Received: 12 July 2021 | Revised: 9 March 2022 | Accepted: 22 March 2022

DOI: 10.1112/blms.12674

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

The diamagnetic inequality for the Dirichlet-to-Neumann operator

A. F. M. ter Elst[1] | **El Maati Ouhabaz**[2]

[1]Department of Mathematics, University of Auckland, Auckland, New Zealand

[2]Institut de Mathématiques de Bordeaux, Université de Bordeaux, Cours de la Libération, Talence, France

Correspondence
El Maati Ouhabaz, Institut de Mathématiques de Bordeaux, Université de Bordeaux, UMR 5251, 351, Cours de la Libération, Talence, 33405, France.
Email: Elmaati.Ouhabaz@math.u-bordeaux.fr

Funding information
Marsden Fund; ANR, Grant/Award Number: ANR-18-CE-0012-01

Abstract

Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz boundary Γ. We define the Dirichlet-to-Neumann operator $\mathcal{N}$ on $L_2(\Gamma)$ associated with a second-order elliptic operator $A = -\sum_{k,j=1}^d \partial_k(c_{kl}\,\partial_l) + \sum_{k=1}^d (b_k\,\partial_k - \partial_k(c_k\cdot)) + a_0$. We prove a criterion for invariance of a closed convex set under the action of the semigroup of $\mathcal{N}$. Roughly speaking, it says that if the semigroup generated by $-A$, endowed with Neumann boundary conditions, leaves invariant a closed convex set of $L_2(\Omega)$, then the 'trace' of this convex set is invariant for the semigroup of $\mathcal{N}$. We use this invariance to prove a criterion for the domination of semigroups of two Dirichlet-to-Neumann operators. We apply this criterion to prove the diamagnetic inequality for such operators on $L_2(\Gamma)$.

MSC 2020
47A07, 47D06, 47D08 (primary)

1 | INTRODUCTION

The well-known diamagnetic inequality states that the semigroup associated with a Schrödinger operator with a magnetic field is pointwise bounded by the free semigroup of the Laplacian. More precisely, let $\vec{a} = (a_1, \dots, a_d)$ be such that each a_k is real valued and locally in $L_2(\mathbb{R}^d)$. Set $H(\vec{a}) = (\nabla - i\vec{a})^*(\nabla - i\vec{a})$. Then the corresponding semigroup $(e^{-tH(\vec{a})})_{t\geqslant 0}$ satisfies

$$|e^{-tH(\vec{a})}f| \leqslant e^{t\Delta}|f|$$

for all $t > 0$ and $f \in L_2(\mathbb{R}^d)$. The same result holds in the presence of a real-valued potential V, that is, with operators $H(\vec{a}) + V$ and $-\Delta + V$.

The diamagnetic inequality plays an important role in spectral theory of Schrödinger operators with magnetic potential. We refer to [16] and references there.

The main objective of the present paper is to prove a similar result for the Dirichlet-to-Neumann operator with magnetic field on the boundary Γ of a Lipschitz domain Ω in $\mathbb{R}^d$. In its simplest case, the diamagnetic inequality we prove says that for all $\vec{a} \in (L_\infty(\Omega, \mathbb{R}))^d$, the solutions of the two problems

$$\begin{cases} \partial_t \operatorname{Tr} u + (\partial_\nu - i\vec{a}\cdot\nu)u = 0 \text{ on } (0,\infty)\times\Gamma \\ (\nabla - i\vec{a})^*(\nabla - i\vec{a})u = 0 \text{ on } (0,\infty)\times\Omega \\ \operatorname{Tr} u = \varphi \end{cases}$$

and

$$\begin{cases} \partial_t \operatorname{Tr} v + \partial_\nu v = 0 \quad \text{on } (0,\infty)\times\Gamma \\ \Delta v = 0 \quad \text{on } (0,\infty)\times\Omega \\ \operatorname{Tr} v = |\varphi| \end{cases}$$

satisfy

$$|u(t,x)| \leqslant v(t,x) \text{ for a.e. } (t,x) \in (0,\infty)\times\Gamma.$$

Here ∂_ν is the normal derivative and ν is the outer normal vector to Ω. We prove more in the sense that we are able to deal with variable and non-symmetric coefficients. To be more precise, we consider $c_{kl}, b_k, c_k, a_0 \in L_\infty(\Omega, \mathbb{C})$ for all $k, l \in \{1, \dots, d\}$ with (c_{kl}) satisfying the usual ellipticity condition. For all $\vec{a} \in (L_\infty(\Omega, \mathbb{R}))^d$ as above we consider the magnetic Dirichlet-to-Neumann operator $\mathcal{N}(\vec{a})$ defined as follows. If $\varphi \in H^{1/2}(\Gamma)$, we solve first

$$\begin{aligned} -\sum_{k,l=1}^{d} (\partial_k - ia_k)(c_{kl}\,(\partial_l - ia_l)\,u) \\ + \sum_{k=1}^{d} (b_k\,(\partial_k - ia_k)u - (\partial_k - ia_k)(c_k\,u)) + a_0\,u = 0 \quad \text{on } \Omega, \\ \operatorname{Tr} u = \varphi, \end{aligned}$$

with $u \in W^{1,2}(\Omega)$ and then define $\mathcal{N}(\vec{a})\varphi$ as the conormal derivative (when it exists as an element of $L_2(\Gamma)$). Formally,

$$\mathcal{N}(\vec{a})\varphi = \sum_{k,l=1}^{d} \nu_k \operatorname{Tr}(c_{kl}\,\partial_l u) - i\sum_{k,l=1}^{d} \nu_k \operatorname{Tr}(c_{kl}\,a_l\,u) + \sum_{k=1}^{d} \nu_k \operatorname{Tr}(c_k\,u).$$

If $(T_{\vec{a}}(t))_{t\geqslant 0}$ denotes the semigroup generated by $-\mathcal{N}(\vec{a})$ on $L_2(\Gamma)$ and if the coefficients c_{kl}, b_k, c_k, a_0 are real valued, then we prove (under an accretivity condition) that

$$|T_{\vec{a}}(t)\varphi| \leqslant T_0(t)|\varphi|$$

for all $t \geqslant 0$ and $\varphi \in L_2(\Gamma)$. In the symmetric case, $c_{kl} = c_{lk}$ and $b_k = c_k = 0$, we obtain as a consequence a trace norm estimate for the eigenvalues of $\mathcal{N}(\vec{a})$ and if the coefficients are Hölder continuous and Ω is of class $C^{1+\varkappa}$ for some $\varkappa > 0$ we obtain that the heat kernel of $\mathcal{N}(\vec{a})$ satisfies a Poisson upper bound on Γ. We also prove other results on positivity (when $\vec{a} = 0$) and L_∞-contractivity of the corresponding semigroup. For example, in the symmetric case $c_{kl} = c_{lk}$, $b_k = c_k$ and a_0 are all real, then the semigroup T_0 is positive if $a_0(x) > -\lambda_0$ for almost every (a.e.) $x \in \Omega$, where λ_0 is the first positive eigenvalue of the elliptic operator

$$-\sum_{k,j=1}^{d} \partial_k (c_{kl}\, \partial_l) + \sum_{k=1}^{d} (b_k\, \partial_k - \partial_k (c_k \cdot)) \tag{1}$$

with Dirichlet boundary conditions. In other words, the quadratic form

$$\mathfrak{a}(u,u) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, (\partial_l u)\, \overline{\partial_k u} + \sum_{k=1}^{d} \int_\Omega \Big(b_k\, (\partial_k u)\, \overline{u} + b_k\, u\, \overline{\partial_k u} \Big) + \int_\Omega a_0\, |u|^2 \tag{2}$$

is positive on $W_0^{1,2}(\Omega)$. It is not clear whether this latter condition remains sufficient for positivity of the semigroup in the non-symmetric case. See Proposition 3.4 and Section 4.

It is worth mentioning that the Dirichlet-to-Neumann operator is an important map which appears in many problems. In particular, it plays a fundamental role in inverse problems such as the Calderón inverse problem. The magnetic Dirichlet-to-Neumann operator also appears in the study of inverse problems in the presence of a magnetic field. We refer to [6] and the references therein.

