LOGIC BLOG 2010

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1. Jan 2010: A downward $GL_1$ set that is not weakly jump traceable

1.1. Original result. Slaman, Greenberg, Kjos-Hanssen, Nies and others, worked at the University of Hawaii at Manoa.

**Definition 1.1.** A is weakly jump traceable (w.j.t.) if there is a function $f \leq_T \emptyset'$ that dominates all functions $\psi$ partial recursive in $A$, in the sense that $f(x) \geq \psi(x)$ for almost all $x$ such that $\psi(x)$ is defined.

It would be sufficient to dominate $J_A$, and is also equivalent to having a finite c.e. trace for $J_A$. This property is closed downward under $\leq_T$, and implies $GL_1$. It is equivalent to the property that $\emptyset'$ is not d.n.c. by $A$, namely, there is $g \leq_T \emptyset'$ such that $g(x) \neq J_A(x)$ for each $x$ [Miller and Ng REF? []). While the original proof of this equivalence used randomness, a direct proof was given by Mingzhong Cai (Mar 2010).

To make $A$ downward $GL_1$, one makes $A$ $GL_1$ and also ensures that

$$\forall B \leq_T A [B \text{ noncomputable } \Rightarrow A \leq_T B \oplus \emptyset'].$$

1.2. New results. This has been improved in May 2010. See Section 4.

1.3. Comments.

2. March 2010: Structures that are computable almost surely

This is a follow-up on a paper by Greenberg, Montalban and Slaman[]. It was done by Kalimullin and Nies in Auckland when Kalimullin visited. Though the proof was found independently, it uses methods from there for a slightly stronger result.

For a countable structure $A$ and a set $Y$, we write $A \leq_T Y$ to denote that some presentation of $A$ (viewed as an atomic diagram) is computable in $Y$. We denote by $\lambda$ the product measure on Cantor space $2^\mathbb{N}$.

**Definition 2.1.** A countable structures is called computable almost surely (or almost computable) if

$$\lambda\{Y : A \leq_T Y\} = 1.$$

2.1. A structure that is computable almost surely, is computable in every $\Pi^1_1$ random.

**Theorem 2.2.** Let $A$ be computable almost surely. Then $A$ is computable in every $\Pi^1_1$ random set.

Note that by Gandy basis theorem there is a $\Pi^1_1$ random $Y \leq_T \mathcal{O}$ such that also $\mathcal{O}^Y \leq_h \mathcal{O}$. In this way we reobtain the result of Greenberg et al.
Proof. Via some pre-agreed encoding, we view a subset of \( \mathbb{N} \) as a diagram of a structure in the language of \( \mathcal{A} \). For an index \( i \in \mathbb{N} \) for a Turing functional, and a rational \( p < 1 \), define a \( \Sigma^1_1 \) class uniformly in \( i, p \) by

\[
S_{i,p} = \{ Y : p < \lambda\{ Z : \Phi^Y_i \cong \Phi^Z_i \}\}.
\]

Clearly if \( S_{i,1/2} \neq \emptyset \) then \( \Phi^Y_i \) is a unique structure up to isomorphism for each member \( Y \).

Note that the relation \( \{\langle i, p, Y \rangle : Y \in S_{i,p}\} \) is \( \Sigma^1_1 \); see the uniform Measure Lemma [28, 9.1.1], which paraphrases [32, 1.11.IV]. Now define a \( \Sigma^1_1 \) equivalence relation on numbers by

\[
e \sim i \iff \exists Y \in S_{i,1/2} \exists Z \in S_{i,1/2} [\Phi^Y_e \cong \Phi^Z_i].
\]

By hypothesis there is an index \( k \) such that \( \lambda\{ Y : \Phi^Y_k \text{ presents } A \} \) is positive, so by the Lebesgue density theorem in the simple version [28, 1.9.4] we may assume that this measure is greater than \( 1/2 \). Now let

\[
C = \bigcup\{ S_{i,p} : 1/2 \leq p < 1, i \in \mathbb{N}, i \sim k \}.
\]

Each \( Y \in C \) computes a representation of \( A \). The class \( C \) is \( \Sigma^1_1 \), and \( C \) is conull, again by the Lebesgue density theorem. Thus it contains each \( \Pi^1_1 \)-random set. \( \square \)

This proof seems to work under the weaker hypothesis that \( \lambda\{ Y : A \leq_h Y \} = 1 \).

2.2. Related questions. Montalban and Nies asked some questions at the end of their survey paper [22]. A Borel structure is one that can be presented on a standard Polish space in such a way that the atomic diagram (including equality) is Borel.

Recall that a relation is Borel iff it is \( \Delta^1_1(Y) \) for some \( Y \subseteq \omega \). The spectrum of a Borel structure \( \mathcal{A} \) is the class of sets \( Y \subseteq \omega \) such that some presentation of \( \mathcal{A} \) is \( \Delta^1_1(Y) \). What can we say about possible spectra? Do the non-hyperarithmetical sets form a spectrum?

3. March 2010: Solovay Functions and \( K \)-triviality

This work was started by Bienvenu, Merkle and Nies. Bienvenu and Nies met at Paris 7. Then they all met at Uni Heidelberg. The final paper [] is quite different from what follows. It extends the results to weak Solovay functions (i.e., computably approximable from above), and argues mostly in terms of prefix-free complexity \( K \). Further, it contains a result relating Solovay functions to the \( C \)-characterization of ML-randomness due to Miller and Yu.

3.1. Introduction. Recall that a set \( A \subseteq \mathbb{N} \) is \( K \)-trivial [3] if there is \( b \) such that

\[
K(A|_n) \leq K(n) + b
\]

for each \( n \). This class was studied in [6, 27].

For any function \( g \) let

\[
\mathcal{C}_g = \{ A : \forall n K(A|_n) \leq^+ g(n) \}.
\]
Thus $C_K$ is the class of $K$-trivials, and $\forall n K(n) \leq^+ g(n)$ implies $C_K \subseteq C_g$.

**Definition 3.1** ([1]). We say that a computable function $g : \mathbb{N} \to \mathbb{N}$ is a Solovay function if $\forall n K(n) \leq^+ g(n)$ and $\exists^\infty n K(n) =^+ g(n)$.

If $g$ is computable, then the condition $\forall n K(n) \leq^+ g(n)$ is equivalent to $\sum_n 2^{-g(n)} < \infty$. Bienvenu and Downey [1] showed that $g$ is a Solovay function if and only if $\sum_n 2^{-g(n)}$ is ML-random.

Our main result is that for every computable function $g$,

\[(*) \quad g \text{ is a Solovay function } \iff C_g \text{ is the class of } K\text{-trivials.}\]

The result follows from two somewhat stronger theorems, corresponding to the implications from left to right, and its converse.

### 3.2. Every Solovay function characterizes the $K$-trivials

We show that for every Solovay function $g$, the class $C_g$ coincides with the $K$-trivials.

**Theorem 3.2.** Suppose $g$ is a Solovay function. If $\forall n K(A |_n) \leq^+ g(n)$ then $A$ is $K$-trivial.

**Proof idea.** After adding a natural number to $g$ if necessary, we may assume that $\forall n K(A |_n) \leq g(n)$. By [1], the left-c.e. real $\alpha = \sum_n 2^{-g(n)}$ is ML-random. Hence, by [2] $\alpha$ is complete for Solovay reducibility $\leq_S$. Let

$$\alpha_s = \sum_{n=0}^{s} 2^{-g(n)}.$$

Roughly speaking, the Solovay completeness of $\alpha$ means the following: if we choose a non-negative rational $q$ at a stage $s$, we can determine a stage $t \geq s$ such that $\alpha_t - \alpha_s \geq Cq$ for some constant $C > 0$ known in advance. However, we have to make sure that the sum of all rationals we choose does not exceed 1.

To show $A$ is $K$-trivial we build a bounded request set (aka KC set) $W$. When we see a description $\cup(v) = v$ at stage $s$, we want to ensure $W$ contains a request $|v| + O(1), A |_v)$. To ensure the weight of $W$ is at most 1, we will account the weight of such a request against the weight of $\cup$-descriptions of $A |_n$ for a whole interval $I_\sigma$ of numbers $n$. For $q = 2^{-|\sigma| - O(1)}$ we find a stage $t \geq s$ as above and let $I_{\sigma} = (s, t]$. Once we have $\cup$-descriptions of $A |_n$ of length at most $g(n)$ for each $n \in I_\sigma$, we can put the request into $W$, because the total weight of these descriptions is at least $q$. We arrange that the intervals associated with different $\sigma$ are disjoint. This implies that $W$ is a bounded request set.

