A FINITE ELEMENT METHOD FOR AN EIKONAL EQUATION MODEL OF MYOCARDIAL EXCITATION WAVEFRONT PROPAGATION*

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Abstract. An efficient finite element method is developed to model the spreading of excitation in ventricular myocardium by treating the thin region of rapidly depolarizing tissue as a propagating wavefront. The model is used to investigate excitation propagation in the full canine ventricular myocardium. An eikonal-curvature equation and an eikonal-diffusion equation for excitation time are compared. A Petrov–Galerkin finite element method with cubic Hermite elements is developed to solve the eikonal-diffusion equation on a reasonably coarse mesh. The oscillatory errors seen when using the Galerkin weighted residual method with high mesh Péclet numbers are avoided by supplementing the Galerkin weights with $C^0$ functions based on derivatives of the interpolation functions. The ratio of the Galerkin and supplementary weights is a function of the Péclet number such that, for one-dimensional propagation, the error in the solution is within a small constant factor of the optimal error achievable in the trial space. An additional no-inflow boundary term is developed to prevent spurious excitation from initiating on the boundary. The need for discretization in time is avoided by using a continuation method to gradually introduce the nonlinear term of the governing equation. A simulation is performed in an anisotropic model of the complete canine ventricular myocardium, with 2355 degrees of freedom for the dependent variable.

Key words. eikonal equation, myocardial excitation, wavefront propagation, Petrov–Galerkin method, Hermite interpolation, numerical continuation

AMS subject classifications. 65N30, 35J60, 35K90

1. Introduction. In developing a computational model of the electrical behavior of the ventricular myocardium, it would be unreasonable to expect to be able to model every microscopic biological process that occurs within and between each and every cell. Such detail in the model is also unnecessary: the ventricular function and the electrical fields induced in the torso are not so much affected by the activity of one ion or one ion channel or even one cell as by the collective activity of many cells. Instead of resolving the small spatial detail of the microscopic processes, the collective macroscopic effect of these processes can be modelled.

The most intense electrical activity is the depolarization of cells, which leads to the activation of the mechanisms that cause the myocardium to contract and the heart to pump. Depolarization occurs quickly and in only a narrow region of cells at a time, so this narrow region can be considered as a propagating excitation wavefront. An eikonal model may be used to approximate the propagation process, describing the motion of the wavefront by the time at which it excites every point in the myocardium.

A finite element method with cubic Hermite elements is developed to determine excitation times on a fairly coarse mesh for large scale simulations. Petrov–Galerkin weighted residual equations, developed in section 2, are supplemented with a no-inflow term, developed in section 3, to prevent spurious excitation on boundaries, and are solved by a continuation method with Newton’s method (section 4). Section 5

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employs the computational method to simulate excitation in the complete ventricular myocardium.

1.1. The bidomain model. As a means for collecting together the microscopic functional elements of the myocardium to model their macroscopic effects, Schmitt [21] suggested the concept of two interpenetrating domains. One domain was to represent the volume-averaged properties of the intracellular contents and their interconnections, and the other domain was to represent the volume-averaged properties of the surrounding extracellular tissue and fluid. These domains were to coexist spatially, and the behavior of current flow between them was to be based on the volume-averaged properties of the cell membrane. This approach is now generally referred to as the bidomain model [12]. The two domains are referred to as the intracellular and extracellular domains. Each is treated as a continuum.

A reaction-diffusion system of equations for the potential \( \phi_e \) in the extracellular domain and the difference in potential \( V_m \) across the membrane between the domains can be derived from conservation of current under the assumptions that capacitive, inductive, and electromagnetic propagative effects within the domains are negligible and that the current in each domain obeys Ohm’s law:

\[
\nabla \cdot (G^i \nabla V_m) = - \nabla \cdot ((G^i + G^e) \nabla \phi_e),
\]

\[
\dot{\phi}_e + c_m \frac{\partial V_m}{\partial t} = - \nabla \cdot (G^e \nabla \phi_e).
\]

The intracellular potential \( \phi_i \) is the sum of the extracellular potential \( \phi_e \) and the transmembrane potential \( V_m \). \( G^i \) and \( G^e \) are intracellular and extracellular **effective conductance tensors**. The fibrous and laminar structure of the myocardium is modelled under the assumptions that the conductivities are orthotropic and that they share the same principal axes, \( a_1, a_2, \) and \( a_3 \), where \( a_1 \) is parallel to the fibers (longitudinal), \( a_2 \) is transverse to the fibers but in the plane of the sheets, and \( a_3 \) is normal to the sheets. \( \dot{\phi}_e \) represents the sum of the (outward) membrane ionic currents per unit tissue volume, and \( c_m \) is the membrane capacitance per unit volume.

It is assumed that the extracellular space is in direct contact with the outside volume. Continuity of the extracellular potential \( \phi_e \) with the potential \( \phi_o \) in the outside volume and conservation of current between the volumes leads to the boundary conditions

\[
\phi_o = \phi_e,\]

\[
n \cdot G^i \nabla (\phi_o + V_m) = 0,
\]

\[
n \cdot G^e \nabla \phi_e = n \cdot j_o \quad \text{on} \; \partial \Omega,
\]

where \( n \) is the unit normal to the boundary and \( j_o \) is the current density in the outside volume [15].

If the intra- and extracellular conductivity tensors were related by a constant scalar factor (equal anisotropy), then system (1.1) could be reduced to a simple monodomain reaction-diffusion equation in one variable:

\[
\dot{\phi}_e + c_m \frac{\partial V_m}{\partial t} = \nabla \cdot (G^m \nabla V_m).
\]

\( G^m \) has the same principal axes as \( G^i \) and \( G^e \), and the reciprocals of its eigenvalues equal the sums of the reciprocals of the intra- and extracellular principal conductivi-
ties. The boundary condition on \( V_m \) would be

\[
(1.5) \quad n \cdot G^e \nabla V_m = -n \cdot G^o \nabla \phi_{o} \quad \text{on } \partial \Omega.
\]

If the anisotropic ratios are not equal, the monodomain equation (1.4) may still be used as an approximation of the bidomain system (1.1). For plane wave propagation in any of the three principal directions, both (1.1) and (1.4) predict the same propagation speeds, but the predicted speeds may differ for intermediate directions.

It is convenient to scale (1.4) so that the parameters give indications of the important spatial and temporal scales. This can be done by dividing the equation by a characteristic conductance per unit volume. During the depolarization phase of the action potential, consideration of the large difference between the activation and inactivation time constants of the dominating fast sodium current leads to the approximation of \( i_{ion} \) as a time-independent function of the transmembrane voltage [5]. That is, \( i_{ion} = i_{ion}(V_m) \). If the transmembrane potential \( V_m \) is near its resting potential \( V_r \), the behavior of the ionic membrane currents can be approximated by assuming the membrane has a passive conductance per unit volume defined by

\[
(1.6) \quad \frac{1}{r_m} := \frac{d i_{ion}}{d V_m}(V_r).
\]

(The symbol "\( := \)" denotes definition.) Multiplying the terms in (1.4) by an average (space-independent) value \( \bar{r}_m \) of \( r_m \) gives

\[
(1.7) \quad \bar{r}_m i_{ion} + r_m \frac{\partial V_m}{\partial t} = \nabla \cdot (M \nabla V_m),
\]

where

\[
(1.8) \quad M := \bar{r}_m G^m \quad \text{and} \quad \tau_m := \bar{r}_m c_{m}
\]

are the coupling tensor, which has dimensions of space squared, and the membrane time constant, which has dimension of time. The eigenvalues of \( M \) are squares of the space constants \( \lambda_1, \lambda_t, \) and \( \lambda_n \) in each of the principal directions. They may be expressed in terms of conductivities using

\[
(1.9) \quad \frac{1}{\lambda^2_1} = \frac{1}{\bar{r}_m} \left( \frac{1}{g_{il}} + \frac{1}{g_{el}} \right), \quad \text{etc.}
\]

These space and time constants are appropriate when the behavior of the tissue is largely passive such as in the early stages of the action potential. The behavior in these stages is important for propagation as it initiates the change in transmembrane potential that leads to activation of the active currents. The time and space constants relevant in the fastest stage of depolarization, however, may be different. The magnitude of the maximum slope of \( i_{ion}(V_m) \) is much larger than the slope at \( V_m = V_r \) used to define \( r_m \) in (1.6). The appropriate multiplier for scaling the system of equations is then smaller than \( \bar{r}_m \), and so the appropriate space and time constants are also smaller. The space constants \( \lambda_1, \lambda_t, \) and \( \lambda_n \) probably provide an indication of the region of influence that the excitation wavefront has, and, together with \( \tau_m \), they provide an upper bound on the relevant spatial and temporal scales.

Solution of the reaction-diffusion equation (1.7) is very computationally demanding due to the important spatial scales being much smaller than the dimensions of
the ventricles. As discussed in [23], the space constants for the passive behavior of canine myocardium are probably $\lambda_1 \approx 0.8\,\text{mm}$ and $\lambda_n < \lambda_t \approx 0.5\,\text{mm}$. Reasonable approximation of the potential would probably require at least 5 degrees of freedom to represent changes over the distance of a space constant. This implies that at least $5^3$ degrees of freedom would be required to represent a volume of about $0.8 \times 0.5 \times 0.5 = 0.2\,\text{mm}^3$. For the full canine ventricular myocardium with a volume of about $0.2 \times 10^6\,\text{mm}^3$, at least $10^8$ degrees of freedom would be needed.