In order to prove the diamagnetic inequality we proceed by invariance of closed convex sets for an appropriate semigroup. This idea appeared already in [14]. Despite the fact that it is an abstract result, the invariance result proved in [14], however, does not seem to apply in an efficient way to the Dirichlet-to-Neumann operator. The reason is that in this setting one has to deal with harmonic lifting (with respect to the elliptic operator) of functions and it is not clear how to describe such a harmonic lifting for complicated expressions (see Section 5 below). What we do is to rely first on a version from [2] of the invariance criterion of [14] and then prove new criteria for invariance of closed convex sets which make a bridge between invariance on $L_2(\Gamma)$ for the Dirichlet-to-Neumann semigroup and invariance on $L_2(\Omega)$ for the semigroup of the elliptic operator with Neumann boundary conditions. The latter is easier to handle. The result is efficient when dealing with the Dirichlet-to-Neumann operator. The diamagnetic inequality is obtained from a domination criterion which is obtained by checking the invariance of the convex set $\{(\varphi,\psi) \in L_2(\Gamma) \times L_2(\Gamma) : |\varphi| \leqslant \psi\}$ for the semigroup $\begin{pmatrix} T_{\vec{a}}(t) & 0 \\ 0 & T_0(t) \end{pmatrix}_{t \geqslant 0}$.

2 | BACKGROUND MATERIAL

The aim of this section is to recall some well-known material on sesquilinear forms and make precise several notations which will be used throughout the paper.

Let $\widetilde{H}$ be a Hilbert space with scalar product $(\cdot,\cdot)_{\widetilde{H}}$ and associated norm $\|\cdot\|_{\widetilde{H}}$. We consider another Hilbert space V which is continuously and densely embedded into $\widetilde{H}$. Let

$$\mathfrak{a} : V \times V \to \mathbb{C}$$

be a sesquilinear form. We assume that $\mathfrak{a}$ is continuous and quasi-coercive. This means, respectively, that there exist constants $M \geqslant 0$, $\mu > 0$ and $\omega \in \mathbb{R}$ such that

$$|\mathfrak{a}(u,v)| \leqslant M \|u\|_V \|v\|_V \text{ and}$$
$$\operatorname{Re} \mathfrak{a}(u,u) + \omega \|u\|_{\widetilde{H}}^2 \geqslant \mu \|u\|_V^2$$

for all $u, v \in V$. It then follows that $\mathfrak{a}$ is a closed sectorial form and hence one can associate an operator $\widetilde{A}$ on $\widetilde{H}$ such that for all $(u,f) \in \widetilde{H} \times \widetilde{H}$ one has

$$u \in D(\widetilde{A}) \quad \text{and} \quad \widetilde{A}u = f$$

if and only if

$$u \in V \quad \text{and} \quad \mathfrak{a}(u,v) = (f,v)_{\widetilde{H}} \text{ for all } v \in V.$$

It is a standard fact that $\widetilde{A}$ is a densely defined (quasi-)sectorial operator and $-\widetilde{A}$ generates a holomorphic semigroup $\widetilde{T} = (\widetilde{T}(t))_{t\geqslant 0}$ on $\widetilde{H}$. See, for example, [12] or [15].

Let now H be another Hilbert space and $j : V \to H$ be a linear continuous map with dense range. Suppose that the form $\mathfrak{a} : V \times V \to \mathbb{C}$ is continuous. Following [2], we say that $\mathfrak{a}$ is j-elliptic if there exist constants $\omega \in \mathbb{R}$ and $\mu > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) + \omega \|j(u)\|_H^2 \geqslant \mu \|u\|_V^2$$

for all $u \in V$. In this case, there exists an operator A, called the operator associated with $(\mathfrak{a}, j)$, defined as follows. For all $(\varphi, \psi) \in H \times H$ one has

$$\varphi \in D(A) \quad \text{and} \quad A\varphi = \psi$$

if and only if

$$\text{there exists a } u \in V \text{ such that } \begin{cases} j(u) = \varphi \text{ and} \\ \mathfrak{a}(u,v) = (\psi, j(v))_H \text{ for all } v \in V. \end{cases}$$

Then A is well defined and $-A$ generates a holomorphic semigroup $T = (T(t))_{t\geqslant 0}$ on H. (See [2] Theorem 2.1.)

We illustrate these definitions by two important examples in which we define the Dirichlet-to-Neumann operator and the magnetic Dirichlet-to-Neumann operator on the boundary of a Lipschitz domain.

Example 2.1 (The Dirichlet-to-Neumann operator). In this example we construct the Dirichlet-to-Neumann operator in a general setting of complex coefficients. Let Ω be a bounded Lipschitz domain of $\mathbb{R}^d$ with boundary Γ. We denote by $\operatorname{Tr} : W^{1,2}(\Omega) \to L_2(\Gamma)$ the trace operator. Let $c_{kl}, b_k, c_k, a_0 \in L_\infty(\Omega, \mathbb{C})$ for all $k, l \in \{1, \dots, d\}$. We assume the usual ellipticity condition: There exists a constant $\mu > 0$ such that

$$\operatorname{Re} \sum_{k,l=1}^{d} c_{kl}(x)\, \xi_k\, \overline{\xi_l} \geqslant \mu\, |\xi|^2$$

for all $\xi \in \mathbb{C}^d$ and almost all $x \in \Omega$. Define the sesquilinear form $\mathfrak{a}: W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(u,v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\,(\partial_l u)\,\overline{\partial_k v} + \sum_{k=1}^{d} \int_\Omega \Big(b_k\,(\partial_k u)\,\overline{v} + c_k\,u\,\overline{\partial_k v}\Big) + \int_\Omega a_0\,u\,\overline{v}. \tag{3}$$

It is a basic fact that the form $\mathfrak{a}$ is continuous and quasi-coercive. We denote by A the operator associated with $\mathfrak{a}$ on $L_2(\Omega)$. Define the operator $\mathcal{A}: W^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ by

$$\langle \mathcal{A}u, v\rangle_{W^{-1,2}(\Omega)\times W^{1,2}_0(\Omega)} = \mathfrak{a}(u,v).$$

Let $u \in W^{1,2}(\Omega)$ with $\mathcal{A}u \in L_2(\Omega)$ and $\psi \in L_2(\Gamma)$. Then we say that u has weak conormal derivative ψ if

$$\mathfrak{a}(u,v) - (\mathcal{A}u, v)_{L_2(\Omega)} = (\psi, \operatorname{Tr} v)_{L_2(\Gamma)}$$

for all $v \in W^{1,2}(\Omega)$. By the Stone–Weierstraß theorem the trace space $\operatorname{Tr}(W^{1,2}(\Omega))$ is dense in $L_2(\Gamma)$. Hence the function ψ is unique and we write $\partial^{\mathfrak{a}}_\nu u = \psi$. Formally,

$$\partial^{\mathfrak{a}}_\nu u = \sum_{k,l=1}^{d} \nu_k \operatorname{Tr}(c_{kl}\,\partial_l u) + \sum_{k=1}^{d} \nu_k \operatorname{Tr}(c_k\,u),$$

where $(\nu_1, \dots, \nu_d)$ is the outer normal vector to Ω. Suppose now that 0 is not in the spectrum of $\mathcal{A}$ endowed with Dirichlet boundary conditions (that is, the form $\mathfrak{a}$ is taken on $V = W^{1,2}_0(\Omega)$). Then we say that $u \in W^{1,2}(\Omega)$ is *$\mathcal{A}$-harmonic* if

$$\mathfrak{a}(u,v) = 0$$

for all $v \in W^{1,2}_0(\Omega)$. Since 0 is not in the spectrum of the Dirichlet operator, for all $\varphi \in H^{1/2}(\Gamma)$ there exists a unique $\mathcal{A}$-harmonic $u \in H^1(\Omega)$ such that $\operatorname{Tr} u = \varphi$. We then define on $L_2(\Gamma)$ the form $\mathfrak{b}: H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \to \mathbb{C}$ by

$$\mathfrak{b}(\varphi, \xi) := \mathfrak{a}(u,v), \tag{4}$$

where $u, v \in W^{1,2}(\Omega)$ are $\mathcal{A}$-harmonic with $\operatorname{Tr} u = \varphi$ and $\operatorname{Tr} v = \xi$, respectively. One proves that the form $\mathfrak{b}$ is continuous, sectorial and closed. The associated operator $\mathcal{N}$ is the Dirichlet-to-Neumann operator. For more details see [8] Section 2, [9] Section 2 or [5]. The operator $\mathcal{N}$ is interpreted as follows. For all $\varphi \in H^{1/2}(\Gamma)$, one solves the Dirichlet problem

$$-\sum_{k,l=1}^{d} \partial_k(c_{kl}\,\partial_l\,u) + \sum_{k=1}^{d} (b_k\,\partial_k u - \partial_k(c_k\,u)) + a_0\,u = 0 \quad \text{weakly in } \Omega,$$

$$\operatorname{Tr} u = \varphi$$

with $u \in W^{1,2}(\Omega)$ and if u has a weak conormal derivative, then $\varphi \in D(\mathcal{N})$ and $\mathcal{N}\varphi = \partial^{\mathfrak{a}}_\nu u$. Alternatively, let $j := \operatorname{Tr}$, $\widetilde{H} = L_2(\Omega)$ and $H = L_2(\Gamma)$. Suppose in addition that $\mathfrak{a}$ is j-elliptic, that is,

suppose that $\operatorname{Re} a_0$ is large enough. Then one checks easily that $\mathcal{N}$ is the operator associated with $(\mathfrak{a}, j)$.