**Details.** We first describe the mechanism to increase the approximation to $\alpha$. We view the $d$-th partial computable function $\phi_d$ as a partial map from $\mathbb{N}$ to $\mathbb{Q}_2 \cap [0,1)$, and let $\beta_{d,s} = \max range(\phi_{d,s})$, and let $\beta_d = \lim_s \beta_{d,s}$. Then $(\beta_d)_{d \in \mathbb{N}}$ is an effective listing of the left-c.e. reals in $[0,1]$. Let $\gamma = \sum_d 2^{-d} \beta_d$. Since $\gamma \leq_S \alpha$, by the definition there is a partial computable $\psi$ defined on $[0, \alpha) \cap \mathbb{Q}_2$ and a constant $k$ such that for each binary rational $q \in [0, \alpha)$ we have $\psi(q) \downarrow$ and

$$0 < \gamma - \psi(q) < 2^{k-1}(\alpha - q).$$
We build a left-c.e. real \( \eta \in [0, 1) \). The construction has a parameter \( d \in \mathbb{N} \). We think of \( \beta_d \) as being \( \eta \). By the recursion theorem, in the verification we can choose such a \( d \). (However, we must ensure that \( \eta \in [0, 1) \) for any parameter \( d \).

Let \( \gamma_s = \sum_d 2^{-d} \beta_d^{s_d} \).

**Construction of a left-c.e. real \( \eta \) and intervals \( I_\sigma \) for all strings \( \sigma \in \text{dom}(U) \).** The construction has a parameter \( d \). It takes place in stages \( u \). If ever we see that \( \eta_u < \beta_d,u \) we immediately stop the construction. Thus \( \eta = \eta_u \) and \( d \) cannot be a fixed point.

Let \( s \) be the current stage.

1. **Wait for \( r \geq s \) such that \( \gamma_r \geq \psi(\alpha_s) \).**
2. **Let \( \sigma \) be least such that \( \cup_s(\sigma) \downarrow \) but \( I_\sigma \) is not defined yet. Define \( \eta_{r+1} = \eta_r + 2^{-|\sigma|} \). Wait for \( t \geq r \) such that \( \alpha_t - \alpha_r \geq 2^{-d-k-|\sigma|} \) (the constant \( k \) associated with \( \gamma \leq s \) \( \alpha \) was defined above).**
3. **Declare \( I_\sigma = (s, t) \). GOTO (1).**

**Verification.** Clearly \( \eta \leq \Omega < 1 \). Thus by the recursion theorem we can choose \( d \) such that \( \eta = \beta_d \). By definition the wait in (1) always terminates.

**Claim 1.** The wait in (2) always terminates.

Since \( \gamma_r \geq \psi(\alpha_s) \), we have

\[
2^{k-1}(\alpha - \alpha_s) > \gamma - \psi(\alpha_s) \geq \gamma - \gamma_r \geq 2^{-d}(\beta_d - \beta_{d,r}).
\]

Since \( \eta_r \geq \beta_{d,r} \) and we add \( 2^{-|\sigma|} \) to \( \eta \) at stage \( r \), there is a stage \( t' > r \) such that \( \beta_{d,t'} - \beta_{d,r} \geq 2^{-|\sigma|}-1 \). Thus \( \alpha_t - \alpha_r \geq 2^{-d-k-|\sigma|} \) for some \( t > r \).

**Claim 2.** If \( \sigma \in \text{dom}(U) \) then \( I_\sigma \) is defined.

Suppose this holds inductively for all \( \tau < \sigma \). So we can pick stage \( s_0 \) such that no such string is processed at a stage \( > s_0 \). Suppose \( s \geq s_0 \), \( \cup_s(\sigma) \) is defined and by Claim 1 the construction goes back to (1) at stage \( s \). Then we can choose \( \sigma \) if \( I_\sigma \) is still undefined.

**Claim 3.** \( A \) is \( K \)-trivial.

We define a bounded request set \( W \). For each string \( \sigma \), if \( n = \cup(\sigma) \) is defined, wait for \( I_\sigma = (s, t) \) to become defined, and do the following: for each string \( x \) of length \( t \), at a stage \( p \) such that \( \forall i \in I_\sigma K_p(x | i) \leq g_t(i) \), put the request \( \langle |\sigma| + d + k, x | n \rangle \) into \( W \).

To see that \( W \) is a bounded request set, firstly note that the total weight of \( U \) descriptions of \( x | i, i \in I_\sigma \), is at least

\[
\sum_{i \in I_\sigma} 2^{-g_t(i)} \geq 2^{-d-k-|\sigma|}.
\]

Secondly, if \( x' \) is a further string of length \( t \) such that we put a request associated with \( U(\sigma) = n \), then \( x'_n \neq x | n \). Note that \( \cup_s(\sigma) = n \) implies \( s > n \). So the descriptions of \( x' | i, i \in I_\sigma \) are different from the ones for \( x | i \). Finally, if \( \sigma \neq \tau \) then \( I_\sigma \) is disjoint from \( I_\tau \), so again the descriptions are different.

If \( \cup(\sigma) = n \) then \( I_\sigma = (s, t) \) is defined by Claim 2. For \( x = A | l \) by the hypothesis on \( A \) we put a request \( \langle |\sigma| + d + k, A | n \rangle \) into \( W \). If \( \sigma \) is shortest such that \( \cup(\sigma) = n \), this shows \( K(A | n) \leq^* |\sigma| = K(n) \).

**3.3. The \( K \)-trivials characterize the Solovay functions.**

**Theorem 3.3.** Let \( g \) be computable function such that \( \forall n K(n) \leq^* g(n) \). Suppose that \( g \) is not a Solovay function. Then \( C_g \) is uncountable.
Proof. By the hypothesis \( \alpha = \sum_i 2^{-g(i)} \) is not ML-random. We will find a computable time bound \( t \) such that \( g(n) - K_{t(n)}(n) \to \infty \). Thereafter we use a result of Figuiera, Nies and Stephan (see [28, Thm 5.2.25]). They show that for every function \( p \) that tends to infinity and is computably approximable from above, there are uncountably many sets \( G \) such that \( \forall n K(G \upharpoonright n) \leq^+ K(n) + p(n) \) (in fact they show this for the function \( p(K(n)) \)). Now let \( p(n) = g(n) - K_{t(n)}(n) \). Every set such that \( \forall n K(G \upharpoonright n) \leq^+ K(n) + p(n) \) is in \( \mathcal{L}_g \).

The time bound \( t \) is obtained through a run time analysis of the proof in [1] that \( \alpha = \sum_i 2^{-g(i)} \) not ML-random implies \( g(n) - K(n) \to \infty \). First we paraphrase the original argument. Effectively in a string \( \tau \) we build a bounded request set \( L_\tau \). We wait for \( \mathbb{U}(\tau) \downarrow = x \), and let \( r = |x| \). For each \( n \) such that \( 0.x \leq \sum_{i \leq n} 2^{-g(i)} < 0.x + 2^{-r} \), we put the request \( \langle g(n) - r, n \rangle \) into \( L_\tau \). Clearly \( L_\tau \) is a bounded request set effectively in \( \tau \). So we can effectively in \( \tau \) obtain a prefix-free machine \( M_\tau \) for \( L_\tau \) according to the machine existence theorem.

Let \( M \) be the prefix-free machine given by \( M(\tau \sigma) \simeq M_\tau(\sigma) \). Given \( b \) choose \( r \) such that \( K(\alpha \upharpoonright r) < r - b \). Thus we can pick \( \tau \) of length \( < r - b \) such that \( \mathbb{U}(\tau) = \alpha \upharpoonright r = x \). Then for all \( n \) such that \( 0.x \leq \sum_{i \leq n} 2^{-g(i)} \), there is \( \sigma \) such that \( M(\tau \sigma) = n \) and \( |\sigma| = g(n) - r \). Hence \( K(n) \leq^+ r - b + g(n) - r = g(n) - b \).

For large \( n \) the time to compute \( \mathbb{U}(\tau) \) is negligible, so the time to verify \( M(\tau \sigma) = n \) is dominated by the time to compute \( \sum_{i \leq n} 2^{-g(i)} \) and the delay introduced by the machine existence theorem, and for \( \mathbb{U} \) to simulate \( M \). Thus we have \( K_{t(n)}(n) \leq^+ g(n) - b \) for some computable \( t \). \( \square \)

4. May 2010 : Cooper’s jump inversion and weak jump traceability

Lempp, Miller, Ng and Yu worked at the University of Wisconsin-Madison.

