical method to show the incomputability of the outputs generated in Section 3.

4.1 Why do we need testing?

Why should we be interested in answering the above question? After all, incomputability is established by mathematical proof, so why would we need experimental corroboration, a weaker argument? An example is a random number generator certified (by a mathematical proof) always to produce an incomputable infinite sequence of random bits. Indeed, the mathematical proof certifying incomputability is part of a mathematical model which uses certain physical assumptions; its veracity rests on those assumptions. The fact that each individual assumption is reasonable does not automatically guarantee that globally, the set of assumptions is also reasonable. Experimental testing is essential not only for corroborating the conclusion of the proof but also for supporting the adequacy of the model. Furthermore, thorough testing allows one to detect any issues with assumptions made in the theoretical analysis of a device or its practical deployment.

Can we test incomputability with a statistical test, that is, with a method of statistical inference to decide whether the data at hand sufficiently supports a particular hypothesis? The answer is *negative*. Intuitively, this is a consequence of the "asymptotic" nature of the notion of computability and its negation: *finite variations do not change them.* For example, if the sequence $x_1x_2\dots x_n\dots$ is computable (incomputable), then the sequences $y_1y_2\dots y_m x_1x_2\dots x_n\dots$ and $x_kx_{k+1}\dots x_m\dots$ are also computable (incomputable) for every string $y_1y_2\dots y_m$ and positive integer k. For example, the Champernowne binary sequence [15]

$$0, 1, 00, 01, 10, 11, 000, \dots$$

obtained by concatenating all binary strings in shortlex order.[1] This sequence is computable and *normal*, i.e. its digits are uniformly distributed: all digits are equally likely, all pairs of digits are equally likely, all triplets of digits are equally likely, and so on. Normality is a "symptom" of randomness, and computability is a "symptom" of non-randomness. The Champernown sequence shows that these symptoms can be compatible; no statistical test can detect its computability, hence non-randomness.

Does this mean that incomputability cannot be "experimentally tested"? Of course, no. In what follows, we will describe such a test used in assessing the quality of outputs of quantum random generators, [12, 2].

[1]Strings are first sorted by increasing length, and strings of the same length are sorted into lexicographical order: $0, 1; 00, 01, 10, 11; 000, 001, \dots 111; \dots$

4.2 Theory

We continue with a topic apparently unrelated to the question discussed in this section: testing of primality of positive integers. Primality is considered computationally *easy* because there exist polynomial algorithms in the size of the input to solve it; the first such algorithm was proposed in 2004 [6]. However, every known primality polynomial algorithm is "practically slow", so probabilistic algorithms[2] are instead used [50].[3]

The practical failure of polynomial primality tests motivated the search for probabilistic algorithms for primality [40, 45, 47, 48, 50, 50]. To test the primality of a positive integer n, the Solovay-Strassen primality test generates the first k natural numbers uniformly distributed between 1 and $n-1$, inclusive, and, for each $i \in \{i_1, \ldots, i_k\}$ checks "quickly" the validity of a predicate $W(i,n)$ based on Euler's criterion (called the Solovay-Strassen predicate). If $W(i,n)$ is true then "i is a witness of n's compositeness"; hence n is certainly not prime. Otherwise, the test is inconclusive. In this case, the probability that n is prime is greater than $1 - 2^{-k}$. This result is based on the fact that *at least half* the i's between 1 and $n-1$ satisfy $W(i,n)$ if n is composite, and *none* of them satisfy $W(i,n)$ if n is prime [49].

In detail, we first define the Solovay-Strassen predicate $W(i,n)$ by

$$\left(\frac{i}{n}\right) i^{(n-1)/2} \not\equiv 1 \mod n,$$

where $\left(\frac{i}{n}\right)$ is the Jacobi symbol[4] with $i \in \mathbb{N}, i < n-1$.