Example 2.2 (The magnetic Dirichlet-to-Neumann operator). Adopt the notation and assumptions as in Example 2.1. Let $\vec{a} := (a_1, \dots, a_d)$ be such that $a_k \in L_\infty(\Omega, \mathbb{R})$ for all $k \in \{1, \dots, d\}$. Set

$$D_k := \partial_k - i a_k$$

for all $k \in \{1, \dots, d\}$. We define as above $\mathfrak{a}(\vec{a})\colon\ W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{a}(\vec{a})(u, v) = \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, (D_l u)\, \overline{D_k v} + \sum_{k=1}^{d} \int_\Omega \Big(b_k\, (D_k u)\, \overline{v} + c_k\, u\, \overline{D_k v} \Big) + \int_\Omega a_0\, u\, \overline{v}. \tag{5}$$

Note that this form has the same expression as in (3) except that ∂_k is now replaced by $D_k = \partial_k - i a_k$. If one expands D_k, then one can rewrite (5) in the form of (3), but with different coefficients b_k, c_k and a_0. Now one can define exactly as above the associated operator $A(\vec{a})$ on $L_2(\Omega)$ as well as the magnetic Dirichlet-to-Neumann operator $\mathcal{N}(\vec{a})$. Formally, if $u \in W^{1,2}(\Omega)$ is $\mathcal{A}(\vec{a})$-harmonic with trace $\operatorname{Tr} u = \varphi$, then

$$\mathcal{N}(\vec{a})\varphi = \partial_\nu^{\mathfrak{a}(\vec{a})}\, u = \sum_{k,l=1}^{d} \nu_k \operatorname{Tr}(c_{kl}\, \partial_l u) - i \sum_{k,l=1}^{d} \nu_k \operatorname{Tr}(c_{kl}\, a_l\, u) + \sum_{k=1}^{d} \nu_k \operatorname{Tr}(c_k\, u).$$

3 | INVARIANCE OF CLOSED CONVEX SETS

As previously, we denote by $\widetilde{H}$ and V two Hilbert spaces such that V is continuously and densely embedded into $\widetilde{H}$. Let $\mathfrak{a}\colon\ V \times V \to \mathbb{C}$ be a quasi-coercive and continuous sesquilinear form. We denote by $\widetilde{A}$ the corresponding operator and $\widetilde{S} = (\widetilde{S}(t))_{t \geqslant 0}$ the semigroup generated by $-\widetilde{A}$ on $\widetilde{H}$.

Let $\widetilde{C}$ be a non-empty closed convex subset of $\widetilde{H}$ and $\widetilde{P}\colon\ \widetilde{H} \to \widetilde{C}$ the corresponding projection. We recall the following invariance criterion (see [14], [15] Theorem 2.2 or [13] Theorem 2.1).

Theorem 3.1. *The following conditions are equivalent.*

(i) *The semigroup $\widetilde{S}$ leaves invariant $\widetilde{C}$, that is, $\widetilde{S}(t)\widetilde{C} \subset \widetilde{C}$ for all $t \geqslant 0$.*
(ii) *$\widetilde{P}V \subset V$ and $\operatorname{Re} \mathfrak{a}(\widetilde{P}u, u - \widetilde{P}u) \geqslant 0$ for all $u \in V$.*

If $\mathfrak{a}$ is accretive, then the previous conditions are equivalent to

(iii) *$\widetilde{P}V \subset V$ and $\operatorname{Re} \mathfrak{a}(u, u - \widetilde{P}u) \geqslant 0$ for all $u \in V$.*

The invariance theorem was first proved in [14] but without Assertion (ii). It is stated in [15], Theorem 2.2, in the case of accretive forms but the proof given there for the equivalence of (i) and (ii) can be adapted to remove accretivity. We also refer to [13] for the Assertion (ii) without accretivity. Note that the implication (ii)⇒(i) is proved in [1], Theorem 2.2, in a general setting of non-autonomous quasi-coercive forms with a non-homogeneous term.

Let now H be a Hilbert space and $j : V \to H$ a bounded linear map with dense range. We assume that $\mathfrak{a}$ is j-elliptic and denote by A the operator associated with $(\mathfrak{a}, j)$. The semigroup generated by $-A$ on H is denoted by $S = (S(t))_{t\geqslant 0}$.

We consider a non-empty closed convex set C of H and denote by $P : H \to C$ the projection. In the context of j-elliptic forms, the previous theorem has the following reformulation (see [2], Proposition 2.9).

Proposition 3.2. *Suppose that $\mathfrak{a}$ is accretive and j-elliptic. Then the following conditions are equivalent.*

(i) *C is invariant for S.*
(ii) *For all $u \in V$, there exists a $w \in V$ such that $P(j(u)) = j(w)$ and $\operatorname{Re}\mathfrak{a}(w, u - w) \geqslant 0$.*
(iii) *For all $u \in V$, there exists a $w \in V$ such that $P(j(u)) = j(w)$ and $\operatorname{Re}\mathfrak{a}(u, u - w) \geqslant 0$.*

The following invariance criterion is implicit in [2]. It allows to obtain invariance of a closed convex set C in H for the semigroup S from the invariance of a certain closed convex set $\widetilde{C}$ for the semigroup $\widetilde{S}$ in $\widetilde{H}$.

Proposition 3.3. *Let $\widetilde{C}$ and C be non-empty closed convex sets of $\widetilde{H}$ and H with corresponding projections $\widetilde{P}$ and P, respectively. Assume $\mathfrak{a}$ is accretive and j-elliptic. Suppose that the convex set $\widetilde{C}$ is invariant for the semigroup $\widetilde{S}$ and that*

$$P \circ j = j \circ \widetilde{P} \quad \text{on } V. \tag{6}$$

Then the convex set C is invariant for the semigroup S.

Proof. First, note that the term in the right-hand side of condition (6) makes sense because of the fact that $\widetilde{P}V \subset V$ by Theorem 3.1 and $j : V \to H$.

Let now $u \in V$ and define $w = \widetilde{P}u$. Then $w \in V$ and $Pj(u) = j(\widetilde{P}u) = j(w)$. Moreover,

$$\operatorname{Re}\mathfrak{a}(w, u - w) = \operatorname{Re}\mathfrak{a}(\widetilde{P}u, u - \widetilde{P}u) \geqslant 0$$

by Theorem 3.1 and the assumption that $\widetilde{C}$ is invariant for the semigroup $\widetilde{S}$. We conclude by Proposition 3.2 that C is invariant for S. □

There are interesting situations where one would like to relax the accretivity assumption in the previous results. A typical situation is when one applies the above criteria to positivity of the Dirichlet-to-Neumann semigroup. For example, if one considers the form given by (2) with $a_0 = \lambda \in \mathbb{R}$, then the accretivity (on $W^{1,2}(\Omega)$) holds only if $\lambda \geqslant 0$. The accretivity on $W^{1,2}_0(\Omega)$, however, holds if $\lambda \geqslant -\lambda_0$, where λ_0 is the first (positive) eigenvalue of the elliptic operator given in (1) with Dirichlet boundary conditions. It is then of interest to know whether one can replace accretivity in the previous results by accretivity on $W^{1,2}_0(\Omega)$ only. In the light of Theorem 3.1, one would expect to have equivalence of (i) and (ii) in Proposition 3.2 in general. It turns out that this is true if the form $\mathfrak{a}$ is symmetric. We do not know whether the same result holds in the case of non-symmetric forms.

Before stating the results we need some notation and assumptions. Set

$$V(\mathfrak{a}) = \{u \in V : \mathfrak{a}(u, v) = 0 \text{ for all } v \in \ker j\}.$$

Clearly $V(\mathfrak{a})$ is closed in V. In Example 2.1 the space $V(\mathfrak{a})$ coincides with the space of $\mathcal{A}$-harmonic functions. We assume that

$$V = V(\mathfrak{a}) \oplus \ker j \tag{7}$$

as vector spaces. In addition, we assume that there exist $\omega \in \mathbb{R}$ and $\mu > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) + \omega \|j(u)\|_H^2 \geqslant \mu \|u\|_V^2 \tag{8}$$

for all $u \in V(\mathfrak{a})$. (Loosely speaking, the j-ellipticity holds only on $V(\mathfrak{a})$.)

