Nonmetrisable Manifolds

Sina Ruth Greenwood

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Abstract

Nyikos has defined a tree, denoted Υ, associated with any given Type I space. This thesis examines the properties of an Υ-tree if the space is a Type I nonmetrisable manifold. It is shown that a tree, T, is an Υ-tree of a Type I manifold iff T is a well-pruned ω_1-tree. Furthermore, if T is any well-pruned ω_1-tree, there are $2^{\aleph_1}$ different Type I manifolds for which T is the Υ-tree. The relationships between the properties of a Type I manifold and the properties of its Υ-tree are examined. It is shown that whenever a Type I manifold contains a copy of ω_1, its Υ-tree must contain an uncountable branch. The thesis then addresses the problem of whether or not an arbitrary tree T admits a Type I manifold which is ω_1-compact. If T does not contain an uncountable antichain, or a Suslin subtree, then there exists a Type I manifold with Υ-tree T. If T contains an uncountable antichain, then whether there exists an ω_1-compact Type I manifold with Υ-tree T is undecidable. $(*)$ implies there does not exist such a tree while $\diamondsuit$ implies that there does. If we assume $\clubsuit^+$, then at least one such manifold exists. $\diamondsuit$ also implies that if T contains a Suslin subtree, then there exists an ω_1-compact manifold with Υ-tree T.

Nyikos has recently defined a Type II space. We may associate an Υ-tree with such a space. This thesis shows that a Tree, T, admits a Type II manifold iff T has height not greater than ω_1, and each level has cardinality no greater than $\mathfrak{c}$.

The final chapter examines the relationship between microbundles and fibre bundles over nonmetrisable manifolds. In 1964 Milnor defined the notion of a microbundle. He ceased developing the theory of microbundles when later in the same year Kister showed that a microbundle over a metrisable manifold is equivalent to a fibre bundle. This thesis proves that the tangent microbundle over a manifold is a fibre

bundle iff the manifold is metrisable. As a consequence of this we obtain further properties equivalent to metrisability in a manifold.

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