1.2. An eikonal approach. Given the difficulty in the numerical solution of a reaction-diffusion equation for transmembrane potential, a governing equation is sought for the motion of the excitation wavefront. It is expected that the speed of propagation can be assumed to vary more slowly and over much larger spatial scales than the transmembrane potential. This assumption is probably reasonable most of the time, but there are abrupt spatial changes in propagation speed where a wavefront collides with the boundary or another wavefront. The fine details of the wavefront shape in these small collision regions are not, however, expected to have much influence on the overall ventricular function.

The wavefront motion can be described by the excitation time $u(x)$, defined as the time at which the wavefront passes through the point $x$ (or, more specifically, the time at which the transmembrane potential at that point crosses the value midway between its resting and plateau potentials). The position of the wavefront at any time $t$ is then given by the surface along which $u(x) = t$, and the excitation time can be described numerically on a stationary mesh. A governing equation for $u$ is referred to as an eikonal equation.

Many myocardial excitation models have been based on Huygens' principle (reviewed in [19]) and are effectively approximating an eikonal equation. In such models, the heart is represented by a matrix of cells or grid points. At fixed time intervals after any cell is excited, its quiescent neighboring cells are excited. The time interval before excitation of each neighboring cell depends on the distance to the cell and the propagation speed for that direction. This method requires little computational effort but has the disadvantage that the numerical treatment of the eikonal equation is very low order and propagation can occur in only a finite number of directions. The result is that the wavefronts generated are polyhedral instead of ellipsoidal.

More accurate numerical solutions for excitation wavefront propagation have been obtained using wavefront propagation equations derived from the reaction-diffusion equation (1.7) under the assumption that the profile of the depolarization upstroke varies slowly in space.

An alternative approach for describing wavefront propagation is to use a function $\varphi(x, t)$, defined so that, at any time $t$, the level set of points $x$ such that $\varphi(x, t) = 0$ gives the position of the wavefront at that time (see [22]). Keener [13] derived an equation for $\varphi$ from (1.7) by selecting a moving coordinate system such that $V_m$ is a function of only a spatial variable normal to the wavefront and then requiring the current conservation equation to be satisfied at the wavefront. The resulting equation,

$$\left[ c_0 + \nabla \cdot \left( \frac{M \nabla u}{\sqrt{\nabla u \cdot M \nabla u}} \right) \right] \sqrt{\nabla \varphi \cdot M \nabla \varphi} = \tau_m \frac{\partial \varphi}{\partial t},$$

is parabolic and time-dependent. $\varphi$ has a physical interpretation only at its zero contour, so the selection of initial conditions is unclear. If, however, $\varphi(x, t)$ is chosen
to be \( t - u(\mathbf{x}) \), then (1.10) reduces to a parabolic eikonal equation for excitation time:
\[
(1.11) \\
\frac{c_0}{\sqrt{\nabla u \cdot M \nabla u}} - \nabla u \cdot M \nabla u \nabla \left( \frac{M \nabla u}{\sqrt{\nabla u \cdot M \nabla u}} \right) = \tau_m.
\]

The numerical solution to (1.10) was found using finite difference discretizations in space and time. Second-order central differences were initially used for the spatial discretization [13], but [14] later replaced these with first-order upwind differences to stabilize the numerical solution.

An elliptic eikonal equation was derived by Colli Franzone, Guerri, and Rovida [8] and Colli Franzone, Guerri, and Tentoni [9] using singular perturbation techniques. The equivalent eikonal equation for reaction-diffusion equation (1.7) is
\[
(1.12) \\
c_0 \sqrt{\nabla u \cdot M \nabla u} - \nabla \cdot (M \nabla u) = \tau_m.
\]

As the equation is elliptic, a boundary condition is required around the entire boundary. Without a model of the surrounding tissue, it is not possible to predict the current flux from the outside domain, and thus the boundary condition (1.5) for the reaction-diffusion system is not helpful. However, experimental evidence suggests that epicardial isochrones are unaffected by surrounding conducting volumes [11]. Without a surrounding volume, boundary condition (1.5) leads to the simple no-flux boundary condition
\[
(1.13) \\
\mathbf{n} \cdot M \nabla u = 0,
\]
where \( \mathbf{n} \) is the unit normal to the boundary.

In their numerical solution of the eikonal equation, Colli Franzone and Guerri [6] added a time derivative term to give a related parabolic equation in space and time. The time-dependent equation for (1.12) is
\[
(1.14) \\
\frac{\partial \hat{u}}{\partial t} + c_0 \sqrt{\nabla \hat{u} \cdot M \nabla \hat{u}} - \nabla \cdot (M \nabla \hat{u}) = \tau_m.
\]
The steady-state solution for \( \hat{u}(\mathbf{x}, t) \) is the excitation time \( u(\mathbf{x}) \). To find this solution, spatial discretization was performed using finite element–like integrals of quantities calculated by finite differences, and a finite difference scheme was used to step through time until \( \hat{u} \) approached its limiting value. The spatial discretization was later modified [7] so that traditional finite element integrals were used for most terms, but a first-order upwind finite difference was used for the first-order spatial derivatives. A purely explicit finite difference scheme in time gave a method that was similar to Jacobi successive overrelaxation. In order to avoid instability, the time step (or relaxation parameter) had to be small, and thus convergence was very slow.

When compared to the reaction-diffusion equation (1.7), the eikonal equations (1.11) and (1.12) have the advantages that the domain is reduced by one dimension (because the dependent variable is no longer a function of time) and that the important spatial scales are much larger. In order to make use of these advantages, a numerical method needs to be found that requires only a spatial discretization and will work effectively when this discretization is reasonably coarse.

Both the level set and relaxation methods discussed above fail to take advantage of the fact that excitation time depends only on spatial position. The use of either of the time-dependent equations (1.10) or (1.14) increases the size of the domain by one dimension. For this reason, the method investigated here uses numerical
continuation, with Newton's method applied directly to a spatial discretization of an eikonal equation, to converge from an initial guess to the solution. Each Newton iteration requires not much more work than that required in an iteration of an implicit time stepping scheme for either (1.10) or (1.14), yet makes a considerably better attempt to go directly to the required solution.

1.3. Interpretation and comparison of eikonal equations. Interpretations of the two suggested eikonal equations (1.11) and (1.12) for wavefront propagation can be made from each of the terms involved.

The contours of \( u \) give the positions of the wavefront at time \( t = u \). The gradient of \( u \) at any point along one of these contours is therefore normal to that wavefront surface and has magnitude equal to the reciprocal of the speed of that point on the wavefront. That is,

\[
\nabla u = \frac{1}{\theta} p,
\]

where \( \theta \) is the local wavefront speed and \( p \) is the unit normal to the wavefront pointing away from depolarized tissue. A space constant \( \rho \) in the direction of propagation \( p \) may be calculated from the square root of the component of the coupling tensor in that direction:

\[
\rho := \sqrt{p \cdot M p}.
\]

The first term in both governing equations (1.11) and (1.12) is a nonlinear advection term, which may be written as

\[
c_0 \sqrt{\nabla u \cdot M \nabla u} = c_0 \frac{\rho}{\theta}.
\]

This term is an anisotropic generalization of the left-hand side of the standard eikonal equation \( |\nabla u| = 1 \) and is a function of the local speed of the wavefront surface.

The second term in the parabolic equation (1.11) may be written as

\[
\sqrt{\nabla u \cdot M \nabla u} \nabla \left( \frac{M \nabla u}{\sqrt{\nabla u \cdot M \nabla u}} \right) = \frac{\rho}{\theta} \nabla \left( \frac{1}{\rho} M p \right) = \frac{\rho}{\theta} \kappa,
\]

where \( \kappa \) is an anisotropic generalization of the mean curvature. It is positive when the wavefront is convex if viewed from ahead of the wavefront. The parabolic equation is therefore called an eikonal-curvature equation.

Using expressions (1.17) and (1.18) in the eikonal-curvature equation (1.11) gives

\[
\frac{\tau_m}{\rho} \theta = c_0 - \kappa.
\]

For a given propagation direction, this equation states that the speed of the wavefront is a linear function of its anisotropic mean curvature. Propagation is faster when the wavefront is concave, and slower when it is convex. This reflects the dependency of tissue depolarization on the diffusion of charge from already depolarized tissue. If there is more depolarized tissue in close proximity to a region of quiescent tissue, then that region will be depolarized faster.

If there is no curvature, the speed of propagation is \( c_0 \) space constants per time constant. The constant \( c_0 \) is therefore the dimensionless propagation speed for a planar wavefront in homogeneous tissue.
The second term in the eikonal-curvature equation may also be expressed as

\[
\sqrt{\nabla u \cdot M \nabla u} \nabla \left( \frac{M \nabla u}{\sqrt{\nabla u \cdot M \nabla u}} \right) = \nabla \cdot (M \nabla u) - \nabla \sqrt{\nabla u \cdot M \nabla u} \cdot \frac{M \nabla u}{\sqrt{\nabla u \cdot M \nabla u}}
\]

where the right-hand side is an anisotropic generalization of the Laplacian of \( u \) minus the component of this term in the direction of propagation. The eikonal-curvature equation is parabolic, as it lacks this second derivative in the direction of propagation. Propagation is effectively determined only by information at the wavefront. It is unaffected by boundaries or approaching wavefronts until a collision occurs.

The eikonal equation (1.12) is elliptic, as it contains the full generalized Laplacian. Although it is difficult to comprehend diffusion of excitation time, it is not too surprising that there is a Laplacian in the governing equation, as the propagation process depends heavily on the diffusion of charge. The elliptic eikonal equation is therefore called an eikonal-diffusion equation. Under this equation, propagation speed depends not only on information at the wavefront but also on the activity of the surrounding tissue. The constant \( c_0 \) is still the dimensionless speed of steady planar wavefront propagation in infinite homogeneous tissue.