Under these two assumptions, one can define as previously the operator A associated with $(\mathfrak{a}, j)$ and A is m-sectorial (see [2] Corollary 2.2). We denote again by S the semigroup generated by $-A$ on H. Then we have the following version of Proposition 3.2 in which we relax the accretivity assumption to be valid only on $\ker j$. Note that we always assume that $j : V \to H$ is continuous and has dense range.

Proposition 3.4. *Let C be a non-empty closed convex set of H with corresponding projection P. Suppose that the form $\mathfrak{a}$ is symmetric and satisfies (7) and (8). Suppose in addition that*

$$\mathfrak{a}(u,u) \geqslant 0 \quad \text{for all } u \in \ker j. \tag{9}$$

Then the following conditions are equivalent.

(i) *C is invariant for S.*
(ii) *For all $u \in V$ there exists a $w \in V$ such that $Pj(u) = j(w)$ and $\operatorname{Re} \mathfrak{a}(w, u-w) \geqslant 0$.*

Remark 3.5. The implication (i)⇒(ii) remains valid without the symmetry assumption of the form $\mathfrak{a}$ and without the accretivity assumption (9).

Proof of Proposition 3.4. In H define the form $\mathfrak{a}_c : j(V(\mathfrak{a})) \times j(V(\mathfrak{a})) \to \mathbb{C}$ by

$$\mathfrak{a}_c(j(u), j(v)) := \mathfrak{a}(u,v)$$

for all $u, v \in V(\mathfrak{a})$. We provide $j(V(\mathfrak{a}))$ with the norm carried over from $V(\mathfrak{a})$ by j. It is easy to see that the form $\mathfrak{a}_c$ is well defined, continuous and quasi-coercive. Its associated operator is again A (see [2] Theorem 2.5 and one can easily replace the j-ellipticity there by (8)). Now we can apply Theorem 3.1 in which the equivalence of the first two assertions does not use accretivity.

'(i)⇒(ii)'. By Theorem 3.1 we have $P(j(V(\mathfrak{a}))) \subset j(V(\mathfrak{a}))$. Let $u \in V$. By (7) there exists a $u' \in V(\mathfrak{a})$ such that $j(u) = j(u')$. Hence there is a $w \in V(\mathfrak{a})$ such that $Pj(u') = j(w)$. Then $Pj(u) = Pj(u') = j(w)$. In addition, since $u - u' \in \ker j$ and $w \in V(\mathfrak{a})$, we have

$$\begin{aligned}
\operatorname{Re} \mathfrak{a}(w, u-w) &= \operatorname{Re} \mathfrak{a}(w, u-u') + \operatorname{Re} \mathfrak{a}(w, u'-w) \\
&= \operatorname{Re} \mathfrak{a}(w, u'-w) \\
&= \operatorname{Re} \mathfrak{a}_c(j(w), j(u')-j(w)) \\
&= \operatorname{Re} \mathfrak{a}_c(Pj(u'), j(u')-Pj(u')) \\
&\geqslant 0,
\end{aligned}$$

where we use again Theorem 3.1 in the last step. This gives Condition (ii). We observe that the symmetry assumption is not used here.

'(ii)⇒(i)'. Let $\varphi := j(u) \in D(\mathfrak{a}_c)$, where $u \in V(\mathfrak{a})$. By (ii) there exists a $w \in V$ such that $Pj(u) = j(w)$ and $\operatorname{Re}\mathfrak{a}(w, u-w) \geqslant 0$. By (7) there is a $w' \in V(\mathfrak{a})$ such that $j(w) = j(w')$. Then $P\varphi = Pj(u) = j(w) = j(w') \in D(\mathfrak{a}_c)$. Next

$$\begin{aligned}\operatorname{Re}\mathfrak{a}_c(P\varphi, \varphi - P\varphi) &= \operatorname{Re}\mathfrak{a}_c(j(w'), j(u) - j(w'))\\ &= \operatorname{Re}\mathfrak{a}(w', u - w')\\ &= \operatorname{Re}\mathfrak{a}(w' - w, u - w') + \operatorname{Re}\mathfrak{a}(w, u - w')\\ &= \operatorname{Re}\mathfrak{a}(w, u - w').\end{aligned}$$

Here we use

$$\operatorname{Re}\mathfrak{a}(w' - w, u - w') = \operatorname{Re}\mathfrak{a}(u - w', w' - w) = 0$$

by the symmetry of $\mathfrak{a}$ and the facts that $u - w' \in V(\mathfrak{a})$ and $w' - w \in \ker j$. Now, by Condition (ii) one deduces that

$$\begin{aligned}\operatorname{Re}\mathfrak{a}(w, u - w') &= \operatorname{Re}\mathfrak{a}(w, u - w) + \operatorname{Re}\mathfrak{a}(w, w - w')\\ &\geqslant \operatorname{Re}\mathfrak{a}(w, w - w').\end{aligned}$$

On the other hand, $\operatorname{Re}\mathfrak{a}(w', w - w') = 0$ since $w' \in V(\mathfrak{a})$ and $w - w' \in \ker j$. Therefore

$$\begin{aligned}\operatorname{Re}\mathfrak{a}(w, w - w') &= \operatorname{Re}\mathfrak{a}(w - w', w - w') + \operatorname{Re}\mathfrak{a}(w', w - w')\\ &= \operatorname{Re}\mathfrak{a}(w - w', w - w')\\ &\geqslant 0,\end{aligned}$$

where we use the accretivity assumption on $\ker j$. Hence we proved that

$$\operatorname{Re}\mathfrak{a}_c(P\varphi, \varphi - P\varphi) \geqslant 0.$$

Using again Theorem 3.1(ii)⇒(i) we conclude that C is invariant for S. □

Now we have the following version of Proposition 3.3 with an identical proof, except that now we apply Proposition 3.4 instead of Proposition 3.2.

Corollary 3.6. *Let $\widetilde{C}$ and C be non-empty closed convex sets of $\widetilde{H}$ and H with corresponding projections $\widetilde{P}$ and P, respectively. Assume that the form $\mathfrak{a}$ is symmetric and satisfies (7) and (8). Suppose in addition that $\mathfrak{a}(u,u) \geqslant 0$ for all $u \in \ker j$. Suppose that the convex set $\widetilde{C}$ is invariant for the semigroup $\widetilde{S}$ and that*

$$P \circ j = j \circ \widetilde{P}.$$

Then the convex set C is invariant for the semigroup S.

We conclude this section by mentioning that one may consider the Condition (ii) in Theorem 3.1, Proposition 3.2 and Proposition 3.4 on a dense subset of V as in [15], Theorem 2.2.

4 | POSITIVITY AND L_∞-CONTRACTIVITY

The criteria in the previous section turn out to be simple and effective in applications. We illustrate this by proving positivity and L_∞-contractivity of the semigroup generated by the Dirichlet-to-Neumann operator $\mathcal{N}$ described in Example 2.1 of Section 2 under a mild additional condition. This mild condition is that there is a $\mu > 0$ such that

$$\operatorname{Re} \mathfrak{a}(u,u) \geqslant \mu \|\nabla u\|^2_{L_2(\Omega)} \tag{10}$$

for all $u \in W^{1,2}(\Omega)$. This condition is valid if $\operatorname{Re} a_0$ is large enough. It is a standard fact that there is a $\mu' > 0$ such that

$$\int_\Omega |\nabla u|^2 + \int_\Gamma |\operatorname{Tr}(u)|^2 \geqslant \mu' \|u\|^2_{W^{1,2}(\Omega)}$$

for all $u \in W^{1,2}(\Omega)$. From this and (10), it follows that $\mathfrak{a}$ is j-elliptic with $j = \operatorname{Tr}$. Then we have the following consequence of Proposition 3.3.

