# LOGIC BLOG 2010

Contents

## 1. Jan 2010: a downward $\mathrm{GL}_1$ set that is not weakly jump traceable

### 1.1. **Original result.**
Slaman, Greenberg, Kjos-Hanssen, Nies and others, worked at the University of Hawaii at Manoa.

**Definition 1.1.** *$A$ is* weakly jump traceable *(w.j.t.) if there is a function $f \le_{\mathrm{T}} \emptyset'$ that dominates all functions $\psi$ partial recursive in $A$, in the sense that $f(x) \ge \psi(x)$ for almost all $x$ such that $\psi(x)$ is defined.*

It would be sufficient to dominate $J^A$, and is also equivalent to having a finite c.e. trace for $J^A$. This property is closed downward under $\le_{\mathrm{T}}$, and implies $\mathrm{GL}_1$. It is equivalent to the property that $\emptyset'$ is not d.n.c. by $A$, namely, there is $g \le_{\mathrm{T}} \emptyset'$ such that $g(x) \neq J^A(x)$ for each $x$ [Miller and Ng REF? []]. While the original proof of this equivalence used randomness, a direct proof was given by Mingzhong Cai (Mar 2010).

To make $A$ downward $\mathrm{GL}_1$, one makes $A$ $\mathrm{GL}_1$ and also ensures that

$$\forall B \le_{\mathrm{T}} A[\, B \,\text{noncomputable} \ \Rightarrow A \le_{\mathrm{T}} B \oplus \emptyset'.\,].$$

### 1.2. **New results.**
This has been improved in May 2010. See Section 4.

### 1.3. **Comments.**

## 2. March 2010: Structures that are computable almost surely

This is a follow-up on a paper by Greenberg, Montalban and Slaman []. It was done by Kalimullin and Nies in Auckland when Kalimullin visited. Though the proof was found independently, it uses methods from there for a slightly stronger result.

For a countable structure $\mathcal{A}$ and a set $Y$, we write $\mathcal{A} \le_{\mathrm{T}} Y$ to denote that some presentation of $\mathcal{A}$ (viewed as an atomic diagram) is computable in $Y$. We denote by $\lambda$ the product measure on Cantor space $2^{\mathbb{N}}$.

**Definition 2.1.** *A countable structures is called* computable almost surely *(or almost computable) if*

$$\lambda\{Y \colon \mathcal{A} \le_{\mathrm{T}} Y\} = 1.$$

### 2.1. **A structure that is computable almost surely, is computable in every $\Pi^1_1$ random.**

**Theorem 2.2.** *Let $\mathcal{A}$ be computable almost surely. Then $\mathcal{A}$ is computable in every $\Pi^1_1$ random set.*

Note that by Gandy basis theorem there is a $\Pi^1_1$ random $Y \le_{\mathrm{T}} \mathcal{O}$ such that also $\mathcal{O}^Y \le_h \mathcal{O}$. In this way we reobtain the result of Greenberg et al.

*Proof.* Via some pre-agreed encoding, we view a subset of $\mathbb{N}$ as a diagram of a structure in the language of $\mathcal{A}$. For an index $i \in \mathbb{N}$ for a Turing functional, and a rational $p < 1$, define a $\Sigma^1_1$ class uniformly in $i, p$ by

$$\mathcal{S}_{i,p} = \{Y\colon\ p < \lambda\{Z\colon\ \Phi_i^Y \cong \Phi_i^Z\}.$$

Clearly if $\mathcal{S}_{i,1/2} \neq \emptyset$ then $\Phi_i^Y$ is a unique structure up to isomorphism for each member $Y$.

Note that the relation $\{\langle i, p, Y\rangle\colon\ Y \in \mathcal{S}_{i,p}\}$ is $\Sigma^1_1$; see the uniform Measure Lemma [28, 9.1.1], which paraphrases [32, 1.11.IV]. Now define a $\Sigma^1_1$ equivalence relation on numbers by

$$e \sim i \leftrightarrow \exists Y \in \mathcal{S}_{e,1/2} \exists Z \in \mathcal{S}_{i,1/2}\, [\Phi_e^Y \cong \Phi_i^Z].$$

By hypothesis there is an index $k$ such that $\lambda\{Y\colon\ \Phi_k^Y \text{ presents } \mathcal{A}\}$ is positive, so by the Lebesgue density theorem in the simple version [28, 1.9.4] we may assume that this measure is greater than 1/2. Now let

$$\mathcal{C} = \bigcup\{\mathcal{S}_{i,p}\colon\ 1/2 \leq p < 1, i \in \mathbb{N}, i \sim k\}.$$

Each $Y \in \mathcal{C}$ computes a representation of $\mathcal{A}$. The class $\mathcal{C}$ is $\Sigma^1_1$, and $\mathcal{C}$ is conull, again by the Lebesgue density theorem. Thus it contains each $\Pi^1_1$-random set. □

This proof seems to work under the weaker hypothesis that $\lambda\{Y\colon\ \mathcal{A} \leq_h Y\} = 1$.

2.2. **Related questions.** Montalban and Nies asked some questions at the end of their survey paper [22]. A *Borel structure* is one that can be presented on a standard Polish space in such a way that the atomic diagram (including equality) is Borel.

Recall that a relation is Borel iff it is $\Delta^1_1(Y)$ for some $Y \subseteq \omega$. The *spectrum* of a Borel structure $\mathcal{A}$ is the class of sets $Y \subseteq \omega$ such that some presentation of $\mathcal{A}$ is $\Delta^1_1(Y)$. What can we say about possible spectra? Do the non-hyperarithmetical sets form a spectrum?

## 3. March 2010: Solovay Functions and $K$-triviality

This work was started by Bienvenu, Merkle and Nies. Bienvenu and Nies met at Paris 7. Then they all met at Uni Heidelberg. The final paper [] is quite different from what follows. It extends the results to weak Solovay functions (i.e., computably approximable from above), and argues mostly in terms of prefix-free complexity $K$. Further, it contains a result relating Solovay functions to the $C$-characterization of ML-randomness due to Miller and Yu.