We give a (hopefully) more comprehensible proof of Cooper’s jump inversion theorem, and use the same ideas to construct a minimal GL1 degree which is not weakly jump traceable. The question of whether BGL1 was equal to weak jump traceability was first raised in Ng [23]. We can extend Cooper’s jump inversion to get the analogous jump inversion result in the tt-degrees, thus obtaining a superhigh set of minimal Turing degree.

Let us turn to Cooper’s jump inversion. Fix \( C \geq_T \emptyset' \), and we need to construct a set \( A \) such that \( A' \leq_T C \leq_T A \oplus \emptyset' \). We force with partial (modified) splitting trees, and ensure that both \( C \) and \( A \oplus \emptyset' \) can recover the construction. Since we do not have \( \emptyset'^{\omega} \) for recovering the construction, the splitting trees clearly have to be partial, as in Sack’s version. As we’re forcing longer initial segments of \( A \) we may change our mind on how the \( e^{th} \) minimality requirement \( R_e \) is to be met. If we change our mind on some \( R_e \) then it is because we have reached a dead end in an \( i \)-splitting tree for some \( i \leq e \), which is finite injury. Unfortunately this is not immediately compatible with forcing the jump, since we have to decide the jump \( A'(c) \) with oracle \( \emptyset' \) (together with \( A \) or \( C \)). If we are not able to force the jump on a current splitting tree, say \( T_e \), then this is ok as long as we make \( A \) stay within \( T_e \). Unfortunately \( T_e \) may not be a total tree,
and we may have $A$ leave $T_e$ through a dead branch. In this case we may be able to have $e \in A'$, which is bad for deciding the jump. 

The solution is to modify the partial computable splitting trees so that this does not happen; if we are unable to force the jump within $T_e$, then we ensure that we will never be able to force the jump even if we should leave $T_e$ through a dead end. To construct $T_0$, we define $T_0(\emptyset) = \emptyset$. Assume inductively that $T_0(\sigma)$ has been defined. We search for either

(i) the first pair of 0-splits $(\tau_0, \tau_1)$ extending $T_0(\sigma)$, or 
(ii) for the first $j$ found such that $J^*(j) \downarrow$ for $\tau \supset T_0(\sigma)$. 

If neither is ever found, then $T_0(\sigma)$ is a dead end, i.e. $T_0(\sigma 0) \uparrow, T_0(\sigma 1) \uparrow$. Otherwise if (i) is found first then we extend $T_0(\sigma i) = \tau_i$. If (ii) is found first with $j_0$ we extend $T_0(\sigma) = \tau$. In this case we repeat by searching for (i) or (ii) above the new $T_0(\sigma)$, except that we now limit the search in (ii) to all $j < j_0$.

That is, $T_0$ will search for both 0-splits, and jumps. If it finds a place it can force the jump before it finds a split, it grabs it immediately. Subsequently it will search for splits or jump computations on smaller inputs, until it finds the next split, where the counter $j_0$ is reset. $T_{e+1}$ is defined similarly, except it restricts its search to within the convergent part of $T_e$. To construct $A$, we force the jump at even stages, and code $C$ at odd stages. At stage 0 we start at $A_0 = \emptyset$, and ask if we can force $J(0)$ within $T_0$. Since $T_0$ is partial computable, the existence is a $\Sigma^0_0$ question. If the answer is yes, we extend $A$ to it while staying within $T_0$. If the answer is no, we do nothing to $A$.

Note that in this case we cannot have $J^A(0) \downarrow$ because otherwise the use has to extend a dead end of $T_0$, say $A \supset T_0(\sigma)$, but by the definition of $T_0$ we would have extended $T_0(\sigma)$ to this segment of $A$. At stage 1 we code $C(0)$ in the usual way. At stage 2 if we are still within $T_0$, then we ask if $J(1)$ can be forced in $T_1$. If yes, we grab it, otherwise we claim that $J^A(1) \uparrow$: If $J^*(1) \downarrow$ for some $\tau \supset A$, then $\tau$ cannot be on $T_1$. It cannot be on $T_0$ and extend a dead end of $T_1$, because otherwise we would have extended $T_1$ to include $\tau$, since inductively we assume that we have forced $J^A(0)$. Finally it cannot extend a dead end of $T_0$ because otherwise we would have extended $T_0$ to include $\tau$ because we assume that we have forced $J^A(0)$. The rest of the construction proceeds similarly.

It is clear that both $C$ and $A \oplus \emptyset'$ can recover the construction, giving us our desired result. To make $A$ not wjt we force the jump at even stages, and at odd stages we satisfy the $i^{th}$ non-wjt requirement. Given any partial computable tree $T$ and any $\sigma$ such that $T(\sigma) \downarrow$, there is (effectively) an index $i$ such that $J^{T(\sigma 0^{i+1})}(i) \downarrow = k$. Now at odd stages, assume we have $\tau \subset A$, and we are committed to staying on $T_0, \ldots, T_k$. Assume for simplicity that we only have to stay on $T_0$, and $\tau = T_0(\sigma)$. We ask $\emptyset''$ if some $f$ is $\Delta^0_2$. If the answer is no, we extend $A$ trivially to $T(\sigma 1)$. Otherwise we can find a large enough $k$ such that $J^{T_0(\sigma 0^{k+1})}(i) > f(i)$ and make $A$ extend $T_0(\sigma 0^{k+1})$. If on the other hand $T_0(\sigma 0^{k+1}) \uparrow$ we know that $T_0(\sigma 0^j) \uparrow$ for some least $j \leq k$, and we can then make $A$ extend the dead end $T_0(\sigma 0^j)$, and try to meet the same non-wjt requirement with a different $i$. Since we no longer have to care about $T_0$, this second try for meeting non-wjt will succeed. Finally we see that $A \oplus \emptyset'$ can recover the construction. It will be
able to figure out what we did during the odd stages: the number $k$ such that $A$ extend $T_0(\sigma 0^k 1)$, or the number $j$ where $A$ left $T_0$ through a dead end $T_0(\sigma 0^j)$.

Finally it is easy(?) to see that we can adapt the above proof (similar to Mohrherr) and make the jump inversion hold for tt-degrees.

5. May 2010: The collection of weakly 2-random reals is not $\Sigma^0_3$

Joint work of Lempp, Miller, Ng, Turetsky and Yu.

**Lemma 5.1.** For ever recursive tree $T$, there is a generalized Martin-Löf test $\{V_n\}_{n \in \omega}$ so that for any $\sigma$, if $[\sigma] \cap [T]$ is not empty, then $[\sigma] \cap [T] \cap \bigcap_n V_n$ is not empty.

*Proof.* To be added. \(\square\)

Now suppose that the collection of weakly 2-random reals $A$ is a $\Sigma^0_3$-set. So there is a sequence open sets $\{G_{i,j}\}_{i,j \in \omega}$ so that $A = \bigcup_i \bigcap_j G_{i,j}$. Let $T$ be a recursive tree such that $[T] = \{x \mid \forall n (x \upharpoonright n \in T)\}$ is not empty and only contains 1-random reals.

**Lemma 5.2.** There is a finite $\sigma \in T$ such that $[\sigma] \cap [T] \neq \emptyset$ but $[\sigma] \cap [T] \cap (\bigcap_j G_{0,j}) = \emptyset$.

*Proof.* Suppose not. Then let $\{V_n\}_{n \in \omega}$ be a generalized Martin-Löf test as in Lemma 5.1. Define $\sigma_0 = \emptyset$. For any $i \geq 0$, By Lemma 5.1, let $\sigma_{i+1} \succ \sigma_i$ such that $[\sigma_i] \cap [T] = \emptyset$ and $[\sigma_{i+1}] \cap [T] \subseteq V_i$. Then $[\sigma_{i+1}] \cap [T] \cap (\bigcap_j G_{0,j}) = \emptyset$.