Corollary 4.1. *Suppose (10) and that the coefficients c_{kl}, b_k, c_k and a_0 are all real valued for all $k, l \in \{1, \dots, d\}$. Then the semigroup S generated by (minus) the Dirichlet-to-Neumann operator $\mathcal{N}$ is positive.*

Proof. It follows from [15], Theorem 4.2, that the semigroup $\widetilde{S}$ generated by $-A$ on $L_2(\Omega)$ is positive. Therefore $\widetilde{S}$ leaves invariant the closed convex set $\widetilde{C} := \{u \in L_2(\Omega) : u \geqslant 0\}$. The projection onto $\widetilde{C}$ is $\widetilde{P}u = (\operatorname{Re} u)^+$. Now we choose $C := \{\varphi \in L_2(\Gamma) : \varphi \geqslant 0\}$. Then $P\varphi = (\operatorname{Re}\varphi)^+$. It is clear that (6) is satisfied and hence C is invariant for S by Proposition 3.3. This latter property means that S is positive. □

Regarding the positivity proved above a remark is in order. We have assumed (10) in order to ensure j-ellipticity and define the Dirichlet-to-Neumann operator using the $(\mathfrak{a}, j)$ technique as explained in Section 2. The condition (10) is however not true for general (too negative) a_0. On the other hand, for general $a_0 \in L_\infty(\Omega)$ one can still define the Dirichlet-to-Neumann operator using the form (4) under the sole condition that the elliptic operator with Dirichlet boundary conditions is invertible on $L_2(\Omega)$. If $\mathfrak{a}$ is symmetric, then we apply Proposition 3.6 instead of Proposition 3.3 and obtain that the Dirichlet-to-Neumann semigroup S is positive if the form $\mathfrak{a}$ is accretive on $W^{1,2}_0(\Omega)$. In particular, if $c_{kl} = c_{lk}$ and $b_k = c_k$ for all $k, l \in \{1, \dots, d\}$ then S is positive as soon as $a_0 + \lambda^D_1 > 0$ a.e. on Ω, where λ^D_1 is the first eigenvalue of the operator

$$-\sum_{k,l=1}^d \partial_l(c_{kl}\,\partial_k) + \sum_{k=1}^d (b_k\,\partial_k - \partial_k(c_k\,\cdot))$$

subject to the Dirichlet boundary conditions. Note that if the condition $a_0 + \lambda^D_1 > 0$ a.e. on Ω is not satisfied, the semigroup S might not be positive. See [7].

Concerning the L_∞-contractivity of the Dirichlet-to-Neumann semigroup S we have the following result.

Corollary 4.2. *Suppose in addition to (10) that* $\operatorname{Re} a_0 \geqslant 0$ *a.e. on* Ω. *Suppose also that* c_{kl}, b_k *and* ic_k *are real valued for all* $k, l \in \{1, \dots, d\}$. *Then the semigroup* S *is* L_∞*-contractive.*

Proof. Under the assumptions of the corollary, the semigroup $\widetilde{S}$ is L_∞-contractive by Theorem 4.6 in [15]. This means that $\widetilde{S}$ leaves invariant the closed convex set given by $\widetilde{\mathcal{C}} := \{u \in L_2(\Omega) : |u| \leqslant 1\}$. The projection onto $\widetilde{\mathcal{C}}$ is $\widetilde{P}u = (1 \wedge |u|) \operatorname{sign} u$. We choose $\mathcal{C} := \{\varphi \in L_2(\Gamma) : |\varphi| \leqslant 1\}$. Then $P\varphi = (1 \wedge |\varphi|) \operatorname{sign} \varphi$. Since $\operatorname{Tr}((1 \wedge |u|) \operatorname{sign} u) = (1 \wedge |\operatorname{Tr} u|) \operatorname{sign}(\operatorname{Tr} u)$ the condition (6) is satisfied and hence $\mathcal{C}$ is invariant for S by Proposition 3.3. This proves that S is L_∞-contractive. □

Here the sign function is defined by $\operatorname{sign} z = \frac{z}{|z|}$ if $z \neq 0$ and $\operatorname{sign} 0 = 0$.

A consequence of the previous corollary is that the semigroup S can be extended to a holomorphic semigroup on $L_p(\Gamma)$ for all $p \in (2, \infty)$. For all $p \in (1, 2)$ one may argue by duality by applying the corollary to the adjoint operator.

5 | A DOMINATION CRITERION

This section is devoted to a domination criterion for semigroups such as those generated by Dirichlet-to-Neumann operators. Although one can find in the literature several criteria for the domination in terms of sesquilinear forms (see [14] or Chapter 2 in [15]) their application to Dirichlet-to-Neumann operators is difficult since one has to deal with harmonic lifting of functions such as $\varphi \operatorname{sign} \psi$ with $\varphi, \psi \in H^{1/2}(\Gamma)$ such that $|\varphi| \leqslant |\psi|$ (see Theorem 5.3 below). In contrast to general criteria in [14] we shall focus on operators such as the Dirichlet-to-Neumann operator and make a link between the domination in $L_2(\Gamma)$ and the domination in $L_2(\Omega)$. In a sense, we obtain the domination in $L_2(\Gamma)$ for the semigroup generated by (minus) the Dirichlet-to-Neumann operator from the domination in $L_2(\Omega)$ of the corresponding elliptic operator with Neumann boundary conditions.

We start by fixing some notation. Let $\widetilde{H} := L_2(\widetilde{X}, \tilde{\nu})$ and $H = L_2(X, \nu)$, where $(\widetilde{X}, \tilde{\nu})$ and (X, ν) are σ-finite measure spaces. Let U and V be two Hilbert spaces which are densely and continuously embedded into $\widetilde{H}$. We consider two sesquilinear forms

$$\mathfrak{a} : U \times U \to \mathbb{C} \quad \text{and} \quad \mathfrak{b} : V \times V \to \mathbb{C}$$

which are continuous, accretive and quasi-coercive. We denote by $\widetilde{A}$ and $\widetilde{B}$ their associated operators, respectively. Let $j_1 : U \to H$ and $j_2 : V \to H$ be two bounded operators with dense ranges. We assume that $\mathfrak{a}$ is j_1-elliptic and $\mathfrak{b}$ is j_2-elliptic and denote by A and B the operators associated with $(\mathfrak{a}, j_1)$ and $(\mathfrak{b}, j_2)$, respectively. Finally, we denote by $\widetilde{T} = (\widetilde{T}(t))_{t \geqslant 0}$ and $\widetilde{S} = (\widetilde{S}(t))_{t \geqslant 0}$ the semigroups generated by $-\widetilde{A}$ and $-\widetilde{B}$ on $\widetilde{H}$ and $T = (T(t))_{t \geqslant 0}$ and $S = (S(t))_{t \geqslant 0}$ the semigroups generated by $-A$ and $-B$ on H, respectively. Then under suitable assumptions we have transference of domination.

Theorem 5.1. *Adopt the above notation and assumptions. Further suppose the following.*

(I) *$\widetilde{T}$ is dominated by $\widetilde{S}$, that is,*

$$|\widetilde{T}(t)f| \leqslant \widetilde{S}(t)|f|$$

for all $t \geqslant 0$ and $f \in \widetilde{H}$.

(II) *The maps j_1 and j_2 satisfy the four properties in Hypothesis 5.4.*

Then T is dominated by S, that is,

$$|T(t)\varphi| \leqslant S(t)|\varphi|$$

for all $t \geqslant 0$ and all $\varphi \in H$.

In light of Proposition 3.4 and Corollary 3.6 the accretivity assumption can be relaxed if the forms $\mathfrak{a}$ and $\mathfrak{b}$ are symmetric. We leave the details to the interested reader.

The following definition was introduced in [14].

Definition 5.2. We say that U is an *ideal* of V if

- $u \in U \Rightarrow |u| \in V$ and
- if $u \in U$ and $v \in V$ are such that $|u| \leqslant |v|$, then $v \operatorname{sign} u \in U$.

We also recall the following criterion for the domination (see [14] or [15] Theorem 2.21).

Theorem 5.3. *Suppose that the semigroup $\widetilde{S}$ is positive. The following conditions are equivalent.*

(i) *$\widetilde{T}$ is dominated by $\widetilde{S}$.*
(ii) *U is an ideal of V and $\operatorname{Re}\mathfrak{a}(u, |v| \operatorname{sign} u) \geqslant \mathfrak{b}(|u|, |v|)$ for all $(u, v) \in U \times V$ such that $|u| \leqslant |v|$.*
(iii) *U is an ideal of V and $\operatorname{Re}\mathfrak{a}(u, v) \geqslant \mathfrak{b}(|u|, |v|)$ for all $(u, v) \in U \times V$ such that $u\overline{v} \geqslant 0$.*

Since we assume in Theorem 5.1 that $\widetilde{T}$ is dominated by $\widetilde{S}$, it is then a consequence of Theorem 5.3 that U is an ideal of V. In particular, all the quantities appearing in the following properties are well defined.

Hypothesis 5.4. *Assume*

- *$j_2(\operatorname{Re} v) = \operatorname{Re} j_2(v)$ for all $v \in V$;*
- *$j_2(v_1 \vee v_2) = j_2(v_1) \vee j_2(v_2)$ for all $v_1, v_2 \in V$ which are real valued;*
- *$j_2(|u|) = |j_1(u)|$ for all $u \in U$; and*
- *$j_1(v \operatorname{sign} u) = j_2(v) \operatorname{sign}(j_1(u))$ for all $(u, v) \in U \times V$ such that $0 \leqslant v \leqslant |u|$.*

Note that the first two properties use the fact that semigroup $\widetilde{S}$ is positive and hence $\operatorname{Re} u, (\operatorname{Re} u)^+ \in V$ for all $u \in V$. This implies that $v_1 \vee v_2 \in V$ for all real-valued $v_1, v_2 \in V$.