3.1. **Introduction.** Recall that a set $A \subseteq \mathbb{N}$ is $K$-trivial [3] if there is $b$ such that

$$K(A\restriction_n) \leq K(n) + b$$

for each $n$. This class was studied in [6, 27].

For any function $g$ let

$$\mathcal{C}_g = \{A\colon\ \forall n K(A\restriction_n) \leq^+ g(n)\}.$$

Thus $\mathcal{C}_K$ is the class of $K$-trivials, and $\forall n\, K(n) \leq^+ g(n)$ implies $\mathcal{C}_K \subseteq \mathcal{C}_g$.

**Definition 3.1** ([1]). *We say that a computable function* $g\colon \mathbb{N} \to \mathbb{N}$ *is a* Solovay function *if* $\forall n\, K(n) \leq^+ g(n)$ *and* $\exists^\infty n\, K(n) =^+ g(n)$.

If $g$ is computable, then the condition $\forall n\, K(n) \leq^+ g(n)$ is equivalent to $\sum_n 2^{-g(n)} < \infty$ . Bienvenu and Downey [1] showed that $g$ is a Solovay function if and only if $\sum_n 2^{-g(n)}$ is ML-random.

Our main result is that for every computable function $g$,

$(\star)$ $\quad g$ is a Solovay function $\Longleftrightarrow$ $\mathcal{C}_g$ is the class of $K$-trivials.

The result follows from two somewhat stronger theorems, corresponding to the implications from left to right, and its converse.

3.2. **Every Solovay function characterizes the $K$-trivials.** We show that for every Solovay function $g$, the class $\mathcal{C}_g$ coincides with the $K$-trivials.

**Theorem 3.2.** *Suppose $g$ is a Solovay function. If $\forall n\, K(A\restriction_n) \leq^+ g(n)$ then $A$ is $K$-trivial.*

*Proof idea.* After adding a natural number to $g$ if necessary, we may assume that $\forall n\, K(A\restriction_n) \leq g(n)$. By [1], the left-c.e. real $\alpha = \sum_n 2^{-g(n)}$ is ML-random. Hence, by [**?**] $\alpha$ is complete for Solovay reducibility $\leq_S$. Let

$$\alpha_s = \textstyle\sum_{n=0}^{s} 2^{-g(n)}.$$

Roughly speaking, the Solovay completeness of $\alpha$ means the following: if we choose a non-negative rational $q$ at a stage $s$, we can determine a stage $t > s$ such that $\alpha_t - \alpha_s \geq Cq$ for some constant $C > 0$ known in advance. However, we have to make sure that the sum of all rationals we choose does not exceed 1.

To show $A$ is $K$-trivial we build a bounded request set (aka KC set) $W$. When we see a description $\mathbb{U}(\sigma) = v$ at stage $s$, we want to ensure $W$ contains a request $\langle |\sigma| + O(1), A\restriction_v\rangle$. To ensure the weight of $W$ is at most 1, we will account the weight of such a request against the weight of $\mathbb{U}$-descriptions of $A\restriction_n$ for a whole interval $I_\sigma$ of numbers $n$. For $q = 2^{-|\sigma|-O(1)}$ we find a stage $t > s$ as above and let $I_\sigma = (s, t]$. Once we see $\mathbb{U}$-descriptions of $A\restriction_n$ of length at most $g(n)$ for each $n \in I_\sigma$, we can put the request into $W$, because the total weight of these descriptions is at least $q$. We arrange that the intervals associated with different $\sigma$ are disjoint. This implies that $W$ is a bounded request set.

*Details.* We first describe the mechanism to increase the approximation to $\alpha$. We view the $d$-th partial computable function $\phi_d$ as a partial map from $\mathbb{N}$ to $\mathbb{Q}_2 \cap [0, 1)$. Let $\beta_{d,s} = \max \mathrm{range}\,(\phi_{d,s})$, and let $\beta_d = \lim_s \beta_{d,s}$. Then $(\beta_d)_{d\in\mathbb{N}}$ is an effective listing of the left-c.e. reals in $[0, 1]$. Let $\gamma = \sum_d 2^{-d}\beta_d$. Since $\gamma \leq_S \alpha$, by the definition there is a partial computable $\psi$ defined on $[0, \alpha) \cap \mathbb{Q}_2$ and a constant $k$ such that for each binary rational $q \in [0, \alpha)$ we have $\psi(q)\downarrow$ and

$$0 < \gamma - \psi(q) < 2^{k-1}(\alpha - q).$$

We build a left-c.e. real $\eta \in [0,1)$. The construction has a parameter $d \in \mathbb{N}$. We think of $\beta_d$ as being $\eta$. By the recursion theorem, in the verification we can choose such a $d$. (However, we must ensure that $\eta \in [0,1)$ for any parameter $d$.)

Let $\gamma_s = \sum_d 2^{-d}\beta_{d,s}$.

*Construction of a left-c.e. real $\eta$ and intervals $I_\sigma$ for all strings $\sigma \in \mathrm{dom}(\mathbb{U})$.* The construction has a parameter $d$. It takes place in stages $u$. If ever we see that $\eta_u < \beta_{d,u}$ we immediately stop the construction. Thus $\eta = \eta_u$ and $d$ cannot be a fixed point.

Let $s$ be the current stage.

(1) WAIT for $r \geq s$ such that $\gamma_r \geq \psi(\alpha_s)$.
(2) Let $\sigma$ be least such that $\mathbb{U}_s(\sigma)\downarrow$ but $I_\sigma$ is not defined yet. Define $\eta_{r+1} = \eta_r + 2^{-|\sigma|}$. WAIT for $t \geq r$ such that $\alpha_t - \alpha_s \geq 2^{-d-k-|\sigma|}$ (the constant $k$ associated with $\gamma \leq_S \alpha$ was defined above).
(3) Declare $I_\sigma = (s,t]$. GOTO (1).

*Verification.* Clearly $\eta \leq \Omega < 1$. Thus by the recursion theorem we can choose $d$ such that $\eta = \beta_d$. By definition the wait in (1) always terminates.