It is interesting to investigate three-dimensional analytic solutions to these two governing equations for a wavefront spreading out from the origin in an infinite homogeneous domain. There exist solutions that may be written as functions of only the dimensionless distance from the origin,

\[
r := \sqrt{x \cdot M^{-1} x}.
\]

The solutions describe ellipsoidal wavefronts having the same principal axes as the coupling tensor. Both eikonal equations predict that an initial wavefront of this shape will retain the same shape as it propagates. Under the eikonal-curvature equation, the propagation speed \( \theta = \rho / \frac{\partial n}{\partial r} \) satisfies

\[
\frac{\tau_m}{\rho} \theta = c_0 \frac{r - \frac{2}{c_0}}{r},
\]

and under the eikonal-diffusion equation,

\[
\frac{\tau_m}{\rho} \theta = c_0 \frac{r^2}{r^2 + \frac{2}{c_0} r + \frac{2}{c_0^2}}.
\]

For very large \( r \), both equations predict that the ellipsoid grows at the same constant speed, but for small \( r \) the equations differ in the way they predict propagation under large curvature. The eikonal-curvature equation has a change in propagation direction at \( r = \frac{2}{c_0} \), suggesting that the initially depolarized region must have a radius of at least \( \frac{2}{c_0} \) space constants in order for the region to be able to supply enough current to surrounding tissue for propagation to proceed. If the initially depolarized region is smaller than this threshold size, then the equation predicts that the wavefront will retreat and the region will repolarize. The eikonal-diffusion equation, on the other hand, predicts a zero propagation speed only at the origin, suggesting that if enough current has been injected into the tissue to depolarize a region of tissue, then propagation will proceed however small this region may be.

As a wavefront approaches a no-flux boundary or another approaching wavefront, there is less quiescent tissue to drain current from the depolarizing tissue, and thus
the reaction-diffusion model predicts an increase in propagation speed. The eikonal-curvature equation, however, does not include any effects of the boundary or collision on wavefront propagation. Solutions to the eikonal-diffusion equation, on the other hand, much more closely approximate the variations in propagation speed due to another approaching wavefront [8]. The eikonal-diffusion equation is selected as the governing eikonal equation for the work presented here and shall be referred to simply as the eikonal equation.

1.4. Solution spaces and interpolation. In the finite element method the numerical solution $U(x)$ is represented by a linear combination of known interpolation functions $\psi_i(x)$:

(1.24) \[ U(x) := U_i \psi_i(x). \]

The finite element method selects the unknown parameters $U_i$ in an attempt to approximate the exact solution $u(x)$. The domain is divided into a number of elements so that, within each element, $U$ depends on only a subset of the parameters. For each element, a local coordinate system $\xi$ is defined, and thus, within that element,

(1.25) \[ U(x(\xi)) := U_{\nu(e,j)} \Psi_j(\xi), \]

where $\Psi_j(\xi)$ are the element’s local basis functions, and $\nu(e,j)$ is a known function mapping the local parameter $j$ in element $e$ to its corresponding global parameter. The interpolation functions $\psi_i(x(\xi))$ are therefore equal to corresponding basis functions $\Psi_j(\xi)$ in elements influenced by $U_i$ and zero elsewhere.

Cubic Hermite elements are used here for discretization of the geometry and dependent variables. One-dimensional basis functions are cubic polynomials that interpolate the value and first derivative of $U$ at the two adjacent nodes. Multidimensional basis functions are obtained from tensor products of the one-dimensional functions. Hermite elements have the advantage over cubic Lagrange elements that all nodes lie on element vertices, and thus parameters can be shared by surrounding elements, and a high-order interpolation is achieved with fewer parameters. This also provides first derivative continuity ($C^1$) in $U$.

Because the exact solution $u$ satisfies an elliptic differential equation with predominantly smooth space-dependent coefficients and boundary conditions, it is expected to be sufficiently smooth for the first derivative continuity of the interpolation. Across any surface where the coefficients of the equation are not sufficiently smooth, similar interpolation can be used but with the elements on opposing sides of the surface using separate derivative parameters.

The time and location of excitation wavefront initiation is specified by Dirichlet boundary conditions for excitation time $u$ on $\Gamma_D$, where $\Gamma_D$ denotes the portion of the boundary in which $u$ is known from the initiation process. These boundary conditions are enforced by specifying the values of the parameters $U_j$ that describe $U$ on $\Gamma_D$. The set $D$ is defined as the list of indices $j$ for these parameters $U_j$ and their corresponding interpolation functions $\psi_j$. The set $N$ is defined as the list of indices $j$ for the remaining parameters, which do not influence the value of $U$ on $\Gamma_D$ and are free to be determined by the finite element method.

The trial space $S^h_D$ is defined as the space of possible numerical solutions $U$ sat-
isfying the Dirichlet boundary conditions,

\[(1.26a) \quad S_D^h := \left\{ V : \exists v_j \in \mathbb{R} \text{ for } j \in N \text{ such that } (s.t.) \quad V = \sum_{j \in D} U_j \psi_j + \sum_{j \in N} v_j \psi_j \right\}, \]

and the space \(S_D^h\) is defined as the space of possible variations in the approximation,

\[(1.26b) \quad S_0^h := \left\{ V : \exists v_j \in \mathbb{R} \text{ for } j \in N \text{ s.t. } V = \sum_{j \in N} v_j \psi_j \right\}. \]

The exact solution \(u\) is expected to lie in the Sobolev space \(H^1(\Omega)\). The spaces \(H_D^1\) and \(H_{D_0}^1\) are defined for the exact solution in a manner similar to that for \(S_D^h\) and \(S_0^h\) for the approximate solution:

\[(1.27a) \quad H_D^1 := \{ v \in H^1(\Omega) : v = u \text{ on } \Gamma_D \}, \]

\[(1.27b) \quad H_{D_0}^1 := \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \}. \]

Note that for either Lagrange or Hermite interpolation \(S_0^h \subset H_{D_0}^1\), and, assuming \(\sum_{j \in D} U_j \psi_j = u \) on \(\Gamma_D\), \(S_D^h \subset H_D^1\). Under these conditions we also have \(u - U \in H_{D_0}^1\).

2. A Petrov-Galerkin finite element method. The space constants in the coupling tensor \(M\) for myocardium are several times smaller than the dimensions of the tissue, and thus the advection term in the eikonal equation tends to dominate the diffusion term. Care must be taken in selecting a spatial discretization to prevent oscillatory errors such as those that can occur in numerical solution of the steady-state linear advection-diffusion equation (e.g., [24]). A Petrov-Galerkin finite element method that avoids this problem is developed here for eikonal equation (1.12).

The general Petrov-Galerkin finite element method for determining an approximation \(U\) for \(u\) may be formulated as finding \(U \in S_D^h\) such that

\[(2.1) \quad B(U, W) = \langle \tau_m, W \rangle, \quad \forall W \in T^h, \]

where \(T^h\) is the test space,

\[(2.2) \quad B(v, w) := \langle c_0 \nabla v \cdot M \nabla v, w \rangle + \langle M \nabla v, \nabla w \rangle, \]

and \(\langle \cdot, \cdot \rangle\) denotes the inner product over the domain \(\Omega\).

In this section, a means for estimating the quality of a test space \(T^h\) is described, and a set of weighting functions, which form a basis for \(T^h\), is selected on the grounds of keeping the expected error in the solution to a minimum and facilitating numerical solution of the resulting weighted residual equations.

2.1. Approximate symmetrization. The performance of the Galerkin finite element method is poor when diffusion is small due to the asymmetric nature of \(B(\cdot, \cdot)\). The object of selecting a Petrov-Galerkin scheme is to choose a mapping from \(S_0^h\) to \(T^h\) so that it compensates for this asymmetry. Barrett and Morton [2] showed how an error bound can be derived for a test space \(T^h\) if \(B(\cdot, \cdot)\) is bilinear. This form is bilinear for the eikonal equation if and only if propagation is in only one direction, but analysis of this simple case leads to one-dimensional weighting functions that can
be extended to higher dimensions. A summary of the key points in the error bound derivation follows.

If $T^h \subset H^1_{D_0}$, then the exact solution $u$ satisfies the same weighted residual equations (2.1) as the numerical solution $U$. Therefore, if $B(\cdot, \cdot)$ is bilinear, the error $u - U$ satisfies the orthogonality property

$$B(u - U, W) = 0 \quad \forall W \in T^h. \tag{2.3}$$

The convergence properties implied by this orthogonality property depend on $T^h$.