Obviously, the properties in Hypothesis 5.4 are satisfied if $U = V = W^{1,2}(\Omega)$, $H = L_2(\Gamma)$ and $j_1 = j_2 = \operatorname{Tr}$.

Proof of Theorem 5.1. We follow an idea from [14] and view the domination as the invariance of a closed convex set by an appropriate semigroup. Define $\widehat{H} := \widetilde{H} \times \widetilde{H} = L_2(\widetilde{X}, \tilde{\nu}) \times L_2(\widetilde{X}, \tilde{\nu})$ and

consider the closed convex set

$$\widehat{C} := \{(u,v) \in \widehat{H} : |u| \leqslant v\}.$$

The projection onto $\widehat{C}$ is given by

$$\widehat{P}(u,v) = \frac{1}{2}\big([|u| + |u| \wedge \operatorname{Re} v]^+ \operatorname{sign} u, [|u| \vee \operatorname{Re} v + \operatorname{Re} v]^+\big). \tag{11}$$

See [14] or [15] (2.7). We also define $\hat{\jmath} : U \times V \to H \times H$ by

$$\hat{\jmath}(u,v) := (j_1(u), j_2(v)).$$

Since j_1 and j_2 are bounded with dense ranges it is clear that $\hat{\jmath}$ is bounded and has a dense range.

Next define the sesquilinear form $\mathfrak{c} : (U \times V) \times (U \times V) \to \mathbb{C}$ by

$$\mathfrak{c}((u_0,v_0),(u_1,v_1)) := \mathfrak{a}(u_0,u_1) + \mathfrak{b}(v_0,v_1).$$

This form is quasi-coercive, accretive and continuous. Its associated operator is

$$\begin{pmatrix} \widetilde{A} & 0 \\ 0 & \widetilde{B} \end{pmatrix}$$

and the corresponding semigroup on $\widehat{H}$ is

$$\begin{pmatrix} \widetilde{T} & 0 \\ 0 & \widetilde{S} \end{pmatrix} = \begin{pmatrix} \widetilde{T}(t) & 0 \\ 0 & \widetilde{S}(t) \end{pmatrix}_{t \geqslant 0}.$$

We next show that $\mathfrak{c}$ is $\hat{\jmath}$-elliptic. Indeed, if $(u,v) \in U \times V$, then

$$\begin{aligned} \operatorname{Re} \mathfrak{c}((u,v),(u,v)) + \omega \|\hat{\jmath}(u,v)\|^2_{H\times H} &= \operatorname{Re} \mathfrak{a}(u,u) + \omega \|j_1(u)\|^2_H + \operatorname{Re} \mathfrak{b}(v,v) + \omega \|j_2(v)\|^2_H \\ &\geqslant \mu(\|u\|^2_U + \|v\|^2_V), \end{aligned}$$

where we use that $\mathfrak{a}$ is j_1-elliptic and $\mathfrak{b}$ is j_2-elliptic with some constants ω_1, ω_2 and $\mu_1, \mu_2 > 0$ and then we take $\omega = \max(\omega_1, \omega_2)$ and $\mu = \min(\mu_1, \mu_2)$. Recall that A and B are the operators associated with $(\mathfrak{a}, j_1)$ and $(\mathfrak{b}, j_2)$, respectively. Denote by C the operator associated with $(\mathfrak{c}, \hat{\jmath})$.

We shall show that

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \tag{12}$$

In order to prove this we use the definition of the associated operator. Let $(\varphi, \psi) \in D(C)$ and write $(\eta, \chi) = C(\varphi, \psi)$. This means that there exists $(u,v) \in U \times V$ such that

$$\hat{\jmath}(u,v) = (\varphi, \psi) \text{ and} \tag{13}$$

$$\mathfrak{c}((u,v),(w,z)) = ((\eta,\chi),\hat{j}(w,z))_{H\times H} \text{ for all } (w,z)\in U\times V. \tag{14}$$

The equality in (14) reads as

$$\mathfrak{a}(u,w)+\mathfrak{b}(v,z) = (\eta, j_1(w))_H + (\chi, j_2(z))_H$$

for all $w\in U$ and $z\in V$. Taking $z=0$ in the last equality and using (13) yields $\varphi = j_1(u)$ and $\mathfrak{a}(u,w) = (\eta, j_1(w))_H$ for all $w\in U$. This means that $\varphi\in D(A)$ and $A\varphi=\eta$. Similarly, $\psi\in D(B)$ and $B\psi=\chi$. Hence

$$(\varphi,\psi)\in D\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix} \quad\text{and}\quad C(\varphi,\psi) = \begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}(\varphi,\psi).$$

We have proved that $\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}$ is an extension of C. The converse inclusion is similar and we obtain (12).

We conclude from equality (12) that the semigroup generated by $-C$ is given by

$$\begin{pmatrix} T & 0\\ 0 & S\end{pmatrix} = \begin{pmatrix} T(t) & 0\\ 0 & S(t)\end{pmatrix}_{t\geqslant 0}.$$

Now we consider the closed convex subset of $H\times H$ defined by

$$\mathcal{C} := \{(\varphi,\psi)\in H\times H : |\varphi|\leqslant\psi\}.$$

Similarly to (11), the projection onto $\mathcal{C}$ is given by

$$P(\varphi,\psi) = \frac{1}{2}\big([|\varphi|+|\varphi|\wedge \operatorname{Re}\psi]^+ \operatorname{sign}\varphi, [|\varphi|\vee \operatorname{Re}\psi + \operatorname{Re}\psi]^+\big).$$

It follows easily from Hypothesis 5.4 that $P\circ\hat{j} = \hat{j}\circ\widehat{P}$. Since the domination of $\widetilde{T}$ by $\widetilde{S}$ means that the semigroup $\begin{pmatrix}\widetilde{T} & 0\\ 0 & \widetilde{S}\end{pmatrix}$ leaves invariant the convex $\widehat{\mathcal{C}}$, we conclude by Proposition 3.3 that the semigroup $\begin{pmatrix} T & 0\\ 0 & S\end{pmatrix}$, generated by $-C$ on $H\times H$, leaves invariant the convex set $\mathcal{C}$. The latter property means again that T is dominated by S. This proves the theorem. □

6 | THE DIAMAGNETIC INEQUALITY

In this section we prove the diamagnetic inequality for the Dirichlet-to-Neumann operator. This will be obtained by applying Theorem 5.1.

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^d$ with boundary Γ. Let $\vec{a} = (a_1,\dots,a_d)$ be such that $a_k\in L_\infty(\Omega,\mathbb{R})$ for all $k\in\{1,\dots,d\}$. We consider the magnetic Dirichlet-to-Neumann operator $\mathcal{N}(\vec{a})$ on $L_2(\Gamma)$ and the Dirichlet-to-Neumann operator $\mathcal{N}$ corresponding to $\vec{a}=0$ (see Examples 2.1 and 2.2 in Section 2). We denote by $T_{\vec{a}} = (T_{\vec{a}}(t))_{t\geqslant 0}$ and $T=(T(t))_{t\geqslant 0}$ the semigroups generated by $-\mathcal{N}(\vec{a})$ and $-\mathcal{N}$ on $L_2(\Gamma)$, respectively. We have the following domination.

Theorem 6.1. *Suppose that c_{kl}, b_k, c_k, a_0 and a_k are real valued for all $k, l \in \{1, \dots, d\}$. Suppose in addition that the form $\mathfrak{a}$ in (3) is accretive and j-elliptic with $j = \mathrm{Tr}$. Then $T_{\vec{a}}$ is dominated by T on $L_2(\Gamma)$. That is,*

$$|T_{\vec{a}}(t)\varphi| \leqslant T(t)|\varphi|$$

for all $t \geqslant 0$ and $\varphi \in L_2(\Gamma)$.

Proof. We apply Theorem 5.1 with $\widetilde{H} = L_2(\Omega), U = V = W^{1,2}(\Omega)$ and $H = L_2(\Gamma)$. Set $j_1 = j_2 = \mathrm{Tr}$. It is clear that the four properties in Hypothesis 5.4 are satisfied. Therefore Theorem 6.1 follows immediately from Theorem 5.1 and the next result, Proposition 6.2, on the domination in $L_2(\Omega)$. □

Denote by $\widetilde{A}(\vec{a})$ and $\widetilde{A} = \widetilde{A}(0)$ the elliptic operators in $L_2(\Omega)$ associated with the forms defined by (5) and (3) on $W^{1,2}(\Omega)$. We denote by $\widetilde{T_{\vec{a}}}$ and $\widetilde{T}$ the semigroups generated by $-\widetilde{A}(\vec{a})$ and $-\widetilde{A}$ on $L_2(\Omega)$, respectively.