**Claim 1.** *The wait in (2) always terminates.*
Since $\gamma_r \geq \psi(\alpha_s)$, we have

$$2^{k-1}(\alpha - \alpha_s) > \gamma - \psi(\alpha_s) \geq \gamma - \gamma_r \geq 2^{-d}(\beta_d - \beta_{d,r}).$$

Since $\eta_r \geq \beta_{d,r}$ and we add $2^{-|\sigma|}$ to $\eta$ at stage $r$, there is a stage $t' > r$ such that $\beta_{d,t'} - \beta_{d,r} \geq 2^{-|\sigma|-1}$. Thus $\alpha_t - \alpha_s \geq 2^{-d-k-|\sigma|}$ for some $t > r$.

**Claim 2.** *If $\sigma \in \mathrm{dom}(\mathbb{U})$ then $I_\sigma$ is defined.*
Suppose this holds inductively for all $\tau < \sigma$. So we can pick stage $s_0$ such that no such string is processed at a stage $> s_0$. Suppose $s \geq s_0$, $\mathbb{U}_s(\sigma)$ is defined and by Claim 1 the construction goes back to (1) at stage $s$. Then we can choose $\sigma$ if $I_\sigma$ is still undefined.

**Claim 3.** *$A$ is $K$-trivial.*

We define a bounded request set $W$. For each string $\sigma$, if $n = \mathbb{U}(\sigma)$ is defined, wait for $I_\sigma = (s,t]$ to become defined, and do the following: for each string $x$ of length $t$, at a stage $p$ such that $\forall i \in I_\sigma\, K_p(x\restriction_i) \leq g_t(i)$, put the request $\langle |\sigma| + d + k, x\restriction_n\rangle$ into $W$.

To see that $W$ is a bounded request set, firstly note that the total weight of $\mathbb{U}$ descriptions of $x\restriction_i$, $i \in I_\sigma$, is at least $\sum_{i\in I_\sigma} 2^{-g(i)} \geq 2^{-d-k-|\sigma|}$. Secondly, if $x'$ is a further string of length $t$ such that we put a request associated with $\mathbb{U}(\sigma) = n$, then $x'\restriction_n \neq x\restriction_n$. Note that $\mathbb{U}_s(\sigma) = n$ implies $s > n$. So the descriptions of $x'\restriction_i$, $i \in I_\sigma$ are different from the ones for $x\restriction_i$. Finally, if $\sigma \neq \tau$ then $I_\sigma$ is disjoint from $I_\tau$, so again the descriptions are different.

If $\mathbb{U}(\sigma) = n$ then $I_\sigma = (s,t]$ is defined by Claim 2. For $x = A\restriction_t$ by the hypothesis on $A$ we put a request $\langle |\sigma| + d + k, A\restriction_n\rangle$ into $W$. If $\sigma$ is shortest such that $\mathbb{U}(\sigma) = n$, this shows $K(A\restriction_n) \leq^+ |\sigma| = K(n)$. □

### 3.3. The $K$-trivials characterize the Solovay functions.

**Theorem 3.3.** *Let $g$ be computable function such that $\forall n\, K(n) \leq^+ g(n)$. Suppose that $g$ is not a Solovay function. Then $\mathcal{C}_g$ is uncountable.*

*Proof.* By the hypothesis $\alpha = \sum_i 2^{-g(i)}$ is not ML-random. We will find a computable time bound $t$ such that $g(n) - K_{t(n)}(n) \to \infty$. Thereafter we use a result of Figueira, Nies and Stephan (see [28, Thm 5.2.25]). They show that for every function $p$ that tends to infinity and is computably approximable from above, there are uncountably many sets $G$ such that $\forall n\, K(G\upharpoonright_n) \le^+ K(n) + p(n)$ (in fact they show this for the function $p(K(n))$). Now let $p(n) = g(n) - K_{t(n)}(n)$. Every set such that $\forall n\, K(G\upharpoonright_n) \le^+ K(n) + p(n)$ is in $\mathcal{C}_g$.

The time bound $t$ is obtained through a run time analysis of the proof in [1] that $\alpha = \sum_i 2^{-g(i)}$ not ML-random implies $g(n) - K(n) \to \infty$. First we paraphrase the original argument. Effectively in a string $\tau$ we build a bounded request set $L_\tau$. We wait for $\mathbb{U}(\tau)\downarrow = x$, and let $r = |x|$. For each $n$ such that $0.x \le \sum_{i\le n} 2^{-g(i)} < 0.x + 2^{-r}$, we put the request $\langle g(n) - r, n\rangle$ into $L_\tau$. Clearly $L_\tau$ is a bounded request set effectively in $\tau$. So we can effectively in $\tau$ obtain a prefix-free machine $M_\tau$ for $L_\tau$ according to the machine existence theorem.

Let $M$ be the prefix-free machine given by $M(\tau\sigma) \simeq M_\tau(\sigma)$. Given $b$ choose $r$ such that $K(\alpha\upharpoonright_r) < r - b$. Thus we can pick $\tau$ of length $< r - b$ such that $\mathbb{U}(\tau) = \alpha\upharpoonright_r = x$. Then for all $n$ such that $0.x \le \sum_{i\le n} 2^{-g(i)}$, there is $\sigma$ such that $M(\tau\sigma) = n$ and $|\sigma| = g(n) - r$. Hence $K(n) \le^+ r - b + g(n) - r = g(n) - b$.

For large $n$ the time to compute $\mathbb{U}(\tau)$ is negligible, so the time to verify $M(\tau\sigma) = n$ is dominated by the time to compute $\sum_{i\le n} 2^{-g(i)}$ and the delay introduced by the machine existence theorem, and for $\mathbb{U}$ to simluate $M$. Thus we have $K_{t(n)}(n) \le^+ g(n) - b$ for some computable $t$. □

## 4. May 2010 : Cooper's jump inversion and weak jump traceability

Lempp, Miller, Ng and Yu worked at the University of Wisconsin-Madison.

We give a (hopefully) more comprehensible proof of Cooper's jump inversion theorem, and use the same ideas to construct a minimal $\mathrm{GL}_1$ degree which is not weakly jump traceable. The question of whether $\mathrm{BGL}_1$ was equal to weak jump traceability was first raised in Ng [23]. We can extend Cooper's jump inversion to get the analogous jump inversion result in the tt-degrees, thus obtaining a superhigh set of minimal Turing degree.