So, without loss of generality, we may assume that $[\sigma_{i+1}] \cap [T] \subseteq G_{0,i}$ since $G_{0,i}$ is an open set. Let $x = \bigcup_{i \in \omega} \sigma_i$. Then $x \in \bigcap_n V_n$ and so $x$ is not weakly 2-random. But $x \in \bigcap_j G_{0,j}$, a contradiction. \(\square\)

Let $\{\bigcap_{n \in \omega} U^i_n\}_{i \in \omega}$ be an enumeration of all generalized Martin-Löf tests. By Lemma 5.2, let $\sigma \in T$ such that $[\sigma] \cap [T] \neq \emptyset$ but $[\sigma] \cap [T] \cap (\bigcap_j G_{0,j}) = \emptyset$. Since $[T]$ only contains 1-random reals, $\mu([T] \cap [\sigma]) > 0$. Let $n$ be large enough such that $\mu(U^i_n) < \frac{1}{2}\mu([T] \cap [\sigma])$. Then there is a recursive tree $T_0 \subseteq [T] \cap [\sigma]$ so that $\mu([T_0] \cap U^i_n) > 0$ but $[T_0] \cap U^i_n = \emptyset$. Let $\sigma_0 = \sigma$. By induction, we have $\sigma_{i+1} \succ \sigma_i$, $[\sigma_{i+1}] \cap [T_i] \cap (\bigcap_j G_{i,j}) = \emptyset$, $[T_{i+1}] \subseteq [\sigma_{i+1}] \cap [T_i]$ and $\mu(T_{i+1}) > 0$ such that there is some large $n$ for which $[T_{i+1}] \cap U^{i+1}_n = \emptyset$.

Let $x = \bigcap_{i \in \omega} \sigma_i$. By the construction, $x$ is weakly 2-random but $x \notin \bigcap_j G_{i,j}$ for any $i$, a contradiction.

6. May 2010: There is a perfect set in which any two reals are LR comparable

Added by Liang Yu.

It was shown, in [19], if $x$ and $y$ are random and $x \leq_K y$, then $y \leq_{LR} x$. So it is sufficient to prove the result for $K$-degrees among random reals.

The statement can be formalized as

$$\exists \tau \forall x \forall y \left( T \text{ is a perfect tree} \land (x \in [T] \Rightarrow x \text{ is random }) \land (x \in [T] \land y \in [T] \Rightarrow (x \leq_K y \lor y \leq_K x) \right).$$
where \([T] = \{x \mid \forall n(x|n \in T)\}\).

So the statement is \(\Sigma^1_2\).

Starting with any transitive model \(M \models ZFC\) (say \(L\)), for any cardinal \(\kappa\) in \(M\), we may build a c.c.c. forcing \(P\) so that \(M[G]\) models Martin’s axiom and \(2^{\aleph_0} > \kappa^+\) for any generic set \(G\). By Miller and Yu ([20, Thm 7.4]), there is a chain of \(K\)-degrees among random reals of size \(\kappa^+\). Then the set \(A = \{(x,y) \mid x, y\ \text{are 1-random and } x \leq_K y \lor y \leq_K x\}\) is a Borel set which contains a \(\kappa^+\)-square (i.e. a set \(B \subseteq 2^\omega\) of size \(\kappa^+\) such that \(B \times B \subseteq A\)). If \(\kappa > (\beth_{\omega_1})^M\), then by Shelah’s result (*), there is a chain of \(K\)-degrees among random reals of size \(\kappa^+\). Then the set \(B \subseteq 2^\omega\) of size \(\kappa^+\) such that \(B \times B \subseteq A\). If \(\kappa > (\beth_{\omega_1})^M\), then by Shelah’s result (**), then by Shelah’s result (**) on page 2 of [33],

\[\kappa^+ > \lambda_{\omega_1}(\aleph_0),\]

where \(\lambda_{\omega_1}(\aleph_0)\) is the Hanf number for models of sentences in the infinitary language \(L_{\omega_1, \omega}\) (i.e. the least cardinal \(\kappa\) such that for any sentence \(\phi \in L_{\omega_1, \omega}\) with a model cardinality \(\kappa\), \(\phi\) has models in any infinite cardinalities). Then by Thm 1.15 in the same Shelah paper, there must be a perfect set \(C\) such that \(C \times C \subseteq A\). So there is a perfect chain in the \(K\)-degrees.

By \(\Sigma^1_2\)-absoluteness, the result follows.

7. **JUNE 2010: CLASSES RELATED TO THE \(K\)-TRIVIALS AND THE STRONGLY JUMP TRACEABLES**

By Andre.

7.1. **Cupping \(K\)-trivials by random sets.** We say that \(A\) is \(ML\)-non-cuppable if \(\emptyset' \leq_T A \oplus Z\) implies \(\emptyset' \leq_T Z\) for each ML-random \(Z\). It is a persistent open question [18] whether each \(K\)-trivial \(A\) is ML-noncuppable. Note that any possible ML-random cupping partner of a \(K\)-trivial set \(A\) must be LR-complete (work of Hirschfeldt and Nies). We can use this to rule out cupping partners with certain randomness properties stronger that ML-randomness.

The following proposition, due to [7], shows that such a cupping partner cannot be weakly Demuth random at level \(O(h(m)2^m)\) for any order function \(h\).

**Proposition 7.1.** Suppose \(Z\) is an \(O(h(m)2^m)\)-weak Demuth random set for some order function \(h\). Then \(Z\) is not \(\omega\)-c.e. tracing, and hence not LR-complete.

The next proposition shows that they cannot be Demuth random at level \(O(2^m)\) (note however that the latter notion is somewhat arbitrary because of the \(2^{-m}\) bounds on the measure of the \(m\)-th component of a test). An array computable set is not LR-complete by Barmpalias [2]. Then, by the proof of [28, 3.6.26] we have:

**Proposition 7.2.** There is a Demuth test \((S_m)_{m \in \mathbb{N}}\) as follows:

(a) The version \(S_m\) changes at most \(2^m\) times.

(b) If \(Z\) passes the test (i.e. \(Z \notin S_m\) for a.e. \(m\)), then \(Z\) is array computable.

**Corollary 7.3.** No \(K\)-trivial set is Demuth cuppable.
On the other hand, there is a Turing incomplete c.e. set \( A \) that cups with a 2-random (and hence Demuth random). Simply make \( A \) LR-complete, so that \( \Omega^A \) is 2-random. Use that \( A \uplus \Omega^A \equiv_T A' \). Lots of obvious questions arise here. For instance, how close to computable can a Demuth cuppable c.e. set be? Ng has announced it can be superlow.

### 7.2. Box classes and diamond classes.

For a class \( C \subseteq \mathcal{P}_\mathbb{N} \) let
\[
C^\square = \{ A \mid \forall Y \in C \cap \text{MLR}[A \equiv_T Y] \}.
\]

Let \( C^\Diamond \) denote the collection of c.e. sets in \( C^\square \).

It is known [8] that \((\omega\text{-c.e.})^\square\) is contained in the strongly jump traceables and \((\omega\text{-c.e.})^\Diamond\) coincides with the c.e. strongly jump traceables.

The following definition is taken from [8]. We let \( R = (\omega, <_R)\) be a computable well-order.

**Definition 7.4.** An \( R \)-approximation is a computable function
\[
g = (g_0, g_1) : \omega \times \omega \to \omega \times \omega
\]
such that for each \( x \) and each \( s > 0 \),
\[
(1) \quad g(x, s) \neq g(x, s - 1) \rightarrow g_1(x, s) <_R g_1(x, s - 1).
\]
In this case, \( g_0 \) is a computable approximation of a total \( \Delta^0_2 \) function \( f \). We say that \( g \) is an \( R \)-approximation of \( f \), and that \( f \) is \( R \text{-c.e.} \).

\( Y \) is \( \omega^n \text{-c.e.} \) if its characteristic function is \( R \text{-c.e.} \) for \( R \) the canonical presentation of \( \omega^n \). It is not hard to check that we could as well take the function \( n \to Y |_n \).

Diamondstone, Hirschfeldt, and Nies showed that \((\omega^2\text{-c.e.})^\Diamond\) is a proper subclass of \((\omega\text{-c.e.})^\Diamond\). They gave a direct proof. The following, alternative proof pieces together results from the literature.

**Theorem 7.5.** There is a c.e. set \( A \) below all \( \omega\text{-c.e.} \) random sets, but not below all \( \omega^2\text{-c.e.} \) random sets.

**Proof.**

(1) [28, Theorem 3.6.25] builds a Demuth random \( \Delta^0_2 \) set \( Y \). It is not hard to see that the constructed set \( Y \) is in fact \( \omega^2\text{-c.e.} \). For detail see [8].

(2) Kučera and Nies [15] show that \( A \equiv_T Y \) for c.e. set \( A \) and Demuth random \( Y \) imply that \( A \) is strongly jump traceable.