Proposition 6.2. *Suppose that c_{kl}, b_k, c_k, a_0 and a_k are real valued for all $k, l \in \{1, \dots, d\}$. Then we have the diamagnetic inequality*

$$|\widetilde{T_{\vec{a}}}(t)f| \leqslant \widetilde{T}(t)|f|$$

for all $t \geqslant 0$ and $f \in L_2(\Omega)$.

The proposition is very well known in the case $\Omega = \mathbb{R}^d$, $c_{kl} = \delta_{kl}$ and $b_k = c_k = 0$. For general domains with Neumann boundary conditions (as we do in the previous proposition) and $c_{kl} = \delta_{kl}$, $b_k = c_k = 0$ it was proved in [11]. Note that in our case we do not assume any regularity or symmetry for (c_{kl}). In addition we allow the presence of terms of order 1. The same domination result is also valid, with the same proof, if the operators $\widetilde{A}(\vec{a})$ and $\widetilde{A}$ are endowed with other boundary conditions such as Dirichlet or mixed boundary conditions.

Proof. Note first that since all the coefficients are real valued, the semigroup $\widetilde{T}$ generated by $-\widetilde{A}$ is positive (cf. [15] Corollary 4.3). In particular, $W^{1,2}(\Omega)$ is an ideal of itself (see [14] or [15] Proposition 2.20). It remains to prove that

$$\mathrm{Re}\, \mathfrak{a}(\vec{a})(u, v) \geqslant \mathfrak{a}(|u|, |v|) \tag{15}$$

for all $u, v \in W^{1,2}(\Omega)$ with $u\,\overline{v} \geqslant 0$ and then apply Theorem 5.3. Let $u, v \in W^{1,2}(\Omega)$ with $u\,\overline{v} \geqslant 0$. Then $u\,\overline{v} = |u|\,|v|$ and $(\mathrm{sign}\,\overline{u})\,\mathrm{sign}\, v = 1$ outside the sets where $u = 0$ or $v = 0$. Hence

$$\begin{aligned}\mathrm{Re}\, \mathfrak{a}(\vec{a})(u, v) = \mathrm{Re} \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, (\partial_l u)\, \overline{\partial_k v} + \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, a_l\, \mathrm{Im}(u\, \overline{\partial_k v}) - \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, a_k\, \mathrm{Im}((\partial_l u)\, \overline{v}) \\ + \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, a_l\, a_k\, |u|\, |v| + \sum_{k=1}^{d} \int_\Omega \Big(b_k\, \mathrm{Re}((\partial_k u)\, \overline{v}) + c_k\, \mathrm{Re}(u\, \overline{\partial_k v}) \Big) \\ + \int_\Omega a_0\, |u|\, |v|\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k,l=1}^{d} \int_\Omega c_{kl} \operatorname{Re}((\partial_l u)\ \operatorname{sign}\overline{u}) \operatorname{Re}((\partial_k v)\ \operatorname{sign}\overline{v}) \\
&\quad + \sum_{k,l=1}^{d} \int_\Omega c_{kl} \operatorname{Im}((\partial_l u)\ \operatorname{sign}\overline{u}) \operatorname{Im}((\partial_k v)\ \operatorname{sign}\overline{v}) \\
&\quad + \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, a_l \operatorname{Im}(u\,\overline{\partial_k v}) - \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, a_k \operatorname{Im}((\partial_l u)\,\overline{v}) \\
&\quad + \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, a_l\, a_k\, |u|\,|v| + \sum_{k=1}^{d} \int_\Omega \Big(b_k \operatorname{Re}((\partial_k u)\,\overline{v}) + c_k \operatorname{Re}(u\,\overline{\partial_k v}) \Big) \\
&\quad + \int_\Omega a_0\, |u|\,|v| \\
&= \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, (\partial_l |u|)\, \partial_l |v| + \sum_{k=1}^{d} \int_\Omega (b_k\, (\partial_k |u|)\, |v| + c_k\, |u|\, \partial_k |v|) + \int_\Omega a_0\, |u|\,|v| \\
&\quad + \sum_{k,l=1}^{d} \int_\Omega c_{kl} \operatorname{Im}((\partial_l u)\ \operatorname{sign}\overline{u}) \operatorname{Im}((\partial_k v)\ \operatorname{sign}\overline{v}) \\
&\quad - \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, a_k \operatorname{Im}((\partial_l u)\ \operatorname{sign}\overline{u})\, |v| - \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, a_l \operatorname{Im}((\partial_k u)\ \operatorname{sign}\overline{u})\, |v| \\
&\quad + \sum_{k,l=1}^{d} \int_\Omega c_{kl}\, a_l\, a_k\, |u|\,|v|,
\end{aligned}
$$

where we used the standard fact that

$$\partial_k |u| = \operatorname{Re}((\partial_k u)\ \operatorname{sign}\overline{u}).$$

Moreover, since $u\,\overline{v} \geqslant 0$ we have $\operatorname{Im} \partial_k (u\,\overline{v}) = 0$ and hence

$$-|u| \operatorname{Im}((\partial_k v)\ \operatorname{sign}\overline{v}) = \operatorname{Im}(u\,\overline{\partial_k v}) = -|v| \operatorname{Im}((\partial_k u)\ \operatorname{sign}\overline{u}).$$

So

$$\int_\Omega c_{kl} \operatorname{Im}((\partial_l u)\ \operatorname{sign}\overline{u}) \operatorname{Im}((\partial_k v)\ \operatorname{sign}\overline{v}) = \int_\Omega c_{kl} \operatorname{Im}((\partial_l u)\ \operatorname{sign}\overline{u}) \operatorname{Im}((\partial_k u)\ \operatorname{sign}\overline{u})\, \frac{|v|}{|u|},$$

with the convention that $\operatorname{Im}((\partial_l u)\ \operatorname{sign}\overline{u}) \operatorname{Im}((\partial_k u)\ \operatorname{sign}\overline{u})\, \frac{|v|}{|u|} = 0$ on the set where $u = 0$.

It follows that

$$\begin{aligned}\operatorname{Re}\mathfrak{a}(\vec{a})(u,v) &= \mathfrak{a}(|u|,|v|) + \sum_{k,l=1}^{d}\int_\Omega c_{kl}\operatorname{Im}((\partial_l u)\operatorname{sign}\overline{u})\operatorname{Im}((\partial_k u)\operatorname{sign}\overline{u})\frac{|v|}{|u|}\\ &\quad - \sum_{k,l=1}^{d}\int_\Omega (c_{kl}+c_{lk})a_k\operatorname{Im}((\partial_l u)\operatorname{sign}\overline{u})\,|v| + \sum_{k,l}\int_\Omega c_{kl}\,a_l\,a_k\,|u|\,|v|\\ &= \mathfrak{a}(|u|,|v|) + \int_\Omega Q\,\frac{|v|}{|u|},\end{aligned}$$

where

$$\begin{aligned}Q &= \sum_{k,l=1}^{d} c_{kl}\operatorname{Im}((\partial_l u)\operatorname{sign}\overline{u})\operatorname{Im}((\partial_k u)\operatorname{sign}\overline{u}) - \sum_{k,l=1}^{d}(c_{kl}+c_{lk})\,a_k\,\operatorname{Im}((\partial_l u)\operatorname{sign}\overline{u})\,|u|\\ &\quad + \sum_{k,l=1}^{d} c_{kl}\,a_l\,a_k\,|u|^2.\end{aligned}$$

It remains to prove that $Q \geqslant 0$ to obtain (15).

Set $\xi_k := \operatorname{Im}((\partial_k u)\operatorname{sign}\overline{u})$ for all $k \in \{1,\dots,d\}$, $\xi = (\xi_1,\dots,\xi_d)$ and $C = (c_{kl})_{1\leqslant k,l\leqslant d}$. Then

$$Q = \langle C\xi,\xi\rangle_{\mathbb{R}^d} - \langle (C+C^*)\vec{a},\xi\rangle_{\mathbb{R}^d}\,|u| + \langle C\vec{a},\vec{a}\rangle_{\mathbb{R}^d}\,|u|^2.$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}\langle (C+C^*)\vec{a},\xi\rangle_{\mathbb{R}^d}\,|u| &\leqslant \langle (C+C^*)\vec{a},\vec{a}\rangle_{\mathbb{R}^d}^{1/2}\,|u|\,\langle (C+C^*)\xi,\xi\rangle_{\mathbb{R}^d}^{1/2}\\ &\leqslant \frac{1}{2}\langle (C+C^*)\vec{a},\vec{a}\rangle_{\mathbb{R}^d}\,|u|^2 + \frac{1}{2}\langle (C+C^*)\xi,\xi\rangle_{\mathbb{R}^d}\\ &= \langle C\vec{a},\vec{a}\rangle_{\mathbb{R}^d}\,|u|^2 + \langle C\xi,\xi\rangle_{\mathbb{R}^d}.\end{aligned}$$

This implies that $Q \geqslant 0$ and finishes the proof of the proposition. □

Remark 6.3. We mentioned above that the diamagnetic inequality of Proposition 6.2 is valid with other boundary conditions. Note also that if we add a positive potential V to a_0 in the expression of $\widetilde{A}(\vec{a})$ then we have the same domination by the semigroup of $\widetilde{A}$ (without V). The same domination holds for the corresponding semigroups of the Dirichlet-to-Neumann operators. A particular case of this result was proved in [8] for the Dirichlet-to-Neumann operators associated with $-\Delta + V$ and $-\Delta$ on $L_2(\Gamma)$.