Let us turn to Cooper's jump inversion. Fix $C \ge_T \emptyset'$, and we need to construct a set $A$ such that $A' \le_T C \le_T A \oplus \emptyset'$. We force with partial (modified) splitting trees, and ensure that both $C$ and $A \oplus \emptyset'$ can recover the construction. Since we do not have $\emptyset''$ for recovering the construction, the splitting trees clearly have to be partial, as in Sack's version. As we're forcing longer initial segments of $A$ we may change our mind on how the $e^{th}$ minimality requirement $\mathcal{R}_e$ is to be met. If we change our mind on some $\mathcal{R}_e$ then it is because we have reached a dead end in an $i$-splitting tree for some $i \le e$, which is finite injury. Unfortunately this is not immediately compatible with forcing the jump, since we have to *decide* the jump $A'(e)$ with oracle $\emptyset'$ (together with $A$ or $C$). If we are not able to force the jump on a current splitting tree, say $T_e$ for the $e$-splitting tree, this is ok as long as we make $A$ stay within $T_e$. Unfortunately $T_e$ may not be a total tree, and we may have $A$ leave $T_e$ through a dead branch. In this case we may be able to have $e \in A'$, which is bad for deciding the jump.

The solution is to modify the partial computable splitting trees so that this does not happen; if we are unable to force the jump within $T_e$, then we ensure that we will *never* be able to force the jump even if we should leave $T_e$ through a dead end. To construct $T_0$, we define $T_0(\emptyset) = \emptyset$. Assume inductively that $T_0(\sigma)$ has been defined. We search for either

(i) the first pair of 0-splits $(\tau_0, \tau_1)$ extending $T_0(\sigma)$, or
(ii) for the first $j$ found such that $J^\tau(j)\downarrow$ for $\tau \supset T_0(\sigma)$.

If neither is ever found, then $T_0(\sigma)$ is a dead end, i.e. $T_0(\sigma 0)\uparrow, T_0(\sigma 1)\uparrow$. Otherwise if (i) is found first then we extend $T_0(\sigma i) = \tau_i$. If (ii) is found first with $j_0$ we extend $T_0(\sigma) = \tau$. In this case we repeat by searching for (i) or (ii) above the new $T_0(\sigma)$, except that we now limit the search in (ii) to all $j < j_0$.

That is, $T_0$ will search for both 0-splits, and jumps. If it finds a place it can force the jump before it finds a split, it grabs it immediately. Subsequently it will search for splits or jump computations on smaller inputs, until it finds the next split, where the counter $j_0$ is reset. $T_{e+1}$ is defined similarly, except it restricts its search to within the convergent part of $T_e$. To construct $A$, we force the jump at even stages, and code $C$ at odd stages. At stage 0 we start at $A_0 = \emptyset$, and ask if we can force $J(0)$ within $T_0$. Since $T_0$ is partial computable, the existence is a $\Sigma^0_1$ question. If the answer is yes, we extend $A$ to it while staying within $T_0$. If the answer is no, we do nothing to $A$. Note that in this case we cannot have $J^A(0)\downarrow$ because otherwise the use has to extend a dead end of $T_0$, say $A \supset T_0(\sigma)$, but by the definition of $T_0$ we would have extended $T_0(\sigma)$ to this segment of $A$. At stage 1 we code $C(0)$ in the usual way. At stage 2 if we are still within $T_0$, then we ask if $J(1)$ can be forced in $T_1$. If yes, we grab it, otherwise we claim that $J^A(1)\uparrow$: If $J^\tau(1)\downarrow$ for some $\tau \supset A$, then $\tau$ cannot be on $T_1$. It cannot be on $T_0$ and extend a dead end of $T_1$, because otherwise we would have extended $T_1$ to include $\tau$, since inductively we assume that we have forced $J^A(0)$. Finally it cannot extend a dead end of $T_0$ because otherwise we would have extended $T_0$ to include $\tau$ because we assume that we have forced $J^A(0)$. The rest of the construction proceeds similarly.

It is clear that both $C$ and $A \oplus \emptyset'$ can recover the construction, giving us our desired result. To make $A$ not wjt we force the jump at even stages, and at odd stages we satisfy the $e^{th}$ non-wjt requirement. Given any partial computable tree $T$ and any $\sigma$ such that $T(\sigma)\downarrow$, there is (effectively) an index $i$ such that $J^{T(\sigma 0^k 1)}(i)\downarrow= k$. Now at odd stages, assume we have $\tau \subset A$, and we are committed to staying on $T_0, \cdots, T_k$. Assume for simplicity that we only have to stay on $T_0$, and $\tau = T_0(\sigma)$. We ask $\emptyset''$ if some $f$ is $\Delta^0_2$. If the answer is no, we extend $A$ trivially to $T(\sigma 1)$. Otherwise we can find a large enough $k$ such that $J^{T_0(\sigma 0^k 1)}(i) > f(i)$ and make $A$ extend $T_0(\sigma 0^k 1)$. If on the other hand $T_0(\sigma 0^k 1)\uparrow$ we know that $T_0(\sigma 0^j)$ is a dead end for some least $j \le k$, and we can then make $A$ extend the dead end $T_0(\sigma 0^j)$, and try to meet the same non-wjt requirement with a different $i$. Since we no longer have to care about $T_0$, this second try for meeting non-wjt will succeed. Finally we see that $A \oplus \emptyset'$ can recover the construction. It will be able to figure out what we did during the odd stages: the number $k$ such that $A$ extend $T_0(\sigma 0^k 1)$, or the number $j$ where $A$ left $T_0$ through a dead end $T_0(\sigma 0^j)$.

Finally it is easy(?) to see that we can adapt the above proof (similar to Mohrherr) and make the jump inversion hold for tt-degrees.

## 5. May 2010: The collection of weakly-2-random reals is not $\mathbf{\Sigma}^0_3$

Joint work of Lempp, Miller, Ng, Turetsky and Yu.