If $B_S(\cdot, \cdot)$ is any symmetric continuous coercive bilinear form on $H^1_{D_0} \times H^1_{D_0}$, then, from the Riesz representation theorem, there exists a representer $R_S: H^1_{D_0} \rightarrow H^1_{D_0}$ such that

$$B(v, w) = B_S(v, R_S w) \quad \forall v, w \in H^1_{D_0}. \tag{2.4}$$

Assuming $u - U \in H^1_{D_0}$, this means that the orthogonality property (2.3) may be written as

$$B_S(u - U, R_S W) = 0 \quad \forall W \in T^h. \tag{2.5}$$

The performance of the method depends on how closely $S^h_0$ can be approximated by $R_ST^h$. Define the norm $\|\cdot\|_{B_S}$ such that

$$\|v\|_{B_S}^2 := B_S(v, v). \tag{2.6}$$

If $T^h \subset H^1_{D_0}$ and there exists a constant $\Delta_S \in [0, 1)$ such that

$$\inf_{W \in T^h} \|V - R_S W\|_{B_S} \leq \Delta_S \|V\|_{B_S} \quad \forall V \in S^h_0, \tag{2.7}$$

then it is possible to determine a bound for the error in terms of the optimal error and the constant $\Delta_S$:

$$\|u - U\|_{B_S} \leq \frac{1}{\sqrt{1 - \Delta_S^2}} \inf_{Z \in S^h_0} \|u - Z\|_{B_S}. \tag{2.8}$$

The ratio of this bound on the error to the optimal solution error is therefore described by the error factor $(1 - \Delta^2)$. This factor is 1 if the test space $T^h$ is chosen to be equal to $T^{h*} \subset H^1_{D_0}$ defined such that

$$R_ST^{h*} = S^h_0. \tag{2.9}$$

If the representer $R_S$ is known, then the constant $\Delta_S$ may be calculated for given $S^h_0$ and $T^h$. In the Petrov–Galerkin finite element method, $S^h_0$ and $T^h$ are both of dimension $N$. Define the $N \times N$ matrices $A$, $B$, and $C$ with entries

$$A_{ij} := B_S(R_S w_i, R_S w_j),$$

$$B_{ij} := B_S(R_S w_i, \psi_j) = B(\psi_j, w_i),$$

and

$$C_{ij} := B_S(\psi_i, \psi_j), \tag{2.10}$$

where the weighting functions $w_i$ (usually based on $\psi_i$) form a basis for $T^h$. The error factor $(1 - \Delta^2)$ is the reciprocal of the square root of the smallest eigenvalue $\lambda$ of the generalized eigenvalue problem

$$B^T A^{-1} BV = \lambda CV. \tag{2.11}$$
2.2. One-dimensional optimal weighting functions. Consider a one-dimensional problem on a domain of length $L$ with a Dirichlet boundary condition at $x = 0$ and a Neumann boundary condition at $x = L$ so that wavefront propagation is only in the direction of increasing $x$. The excitation time $u(x)$ is required to satisfy
\begin{align}
(2.12a) \quad c_0 \sqrt{M} u' - Mu'' &= \tau_m \quad \text{on } (0, L), \\
(2.12b) \quad u(0) &= 0, \quad \text{and} \quad Mu'(L) = 0,
\end{align}
where $c_0$ and $M$ are positive constants and $'$ denotes the derivative with respect to $x$.

With propagation in only one direction, the form $B(x, y)$ is bilinear. For constant $c_0$ and $M$, it simplifies to
\begin{equation}
B(v, w) := c_0 \sqrt{M} \langle v', w \rangle + M \langle v', w' \rangle.
\end{equation}
The second inner product is symmetric and can be used for $B_S(x, y)$:
\begin{equation}
B_{S}(v, w) := M \langle v', w' \rangle.
\end{equation}

For this problem the Riesz reprenter $R_S$ may be easily found. From its defining relation (2.4) and definitions of $B(x, y)$ (2.13) and $B_{S}(x, y)$ (2.14),
\[
\left. \left. \frac{c_0}{\sqrt{M}} w + w' - \left( R_S w \right)' \right| \right. = 0 \quad \forall v, w \in H^1_{\text{D}_0}.
\]
With the Neumann boundary condition at $x = L$, $v$ is only confined to be zero at $x = 0$, and thus
\begin{equation}
(R_S w)' = \gamma \gamma w + w' \quad \forall w \in H^1_{\text{D}_0},
\end{equation}
where $\gamma = \frac{c_0}{\sqrt{M}}$. This and the boundary condition $(R_S w)(0) = 0$ due to $R_S w \in H^1_{\text{D}_0}$ uniquely determine $R_S w$ for any given $w$.

If each of the weighting functions $w_i^*$ were chosen such that $R_S w_i^* = \psi_i$, they would form a basis for the optimal test space $T^{\psi}$.

2.3. One-dimensional approximate symmetrization. The expressions for optimal one-dimensional weighting functions $w_i^*$ become rather complicated, particularly for irregular meshes or variable coefficients. Extension to more than one dimension and to the nonlinear eikonal equation does not seem feasible. Instead, therefore, the weighting functions are chosen to be simple combinations of the optimal functions when $\gamma$ approaches $0$ and $\infty$. For the one-dimensional problem (2.12), the weighting functions are
\begin{equation}
w_i := A_0 w_i^0 + A_\infty w_i^\infty,
\end{equation}
where
\begin{equation}
w_i^0 := \psi_i \quad \text{and} \quad w_i^\infty := \gamma^{-1} \psi_i',
\end{equation}
and $A_0$ and $A_\infty$ are functions of the mesh Péclet number,
\begin{equation}
P_e := \frac{c_0}{\sqrt{M}} \frac{\text{d} x}{\text{d} \xi}.
\end{equation}
These weighting functions are local and easily evaluated. With the $C^1$ continuity of cubic Hermite interpolation, they all lie in $H^1_{\text{D}_0}$ except the function corresponding to the derivative at $x = 0$. This will be discussed and corrected below.
2.3.1. Selection of coefficients. The proportionality coefficients $A_0$ and $A_\infty$ are chosen with the intention of making the factor in the error bound (2.8) as small as possible. This contrasts with the work of Christie et al. [4], in which weighting functions were assembled to cancel truncation errors in difference equations for one-dimensional equal-length elements. The error factor depends on the closeness with which $R_S T^h$ approximates $S_0^h$ as measured by the constant $\Delta_S$ in bound (2.7). $(R_S W)'$ is given by expression (2.15), and thus bound (2.7) is equivalent to

\begin{equation}
\inf_{W \in T^h} \|V' - \gamma W - W'\|_{L_2} \leq \Delta_S \|V'\|_{L_2} \quad \forall V \in S_0^h.
\end{equation}

Bounds for error factors in terms of $P_e$ have been obtained for meshes of equal-length one-dimensional linear elements using eigenvalue problem (2.11) (see [17]), but extension to cubic Hermite elements is difficult. Analysis is therefore simplified by considering only the function in $S_0^h$ that is expected to be most poorly approximated by functions in $R_S T^h$.

With cubic Hermite interpolation, each $V'$ for $V \in S_0^h$ is piecewise quadratic with $C^0$ continuity. $V'$ may have discontinuities in derivatives at element boundaries. If advection dominates ($\gamma$ is large), each $V'$ must be approximated by a $W \in T^h$. With Galerkin weights $\psi_i, each W$ is piecewise cubic with $C^1$ continuity and cannot approximate discontinuities in first derivatives. If elements are equally spaced, the function $V \in S_0^h$ with the largest discontinuities in first derivatives of $V'$ relative to $\|V\|_{L_2}$ is

\begin{equation}
\hat{V} := \sum_{j \in N_1} \psi_j,
\end{equation}

where $N_1$ indexes the interpolation functions corresponding to first derivatives. $\hat{V}'$ is orthogonal to every $\psi_i$ except those corresponding to derivatives at the boundaries. This explains the poor performance of the Galerkin method in advection-dominated problems. Of course, on the other hand, the space spanned by derivative weights $\psi'_i$ allows any $V'$ for $V \in S_0^h$ to be represented exactly. If diffusion dominates ($\gamma$ is small), each $V'$ must be approximated by a $W'$ such that $W \in T^h$. Galerkin weights achieve this exactly because $T^h = S_0^h$. With derivative weights, however, each $W'$ is piecewise linear. These $W'$ are therefore orthogonal to the highest frequency (piecewise-quadratic) function $V'$ such that $V \in S_0^h$. The function that cannot be approximated is again $\hat{V}'$.

Here $A_0$ and $A_\infty$ are selected so that $\hat{V}'$ is approximated as closely as possible by $\gamma \hat{W} + \hat{W}'$, where $\hat{W}$ is a simple combination of the $W'$s that provide an exact representation when $P_e$ approaches 0 and $\infty$:

\begin{equation}
\hat{W} := A_0 \hat{V} + A_\infty \gamma^{-1} \hat{V}'.
\end{equation}

The smallest eigenvalue in eigenvalue problem (2.11) is estimated by considering only $\hat{V}$ and $\hat{W}$. This leads to an estimate of the error factor in bound (2.8),

\begin{align*}
\frac{1}{\sqrt{1 - \Delta_S^2}} \approx \frac{\|\hat{V}'\|_{B_S} \|R_S \hat{W}'\|_{B_S}}{B(\hat{V}, \hat{W})} = \frac{\|\hat{V}'\|_{L_2} \|(R_S \hat{W})')\|_{L_2}}{\langle \hat{V}', (R_S \hat{W})' \rangle},
\end{align*}

where, from (2.15) and (2.21),

\begin{equation}
(R_S \hat{W})' = \gamma A_0 \hat{V} + (A_0 + A_\infty) \hat{V}' + \gamma^{-1} A_\infty \hat{V}''.
\end{equation}
If elements are of equal length $h$ and boundary effects are ignored, the integrals may be evaluated to give

$$(2.22) \quad \frac{1}{\sqrt{1 - \Delta_S^2}} \approx \sqrt{\left(\frac{P_e^2}{42} + 1\right) A_0^2 + \left(1 + \frac{60}{P_e^2}\right) A_\infty^2 A_0 + A_\infty}.$$  

This estimate is minimized when

$$(2.23) \quad \frac{A_\infty}{A_0} = \frac{P_e^2 + 42}{42(P_e^2 + 60)} P_e^2.$$  

With coefficients in this optimum ratio,

$$(2.24) \quad \frac{1}{\sqrt{1 - \Delta_S^2}} \approx \sqrt{\frac{P_e^4 + 102P_e^2 + 2520}{P_e^4 + 84P_e^2 + 2520}}.$$  

As expected, the estimate of the error factor approaches 1 as $P_e$ approaches $\infty$ or 0. Its maximum value is $\sqrt{\frac{2\sqrt{70} + 17}{2\sqrt{70} + 14}} \approx 1.05$, which is predicted at $P_e = \sqrt{6\sqrt{70}} \approx 7.1$.