7 | SOME CONSEQUENCES

Let Ω be a bounded open Lipschitz subset of $\mathbb{R}^d$ with boundary Γ, where $d \geqslant 2$. Let $T_{\vec{a}}$ be the semigroup generated by (minus) the magnetic Dirichlet-to-Neumann operator $\mathcal{N}(\vec{a})$ on $L_2(\Gamma)$. Since the trace operator is compact, it follows that the spectrum of $\mathcal{N}(\vec{a})$ is discrete. The first consequence of Theorem 6.1 is as follows.

Corollary 7.1. *Suppose that $c_{kl} = c_{lk} \in L_\infty(\Omega, \mathbb{R})$, $b_k = c_k = 0$ and $a_k \in L_\infty(\Omega, \mathbb{R})$ for all $k, l \in \{1, \dots, d\}$. Suppose also that $a_0 \geqslant 0$. Then there exists a constant $c > 0$, independent of $\vec{a}$, such that*

$$\sum_{k=1}^{\infty} e^{-\lambda_k t} \leqslant c\, t^{-(d-1)}$$

for all $t \in (0, 1]$, where $\lambda_1 \leqslant \lambda_2 \leqslant \dots$ is the sequence of the corresponding eigenvalues of the self-adjoint operator $\mathcal{N}(\vec{a})$.

Proof. As in Theorem 6.1, let T be the semigroup generated by $-\mathcal{N}$. It follows from [9] Lemma 8.2 and [8] Theorem 2.6 that $T(t)$ satisfies

$$\|T(t)\|_{L_1(\Gamma) \to L_\infty(\Gamma)} \leqslant c\, t^{-(d-1)} \tag{16}$$

for all $t \in (0, 1]$. We obtain from this and Theorem 6.1 that

$$\|T_{\vec{a}}(t)\|_{L_1(\Gamma) \to L_\infty(\Gamma)} \leqslant c\, t^{-(d-1)}$$

for all $t \in (0, 1]$. The constant c is independent of t and $\vec{a}$. Next $T_{\vec{a}}(t) L_2(\Gamma) \subset C(\Gamma)$ for all $t > 0$ by [10], Theorem 5.5 or Proposition 5.7. Then [4], Theorem 2.1, implies that $T_{\vec{a}}(t)$ is given by a continuous kernel $K_{\vec{a}}(t, \cdot, \cdot) : \Gamma \times \Gamma \to \mathbb{C}$ in the sense

$$(T_{\vec{a}}(t)\varphi)(w) = \int_\Gamma K_{\vec{a}}(t, z, w)\, \varphi(z)\, d\sigma(z)$$

for all $\varphi \in L_1(\Gamma)$ and $w \in \Gamma$. Then (16) gives

$$|K_{\vec{a}}(t, z, w)| \leqslant c\, t^{-(d-1)} \tag{17}$$

for all $t \in (0, 1]$ and $z, w \in \Gamma$. It is well known that the trace of the operator $T_{\vec{a}}(t)$ coincides with $\int_\Gamma K_{\vec{a}}(t, z, z)\, d\sigma(z)$ and the corollary follows from (17). □

Note that (17) can also be used to obtain some bounds on the counting function of $\mathcal{N}(\vec{a})$. See [3].

The second consequence we mention here is that under additional regularity the estimate (17) on the heat kernel $K_{\vec{a}}$ can be improved into an optimal Poisson bound.

Corollary 7.2. *Let Ω be a bounded domain of class $C^{1+\kappa}$ for some $\kappa > 0$. Suppose also that $c_{kl} = c_{lk} \in C^\kappa(\Omega, \mathbb{R})$, $b_k = c_k = 0$ and $a_k \in L_\infty(\Omega, \mathbb{R})$ for all $k, l \in \{1, \dots, d\}$. Suppose in addition that $a_0 \geqslant 0$ a.e. on Ω. Then there exists a constant $c > 0$ such that*

$$|K_{\vec{a}}(t, z, w)| \leqslant \frac{c\, (t \wedge 1)^{-(d-1)}\, e^{-\lambda_1 t}}{\left(1 + \dfrac{|z - w|}{t}\right)^d}$$

for all $z, w \in \Gamma$ and $t > 0$, where λ_1 is the first eigenvalue of the operator $\mathcal{N}(\vec{a})$.

Proof. The estimate

$$|K_{\vec{a}}(t,z,w)| \leqslant \frac{c\,(t\wedge 1)^{-(d-1)}}{\left(1+\dfrac{|z-w|}{t}\right)^{d}}$$

for all $z,w\in\Gamma$ and $t>0$ follows immediately from Theorem 6.1 and Theorem 1.1 in [9]. The constant c in this estimate is independent of $\vec{a}$.

The improvement upon the factor $e^{-\lambda_1 t}$ for $t\geqslant 1$ can be proved as follows. Define

$$\tilde{c}=\|T_{\vec{a}}(1)\|_{L_2(\Gamma)\to L_\infty(\Gamma)}<\infty.$$

If $t\in[3,\infty)$, then

$$\begin{aligned}|K_{\vec{a}}(t,z,w)| &\leqslant \|T_{\vec{a}}(t)\|_{L_1(\Gamma)\to L_\infty(\Gamma)}\\ &\leqslant \|T_{\vec{a}}(1)\|_{L_2(\Gamma)\to L_\infty(\Gamma)}\,\|T_{\vec{a}}(t-2)\|_{L_2(\Gamma)\to L_2(\Gamma)}\,\|T_{\vec{a}}(1)\|_{L_1(\Gamma)\to L_2(\Gamma)}\leqslant \tilde{c}^2\,e^{-\lambda_1(t-2)}\end{aligned}$$

for all $z,w\in\Gamma$. If $R=\max\{|z-w|:z,w\in\Gamma\}$, then

$$|K_{\vec{a}}(t,z,w)|\leqslant \tilde{c}^2\,e^{2\lambda_1}\,(1+R)^d\,\frac{(t\wedge 1)^{-(d-1)}\,e^{-\lambda_1 t}}{\left(1+\dfrac{|z-w|}{t}\right)^{d}}$$

for all $t\in[3,\infty)$ and $z,w\in\Gamma$. □

Corollary 7.3. *Adopt the notation and assumptions of Corollary 7.2. In addition suppose that $d\geqslant 3$. Then for all $\varepsilon,\tau'\in(0,1)$, $\tau>0$ and $\vec{a}\in\mathbb{R}^d$ there exist $c,\nu>0$ such that*

$$\begin{aligned}&|K_{\vec{a}}(t,z,w)-K_{\vec{a}}(t,z',w')|\\ &\leqslant c\,(t\wedge 1)^{-(d-1)}\left(\frac{|z-z'|+|w-w'|}{t+|z-w|}\right)^{\nu}\frac{1}{\left(1+\dfrac{|z-w|}{t}\right)^{d-\varepsilon}}\,(1+t)^{\nu}\,e^{-\lambda_1 t}\end{aligned}$$

for all $z,w,z',w'\in\Gamma$ and $t>0$ with $|z-z'|+|w-w'|\leqslant\tau\,t+\tau'\,|z-w|$.

Proof. The proof is the same as the proof of Theorem 5.11 in [10], since we now have the Poisson bounds of Corollary 7.2 for the kernel associated to $\mathcal{N}(\vec{a})$. □

ACKNOWLEDGEMENTS

The authors wish to thank the referees for their interesting comments. This work was carried out when the second author was visiting the University of Auckland and the first author was visiting the University of Bordeaux. Both authors wish to thank the universities for hospitalities. The research of A.F.M. ter Elst is partly supported by the Marsden Fund Council from government funding, administered by the Royal Society of New Zealand. The research of E.M. Ouhabaz is partly supported by the ANR project RAGE, ANR-18-CE-0012-01.

Open access publishing facilitated by The University of Auckland, as part of the Wiley - The University of Auckland agreement via the Council of Australian University Librarians.

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The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

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