**Lemma 5.1.** *For ever recursive tree $T$, there is a generalized Martin-Lof test $\{V_n\}_{n\in\omega}$ so that for any $\sigma$, if $[\sigma]\cap[T]$ is not empty, then $[\sigma]\cap[T]\cap\bigcap_n V_n$ is not empty.*

*Proof.* To be added. □

Now suppose that the collection of weakly 2-random reals $A$ is a $\mathbf{\Sigma}^0_3$-set. So there is a sequence open sets $\{G_{i,j}\}_{i,j\in\omega}$ so that $A=\bigcup_i\bigcap_j\bigcup_k G_{i,j}$. Let $T$ be a recursive tree such that $[T]=\{x\mid \forall n(x\upharpoonright n\in T)\}$ is not empty and only contains 1-random reals.

**Lemma 5.2.** *There is a finite $\sigma\in T$ such that $[\sigma]\cap[T]\neq\emptyset$ but $[\sigma]\cap[T]\cap(\bigcap_j G_{0,j})=\emptyset$.*

*Proof.* Suppose not. Then let $\{V_n\}_{n\in\omega}$ be a generalized Martin-Lof test as in Lemma 5.1. Define $\sigma_0=\emptyset$. For any $i\geq 0$, By Lemma 5.1, let $\sigma_{i+1}\succ\sigma_i$ such that $\sigma_{i+1}\cap[T]\neq\emptyset$ and $[\sigma_{i+1}]\cap[T]\subseteq V_i$. Then $[\sigma_{i+1}]\cap[T]\cap(\bigcap_j G_{0,j})\neq\emptyset$. So, without loss of generality, we may assume that $[\sigma_{i+1}]\cap[T]\subseteq G_{0,i}$ since $G_{0,i}$ is an open set. Let $x=\bigcup_{i\in\omega}\sigma_i$. Then $x\in\bigcap_{n\in\omega}V_n$ and so $x$ is not weakly 2-random. But $x\in\bigcap_{j\in\omega}G_{0,j}$, a contradiction. □

Let $\{\bigcap_{n\in\omega}U^j_n\}_{j\in\omega}$ be an enumeration of all generalized Martin-Lof tests. By Lemma 5.2, let $\sigma\in T$ such that $[\sigma]\cap[T]\neq\emptyset$ but $[\sigma]\cap[T]\cap(\bigcap_j G_{0,j})=\emptyset$. Since $[T]$ only contains 1-random reals, $\mu([T]\cap[\sigma])>0$. Let $n$ be large enough such that $\mu(U^0_n)<\frac{1}{2}\mu([T]\cap[\sigma])$. Then there is a recursive tree $T_0\subseteq[T]\cap[\sigma]$ so that $\mu([T_0]>0$ but $[T_0]\cap U^0_n=\emptyset$. Let $\sigma_0=\sigma$. By induction, we have $\sigma_{i+1}\succ\sigma_i$, $[\sigma_{i+1}]\cap[T_i]\cap(\bigcap_j G_{i,j})=\emptyset$, $[T_{i+1}]\subseteq[\sigma_{i+1}]\cap[T_i]$ and $\mu(T_{i+1})>0$ such that there is some large $n$ for which $[T_{i+1}]\cap U^{i+1}_n=\emptyset$.

Let $x=\bigcap_{i\in\omega}\sigma_i$. By the construction, $x$ is weakly 2-random but $x\notin\bigcap_j G_{i,j}$ for any $i$, a contradiction.

## 6. May 2010: There is a perfect set in which any two reals are $LR$ comparable

Added by Liang Yu.

It was shown, in [19], if $x$ and $y$ are random and $x\leq_K y$, then $y\leq_{LR} x$. So it is sufficient to prove the result for $K$-degrees among random reals.

The statement can be formalized as

$$\exists T\forall x\forall y(T \text{ is a perfect tree}$$
$$\wedge(x\in[T]\implies x\text{ is random })\wedge(x\in[T]\wedge y\in[T]\implies(x\leq_K y\vee y\leq_K x))),$$

where $[T] = \{x \mid \forall n(x\restriction_n \in T)\}$.

So the statement is $\Sigma^1_2$.

Starting with any transitive model $M \models ZFC$ (say $L$), for any cardinal $\kappa$ in $M$, we may build a c.c.c. forcing $P$ so that $M[G]$ models Martin's axiom and $2^{\aleph_0} > \kappa^+$ for any generic set $G$. By Miller and Yu ([20, Thm 7.4]), there is a chain of $K$-degrees among random reals of size $\kappa^+$. Then the set $A = \{(x, y) \mid x, y$ are 1-random and $x \leq_K y \vee y \leq_K x\}$ is a Borel set which contains a $\kappa^+$-square (i.e. a set $B \subseteq 2^\omega$ of size $\kappa^+$ such that $B \times B \subseteq A$). If $\kappa > (\beth_{\omega_1})^M$, then by Shelah's result $(*)'_1$ on page 2 of [33],

$$\kappa^+ > \lambda_{\omega_1}(\aleph_0),$$

where $\lambda_{\omega_1}(\aleph_0)$ is the Hanf number for models of sentences in the infinitary language $L_{\omega_1,\omega}$ (i.e. the least cardinal $\kappa$ such that for any sentence $\phi \in L_{\omega_1,\omega}$ with a model cardinality $\kappa$, $\phi$ has models in any infinite cardinalities). Then by Thm 1.15 in the same Shelah paper, there must be a perfect set $C$ such that $C \times C \subseteq A$. So there is a perfect chain in the $K$-degrees.

By $\Sigma^1_2$-absoluteness, the result follows.

## 7. June 2010: Classes related to the $K$-trivials and the strongly jump traceables

By Andre.

7.1. **Cupping $K$-trivials by random sets.** We say that $A$ is *ML-non-cuppable* if $\emptyset' \leq_T A \oplus Z$ implies $\emptyset' \leq_T Z$ for each ML-random $Z$. It is a persistent open question [18] whether each $K$-trivial $A$ is ML-noncuppable. Note that any possible ML-random cupping partner of a $K$-trivial set $A$ must be LR-complete (work of Hirschfeldt and Nies). We can use this to rule out cupping partners with certain randomness properties stronger that ML-randomness.

The following proposition, due to [7], shows that such a cupping partner cannot be weakly Demuth random at level $O(h(m)2^m)$ for any order function $h$.