If $i \geq 2$ and $W(i,n)$ is true, we say that i is an *Euler witness (E-witness)*. If $n > 3$ is an odd composite, and $W(i,n)$ is false for $i \geq 1$, we say n is an *Euler* pseudoprime for the base i or that i is an *Euler liar (E-liar)* for the Solovay-Strassen primality test. In particular, the set $L_{ss}(n)$ of *E-liars* has at most $\frac{\phi(n)}{2}$ elements. Thus, the probability of sampling an *E-liar* when performing the Solovay Strassen test is given by $\beta_n = |L_{ss}(n)|/(n-1)$

The size of $L_{ss}(n)$ varies for different odd composite numbers. Consider the Carmichael numbers, that is, composite positive integers n satisfying the congruence $b^{n-1} \equiv 1 \pmod{n}$ for all integers b relatively prime to n. The largest β_n is found in a subset of Carmichael numbers with $\beta_n = \frac{1}{2}$. A Carmichael number passes a Fermat primality test [18, Section 31.8] to every base relatively

[2] Currently the best runs in time $O((\log\ n)^6)$.

[3] In contrast, factorisation of positive integers is "thought", but not proved, to be a computationally *difficult* problem. Currently, one cannot factorise a positive integer of 500 decimal digits that is the product of two randomly chosen prime numbers. This fact is exploited in the RSA cryptosystem implementing public-key cryptography [46].

[4] If the prime factorisation of the odd number n is $p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$, then $\left(\frac{i}{n}\right) = \left(\frac{i}{p_1}\right)^{a_1} \left(\frac{i}{p_2}\right)^{a_2} \cdots \left(\frac{i}{p_k}\right)^{a_k}$.

prime to the number, but few of them pass the Solovay-Strassen test. Increasingly Carmichael numbers become "rare".[5]

Consider $s = s_0 \dots s_{m-1}$ a binary string (of length m) and n an integer greater than 2. Let k be the smallest integer such that $(n-1)^{k+1} > 2^m - 1$; we can thus rewrite the number whose binary representation is s into base $n-1$ and obtain the unique string $d_k d_{k-1} \dots d_0$ over the alphabet $\{0, 1, \dots, n-2\}$, that is,

$$\sum_{i=0}^{k} d_i (n-1)^i = \sum_{t=0}^{m-1} s_t 2^t.$$

The predicate $Z(s, n)$ is defined by

$$Z(s,n) = \neg W(1 + d_0, n) \wedge \cdots \wedge \neg W(1 + d_{k-1}, n), \tag{4}$$

where W is the Solovay-Strassen predicate.

The digits of s (rewritten in base $n-1$) are used to define the Solovay Strassen predicates. If n is a pseudo-prime for all the bases from s used to construct these predicates, we say that s is a $Z - liar$.

A string s is c-random if $K(s) \geq |s| - c$; $|s|$ is the string length and K is the Kolmogorov complexity [11].

> **Chaitin-Schwartz Theorem. [14]** *For all sufficiently large c, if s is a c-random string of length $(l + 2c)$ and n is an integer whose binary representation is l bits long, then $Z(s, n)$ is true if and only if n is prime.*

This result cannot be used to de-randomise[6] Solovay-Strassen probabilistic algorithm because the set of c-random strings is incomputable.[7] However, the result can be used to model strings from different random number generators to test the quality of long binary strings by comparing their behaviour. In particular, we look at the number of *Z-liars* found by each generator.

4.3 Experimental analysis

Standard statistical tests of randomness focus on properties of the distribution of bits or bit strings within sequences, failing to distinguish between pseudo-random number generators and quantum random number generators. To address this issue, in [2], the ability of random strings to de-randomise the Solovay-Strassen probabilistic test of primality was used to compare the algorithmic randomness of strings generated by a QRNG and those produced by different PRNGs. Despite leading to

[5] There are 1,401,644 Carmichael numbers in the interval $[1, 10^{18}]$.

[6] That is, to transform the probabilistic algorithm into an equivalent deterministic algorithm.

