HOW DOES MATHEMATICAL KNOWLEDGE FOR UNDERGRADUATE TUTORING DEVELOP? ANALYSING WRITTEN REFLECTIONS OF NOVICE TUTORS

Tikva Ovadiya — Oranim College

Igor' Kontorovich — The University of Auckland

Research has become interested in the mathematical knowledge that undergraduate tutoring involves. This study introduces a mechanism that describes how this knowledge can develop through the work of tutoring. The mechanism emerged from an analysis of 24 reflections written by 10 novice tutors on noticeable incidents that took place in their tutorials. The tutors were undergraduate students at advanced stages of their mathematics degrees, and their teaching unfolded as part of an elective course in mathematics education. The mechanism proposes that tutors can find themselves in contingent situations, where their mathematical knowledge is insufficient. To fulfill the emerging pedagogical need, tutors initiated reflexive actions of mathematics learning to prepare for similar contingent situations in their future tutoring.

RATIONALE AND BACKGROUND

Over the last two decades, research in undergraduate mathematics education has become interested in tutors (also referred to as "teaching assistants" and "graduate student instructors"). In many colleges and universities worldwide, tutors are employed by the mathematics departments to contribute to their instruction. The scope of the tutor roles vary from one tertiary context to another, ranging from working in drop-in mathematics support centers to leading regular problem-solving sessions for smaller groups of students enrolled in a course (e.g., Speer et al., 2005). In many countries, tutor-student interactions play a key role in undergraduate mathematics education, meaning that the way in which the former teach can impact how the latter learn (e.g., Kontorovich & Ovadiya, accepted). This raises questions about what mathematical knowledge tutoring involves and how this knowledge develops.

Research into these questions is in its infancy. The emerging findings indicate that undergraduate tutoring "requires mathematical knowledge beyond content knowledge of the course" (Johns & Burks, 2022, p. 2). John and Burks (2022) show that tutors employ variations of the types of knowledge that are familiar from research in teacher education (e.g., in the terms of Ball et al., 2008, knowledge of content, curriculum, and students). But, unlike school teachers, tutors have rarely completed extended educational programs to prepare for the work of teaching. Indeed, tutor training is typically confined to several workshops that are often independent of the disciplinary subject matter (e.g., Speer et al., 2005). Yet, John and Burks (2022) demonstrate that

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tutors with only a few years of experience can hold impressive *mathematical knowledge for tutoring* (*MKTut* hereafter). So, how does this knowledge come about?

One possible answer is that MKTut develops through the work of tutoring. Research provides evidence of school teachers and mathematics educators growing new insights and refining their mathematical knowledge via teaching (e.g., Leikin & Zazkis, 2010). Occasional references to learning through teaching feature in the self-reflections of experienced university lecturers (e.g., Kontorovich, 2021). By analogy, it is reasonable to propose that tutors can learn through teaching as well.

We aim to explore how MKTut develops through the work of tutoring. In this study, we focus on a sub-domain of this knowledge: *specialized content knowledge*—not just "common" knowledge of mathematical facts, but knowledge that enables nimbleness of ideas and practices that are distinctive for mathematical teaching (Ball et al., 2008). To address the aim, we scrutinize written reflections composed by novice tutors about their tutoring of first-year mathematics courses. The use of systematic reflection to investigate and promote professional knowledge for teaching is consistent with research on tutors (e.g., Speer et al., 2005) and school teachers (e.g., Mason, 2002).

THEORETICAL FRAMEWORK

John and Burks (2022) argue that only some aspects of existing frameworks of mathematical knowledge for school teaching are relevant to undergraduate tutoring. Yet, given the developmental stage of this area, we build on the existing frameworks.

Mason (2002) maintains that attention and noticing lie at the heart of all professional practice, teaching included. He conceives attention as a complicated human mechanism, in which noticing is responsible for distinguishing some things from their surroundings and getting them through to the level of awareness. Without this awareness, it is impossible to act on these things, that is, to react to them.

Mason (2002) argues that people notice things insofar as they are unexpected, i.e. contingent. *Contingency* is a dimension in the Knowledge Quartet—a theory concerning the mathematical knowledge that teachers apply in a classroom (Rowland et al., 2015). The dimension concerns situations where a teacher encounters an unanticipated event and is challenged to deviate from their agenda. In Rowland et al.'s (2015) study on elementary classrooms, students' contributions to the lesson constituted the majority of contingencies. These included instances where students provided surprising answers to a question and spontaneous reactions to an activity. The researchers show that the teacher's response to the contingency can be of three kinds: to ignore, to acknowledge but put aside, and to acknowledge and incorporate.