**Proposition 7.1.** *Suppose $Z$ is an $O(h(m)2^m)$-weak Demuth random set for some order function $h$. Then $Z$ is not $\omega$-c.e. tracing, and hence not LR-complete.*

The next proposition shows that they cannot be Demuth random at level $O(2^m)$ (note however that the latter notion is somewhat arbitrary because of the $2^{-m}$ bounds on the measure of the $m$-th component of a test). An array computable set is not LR-complete by Barmpalias []. Then, by the proof of [28, 3.6.26] we have:

**Proposition 7.2.** *There is a Demuth test $(S_m)_{m\in\mathbb{N}}$ as follows:*

(a) *The version $S_m$ changes at most $2^m$ times.*
(b) *If $Z$ passes the test (i.e. $Z \notin S_m$ for a.e. $m$), then $Z$ is array computable.*

**Corollary 7.3.** *No $K$-trivial set is Demuth cuppable.*

On the other hand, there is a Turing incomplete c.e. set $A$ that cups with a 2-random (and hence Demuth random). Simply make $A$ LR-complete, so that $\Omega^A$ is 2-random. Use that $A \oplus \Omega^A \equiv_T A'$. Lots of obvious questions arise here. For instance, how close to computable can a Demuth cuppable c.e. set be? Ng has announced it can be superlow.

7.2. **Box classes and diamond classes.** For a class $\mathcal{C} \subseteq 2^\omega$ let

$$\mathcal{C}^\Box = \{A \colon \forall Y \in \mathcal{C} \cap \mathsf{MLR}[A \leq_{\mathrm{T}} Y]\}.$$

Let $\mathcal{C}^\Diamond$ denote the collection of c.e. sets in $\mathcal{C}^\Box$.

It is known [8] that ($\omega$-c.e.)$^\Box$ is contained in the strongly jump traceables and ($\omega$-c.e.)$^\Diamond$ coincides with the c.e. strongly jump traceables.

The following definition is taken from [8]. We let $R = (\omega, <_R)$ be a computable well-order.

**Definition 7.4.** *An* $R$-approximation *is a computable function*

$$g = \langle g_0, g_1 \rangle : \omega \times \omega \to \omega \times \omega$$

*such that for each $x$ and each $s > 0$,*

$$g(x,s) \neq g(x,s-1) \to g_1(x,s) <_R g_1(x,s-1). \quad (1)$$

*In this case, $g_0$ is a computable approximation of a total $\Delta^0_2$ function $f$. We say that $g$ is an $R$-approximation of $f$, and that $f$ is $R$-c.e..*

$Y$ is $\omega^n$-c.e. if its characteristic function is $R$-c.e. for $R$ the canonical presentation of $\omega^n$. It is not hard to check that we could as well take the function $n \to Y\restriction_n$.

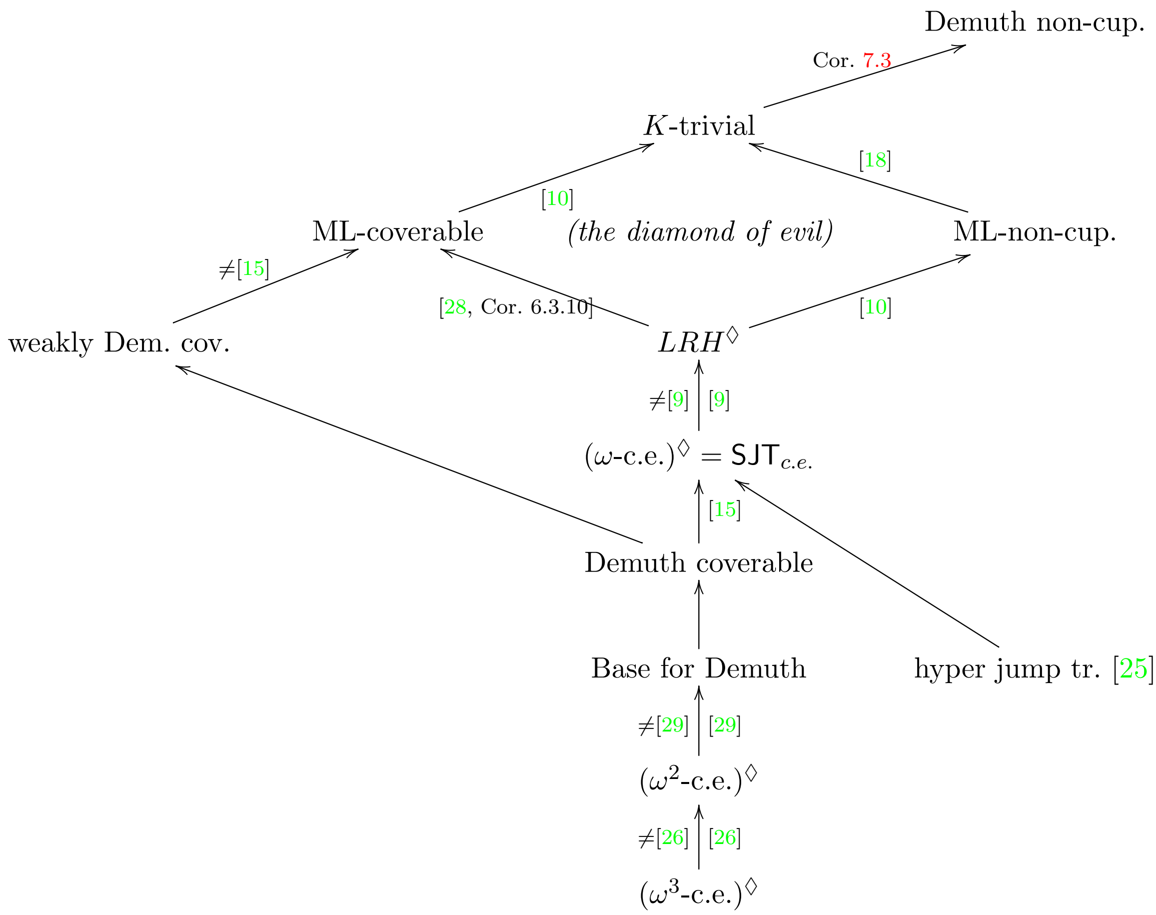
Diamondstone, Hirschfeldt, and Nies showed that ($\omega^2$-c.e.)$^\Diamond$ is a proper subclass of ($\omega$-c.e.)$^\Diamond$. They gave a direct proof. The following, alternative proof pieces together results from the literature.