[7] In fact, highly incomputable [11]: no infinite set of c-random is computable.

mostly inconclusive results, the tests conducted showed some advantages offered by a 3D-QRNG against PRNGs with respect to the randomness of its outputs.

The following test, called the fourth Chaitin-Schwartz-Solovay-Strassen test (CSS4) in [2], showed the highest potential for distinguishing between sources of random strings. Recall that the crucial fact is that the set of c-random strings is (highly) *incomputable*.

We construct the Chaitin-Schwartz predicate $Z(s, n)$ from (4) and generate a pool of Solovay-Strassen predicates composed of the digits s in base $n - 1$. Then, we fix $c = l - 1$ where l is the l-bit binary representation of n and sample s from chunks of $l(l + 2c)$ bits in order to look for *Z-liars* generated by a set of bases for the predicates extracted from the string s.

In [2], Carmichael numbers were used in the majority of the tests. However, despite Carmichael numbers having a larger $L_{ss}(n)$, it is difficult to find *Z-liars* due to the length of their binary representation. For example, for the smallest Carmichael number more than 70×2^{32} bits would need to be read to find a *Z-liar* since the Solovay-Strassen test guarantees a predicate is true with a probability of at least one-half when n is a composite number. For smaller numbers we expect see to a larger number of *Z-liars*. Thus, for this test, only odd composite numbers less than 50 were used for each round, and the process was repeatedly parsed through each string with an incremental bit offset.

Recently in [30], a similar approach was taken by applying these tests to a different set of PRNGs and two different QRNGs with a larger set of numbers; each string tested had a length of 2^{26}. Once again, the QRNGs showed no clear advantage over the PRNGs. Moreover, the difficulty of finding *Z-liars* led to a similar limitation in terms of numbers tested; *Z-liars* were only observed for composites $n \leq 25$. Still, an essential characteristic of this test was confirmed: its sensitivity to the size of the pool of unique bases extracted from the random strings. No *Z-liars* were recorded when a repetitive structure generated by their sampling process was present. For this reason, we have a variation of this test was performed.

We tested two PRNGs and a QRNG: the Python3 Mersenne Twister-based generator, the hashing function SHA3, considered a "cryptographically secure PRNG" and the 3D-QRNG described in this paper.

Since the number of Solovay-Strassen tests increases with longer binary representations, the probability of observing a *Z-liar* becomes smaller, so a large pool of unique bases was required to detect a significant number of *Z-liars* [2]. Thus, we prepared ten sets of strings of size 2^{32} for each generator and applied the shifting process described in [2] for the test. The average number of *Z-liars* over the composite numbers less than 50 was taken as the metric. Despite only detecting *Z-liars* for composites up to 25, there was a noticeable difference between sources for the numbers 9 and 15. For these numbers, from our predicate construction, we have

Composite number tested	9	15	21	25	27	33	35	39	45	49
sha3	265.6	60.3	0	0.2	0	0	0	0	0	0
python3	260.1	58	0	0.3	0	0	0	0	0	0
qutrits	536.4	131.9	0	0.2	0	0	0	0	0	0

Table 1: Average number of *Z-liars* sampled by composite number tested (over 10 strings of length 2^{32})

that a minimum of 40×2^{13} bits and 40×2^{10} bits are needed for a c-random string to have a chance of finding a *Z-liar*.