According to Leikin and Zazkis (2010), contingent situations in secondary-school classrooms can lead teachers to develop new mathematical ideas. Considering teaching as a partially improvised activity (Rowland et al., 2015), we propose that contingencies can take place in university tutorials as well.

Noticing can stem from a disturbance, for instance, when a teacher experiences a need for a certain piece of professional knowledge. Harel (2008) uses *intellectual need* to refer to circumstances where "disciplinary knowledge [is] born out of people's current knowledge through engagement in problematic situations conceived as such by them" (p. 898). In these cases, one's existing state of knowledge is insufficient or inadequate, and additional knowledge must be acquired to reach an equilibrium.

Stylianides and Stylianides (2022) reframe the construct of intellectual need for the case of prospective teachers. In the context of proof teaching and learning, the researchers use a *pedagogical need* to capture teacher's readiness to develop conceptualizations of proof that are new to them. We see no reason to confine this reframing to a particular mathematical topic. Accordingly, with a pedagogical need, we refer to a teacher's openness to develop a new piece of mathematical knowledge for teaching. Indeed, Stylianides and Stylianides (2022) stress that pedagogical needs are linked to the teacher's perceptions of how relevant the new piece of knowledge is to their teaching context.

Noticing can be developed through disciplined reflection on professional experiences (Mason, 2002). Disciplined reflection should not be confused with Schön's (1987) *reflection-on-action*—an umbrella term that includes "anything from vaguely thinking back over what happened, to […] calling upon theories to explain and justify [it]" (Mason, 2002, p. 15). We use the term *reflexion* to stress the disciplined aspect of one's reflection. This includes careful documentation of an incident, while aspiring to avoid judgements and implicit assumptions, and successive introspection of the incident with a deep inward gaze. Mason argues that such monitoring of the incidents of the past can prepare teachers to *reflect-through-action*, that is, to become aware of, and prepared to, modify their practice in the midst of that practice.

METHOD

Our data came from "Mathematics learning through teaching"—a mathematics education course (*MathEd* hereafter) that was offered in the mathematics department at a New Zealand university. The course was not required by any particular program. It mostly attracted undergraduates in the last semesters of their mathematics majors, and who were interested in educational issues.

The central activity of the MathEd course was tutoring in "bridging" (pre-academic) non-credit courses and first-year courses for non-mathematics majors. The MathEd students (*tutors* hereafter) were allocated to groups of up to 25 students, and they led, in pairs, ten one-hour tutorial sessions throughout a semester. The tutors were expected to assist the students with the course content, by supporting their autonomous work on sets of problems. The problems were provided by the course lecturers.

After each tutorial, the tutors were expected to submit a written reflection-on-action, where they accounted for a specific incident that had drawn their attention in the tutorial. This was part of their MathEd coursework. The reflection guidelines asked the tutors to provide a detailed description of an incident they considered significant and encouraged critical questioning of their in-the-moment actions. The tutors were also encouraged to formulate inferences that would be useful for their further teaching. Every week, selected reflections were shared and discussed in the MathEd course.

Over three semesters, we collected hundreds of reflections that focused on myriad of issues. At the first stage, we reviewed each of them to identify those that referred to tutors' learning of mathematics. The process converged to 24 reflections written by ten tutors. These reflections became our data corpus.

The reflections underwent inductive analysis, driven by the question, "how did the incident that the tutors noticed spur the development of their specialized content knowledge for tutoring (*SCKTut*, hereafter)?" We iteratively compared between tutors' reflections, while attending to the differences and similarities between the described incidents and tutors' actions. These comparisons gave rise to initial elements of a mechanism that conceptually connects between the tutorial incidents, the follow-up activity, and tutors' knowledge development. The emerging categories and conceptualizations were applied to the whole data corpus to ensure that they account for the key aspects that the tutors stressed in their reflections.

FINDINGS

We open with the presentation of the mechanism that emerged from the data analysis. Then, we illuminate some of its components with excerpts from a single reflection.

SCKtut development through tutoring: An overview

In accordance with our assumption, many tutors' reflections described contingent incidents that unfolded in tutorial classrooms. Somewhat similarly to Rowland et al. (2015), tutorial contingencies included surprising questions that the students asked and mathematical challenges that they faced when working on the assigned problems. The former pertained to situations where students experienced some intellectual need and turned to tutors with a request to fulfill it. Not all contingencies of the latter type involved the tutors directly. For instance, one of the tutors wrote,

> I overheard a discussion in one of my groups where one of the students stated that "a line and a plane can be non-parallel and not intersect in 3 dimensions." This caught my attention because as far as I know […] this was impossible. I was curious about this student's "non-parallel non-intersecting line and plane" so I asked him if he could elaborate further.

This quote illustrates that the tutors not only coped with contingencies that the students presented to them, but also chose to get involved in the contingencies that they noticed.