**Theorem 7.5.** *There is a c.e. set $A$ below all $\omega$-c.e. random sets, but not below all $\omega^2$-c.e. random sets.*

*Proof.*
(1) [28, Theorem 3.6.25] builds a Demuth random $\Delta^0_2$ set $Y$. It is not hard to see that the constructed set $Y$ is in fact $\omega^2$-c.e. For detail see [8].
(2) Kučera and Nies [15] show that $A \leq_{\mathrm{T}} Y$ for c.e. set $A$ and Demuth random $Y$ imply that $A$ is strongly jump traceable.
(3) Ng [24] proved that the class $\mathsf{SJT}_{c.e.}$ of strongly jump traceable c.e. sets has a $\Pi^0_4$ complete index set.

Now let $Y$ be as in (1). Let $\mathcal{I}(Y)$ be the class of c.e. sets Turing below $Y$. Since $Y$ is low, the class $\mathcal{I}(Y)$ is $\Sigma^0_3$. Thus, since $\mathsf{SJT}_{c.e.}$ coincides with ($\omega$-c.e.)$^\Diamond$, we have ($\omega^2$-c.e.)$^\Diamond \subseteq \mathcal{I}(Y) \subset$ ($\omega$-c.e.)$^\Diamond$. ∎

7.3. **Diagram of classes of c.e. sets, mostly contained in the $K$-trivials.** Here are some classes, all within the c.e. sets. A separation is indicated by the label $\neq$ on the implication arrow. "ML-coverable" and "ML non-cuppable" means covered/not cupped by an *incomplete* ML-random.




*Added Dec 2011:* Day and Miller 2011, relying on a result on non-density of Martin-Loef random sets in Pi-01 classes by Bienvenu Hoelzl, Miller and Nies in STACS 2012, have shown that a set is K-trivial if and only if it cannot be cupped above the halting problem with an incomplete Martin-Loef random set.

Diamondstone, Greenberg, and Turetsky have shown that for any set $A$, c.e. or not, $A$ is below each $\omega$-c.e. ML-random iff $A$ is s.j.t.

## 8. July 2010: Randomness extraction, Mises-Wald-Church stochastic sets, Medvedev reducibility, and effective packing dimension

By Bjørn. We show that there is no Turing reduction procedure that does substantially better than a majority function in extracting randomness from a set of integers that lies a small asymptotic Hamming distance away from a random set [14]. A consequence of the proof is as follows. Let MWC denote the class of Mises-Wald-Church stochastic sets, let $\mathrm{DIM}_p$ denote the class

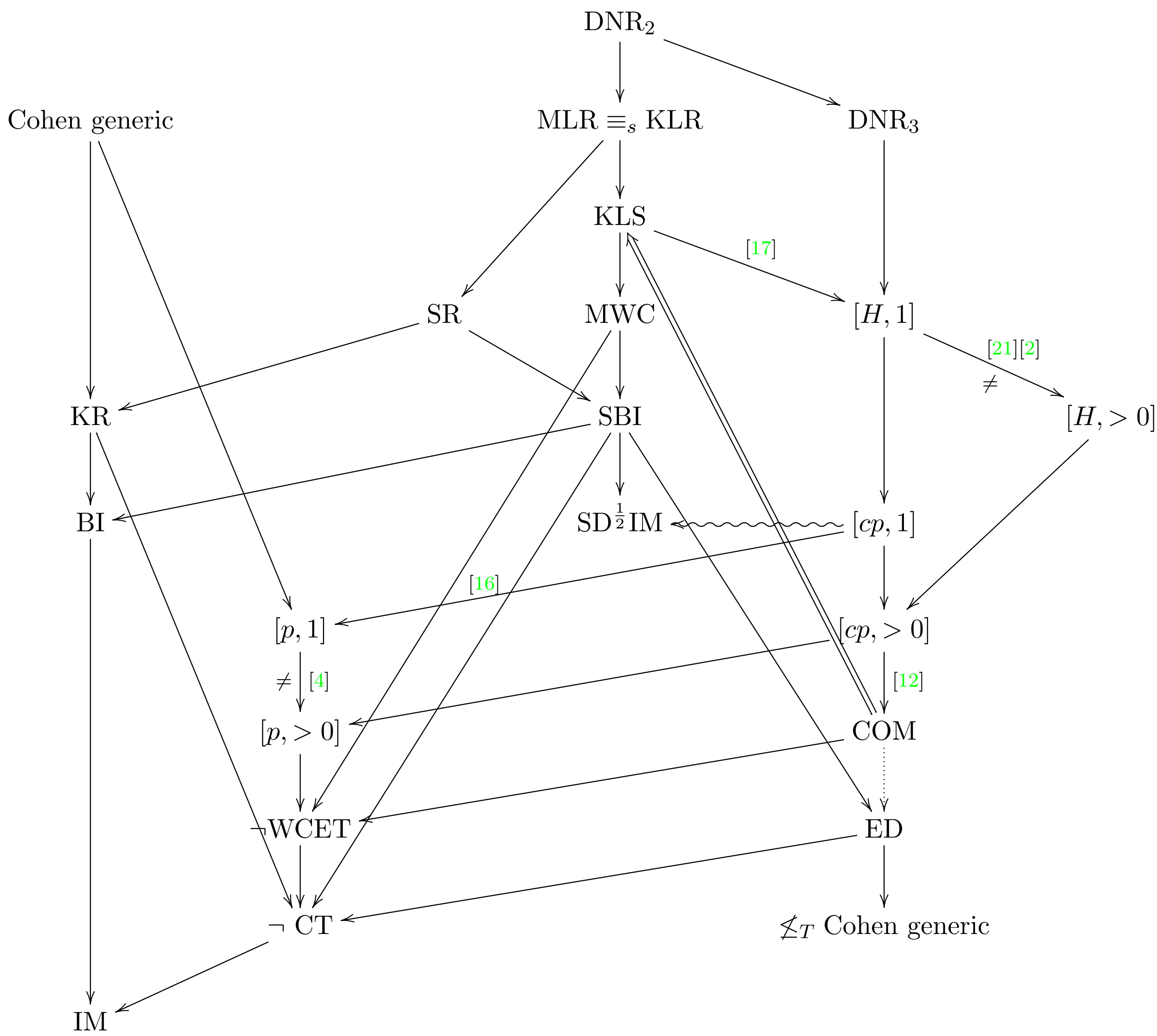



Figure 1. Some Medvedev degrees. The wavy arrow indicates a non-implication proved in [14]. For definitions see Figures 2 and 3.

of sets of effective packing dimension 1, and let COM denote the class of complex sets (in the sense of Kjos-Hanssen, Merkle, and Stephan). Let $\leq_s$ and $\leq_w$ denote strong (i.e. Medvedev) and weak (i.e. Muchnik) reducibility of mass problems.