2.3.2. Dirichlet boundaries. The weighting function corresponding to the derivative at $x = 0$, where the Dirichlet boundary condition is applied, is nonzero on that boundary. This means that the weighted residual equations (2.1) are not satisfied if the exact solution $u$ is substituted for $U$, and so the error orthogonality property (2.3) does not hold. This is corrected by changing the definition of the derivative term $w_i^\infty$ to include a multiplier $\zeta \in H_{D_0}^1$ so that all weighting functions lie in $H_{D_0}^1$:

$$(2.25) \quad w_i^\infty := \zeta \gamma^{-1} \psi_i.'$$  

The multiplier is chosen to be an exponential ramp,

$$(2.26) \quad \zeta := \frac{1 - e^{-\gamma x}}{1 - e^{-\gamma h}},$$

in the element adjacent to $x = 0$, and one elsewhere, so that for large $\gamma$ the behavior of $w_i^\infty$ near $x = 0$ is similar to that of the optimal weighting functions $w_i^*$ in section 2.2. These weighting functions still become optimal when $\gamma \to \infty$.

2.3.3. Verification of error estimates. The estimates of the optimal ratio of $A_\infty$ to $A_0$ (2.23) and of the error factor (2.24) rely on the assumption that $\hat{V}$ is the function in $S_0^h$ that is most poorly approximated by functions in $R_5 T^h$. To investigate the validity of this assumption, error factors were calculated from the smallest eigenvalues of problem (2.11) with the full trial and test spaces for various $P_e$ and numbers of elements. The weighting functions $w_i$ in (2.16) were defined using (2.25) for $w_i^\infty$ and (2.17) for $w_i^0$. $\frac{A_\infty}{A_0}$ was given by (2.23). The resulting error factors are compared with estimates from (2.24) in Figure 2.1.

In all cases investigated, the calculated error factors approached the estimate (2.24) as the number of elements became large. The eigenvector of (2.11) corresponding to the smallest eigenvalue was, in each case, dominated by components corresponding to $w_j$ for $j \in N^1$, which were almost constant in the middle of the domain but smaller nearer the boundaries. This affirms that, without boundary effects, $\hat{V}$ is indeed the most poorly approximated function in $S_0^h$. Near boundaries, $\hat{V}$ can be approximated better, but the estimate (2.24) based on $\hat{V}$ and ignoring boundary effects appears to provide a good upper bound on the error factor.
2.3.4. Variable lengths and coefficients. The terms in the approximately optimal weighting function (2.16) are optimal weights when $P_e$ approaches 0 and $\infty$, even with unequally spaced elements and variable equation coefficients, but the ratio $\frac{A_\infty}{A_0}$ given in (2.23) is based on constant element lengths and coefficients. With variable lengths or coefficients, the best choices of $A_0$ and $A_\infty$ are no longer constant. It is assumed that weighting functions (2.16) are still close to optimal if the ratio (2.23) is used, but the variation over $x$ in the total magnitude of $A_0$ and $A_\infty$ is yet to be determined. Note that with weighting functions (2.16) and propagation only in the direction of increasing $x$, if boundary effects are ignored, the error orthogonality property (2.3) may be written as

$$\langle u' - U', A_0 \gamma MV \rangle + \langle u' - U', (A_0 + A_\infty)MV' \rangle + \langle u' - U', A_\infty \gamma^{-1}MV'' \rangle = 0$$

$\forall V \in S_0^b$.

If we aim for a small error in the sense of the $\| \cdot \|_{B_S}$ norm, then the left-hand side should resemble $B_S(u - U, V)$. The second term is therefore the desirable term, and its dominance is achieved by appropriate selection of $\frac{A_\infty}{A_0}$ in (2.23). The second term is equivalent to $B_S(u - U, V)$ if

$$A_0 + A_\infty := 1.$$

2.4. Extension to three dimensions. For modelling the excitation of the heart, the definitions of the terms in the weighting functions need to be extended to the three-dimensional case with wavefronts travelling in any direction. Weighting functions are still based on the simple combination (2.16) of terms selected for their performance when $P_e$ approaches 0 and $\infty$, but the Riesz representer theory of section 2.1 can no longer be applied because the form $B(\cdot, \cdot)$ defined in (2.2) is no longer bilinear.

2.4.1. Selection of weight terms. When $P_e \to 0$, $B(\cdot, \cdot)$ becomes bilinear, and thus $w^0_i$ are defined as the optimal weights in the sense of $\| \cdot \|_{B_S}$, which are still $\psi_i$. When the advection term is present, its nonlinearity means that the techniques
used in section 2.2 can no longer be used to find a weight that guarantees minimum error in the sense of the $\| \cdot \|_{B_2}$ norm. However, when $P_e \to \infty$, making the residual $c_0 \sqrt{\nabla u \cdot M \nabla u} - \tau_m$ orthogonal to the least squares weight $\frac{\nabla U \cdot M \nabla \psi_i}{c_0 \sqrt{\nabla u \cdot M \nabla u}}$ minimizes

$$\left\| \frac{\sqrt{\nabla U \cdot M \nabla u} - \tau_m}{c_0} \right\|_{L_2} = \left\| \frac{\sqrt{\nabla U \cdot M \nabla u} - \sqrt{\nabla u \cdot M \nabla u}}{c_0} \right\|_{L_2}.$$ 

These weighting functions compare with the derivative weighting functions of Brooks and Hughes [3] in their solution of the multidimensional steady-state linear advection-diffusion equation with linear elements.

As in the one-dimensional case, a multiplier $\zeta \in H^2_{D_0}$ is included with the least squares weight to ensure that the weighting functions are zero on Dirichlet portions of the boundary. $\zeta$ is based on the one-dimensional expression (2.26) and is defined by

$$\zeta := \frac{1 - \exp \left( -\frac{P_e \rho_\zeta}{k_\zeta} \right)}{1 - \exp \left( -\frac{P_e}{k_\zeta} \right)},$$

where $k_\zeta$ is a constant and $p_\zeta$ is a simple nonnegative function in $H^1_{D_0}$. In elements adjacent to Dirichlet boundaries, $p_\zeta$ is a polynomial of $\zeta$; in other elements, $p_\zeta = 1$. This means that $\zeta$ is equal to one over most of the domain, and thus most weights are unaffected by the multiplier. Near Dirichlet boundaries, the weights have similar behavior to one-dimensional optimal weighting functions for large $P_e$ if $p_\zeta$ increases from zero at the boundary with slope $[\nabla_\zeta p_\zeta] = k_\zeta$. A cubic interpolation is used for $p_\zeta$, and $k_\zeta$ is set to 3. Away from Dirichlet boundaries, nodal values of $p_\zeta$ are set to 1, and derivatives to 0. On Dirichlet boundaries, nodal values are 1, and derivatives are set so that the slope of $p_\zeta$ at the boundary is as close to 3 as possible.

The least squares term is discontinuous at wavefront collisions, which makes it difficult to design an integration scheme such that the residuals in the resulting discrete system of nonlinear equations are continuous with respect to the nodal parameters $U_j$. In order to keep the integration scheme simple, the smooth term,

$$w_\zeta := \frac{\nabla U \cdot M \nabla \psi_i}{\sqrt{1 - \alpha_\zeta^2}} c_0^2 \sqrt{\nabla U \cdot M \nabla U} + \alpha_\zeta \frac{\tau_m^2}{2},$$

is used instead with the constant $\alpha_\zeta \in (0, 1)$. This term is close to the least squares term when advection dominates and $U$ is close to $u$. At a collision, however, the denominator remains greater than zero, and so the term vanishes. The best value for $\alpha_\zeta$ has not been thoroughly investigated, but $\alpha_\zeta = \frac{1}{4}$ seems to work well.

2.4.2. Mesh Péclet number. The one-dimensional expression for $P_e$ in (2.18) included the equation space constant $\sqrt{M}$ and an element spatial scale $\frac{dx}{df}$. In more than one dimension, these quantities are not scalar, and thus the intention is to base $P_e$ on suitable space constants in the direction of propagation.

It is convenient to define at each point in space a dimensionless natural coordinate system $\psi$ in which the coupling tensor $M$ transforms to the identity matrix and the advection term becomes isotropic:

$$c_0 \sqrt{\nabla U \cdot M \nabla U} \equiv c_0 |\nabla_\psi U|,$$
where \( \nabla_{\mathbf{v}} \) denotes the gradient operator with respect to \( \mathbf{v} \) coordinates. The one-dimensional scalar ratio of \( \sqrt{M} \) to \( \frac{dx}{d\tau} \) corresponds to a multidimensional tensor \( \nabla_{\mathbf{v}} \xi \). A scalar quantity is selected from this using the rate of change of \( \xi \) arc length with respect to \( \mathbf{v} \) arc length in the direction of propagation:

\[
(2.30) \quad \left| \left( \frac{\nabla_{\mathbf{v}} U}{|\nabla_{\mathbf{v}} U|} \cdot \nabla_{\mathbf{v}} \right) \xi \right|.
\]

A smooth \( P_e \) is defined by

\[
(2.31) \quad P_e := \frac{c_0 \sqrt{(1 - \alpha_\infty)c_0^2 |\nabla_{\mathbf{v}} U|^2 + \alpha_\infty \tau_m^2}}{\sqrt{(1 - \alpha_\infty)c_0^2 (|\nabla_{\mathbf{v}} U \cdot \nabla_{\mathbf{v}}| \xi|^2 + \alpha_\infty \tau_m^2 \mu_\xi}.
\]

where

\[
(2.32) \quad \mu_\xi := \frac{\partial \xi_m}{3 \partial v_p} \frac{\partial \xi_m}{\partial v_p}.
\]

(which is an average of the diagonal elements of the coupling tensor in the \( \xi \) coordinate system).