All reflections in the data corpus described situations where tutors found themselves in a pedagogical need for a certain piece of content or specialized content knowledge. In these situations, the tutors' state of mathematical knowledge was either insufficient or inadequate to handle the contingency "on their feet". For instance, a tutor could explain the problem solution or justify a particular move, but only with advanced mathematics that went beyond the scope of the particular tutorial. Such situations entailed a particular type of reflection-on-action (or reflection on a struggle to execute an appropriate instructional action, in this case), where the tutors autonomously pursued the development of their mathematical knowledge. Examples of this pursuit included re-solving the focal problems after the tutorial, consulting with the relevant literature and online resources, and seeking assistance from mathematically versed others (e.g., other tutors, course lecturers). The mathematics in the focus of these actions was the "piece of the puzzle" that the tutors were missing in the contingency. We refer to this activity of the tutors as *reflexive actions of mathematics learning*. In all collected reflections, the tutors maintained that the learning actions resulted in the successful development of the target mathematics.

The learning actions exemplified above took place after the tutorials. However, some reflections referred to reflexive actions that the tutors managed to take "on the fly" to navigate the contingency as it unfolded. For instance, one tutor described his struggle to explain the transition from $\frac{1}{\frac{5x-1}{x}}$ to $\frac{x}{5x-1}$ in a first-semester course. The tutorial was dedicated to inverse functions, and the tutor assumed that the students will be fluent in fraction manipulation. When he "stopped to think", one of the students suggested to represent the fraction as $1 \div \frac{5x-1}{x}$. The tutor acknowledged the idea and incorporated it in their solution to produce an elaborated explanation (cf. Rowland et al., 2015). In other reflections, the tutors described how they asked the second co-tutor to weigh in. In such cases, the peer tutor took charge and resolved the contingency. These reflections attest to a high level of reflection-through-action that the tutors demonstrated by being aware of classroom resources and using them in-the-moment to address their pedagogical needs. The reflections also depicted these reflexive actions as affording the tutors a chance to advance their mathematical knowledge.

A reflection on one learning journey

We use excerpts from a reflection of Ann (pseudonym), who tutored a "bridging" course. This reflection serves two purposes: (i) to show that a successful resolution of a contingency in a classroom can still entail reflexive development of SCKtut; and (ii) to introduce a new type of reflection that we discerned in tutors' reflections.

After the first tutorial on the concept of functions, Ann submitted a reflection that revolved around the following problem: "For $f(x) = x^2 - 2$ evaluate: (a) $f(2)$; (b) $f(2-x)$; (c) $f(x+h) - f(x)$". This is how Annie described the focal incident:

> During the tutorial, I had more than three students ask me how to solve (b) and (c). I tried to explain this by telling them that function is like a factory. The variable x is the input, and $x^2 - 2$ is the machine. But they told me they didn't understand it at all. So I added more content to my explanation and said that this is a factory that makes apple pies, whatever [is] in the brackets is the apple we need to put in the machine to make an apple pie. So to solve (b) we just use $2 - x$ to replace x that in the function.

Ann wrote that "all students got it", referring to her second explanation. The incident still made her reflect on the two explanations that she provided. In her words,

> After the tutorial I was thinking about what's wrong with my first explanation. […] The key issue here is their failure to understand that '$2-x$' here is a variable. Why my second explanation made them understood it, maybe, because I told them that whatever is in the brackets is an apple. You don't need to think about how to deal with 2, just circle everything in the bracket and put them into formula and replace the x. In fact, I told them '$2-x$' is the input variable.

Ann wrote that she searched for the notion of variable in the mathematics encyclopedia and found out that,

> 'In elementary mathematics, a variable is an alphabetic character representing a number, called the value of the variable, which is either arbitrary, not fully specified, or unknown.' It means variable is not referring to x, y, z it just means not fully specified or unknown.
>
> So I think next time if someone ask me about a similar problem. I will ask them to tell me what the meaning of variable. Is that mean x, y, z? Can a, b, c be variable as well? I will ask them to think about this and refer back to the definition of function. And then I will use my weird apple pie example to help them understand the definition. And also I need to give them chance to tell me what they don't understand about the question and refer their issue back to definition.

In the MathEd lesson, Ann confirmed that the incident was contingent to her. Before the tutorial, she presumed that finding $f(2)$ will prepare the students for the remaining parts of the problem. In spite the contingency, Ann succeeded in presenting a general approach to the problem solution. She even managed to elaborate on it when the students sought additional explanations. In other words, Ann successfully reflected-through-action and satisfied the intellectual need that the students presented her with.