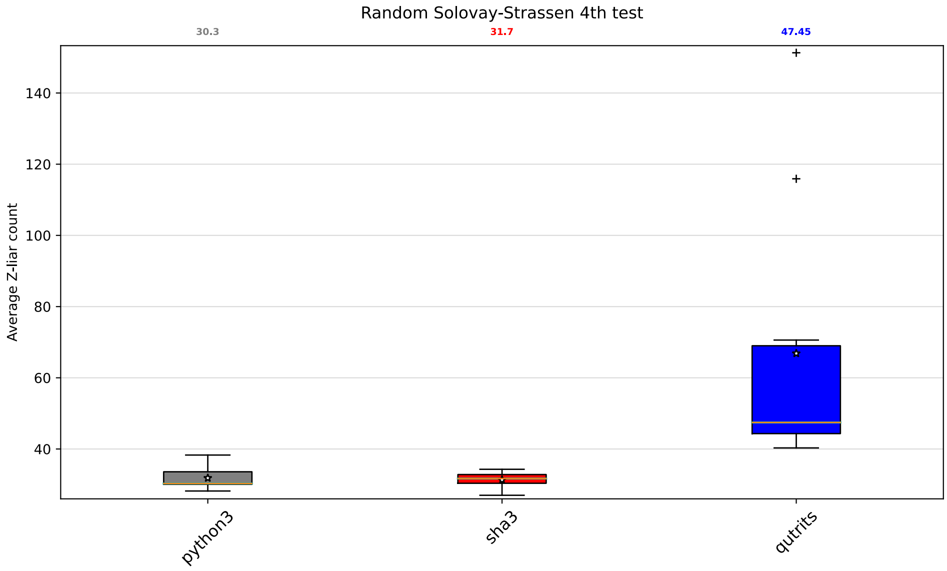

Figure 3: Fourth Chaitin-Schwartz-Solovay-Strassen test: distribution of the average *Z-liar* counts for all odd composite numbers less than 50

The occurrence of patterns in long enough sequences of random events is inevitable. Since a lower quality of randomness increases the rate at which this occurs, the gap between the number of unique bases extractable between RNGs with different qualities of randomness widens. Thus, given long enough strings, we can observe this behaviour. Since many unique bases are required to increase the likelihood of finding *Z-liars*, from Figure 1, we see the advantage offered by a 3D-QRNG generator over other alternative sources of randomness.

In order to analyse the statistical significance of these results, we conducted the non-parametric and distribution-free two-sample Kolmogorov–Smirnov test. This test identifies if two datasets differ significantly without any prior assumption about an underlying distribution. To this end, we say that the difference between two datasets is statistically significant if the p-value obtained through this test is less

	sha3	qutrits
python3	0.9780	0.0047
sha3		0.0047

Table 2: Kolmogorov-Smirnov test p-values for the fourth Chaitin-Schwartz-Solovay-Strassen test with the Z-liar count metric

than 0.005. This critical p-value is chosen to reduce the chance of false positives as well as allow us to provide a direct comparison with results from [2].

We note that there is a significant difference between the 3D-QRNG qutrits and the PRNGs. A similar behaviour was revealed in [2], where despite the non-conclusive results of the fourth Chaitin-Schwartz-Solovay-Strassen test, the Kolmogorov-Smirnoff test showed that the difference between a 3D-QRNG and the other PRNGs is statistically relevant. The outcomes of the fourth Chaitin-Schwartz-Solovay-Strassen test presented here show a stronger advantage of 3D-QRNGs over PRNGs.

5 Conclusions

This article uses a located form of the Kochen-Specker Theorem to derive a physical realisation of a class of 3D-QRNGs by means of a superconducting transmon. The sequences produced by these 3D-QRNGs are strongly incomputable, a property that no other QRNG provides to date. Furthermore, we have used a non-statistical randomness test to probe experimentally the incomputability of its generated long strings: for the first time, a provable advantage over the best PRNGs was found. This result has been achieved by using the Chaitin-Schwartz Theorem to probe the "usefulness" of generated quantum random bits, a form of Einstein's justification.

These results highlight the real effects of incomputability in quantum systems and complement the theoretical certification via value indefiniteness of the class of QRNGs implemented. Furthermore, the experimental results confirm and complement incomputability and, quite significantly, the choice of physical assumptions in the theoretical part.

Finally, there is a strong motivation for developing alternative tests capable of probing at algorithmic properties of randomness that better suit a wide range of applications where the quality of randomness needs to be assessed quickly or dynamically.

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