2.4.3. Discontinuous derivatives of \( U \). Expressions (2.23) and (2.27) for the coefficients \( A_0 \) and \( A_\infty \) are only useful if \( U \) is C\(^1\) continuous. There are, however, places in the ventricular myocardium where \( u \) is not expected to be C\(^1\) continuous [23]. The coefficients in (2.23) and (2.27) and \( w_i^{\infty} \) in (2.29) depend on first derivatives of \( U \), so, with only C\(^0\) continuity in \( U \), the weighting functions (2.16) may be discontinuous. To retain continuity in the weights, \( A_0 \) is made constant and \( p_\xi \) is set to zero on interelement boundaries where C\(^1\) continuity in \( U \) is not expected. The nodal values of \( p_\xi \) on interelement boundaries without C\(^1\) continuity are set in the same manner as if they were on Dirichlet boundaries. In this way, \( A_0w_i^0 \) retains the C\(^0\) continuity of the interpolation functions, and \( A_\infty w_i^{\infty} \) approaches zero at interelement boundaries where derivatives of \( U \) are not expected to be continuous.

With constant \( A_0 \), keeping the ratio of \( A_\infty \) to \( A_0 \) similar to the one-dimensional optimal ratio (2.23) would mean that for large \( P_e \), the weights would be heavily dependent on the direction of propagation, making the weighted residual equations very nonlinear. Instead, \( A_0 \) and \( A_\infty \) are defined by

\[
(2.33) \quad A_0 := 1 \quad \text{and} \quad A_\infty := \frac{P_e^2}{k_{lim} P_e + 50},
\]

where the constant \( k_{lim} \) determines the maximum magnitude of the derivative term. It is chosen to be 2 (discussed below). The weighting functions are therefore given by the sum of Galerkin and supplementary weighting functions,

\[
(2.34) \quad w_i = \psi_i + \bar{w}_i,
\]

where the supplementary weighting functions are defined by

\[
(2.35) \quad \bar{w}_i := \xi \frac{P_e}{2P_e + 50} \frac{c_0 \nabla_{\mathbf{v}} U \cdot \nabla_{\mathbf{v}} \psi_i}{\sqrt{(1 - \alpha_\infty)c_0^2 (|\nabla_{\mathbf{v}} U \cdot \nabla_{\mathbf{v}}| \xi|^2 + \alpha_\infty \tau_m^2 \mu_\xi}.}
\]
For large $P_e$, the magnitude of the supplementary weighting functions, which are dependent on $U$, is similar to that of the Galerkin weighting functions, which are independent of $U$. This reduces the effects of nonlinearity in the weighted residual equations, facilitating their solution.

To estimate the error introduced by not using the optimal ratio of $A_\infty$ to $A_0$ (2.23), one-dimensional error factor estimates for ratios from (2.33) and (2.23) are compared in Figure 2.2. These error factors are calculated from expression (2.22), which is based on constant equation coefficients and assumes a large number of equal-length one-dimensional elements. The maximum predicted error factor with (2.33) is less than two percent greater than the maximum with the optimal ratio (2.23). The constant $k_{lim}$ in (2.33) was set to 2 to keep the nonlinear term in the weight as small as possible while not significantly increasing the expected error factor.

3. No-inflow boundary condition. For large Péclet numbers, the no-flux boundary condition derived from the diffusion of charge is not necessarily enough to sufficiently constrain the solution.

3.1. Inflow boundaries. The lack of stability under large Péclet numbers of the Petrov–Galerkin method developed in the previous section is demonstrated in the situation shown in Figure 3.1. The tissue is stimulated in such a way that the wavefront is initially concave (when viewed from inactive tissue). Note that for $P_e = 10$ the curvature of the wavefront reduces as it propagates across the tissue, but for $P_e = 100$ the curvature increases.

The nature of the solution for $P_e = 100$ is in some ways quite reasonable. The residual in the eikonal equation (1.12) is very small. An inwardly propagating circular wavefront becomes a smaller circle, so an initially concave wavefront becomes more concave. The problem with the solution is that the no-flux boundary condition (1.13) is not satisfied.

The no-flux boundary condition is not very well satisfied on the boundary at the right-hand end of the tissue in the solution for $P_e = 10$ either. Such boundaries where the wavefront extinguishes shall be referred to as outflow boundaries. The boundary condition at these boundaries only affects a small boundary layer of tissue, so failure to satisfy the boundary condition does not introduce much error into the
solution. The boundaries where the wavefronts enter the domain shall be called *inflow* boundaries. The boundary at the left-hand end of the tissue is an inflow boundary because tissue is stimulated on this boundary. In the $P_e = 10$ solution, the no-flux boundary condition on the other boundaries is satisfied very well.

In the $P_e = 100$ solution, the no-flux boundary condition on the boundaries at the top and bottom of the domain is not satisfied. The boundary condition (1.13) is derived from prevention of diffusion of charge across the boundary. For large Péclet numbers, diffusion effects are small, and so the emphasis on satisfying the boundary condition is small. Because the discretization does not allow $U$ to exactly represent $u$, the numerical method selects a solution that closely satisfies the eikonal equation but almost ignores the boundary condition. As the effects of diffusion are small, the propagation speed should be almost unaffected by curvature and should be almost equal to the unit plane wave speed. This is reflected in the solution through the magnitude of the gradient of activation time which is close to one over the entire domain. Note, however, that the average propagation speed along the top and bottom edges of the domain is about 1.1. This is due to the fact that the method does not recognize that tissue needs to be excited by other excited tissue. It is assumed that the propagation direction is normal to the wavefront, but, because the boundary condition is not strongly enforced, the wavefront normal is not parallel to the boundary. The wavefront is propagating from outside the boundary into the domain, and the boundary is an inflow boundary. Tissue is being excited by nonexistent tissue outside the boundary.

Without a mechanism to prevent wavefronts from entering the domain through unwanted inflow boundaries, excitation can initiate at arbitrary points on the boundary and totally corrupt the numerical solution. This problem occurs when the diffusion term becomes insignificant, and thus the nature of propagation without diffusion is now investigated to determine a prevention mechanism.

### 3.2. Propagation without diffusion

For large Péclet numbers the numerical scheme behaves as if it is solving the eikonal equation without a diffusion term and without the associated no-flux boundary condition. Without these, the solution to the eikonal equation (1.12) is not unique. To reflect the fact that tissue must be excited by neighboring tissue, the governing equation should instead be

\[
\sup_{a \in A(x)} \left\{ \lim_{\alpha \to 0} \frac{u(x) - u(x - \alpha a)}{\alpha} \right\} = \tau_m(x) \quad \forall x \in \Omega - \Gamma_D.
\]
where, for $m$ dimensions,

$$A(x) := \{ \mathbf{a} \in \mathbb{R}^m : \mathbf{a} \cdot \mathbf{M}^{-1} \mathbf{a} = c_0^2 ; \exists \alpha \in \mathbb{R} \text{ s.t. } \alpha > 0, \ x - \alpha \mathbf{a} \in \Omega \}. \tag{3.2}$$

Restricting the vectors $\mathbf{a}$ to the set $A(x)$ determines the directions in which propagation can occur at the point $x$ and the propagation speeds for these directions.

In regions where the solution is smooth enough, this governing equation is equivalent to

$$\sup_{\mathbf{a} \in A} \{ \mathbf{a} \cdot \nabla u \} = \tau_m. \tag{3.3}$$

(If $A$ is replaced with $A_{\Omega^0}$ defined below, this equation is a special case of that used by Falcone, Giorgi, and Loreti [10] in their analysis of front propagation problems.)

For a point not on the boundary, the definition of $A$ simplifies to

$$A_{\Omega^0} := \{ \mathbf{a} \in \mathbb{R}^m : \mathbf{a} \cdot \mathbf{M}^{-1} \mathbf{a} = c_0^2 \}, \tag{3.4}$$

and the supremum in (3.3) occurs when

$$\mathbf{a} = c_0 \frac{\mathbf{M} \nabla u}{\sqrt{\nabla u \cdot \mathbf{M} \nabla u}}. \tag{3.5}$$

Away from the boundaries, therefore, (3.3) is equivalent to the eikonal equation (1.12) without a diffusion term.

Without diffusion, there is no Neumann boundary condition, but the notation $\Gamma_N$ will be used for $\partial \Omega - \Gamma_D$, the portion of the boundary in which no Dirichlet boundary condition is applied. For a point on this portion of the boundary, $A$ is equivalent to

$$A_{\partial \Omega} := \{ \mathbf{a} \in \mathbb{R}^m : \mathbf{a} \cdot \mathbf{M}^{-1} \mathbf{a} = c_0^2, \ \mathbf{n} \cdot \mathbf{a} \geq 0 \}, \tag{3.5}$$

where $\mathbf{n}$ is the unit outward-pointing normal to the boundary.