Ann's *resolution* of the contingency engendered a posteriori pedagogical need to understand "what's wrong with my first explanation" and what in the second explanation made students "get it". Drawing on her mathematical knowledge, Ann connected students' intellectual need and her "apple" metaphor to the concept of variable. She followed with further reflexive action of turning to the literature to clarify a formal concept definition. We note a qualitative difference between a more conceptual approach to variables that Ann presented in her reflection-on-action compared to a rather procedural explanation that she described as providing to her students (i.e., "we just use $2-x$ to replace x"). This change illustrates how reflexive actions can lead a tutor to expand their mathematical knowledge.

In the last part of her reflection, Ann generates a series of questions she may use "if someone asks me a similar problem". Two observations can be made regarding these questions. First, they invite the asker to engage with the notions of variable and function on a conceptual level. Second, they initiate an exchange, providing the asker with opportunities to express, clarify, and reflect their current state of knowledge, while articulating their intellectual needs. The reflection suggested (and Ann later confirmed) that these aspects were not part of the incident that took place in her classroom.

We propose that Ann put her expanded SCKtut in use to *reflect-toward-action*, i.e. to sketch instructional actions that she could undertake when faced with a similar mathematical problem. Broadly speaking, reflection-toward-action prepares tutors for future contingencies in which they may experience similar pedagogical needs. What is interesting, in Ann's case, is that there is no evidence to suggest that she experienced a pedagogical need in the classroom. Indeed, she described this need arising "after the tutorial". This suggests that tutors' reflection-toward-action can be aimed not "just" at coping with similar contingencies, but at becoming ready to react to them in a more aware manner that is faithful to their newly developed insights. Reflection-toward-action featured in eleven reflections in our data corpus.

SUMMARY AND DISCUSSION

Figure 1 offers a visual summary of the proposed mechanism of SCKtut development through the work of tutoring. The mechanism suggests that contingencies that tutors face in their tutorials can engender a range of pedagogical needs. A sub-set of them may spur tutors to initiate reflexive actions of mathematics learning. Some actions are taken to address the contingency "on-the-fly", while others unfold outside the tutorial walls. Notably, our data suggest that tutors can pursue learning-oriented actions even when they think that their reflection-through-action addressed the contingency successfully. Reflexive actions can include reflection-toward-action, in which tutors delineate instructional moves that they may take if similar situations arise in the future.

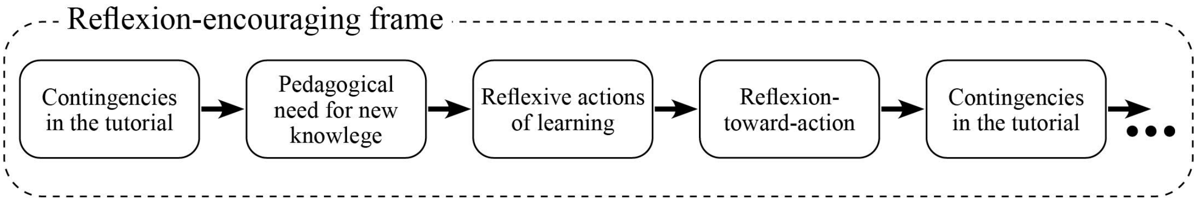

Figure 1: Mechanism of SCKtut development through tutoring

A *reflexion-encouraging frame* is a critical element that contextualizes the proposed mechanism. Indeed, we do not believe that the reflexions that our tutors composed were accidental. They emerged as a response to certain guidelines, and they were part of tutors' coursework. The course attracted a particular student cohort that was led to conduct critical inquiry into mathematics and its education. Overall, the course posited that teaching is an endeavour through which tutors' mathematical knowledge can develop. Within this multi-layered frame, tutors were expected to reflect-on-action in a disciplined manner, and deep introspective reflections were encouraged.

The focal mechanism contributes to research on undergraduate tutoring. The study offers evidence to suggest that tutors can advance their mathematics knowledge through tutoring. In this sense, tutors emerge, not unlike school teachers and teacher educators (e.g., Leikin & Zazkis, 2010). Notably, our tutors were taking their first teaching steps. They tutored first-year and "bridging" (pre-academic) courses, which many tutors initially labelled as "basic" and "easy". This may suggest that a multi-layered reflection frame within which the tutors operated played a key role in their mathematics learning. That said, our findings emerged from self-reflections that the tutors produced as part of coursework. Thus, much more research is needed to understand the complexity of MKTut and its development in different contexts.

Let us consider the MathEd course where soon-to-be mathematics graduates turned into tutors. The course was led by scholars in the didactics of mathematics who were members of the mathematics department. This is not the only department where didacticians and mathematicians work side-by-side. Thus, we propose that MathEd courses offered as part of mathematics programs can provide a promising path for tutor training and for advancing the quality of undergraduate mathematics instruction.

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