(3) Ng [24] proved that the class \( \text{SJT}_{c.e.} \) of strongly jump traceable c.e. sets has a \( \Pi^0_4 \) complete index set.

Now let \( Y \) be as in (1). Let \( \mathcal{I}(Y) \) be the class of c.e. sets Turing below \( Y \). Since \( Y \) is low, the class \( \mathcal{I}(Y) \) is \( \Sigma^0_3 \). Thus, since \( \text{SJT}_{c.e.} \) coincides with \((\omega\text{-c.e.})^\Diamond\), we have \((\omega^2\text{-c.e.})^\Diamond \subseteq \mathcal{I}(Y) \subset (\omega\text{-c.e.})^\Diamond\). \( \square \)
7.3. Diagram of classes of c.e. sets, mostly contained in the $K$-trivials. Here are some classes, all within the c.e. sets. A separation is indicated by the label $\neq$ on the implication arrow. “ML-coverable” and “ML non-cuppable” means covered/not cupped by an incomplete ML-random.

---

**Diagram**

```
K-trivial
\rightarrow
ML-coverable (the diamond of evil)
\rightarrow
ML-non-cup.
\leftarrow
Demuth non-cup.
```

**Diagram**

```
weakly Dem. cov.
\rightarrow
LRH
\leftarrow

\neq[15]
\rightarrow

\neq[10]
\rightarrow

\neq[18]
\rightarrow

\neq[10]
\rightarrow

\neq[18]
```

---

Added Dec 2011: Day and Miller 2011, relying on a result on non-density of Martin-Loef random sets in $\Pi_1^0$ classes by Bienvenu Hoelzl, Miller and Nies in STACS 2012, have shown that a set is K-trivial if and only if it cannot be cupped above the halting problem with an incomplete Martin-Loef random set.

Diamondstone, Greenberg, and Turetsky have shown that for any set $A$, c.e. or not, $A$ is below each $\omega$-c.e. ML-random iff $A$ is s.j.t.

8. July 2010: Randomness extraction, Mises-Wald-Church stochastic sets, Medvedev reducibility, and effective packing dimension

By Bjørn. We show that there is no Turing reduction procedure that does substantially better than a majority function in extracting randomness from a set of integers that lies a small asymptotic Hamming distance away from a random set [14]. A consequence of the proof is as follows. Let MWC denote the class of Mises-Wald-Church stochastic sets, let $\text{DIM}_p$ denote the class...
of sets of effective packing dimension 1, and let COM denote the class of complex sets (in the sense of Kjos-Hanssen, Merkle, and Stephan). Let $\leq_s$ and $\leq_w$ denote strong (i.e. Medvedev) and weak (i.e. Muchnik) reducibility of mass problems.

**Theorem 8.1.** $MWC \preceq_s DIM_p \cap COM$.

**Question 8.2.** Is $MWC \preceq_w DIM_p \cap COM$?
Muchnik above

\[ \rightarrow \text{Not Muchnik above} \]

\[ \rightarrow \rightarrow \text{Medvedev above} \]

\[ \rightarrow \rightarrow \rightarrow \text{Not Medvedev above} \]

**Figure 2. Meaning of arrows.**

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Unabbreviation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>DNR</td>
<td>Diagonally non-recursive function in ( \omega^\omega )</td>
<td></td>
</tr>
<tr>
<td>DNR(_n)</td>
<td>Diagonally non-recursive function in ( \omega^n )</td>
<td></td>
</tr>
<tr>
<td>MLR</td>
<td>Martin-Löf random</td>
<td></td>
</tr>
<tr>
<td>KLR</td>
<td>Kolmogorov-Loveland random</td>
<td></td>
</tr>
<tr>
<td>SR</td>
<td>Schnorr random</td>
<td></td>
</tr>
<tr>
<td>KR</td>
<td>Kurtz random (weakly 1-random)</td>
<td></td>
</tr>
<tr>
<td>MWC</td>
<td>Mises-Wald-Church stochastic</td>
<td>8.3</td>
</tr>
<tr>
<td>KLS</td>
<td>Kolmogorov-Loveland stochastic</td>
<td>8.4</td>
</tr>
<tr>
<td>SBI</td>
<td>Stochastically bi-immune</td>
<td>8.6</td>
</tr>
<tr>
<td>SD(_{1/2})/IM</td>
<td>stochastically dominating for ( p = 1/2 ) and immune</td>
<td>8.7</td>
</tr>
<tr>
<td>BI</td>
<td>bi-immune</td>
<td>8.5</td>
</tr>
<tr>
<td>IM</td>
<td>immune (≡s noncomputable)</td>
<td>8.5</td>
</tr>
<tr>
<td>((H,1))</td>
<td>effective Hausdorff dimension 1</td>
<td>8.9</td>
</tr>
<tr>
<td>((H,&gt;0))</td>
<td>effective Hausdorff dimension &gt; 0</td>
<td>8.9</td>
</tr>
<tr>
<td>((p,1))</td>
<td>effective packing dimension 1</td>
<td>8.9</td>
</tr>
<tr>
<td>((p,&gt;0))</td>
<td>effective packing dimension &gt; 0</td>
<td>8.9</td>
</tr>
<tr>
<td>((cp,1))</td>
<td>complex packing dimension 1</td>
<td>8.9</td>
</tr>
<tr>
<td>((cp,&gt;0))</td>
<td>complex packing dimension &gt; 0</td>
<td>8.9</td>
</tr>
<tr>
<td>COM</td>
<td>complex in the sense of [12]</td>
<td>8.10</td>
</tr>
<tr>
<td>ED</td>
<td>eventually different</td>
<td>8.8</td>
</tr>
<tr>
<td>( \not\in_T ) Cohen generic</td>
<td>not computable from a 2-generic set</td>
<td></td>
</tr>
<tr>
<td>CT</td>
<td>computably traceable</td>
<td></td>
</tr>
<tr>
<td>WCET</td>
<td>weakly c.e. traceable</td>
<td>8.11</td>
</tr>
</tbody>
</table>

**Figure 3. Abbreviations used in Figure 1.**

**Definition 8.3.** An element of \( 2^\omega \) is Mises-Wald-Church (MWC) stochastic if no partial computable monotonic selection rule can select a biased subsequence, i.e., a subsequence where the relative frequencies of 0s and 1s do not converge to 1/2.
Definition 8.4. An element of $2^\omega$ is Kolmogorov-Loveland stochastic if no partial computable (non-monotonic) selection rule can select a biased subsequence, i.e., a subsequence where the relative frequencies of 0s and 1s do not converge to 1/2.

Let $\mathcal{C}$ denote the collection of all infinite computable subsets of $\omega$.

Definition 8.5. A set $X$ is immune if for each $N \in \mathcal{C}$, $N \not\subseteq X$. If $\omega \setminus X$ is immune then $X$ is co-immune. If $X$ is both immune and co-immune then $X$ is bi-immune.

Definition 8.6. A set $X$ is stochastically bi-immune if for each set $N \in \mathcal{C}$, $X \upharpoonright N$ satisfies the strong law of large numbers, i.e.,

$$\lim_{n \to \infty} \frac{|X \cap N \cap n|}{|N \cap n|} = \frac{1}{2}.$$ 

Definition 8.7. Let $0 \leq p < 1$. A sequence $X \in 2^\omega$ is $p$-stochastically dominated if for each $L \in \mathcal{C}$,

$$\limsup_{n \to \infty} \frac{|L \cap n|}{n} > 0 \implies (\exists M \in \mathcal{C}) \ M \subseteq L \ and \ \limsup_{n \to \infty} \frac{|X \cap M \cap n|}{|M \cap n|} \leq p.$$ 

The class of $p$-stochastically dominated sequences is denoted $SD_p$. If $\omega \setminus X \in SD_p$ then we write $X \in SD_p$ and say that $X$ is stochastically dominating.

Definition 8.8. A function $f \in \omega^\omega$ is eventually different (ED) if for each computable function $g \in \omega^\omega$, $\{x : f(x) = g(x)\}$ is finite.

Definition 8.9. The effective Hausdorff dimension of $A \in 2^\omega$ is

$$\liminf_{n \in \omega} \frac{K(A \upharpoonright n)}{n}.$$ 

The complex packing dimension of $A \in 2^\omega$ is

$$\dim_{cp}(A) = \sup \inf_{N \in \mathcal{C}, n \in N} \frac{K(A \upharpoonright n)}{n}.$$ 

The effective packing dimension of $A \in 2^\omega$ is

$$\limsup_{n \in \omega} \frac{K(A \upharpoonright n)}{n}.$$ 

Definition 8.10 ([12]). $A \in 2^\omega$ is complex if there is an order function $h$ with $K(A \upharpoonright n) \geq h(n)$ for almost all $n$.