In order to investigate the nature of the solution to (3.3) near boundaries, consider two points, $x_{\partial \Omega} \in \Gamma_N$ and $x_{\Omega^0} \in \Omega - \partial \Omega$, such that $x_{\Omega^0}$ is an infinitesimal distance from $x_{\partial \Omega}$. As discussed above, the solution at $x_{\Omega^0}$ satisfies

$$c_0 \frac{\mathbf{M} \nabla u}{\sqrt{\nabla u \cdot \mathbf{M} \nabla u}} \cdot \nabla u = \tau_m. \tag{3.5}$$

If the solution is smooth enough in the vicinity of the points, one would expect that $\nabla u(x_{\partial \Omega})$ is equal to $\nabla u(x_{\Omega^0})$ and should satisfy the same equation. This is only consistent with (3.3) if

$$c_0 \frac{\mathbf{M} \nabla u}{\sqrt{\nabla u \cdot \mathbf{M} \nabla u}} \in A_{\partial \Omega}, \tag{3.6}$$

and thus the direction of propagation on $\Gamma_N$ is restricted by

$$\mathbf{n} \cdot \mathbf{M} \nabla u \geq 0. \tag{3.6}$$

If some diffusion is included, it can be assumed that $u$ is smooth enough that the governing equation becomes

$$\sup_{\mathbf{a} \in A} \{ \mathbf{a} \cdot \nabla u \} - \nabla \cdot (\mathbf{M} \nabla u) = \tau_m. \tag{3.7}$$
The limit of the solution to this equation as the diffusion term vanishes satisfies (3.1).

The no-flux boundary condition on $\Gamma_N$ ensures that $M \nabla u$ is either parallel to the boundary or zero. The supremum in (3.7) therefore occurs when $a = c_0 \frac{M \nabla u}{\sqrt{\nabla u \cdot M \nabla u}}$ and (3.7) is equivalent to the eikonal equation (1.12). The limit of the solution to the eikonal equation (1.12) and its no-flux boundary condition (1.13) as the diffusion term vanishes satisfies the diffusionless governing equation (3.1).

3.3. A no-inflow boundary term. Although the exact solution of the eikonal equation (1.12) approaches the solution of the diffusionless propagation equation (3.1), the same is not necessarily true for the numerical solution. Unfortunately, with the Petrov–Galerkin method, when diffusion effects become small, they are swamped by discretization errors. The method behaves as if it were solving an eikonal equation without a diffusion term and without the no-flux boundary condition. Without these, the solution is not unique, and so the scheme becomes unstable. To prevent this, the numerical treatment of the advection term needs to more closely represent the corresponding term in (3.1).

With a finite difference method this is easily done by using an upwind difference scheme [18]. Such schemes can select the grid points used in the difference expressions for the advection term so that the excitation time of each grid point is calculated as the expected time for a wavefront to arrive from neighboring grid points with lower excitation times. As there are only grid points in the domain, the wavefront can only arrive from points in the domain, and there are no unwanted inflow boundaries. None of the so-called upwind finite element methods for steady-state problems provide the same restrictions on the solution. Finite element methods only evaluate the advection term at sample points in the domain, and thus the boundaries have no influence on propagation.

The approach used here to stabilize the Petrov–Galerkin solution of the eikonal equation is to add to the weighted residual equations a boundary integral term that encourages the solution to satisfy the boundary inequality (3.6). If this is satisfied, the supremum in (3.3) occurs when $a = c_0 \frac{M \nabla u}{\sqrt{\nabla u \cdot M \nabla u}}$, and thus the residual in the eikonal equation (1.12) is equivalent to the residual in (3.3).

The satisfaction of boundary inequality (3.6) is encouraged by including a penalty term when it is not satisfied. This penalty term is constructed from minimization of an integral over $\Gamma_N$ of the square of a residual,

$$
\int_{\Gamma_N} \psi^2 \, d\Gamma,
$$

where $\psi$ is a residual that is zero if and only if (3.6) is satisfied, and $A_b$ is a coefficient independent of $U$. The minimum occurs when the derivatives with respect to each parameter $U_i$,

$$
\int_{\Gamma_N} \psi \frac{\partial \psi}{\partial U_i} \, d\Gamma,
$$

are zero. These integrals are added to the left-hand side of the Petrov–Galerkin discrete equations (2.1) to encourage the numerical solution $U$ to satisfy the boundary inequality (3.6).
The natural coordinate system $\nu$ of section 2.4.2 may be used to express inequality (3.6) as

\begin{equation}
\mathbf{n}^\nu \cdot \nabla_\nu u \geq 0,
\end{equation}

where $\mathbf{n}^\nu$ is the unit outward-pointing normal to the boundary in this coordinate system. When $r_b$ was defined as the direct residual in this inequality, the penalty term was found to place too much emphasis on satisfying this inequality at the expense of satisfying the eikonal equation. The expression for $r_b$ was therefore selected to be more closely associated with the advection term. Consider the residual

\begin{equation}
r_b := |\nabla_\nu U| - \sqrt{|\nabla_\nu U|^2 - \min(n^\nu \cdot \nabla_\nu U, 0)^2}.
\end{equation}

If (3.10) is satisfied, this expression is zero. If (3.10) is not satisfied, the expression is essentially the difference between the advection term and what it would be if it were calculated from only the components of $\nabla_\nu U$ in the surface of the boundary. With this residual, the expression $r_b \frac{\partial p}{\partial c_0}$ has a discontinuity when $|\nabla_\nu U| = n^\nu \cdot \nabla_\nu U$, which corresponds to propagation into the domain normal to the boundary. It is not likely that this will occur, but, to ensure that the discrete equations are smooth enough for solution by Newton's method, the modified residual

\begin{equation}
r_b := \sqrt{|\nabla_\nu U|^2 + \alpha_b \frac{\partial p}{\partial c_0^2} - \sqrt{|\nabla_\nu U|^2 - \min(n^\nu \cdot \nabla_\nu U, 0)^2} + \alpha_b \frac{\partial p}{\partial c_0^2}}
\end{equation}

is used. As with $\alpha_{\infty}$ in section 2.4, a value of $\frac{1}{4}$ is used for $\alpha_b$.

The boundary integrands are of similar magnitude to the products of the advection term and the weights in (2.34) and (2.35) if they are multiplied by

\[ p_e^b \left( 1 + \frac{p_e^b}{2p_e^b + 50} \right). \]

To retain the symmetric and positive semidefinite nature of the boundary terms (3.9), an expression that is independent of $U$ is used for the Péclet number:

\begin{equation}
p_e^b := c_0 \left| n^\nu \cdot \frac{\partial \nu}{\partial \xi_n} \right|,
\end{equation}

where $\xi_n$ is the local element coordinate that does not vary over the boundary. The expression is based on the spatial properties in the direction normal to the boundary instead of in the direction of propagation used in (2.31).

Even with the integrands dimensionally consistent, there is still a difference between the dimensions of the boundary and domain integrals in the order of one spatial dimension. An appropriate multiplier needs to be found for the boundary term to balance the emphasis on satisfaction of the eikonal equation and of the boundary inequality. This should reflect the depth of the region of influence that the boundary terms should have. The parameters $U_j$ that are included in the boundary terms have a significant direct influence on the solution over about half an element. If the boundary terms are given a multiplier that resembles half the width of the element, then the equations involving these parameters should put even emphasis on satisfaction of the domain equation and the boundary inequality. The multiplier is chosen to be

\[ \frac{1}{2} |n \cdot \frac{\partial x}{\partial \xi_n}|, \]
so that the width of the element is estimated from information at the boundary. The coefficient $A_b$ in the boundary terms (3.9) is therefore

\begin{equation}
A_b := \frac{P_e^b}{2} \left( 1 + \frac{P_e^b}{2P_e + 50} \right) \left| n \cdot \frac{\partial x}{\partial \xi_n} \right| \text{.}
\end{equation}

The numerical solutions obtained by this modified scheme in solving the test problem of section 3.1 are shown in Figure 3.2. For $P_e = 10$, the solution is very similar to the solution in Figure 3.1(a) obtained without the additional boundary term. For $P_e = 100$, the solution satisfies both the eikonal equation (1.12) and boundary condition (3.6) reasonably well, given the coarse discretization. Although the wavefront is not perpendicular to the top and bottom boundaries, the propagation speeds along these boundaries are sensible, and the satisfaction of (3.6) improves with distance from the initiation point.

4. Summary of the method. The numerical method developed for the simulation of excitation propagation in ventricular myocardium uses Newton’s method to solve a system of weighted residual equations that are sums of the Petrov–Galerkin weighted residuals of section 2.4 and the no-inflow weighted residual of section 3.3. Newton’s method requires a sufficiently good initial guess on which it can iteratively improve. If the diffusion term dominates, the equation is close to linear, and thus almost any initial guess leads to rapid convergence. An initial guess of $U_j = 0 \; \forall j \in N$ is sufficient. If the advection term dominates, however, the significant nonlinearities may prevent the method from converging if the initial guess is not good enough. Approximate solutions to equations with more significant diffusion are used as initial guesses for equations with more advection in a numerical continuation method [1] on the continuum of equations.

\begin{equation}
\alpha_c c_0 \sqrt{\nabla u \cdot M \nabla u} - \nabla \cdot (M \nabla u) = \alpha_c \tau_m \text{.}
\end{equation}

Here $\alpha_c$ is the continuation variable, which is increased from 0 to 1 to transform a diffusion equation into the desired eikonal equation.