Definition 8.11 (Nies [28]). $A \in 2^\omega$ is facile if $K(A \upharpoonright n \upharpoonright n) \leq h(n)$ for all order functions $h$ and almost all $n$. If $A$ is not facile then $A$ is difficult. $A$ is weakly c.e. traceable if for each order function $p$, for all computably bounded functions $f \leq_T A$, there is a c.e. trace for $f$ of size bounded by $p$.

9. Sep 2010: Higher randomness

9.1. The collection of $\Delta^1_1$-random reals is not hyperarithmetically upward closed. Input by Yu.

Define

$$F = \{ x \mid \omega^x_1 = \omega_1^{CK} \land \text{every hyperarithmetic real is recursive in } x\}.$$ 

Then $F$ is a $\Sigma^1_1$ set.
Lemma 9.1 (Folklore). If $\omega^x_1 = \omega^x_1^{CK}$, then there is a real $z \geq_T x$ so that $z \in F$.

Proof. Suppose that $\omega^x_1 = \omega^x_1^{CK}$. The set

$$F_x = \{ z \geq_T x \mid \text{every hyperarithmetic real is recursive in } z \}$$

is a $\Sigma^1_1(x)$ set. Moreover $F_x$ is not empty since $O^x \in F_x$. By Gandy basis theorem, there must be some $z \in F_x$ so that $\omega^x_1 = \omega^x_1 = \omega^x_1^{CK}$. Then $z \in F$. □

Lemma 9.2. (1) For any $x \in F$ and $\Pi^1_1$-random real $z$, $x \not\leq_h z$;

(2) Every $\Pi^1_1$-random real is hyperarithmetically reducible to some real in $F$;

(3) If $x \in F$, then there is a $\Pi^1_1$-random real hyperarithmetically reducible to $x$.

Proof. (1). Suppose that $x \in F$, $z$ is a $\Pi^1_1$-random real and $x \leq_h z$. Since $\omega^x_1 = \omega_1^{CK}$, there exists some recursive ordinal $\alpha$ such that $x \leq \omega^{(\alpha)}$. Since $x \geq_T \emptyset^{(\alpha+1)}$, $\emptyset^{(\alpha+1)} \leq_T z^{(\alpha)}$. But the set $\{ y \mid \emptyset^{(\alpha+1)} \leq_T y^{(\alpha)} \}$ is a $\Delta^1_1$-null set (see [?]). $z$ cannot be $\Delta^1_1$-random, a contradiction.

(2). Immediately from Lemma 9.1.

(3). If $x \in F$, then every $x$-Schnorr random is $\Delta^1_1$-random. Pick up an $x$-Schnorr random real $z \leq_T x'$. So $z$ is $\Delta^1_1$-random. Since $\omega^x_1 = \omega_1^{CK}$, $\omega^x_1 = \omega_1^{CK}$. So $z$ is $\Pi^1_1$-random. □

Corollary 9.3. The collection of the hyperdegrees of $\Delta^1_1$-random reals is not upward closed within the hyperdegrees.

Proof. Any $\Delta^1_1$-random real $x$ with $\omega^x_1 = \omega_1^{CK}$ is $\Pi^1_1$-random. By (2) in Lemma 9.2, there is a $y \in F$ so that $x \leq_h y$. By (1) in Lemma 9.2, $y$ cannot be hyperarithmetically reducible to $\Delta^1_1$-random real. □

9.2. Some trivial observations about $NCR_{\Pi^1_1}$. Input by Yu.

$$NCR_{\Pi^1_1} = \{ x \mid x \text{ is not } \Pi^1_1\text{-random respec to any continuous measure} \}.$$ 

Proposition 9.4. $NCR_{\Pi^1_1} = \{ x \mid x \in L_{\omega^1_1} \}$.

Sketch of the proof.

Lemma 9.5. $NCR_{\Pi^1_1}$ is a thin $\Pi^1_1$-set. So $NCR_{\Pi^1_1} \subseteq \{ x \mid x \in L_{\omega^1_1} \}$.

Proof. Just same as the proof in Reimann and Slaman [30], $NCR_{\Pi^1_1}$ does not contain a perfect subset.

Just same as the proof in Hjorth and Nies [11], there is a $\Pi^1_1$ set $Q \subseteq (2^{\omega_1})^3$ so that for each real $x$ and continuous measure $\mu$, $Q_{\mu,x} = \{ y \mid (\mu, x, y) \in Q \}$ is the largest $\Pi^1_1(x)$ $\mu$-null set. The same as in Reimann and Slaman [31], $NCR_{\Pi^1_1}$ is a $\Pi^1_1$ set. □

Lemma 9.6. If $x \in L_{\omega^1_1}$ and $z \not\geq_h x$, then $z \oplus x \geq_h O^x$.

Proof. Suppose that $x \in L_{\omega^1_1}$ and $z \not\geq_h x$. Then $\omega^x_1 < \omega^z_1$. So $\omega^x_{1 \oplus z} > \omega^z_1$. Thus $z \oplus x \geq_h O^x$. □

Lemma 9.7. If $x \in L_{\omega^1_1}$, then $x \in NCR_{\Pi^1_1}$.
Proof. Given any continuous measure $\mu$. If $x \leq_\beta \mu$, then $x$ obviously is not $\mu$-random. By Lemma 9.6, $x \oplus \mu \geq_\beta \mathcal{O}^\mu$. But $\{z \mid z \oplus \mu \geq \mathcal{O}^\mu\}$ is a $\Pi^1_1$-null set. So $x$ cannot be $\Pi^1_1$-random. □

The proposition follows by the Lemmas above.

Remark: By Reimann and Slaman [31], $NCR_n$ is countable for any $n \in \omega$. This should be true for any recursive ordinal. Then $NCR_{\Pi^1_1}$ puts a limit for their results. By a more involved argument, one can show that every master code belongs to $NCR_{\Delta^1_1}$. So the uncountability of $NCR_{\Delta^1_1}$ is unprovable under $\text{ZFC}$.


Input by Yu.

10.1. Characterizing $\emptyset'$-Schnorr randomness via Martin-Löf randomness.

Definition 10.1. A real $x$ is $\mathbb{L}$-random if for all real $z$ with $z' \leq_T \emptyset'$, $x$ is $z$-random.

This notion was introduced by Mr. Peng. The following result was proved.

Theorem 10.2. Every $\mathbb{L}$-random is $\emptyset'$-Schnorr-random.

The method of the proof is a finite injury argument.

Sketch of the proof:

Proof. We prove that for every $\emptyset'$-Schnorr test $\{U^\emptyset_n\}_{n \in \omega}$, there is a real $z$ with $z' \leq_T \emptyset'$ such that there is $z$-Martin-Löf-test $\{V^z_n\}_{n \in \omega}$ so that $\bigcap_{n \in \omega} V^z_n \supseteq \bigcap_{n \in \omega} U^\emptyset_n$.

Since $\{U^\emptyset_n\}_{n \in \omega}$ is a $\emptyset'$-Schnorr test, there is a recursive function $f : \omega \times 2^{<\omega} \times \omega \to 2$ so that for every $n$ and $\sigma$,

1. $\lim_n f(n, \sigma, s) = 0$ or $1$;
2. $\lim_n f(n, \sigma, s) = 1$ if and only if $\sigma \in U^\emptyset_n$.

We build a low real $z$ and $z$-Martin-Löf test $\{V^z_n\}_{n \in \omega}$ by a full approximation priority argument. We need to satisfy two kinds of requirements:

1. $N_e : \exists^\infty s \Phi^z_{\Phi_e}(e)[s] \downarrow \Rightarrow \Phi^z_e(e) \downarrow$;
2. $P_e : U^\emptyset_{2e} \subseteq V^z_e$.

To satisfy $P_e$, we need to decompose $P_e$ into infinitely many sub-requirements $P_{e,n}$. For every $n, m$, let

$$U^\emptyset_n \upharpoonright m = U^\emptyset_n \cap 2^{<m} = \{\sigma \mid |\sigma| \leq l^m_m \wedge \sigma \in U^\emptyset_n\}$$

where $l^m_m$ is the least number $l$ such that $\mu(U^\emptyset_n \cap 2^l) > 2^{-m}(1 - 2^{-m})$. Notice that since $\{U^\emptyset_n\}_{n \in \omega}$ is a $\emptyset'$-Schnorr test, we may $\emptyset'$-recursive find $l^m_m$ for every $m$ and $n$.