From Petrov–Galerkin weighted residual equations (2.1), no-inflow weighted residual equations (3.9), and governing equation continuum (4.1), the weighted residual
The boundary inequality residual definition (3.12) is used for $r_b$.

When the value of the continuation variable is less than one, the influence of the diffusion term is increased, and thus the supplementary weights $\hat{w}_i$ and the boundary integral coefficient $A_b$ are calculated using the apparent Péclet number. The supplementary weighting functions $\hat{w}_i$ are defined by (2.35), with the Péclet number defined by (cf. (2.31))

\begin{equation}
P_e := \frac{\alpha_c c_0 \sqrt{1 - \alpha_\infty c_0^2} \nabla x U \cdot M \nabla x U + \alpha_\infty \tau m^2}{\sqrt{1 - \alpha_\infty c_0^2} \nabla x U \cdot M \nabla x U + \alpha_\infty \tau m^2 \mu^2}.
\end{equation}

The boundary integral coefficient $A_b$ is defined by (3.14) with the Péclet number defined by (cf. (3.13))

\begin{equation}
P_e^b := \alpha_c c_0 \left| n^r \cdot \frac{\partial v}{\partial x_n} \right|.
\end{equation}

The integrals in (4.2) are evaluated using Gauss–Legendre quadrature schemes. A grid of quadrature points with four points in each direction is used within each element. The system of linear equations for each Newton iteration is solved using the generalized minimum residual (GMRES) iterative solver [20] with a simple diagonal preconditioner and no restarts.

5. Simulation. Numerical simulation of excitation propagation through the full canine ventricular myocardium was performed using the method developed here to solve eikonal equation (1.12). The model of the canine ventricular geometry and the selection of material parameters for the governing equation (1.12) are discussed in [23]. Parameters used were $\lambda_1 = 0.8$ mm, $\lambda_t = \lambda_n = 0.5$ mm, $\tau_m = 3$ ms, and $c_0 = 2.5$. There were 2355 degrees of freedom for the dependent variable. The method was programmed primarily in extended FORTRAN 77 as part of the CMISS (an acronym for Continuum Mechanics, Image analysis, Signal processing and System identification) software package. It was executed on one 195 MHz MIPS R10000 processor of a Silicon Graphics Octane.

A point stimulus site was chosen to match the pacing site used for epicardial-suck activation time recordings by Le Grice [16], so that results could be compared with experimental measurements. This site is on the epicardial surface of the anterior aspect of the left ventricular free wall and located at a distance from the apex about one third of that from apex to base.

Snapshots of wavefront locations from the simulation are presented in Figure 5.1. Epicardial isochrones are similar to those from the experimental recordings for times from about 20 ms to 60 ms after stimulation but start to differ considerably outside this interval. Near the stimulus site, experimental recordings showed much slower propagation in the direction transverse to the fibers. The difference in simulation
Fig. 5.1. Wavefront locations at 20 ms time intervals in a simulation of propagation from an epicardial point stimulus at time 0. For each sample time, two opposing views are shown.

results is probably due both to the coarse discretization and to the inability of the eikonal model to reproduce the transient effects near a stimulus. The distance over which slow initial transverse propagation was observed experimentally is less than one quarter of the element length in this direction. For times greater than 65 ms
after stimulation, experimental recordings showed much earlier epicardial excitation, particularly in the more basal and posterior areas on the left ventricular free wall. The region of latest recorded excitation was at the pulmonary conus, which was excited about 125 ms after stimulation. In simulations, the latest excitation occurred about 180 ms after stimulation in the basal posterior region of the right ventricular free wall. The discrepancy is most likely due to the lack of Purkinje fibre representation in the computational model. If the effects of this fast conduction network are not included in the model, results cannot be expected to be realistic. More realistic simulations are presented in [23].

The solution to the discrete system of equations (4.2) was obtained after seventeen Newton iterations and required just less than four minutes of CPU time. One Newton iteration was performed for each increment of the continuation variable \( \alpha_c \) until it reached one, then four Newton iterations were required before the relative change in the solution reduced to less than \( 10^{-5} \). The time required for each iteration ranged from 10.7 s to 16.9 s. Of this, the time for calculation of the Jacobian was consistently 7.2 s, but the time for solution of the linear system of equations ranged from 3.0 s when diffusion was significant, to 5.3 s when \( \alpha_c \) reached one, to 9.2 s in the final iteration. Most of the remaining 0.5 s in each iteration was spent evaluating the residual in the nonlinear equations.

In the solution of the linear system of equations for each Newton iteration, GMRES iterations were performed until the residual in the linear system was reduced by a factor of \( 10^{-3} \). There was an increasing trend in the number of GMRES iterations required to achieve this, from 111 iterations when diffusion was significant, to 170 when \( \alpha_c \) reached one, to 251 in the final Newton iteration. This suggests that the condition number of the Jacobian may increase as the effect of diffusion decreases.

Although convergence in the solution to the nonlinear system was achieved reasonably easily in this simulation with these material parameters, when \( \alpha_c \) was increased to represent a reduction in the effects of diffusion, convergence could be achieved for \( \alpha_c = 1.06 \) but not for \( \alpha_c = 1.07 \). This means that, if the material parameters were changed so that the relative magnitude of the diffusion term was reduced by more than six percent, artificial diffusion would be required to obtain convergence.

Part of the reason for the inability to achieve convergence when the diffusion term is small may be related to the lack of \( C^1 \) continuity in \( U \) at certain places in the mesh. A close inspection of the wavefront near the apex in Figure 5.1(c) reveals that the front is starting to form a point as it approaches the apex. This feature of the wavefront vanishes when more diffusion is introduced into the equation.

As discussed in section 3.3, when the diffusion effects become very small, the numerical method behaves as if it were solving an eikonal equation without a diffusion term. The appropriate equation to solve in this situation is the diffusionless propagation equation (3.1). Although the discrepancy between this and the eikonal equation was dealt with on boundaries in section 3.3, it was assumed that inside the domain the residuals in the two equations were equivalent. The residuals are only equivalent, however, if first derivatives are continuous. The \( C^1 \) constraint vanishes at the apex because the element widths vanish. Without \( C^1 \) continuity, the eikonal equation admits solutions where tissue is not necessarily excited by neighboring tissue. Wavefronts can initiate and spread out from any point in space where \( C^1 \) continuity is not enforced. This lack of uniqueness in the solution makes the Jacobian for Newton’s method singular and therefore convergence unlikely. If simulations are to be performed with less diffusion, the numerical treatment of the advection term needs to more closely represent its form in (3.1).
If diffusion needs to be added to the equation in order to obtain a stable solution, high-order convergence rates can no longer be expected, and thus one might ask the question of what the advantage of high-order elements might be. To address this question, the cubic Hermite Petrov–Galerkin finite element scheme is compared with a simple finite difference scheme using first-order upwind differences for the advection term. A simple Taylor series analysis of first-order upwind differences shows that the coefficient of numerical diffusion is half the coefficient of the advection term multiplied by the grid point spacing, which resembles \( \frac{1}{2} \epsilon_0 \lambda h \). When doubling the physiological diffusion to stabilize the cubic Hermite scheme, the coefficient of additional diffusion resembles \( \lambda^2 \). In order to make the additional diffusion in the first-order scheme of similar magnitude, the grid point spacing must be given by \( h = \frac{2\lambda}{\epsilon_0} \), or equivalently, the grid Péclet number \( P_e \) must be equal to 2. In the cubic model, the volume average of the geometric mean of the mesh Péclet numbers for each direction is 22, and the maximum Péclet number in any direction at any point is 116. An optimally designed first-order grid would have grid spacings of 0.64 mm in the fibre direction and 0.4 mm in the other directions. The 0.2 \( \times \) 10^6 mm^3 myocardial volume would need to be represented by about 2 \( \times \) 10^6 grid points. This is a factor of about 10^3 greater than the 2355 degrees of freedom in the cubic Hermite mesh.

6. Conclusions. An efficient computational model has been developed for the excitation process in ventricular myocardium. The need to represent the small-scale ionic activity is eliminated by modelling the excitation process as a propagating wavefront of depolarizing tissue.

A Petrov–Galerkin method using cubic Hermite elements has been developed to enable numerical solution of an eikonal equation for excitation time on a reasonably coarse mesh. The method is a weighted residual method with weights that are linear combinations of Galerkin weights and \( C^0 \) continuous supplementary weights based on the derivatives of the interpolation functions in the direction of propagation. For one-directional propagation, the error in the solution is within a small constant factor of the optimal error achievable in the trial space. To estimate the constant factor in the error bound, it was only necessary to consider the function in the trial space with highest frequency first derivative and its corresponding weighting function. A function of the mesh Péclet number was selected for the ratio of the Galerkin and supplementary weights so that this error factor is small for all values of the Péclet number.

For high Péclet numbers, the numerical solution of the eikonal-diffusion equation behaves as if there is no diffusion term. An eikonal equation determines the speed of propagation at each point in space but provides no constraint on the direction of propagation. Without the diffusion term, there is no longer any no-flux boundary condition, and spurious excitation can initiate at any point on the boundary. A no-inflow boundary term has been designed to provide a penalty on such spurious excitation.

Using a continuation method to gradually introduce the nonlinear term of the governing equation, seventeen Newton iterations were required to obtain the solution for a simulation in the full ventricular myocardium. The method showed instabilities when the effect of diffusion was very small, but the level of diffusion required for stability was much less than the level of numerical diffusion that would be introduced in a first-order upwind finite difference scheme with the same number of degrees of freedom.
REFERENCES