Set

$$P_{(e,n)} : U^\emptyset_{2e} \upharpoonright n \subseteq V^z_e.$$
As in the usual finitary injury argument, we build a restriction function $r(e,s) > \phi_{z}^{e}(c)$ for every negative requirement $N_{e}$ at every stage $e$ where $\phi_{z}^{e}(e)$ is the use function of $\Phi_{z}^{e}(e)[s]$. Set
\[ R(e,s) = \sum_{i \leq e} r(i,s). \]

At stage $s$, $N_{e}$ requires attention if $\Phi_{z}^{e}(e)[s] \downarrow$ but $N_{e}$ has not received attention (after initialized).

At every stage $s$, for every $n, m$, let
\[ U_{n}^{m}[s] \cap m = U_{n}^{m}[s] \cap 2^{\omega \cap m}[s] = \{ \sigma \in 2^{\omega} \mid |\sigma| \leq m \} \]
where $l_{m}[s]$ is the least number $l$ such that $\mu(U_{n}^{m}[s] \cap 2^{l}) > 2^{-n}(1 - 2^{-m})$. Obviously $\lim_{s} l_{m}[s] = l_{m}$.

The basic strategy for $P_{(e,n)}$ is: At any stage $s$, for each $e$, there is a follower $\langle e, \sigma, t_{s} \rangle$ attached to $\sigma$. If $\sigma$ enters $U_{e}^{n}[s] \uparrow n$ (i.e. $f(e,\sigma,s) = 1$), then we set $z_{si}(\langle e, \sigma, t_{s} \rangle) = 1$. If $\sigma$ exit $U_{e}^{n}[s] \uparrow n$ (i.e. $f(e,\sigma,s) = 0$), then we set $z_{si}(\langle e, \sigma, t_{s} \rangle) = 0$. So we may define $V_{e}^{z}[s] = \{ \sigma \mid (z_{si}(\langle e, \sigma, t_{s} \rangle) = 1) \}$ and $V_{e}^{z} = \{ \sigma \mid \exists s(z_{si}(\langle e, \sigma, t_{s} \rangle) = 1) \}$.

The rule attributing a follower to $P_{(i,j)}$ at stage $s$: For any $\sigma$ with $l_{j}[s] \geq |\sigma| > l_{j-1}[s]$, we attribute a follower $(i,\sigma, t_{s})$ to $\sigma$ such that $t_{s}$ greater than all the parameters mentioned in the higher priority requirements no later than stage $s$.

$P_{(i,j)}$ requires attention at stage $s$ if $\sigma$ enters $U_{e}^{m}[s] \uparrow j$ but $z_{si}(\langle e, \sigma, t_{s} \rangle) = 0$. Then we intend set $z_{s+i+1}(\langle e, \sigma, t_{s} \rangle) = 1$.

To avoid the conflict between $P_{(i,j_{0})}$ and $P_{(i_{1},j_{1})}$ say $P_{(i_{1},j_{1})} < P_{(i_{1},j_{1})}$, we initialize all the parameters for $P_{(i_{1},j_{1})}$ and set $z_{s+i}(\langle i_{1}, \sigma, t_{s} \rangle) = 0$ for any parameter $(i_{1}, \sigma, t_{s})$ for $P_{(i_{1},j_{1})}$ once upon $P_{(i_{1},j_{0})}$ receives attention. This cannot happen infinitely often by the definition of $f$.

Notice that there are at most $2^{-2^{i}-(j-1)}$ measure of clopen sets put in $V_{i}^{z}$ by $P_{(i,j)}$ for any pair $(i,j)$.

Since $\{ U_{n}^{m} \}_{n \in \omega}$ is a $\emptyset'$-Schnorr test, a usual finite injury argument will show that $N_{e}$ will be injured at most finitely many times for every $e$. So $z$ must be low.

For each $P_{(i,j)}$ with $j \geq i$, there are $j$ many negative requirements $\{ N_{e} \}_{e \leq j}$ having higher priority than $P_{(i,j)}$. For each $e \leq j$, once $N_{e}$ set up a restriction $r(e,s)$, then $P_{(i,j)}$ cannot change its parameters less than $R(e,s)$ anymore until some $P_{(i,j')}$ higher than $N_{e}$ receives attention. So $P_{(i,j)}$ may $j$-times wrongly put clopen sets into $U_{i}^{z}$. The measure of the sum of these mistakes is no more than $j \cdot 2^{-2^{i}-(j+1)}$. Thus
\[ \mu(V_{i}^{z}) \leq \sum_{j \in \omega} (j+1) \cdot 2^{-2^{i}+(j+1)} \leq 2^{-i}. \]

So $\{ V_{i}^{z} \}_{i \in \omega}$ is a $z$-Martin-Löf test. By the definition of $V_{i}^{z}$, for every $i$, $U_{2^{i}}^{\emptyset'} \subseteq V_{i}^{z}$ for every $e$. So $\bigcap_{e \in \omega} U_{2^{i}}^{\emptyset'} \subseteq \bigcap_{e \in \omega} V_{i}^{z}$.

**Corollary 10.3.** For any real $x \geq T \emptyset'$ and $z$, the followings are equivalent:

1. $z$ is $x$-Schnorr random;
For any real \( y \) with \( y' \leq_T x \), \( z \) is weakly-2-random relativized to \( y \);
(3) For any real \( y \) with \( y' \leq_T x \), \( z \) is Martin-Löf-random relativized to \( y \).

Proof. Both (1) \( \implies \) (2) and (2) \( \implies \) (3) are obvious.

We show that (3) \( \implies \) (1). Since \( x \geq_T \emptyset' \), there is a real \( z_0 \leq_T x \) so that \( z_0' \equiv_T x \). Relativizing the proof of Theorem 10.2 to \( z_0 \), every \( z \)-Schnorr random real is Martin-Löf-random relativized to \( y \) for some \( y \) with \( z_0 \leq y \) and \( y' \leq_T x \).

\( \square \)

Obviously another direction of Theorem 10.2 is true. So \( L \)-randomness is the same as \( \emptyset' \)-Schnorr-randomness.

10.2. Lowness properties.

Theorem 10.4. If \( x \) is not low, then there is a \( \emptyset' \)-Schnorr random real which is not \( x \)-random.

The method of the proof is a forcing argument which was based on a couple of results due to Diamondstone, Nies and others. They are:

Theorem 10.5 (Diamondstone [5]). For any pair of low reals \( x \) and \( y \), there is a c.e. low real \( z \) so that every \( z \)-random real is both \( x \)- and \( y \)-random.

Theorem 10.6 (Nies [28]). If \( y \leq_T x' \) and every \( x \)-random is \( y \)-random, then \( y' \leq_T x' \).

And Theorem 5.6.9 in [28].

The forcing is: \( P = (P, \leq) \) where \( P \) is the collection of \( \Pi_0^1(y) \) set of reals having positive measure for some low real \( y \). For \( P_1, P_2 \in P, P_1 \subseteq P_2 \) if and only if \( P_1 \leq P_2 \).

So

Corollary 10.7. Low(Sch(\emptyset'), W2R) = Low(Sch(\emptyset'), ML) = Low(= \{ x \mid x' \equiv_T \emptyset' \}).

11. Uniformly \( \Sigma^0_3 \) index sets

By Frank Stephan. Also see 2012 version of Nies’ book.

[28, Problem 5.3.33] Let \( C \) be an index set for a class of c.e. sets, namely \( e \in C \land W_e = W_i \rightarrow i \in C \). We say that \( C \) is uniformly \( \Sigma^0_3 \) if there is a \( \Pi_2^0 \) predicate \( P \) and a effective (= recursive) sequence \( (e_0, b_0), (e_1, b_1), \ldots \) such that the following three conditions hold:

- \( e \in C \iff \exists b [P(e, b)] \);
- \( P(e_n, b_n) \) for all \( n \);
- For all \( e \in C \) there is an \( n \) with \( W_{e_n} = W_e \).

In other words, \( C \) is the closure, under having the same index, of a projection of a c.e. relation contained in \( P \). For instance, let \( P(e, b) \) be

\[ \forall n \forall s \exists t > s [K_t(W_{e,t} \mid_n) \leq K_s(n) + b]. \]

This shows that the \( K \)-trivials are uniformly \( \Sigma^0_3 \) by a Theorem of Downey, Hirschfeldt, Nies and Stephan (see [28, Thm 5.3.28] for a simpler proof of
that theorem). Also the computables - C-trivials are u’ly $\Sigma^0_3$ by a similar argument.

The problem was whether each $\Sigma^0_3$ class is already uniformly $\Sigma^0_3$.

First an easy counterexample:

1. $\{ e : |W_e| = \infty \}$ has a $\Sigma^0_3$ Index set; in fact $\Pi^0_3$.
2. If this set were uniform $\Sigma^0_3$, there would be an effective sequence $(e_n, b_n)$ and a predicate $P$ such that the following holds:
   
   - $P(e_n, b_n)$ for all $n$;
   - $W_e$ is infinite iff $P(e, b)$ for some $b$;
   - For each infinite $W_e$ there is $n$ such that $W_e = W_{e_n}$.

Thus $n \mapsto W_{e_n}$ is an effective enumeration of all infinite r.e. sets, contradiction.

Next we give a full characterization:

**Theorem.** An index set $C$ is uniformly $\Sigma^0_3$ iff there is a recursive enumeration $e_0, e_1, \ldots$ such that $e \in C \Leftrightarrow \exists n [W_{e_n} = W_e]$.

**Proof.**

The definition directly gives that $e \in C \iff W_e = W_{e_n}$ for some $n$. So every uniformly $\Sigma^0_3$ index set belongs to an r.e. class of r.e. sets. For the converse direction, assume that $C = \{ e : \exists n [W_e = W_{e_n}] \}$ where $e_0, e_1, \ldots$ is an effective sequence of indices. Now let $b_n = n$ for all $n$ and define

$$P(e, b) \iff \forall x \forall s \exists t [t > s \land W_{e,t}(x) = W_{e,b,t}(x)].$$

In other words, $P(e, b)$ is the $\Pi^0_3$ predicate which holds iff $W_e = W_{e_b}$. Now it follows that $e \in C \iff \exists b [P(e, b)]$ and $(e_0, 0), (e_1, 1), \ldots$ is the effective sequence which witnesses together with $P$ that $C$ is uniformly $\Sigma^0_3$.

**Remark.** It is known that there are $\Sigma^0_3$ index sets which do not belong to a uniformly r.e. family of sets. Here some examples:

- The set $\{ e : |W_e| = \infty \}$;
- The set $\{ e : W_e \subseteq A \}$ where $A$ is a non-r.e. $\Pi^0_3$ set;
- The set $\{ e : \exists a \in A [W_e = \{ a \}] \}$ where $A$ is a non-r.e. $\Sigma^0_3$ set.

If $C$ is a $\Sigma^0_3$ index set of a class containing all finite sets then this class is a uniformly r.e. family and $C$ is uniformly $\Sigma^0_3$ as can be seen as follows: Given a formula such that

$$e \in C \iff \exists b \forall c \exists d [\text{Cond}(e, b, c, d)]$$

where $\text{Cond}$ is a recursive predicate, let now

$$W_{f(e, b)} = \{ x \in W_e : \forall c \leq x \exists d [\text{Cond}(e, b, c, d)] \}.$$

The set $W_{f(e, b)}$ is equal to $W_e$ in the case that $e, b$ satisfy $\forall c \exists d [\text{Cond}(e, b, c, d)]$; otherwise $W_{f(e, b)}$ is finite. So every index $f(e, b)$ is in $C$ and the class of sets indexed by $C$ is equal to the family $\{ W_{f(e, b)} : e, b \in \mathbb{N} \}$.

12. **December 2010: Characteristic traceability, and weakly DNC sets**

Freer, Kjos-Hanssen, and Nies worked at the University of Hawai’i.
Definition 12.1. A trace \((T_n)_{n \in \mathbb{N}}\) is a sequence of finite sets. We say that a function \(h\) is a bound for the trace if \(\#T_n \leq h(n)\) for each \(n\). We say \((T_n)_{n \in \mathbb{N}}\) is a trace for function \(f\) if \(f(n) \in T_n\) for each \(n\).

Recall computable traceability [28, 8.2.15]: a trace \((T_n)_{n \in \mathbb{N}}\) is called computable if there is a computable function \(g\) such that \(T_n = D_g(n)\) (strong index) for each \(n\). We say that \(A\) is computably traceable if there is an order function \(h\) such that each function \(f \leq_T A\) has a computable trace with bound \(h\).

A \(\Delta^0_1\)-index for a set \(B \subseteq \mathbb{N}\) is given by a pair of c.e. indices, one for the set, and one for its complement \(\mathbb{N} - B\). This has also been called characteristic index [34], because it is equivalent to having an index for the characteristic function of \(B\).

Definition 12.2. A trace \((T_n)_{n \in \mathbb{N}}\) is called characteristic if there is a computable function \(g\) such that for each \(n\), \(g(n)\) is a characteristic index for \(T_n\).

The following example shows these traces are more general than computable traces.

Example 12.3. Let \(A\) be c.e. via the computable enumeration \((A_s)_{s \in \mathbb{N}}\). Then the modulus function 
\[ f_A(n) = \mu s. [A|_n = A_s|_n] \]
has a characteristic trace with bound \(n + 1\). It has no computable trace unless \(A\) is computable.

Proof. Let \(f_0(n) = n\) for each \(n\). For each stage \(s\), if we have \(y \in A_s - A_{s-1}\) where \(y\) is least, we redefine \(f_s(n) = s\) for all \(n\) such that \(y < n \leq s\). Let \(T_n = \{f_s(n) : s \in \mathbb{N}\}\). Then \((T_n)_{n \in \mathbb{N}}\) is a characteristic trace for \(f\) with bound \(n + 1\).

Any function with a computable trace is dominated by a computable function. This is not possible for the modulus function unless \(A\) is computable. \(\Box\)

12.1. Computably dominated sets and weakly d.n.c. sets. Recall that \(A\) is computably dominated (or of HIF degree) if
\[ \forall f \leq_T A \exists g \text{ computable } \forall n [f(n) < g(n)]. \]

In the definition following apparently weaker property, we only have an inequality.

Definition 12.4. We say \(A\) is weakly d.n.c. if
\[ \forall f \leq_T A \exists g \text{ computable } \forall n [f(n) \neq g(n)]. \]

However, it’s actually the same!

Proposition 12.5. Let \(A\) be weakly d.n.c. Then \(A\) is computably dominated.

Proof. If \(A\) is not computably dominated, there is an increasing function \(r \leq_T A\) such that for any computable \(h\), there are infinitely many \(n\) with \(h(n) \leq r(n)\).

We define inductively a function \(f \leq_T A\) which agrees somewhere with each (total) computable function \(\phi_e\). On input \(x\), with oracle \(A\) compute the least \(e \leq x\) such that
∀y ≤ x φ_{e,r}(x)(y) ↓, and ∀y < x φ_{e,r}(x)(y) ̸=} f(x).

Let f(x) = φ_{e,r}(x). (If there is no e, let f(x) = 0.)

If φ_e is total then the function h(x) = μt.∀y ≤ x φ_{e,t}(y) ↓ is total. So for infinitely many x, r(x) ≥ h(x). So eventually for some x we choose e and ensure f(x) = φ_e(x).

12.2. Characteristic traceability equals computable traceability.

**Definition 12.6.** Let us say A is characteristically traceable if the same definition as above holds for characteristic traces: there is an order function h such that each function f ≤_T A has a characteristic trace with bound h.

Clearly, each characteristically traceable set is weakly d.n.c.: given f ≤_T A, pick a characteristic trace (T_n)_{n∈N}, and let g be a computable function such that g(n) ∈ N − T_n for each n.

**Theorem 12.7.** Each characteristically traceable set A is already computably traceable.

**Proof.** A is computably dominated by Proposition 12.5. Also A is c.e. traceable. Hence A is computably traceable by a simple argument due to [13]. □

13. Questions

13.1. **August 2010: Distribution of a real obtained by tossing a biased coin.** By André.

Let 0 < δ < 1. Imagine we repeatedly toss a biased coin where the probability of tails (0) is δ, and thus of heads (1) is 1 − δ. Let r be the real in [0,1] with binary expansion given by this sequence of coin tosses. Describe the distribution function of r depending on δ, i.e., the function

f_δ(x) = P[r ≤ x].

For instance, f_{0.5}(x) = x.

Here is a plot, in a different scale, for the case δ = 1/3. Looks like it is nondifferentiable at the dyadic rationals –Bjørn

Interesting- something like the Cantor function then. Thanks! Andre

**References**