**Theorem 8.1.** *$MWC \not\leq_s DIM_p \cap COM$.*

**Question 8.2.** *Is $MWC \leq_w DIM_p \cap COM$?*

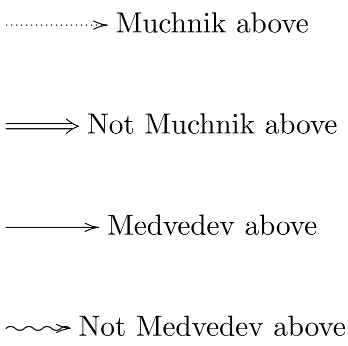



FIGURE 2. Meaning of arrows.

| Abbreviation | Unabbreviation | Definition |
|---|---|---|
| DNR | Diagonally non-recursive function in $\omega^\omega$ | |
| $\mathrm{DNR}_n$ | Diagonally non-recursive function in $n^\omega$ | |
| MLR | Martin-Löf random | |
| KLR | Kolmogorov-Loveland random | |
| SR | Schnorr random | |
| KR | Kurtz random (weakly 1-random) | |
| MWC | Mises-Wald-Church stochastic | 8.3 |
| KLS | Kolmogorov-Loveland stochastic | 8.4 |
| SBI | Stochastically bi-immune | 8.6 |
| $\mathrm{SD}^{1/2}\mathrm{IM}$ | stochastically dominating for $\mathfrak{p} = 1/2$ and immune | 8.7 |
| BI | bi-immune | 8.5 |
| IM | immune ($\equiv_s$ noncomputable) | 8.5 |
| $(H, 1)$ | effective Hausdorff dimension 1 | 8.9 |
| $(H, > 0)$ | effective Hausdorff dimension $> 0$ | 8.9 |
| $(p, 1)$ | effective packing dimension 1 | 8.9 |
| $(p, > 0)$ | effective packing dimension $> 0$ | 8.9 |
| $(cp, 1)$ | complex packing dimension 1 | 8.9 |
| $(cp, > 0)$ | complex packing dimension $> 0$ | 8.9 |
| COM | complex in the sense of [12] | 8.10 |
| ED | eventually different | 8.8 |
| $\not\leq_T$ Cohen generic | not computable from a 2-generic set | |
| CT | computably traceable | |
| WCET | weakly c.e. traceable | 8.11 |

FIGURE 3. Abbreviations used in Figure 1.

**Definition 8.3.** *An element of $2^\omega$ is Mises-Wald-Church (MWC) stochastic if no partial computable monotonic selection rule can select a biased subsequence, i.e., a subsequence where the relative frequences of 0s and 1s do not converge to $1/2$.*

**Definition 8.4.** *An element of $2^\omega$ is Kolmogorov-Loveland stochastic if no partial computable (non-monotonic) selection rule can select a biased subsequence, i.e., a subsequence where the relative frequences of 0s and 1s do not converge to* $1/2$.

Let $\mathfrak{C}$ denote the collection of all infinite computable subsets of $\omega$.

**Definition 8.5.** *A set $X$ is immune if for each $N \in \mathfrak{C}$, $N \not\subseteq X$. If $\omega \setminus X$ is immune then $X$ is co-immune. If $X$ is both immune and co-immune then $X$ is bi-immune.*

**Definition 8.6.** *A set $X$ is stochastically bi-immune if for each set $N \in \mathfrak{C}$, $X \upharpoonright N$ satisfies the strong law of large numbers, i.e.,*

$$\lim_{n\to\infty} \frac{|X \cap N \cap n|}{|N \cap n|} = \frac{1}{2}.$$

**Definition 8.7.** *Let $0 \leq \mathfrak{p} < 1$. A sequence $X \in 2^\omega$ is* $\mathfrak{p}$-stochastically dominated *if for each $L \in \mathfrak{C}$,*

$$\limsup_{n\to\infty} \frac{|L \cap n|}{n} > 0 \quad \Longrightarrow \quad (\exists M \in \mathfrak{C}) \quad M \subseteq L \quad \textit{and} \quad \limsup_{n\to\infty} \frac{|X \cap M \cap n|}{|M \cap n|} \leq \mathfrak{p}.$$

*The class of $\mathfrak{p}$-stochastically dominated sequences is denoted $SD_\mathfrak{p}$. If $\omega \setminus X \in SD_\mathfrak{p}$ then we write $X \in SD^\mathfrak{p}$ and say that $X$ is stochastically dominating.*

**Definition 8.8.** *A function $f \in \omega^\omega$ is* eventually different *(ED) if for each computable function $g \in \omega^\omega$, $\{x : f(x) = g(x)\}$ is finite.*

**Definition 8.9.** *The* effective Hausdorff dimension *of $A \in 2^\omega$ is*

$$\liminf_{n\in\omega} \frac{K(A \upharpoonright n)}{n}.$$

*The* complex packing dimension *of $A \in 2^\omega$ is*

$$\dim_{cp}(A) = \sup_{N\in\mathfrak{C}} \inf_{n\in N} \frac{K(A \upharpoonright n)}{n}.$$

*The* effective packing dimension *of $A \in 2^\omega$ is*

$$\limsup_{n\in\omega} \frac{K(A \upharpoonright n)}{n}.$$

**Definition 8.10** ([12])**.** *$A \in 2^\omega$ is* complex *if there is an order function $h$ with $K(A \upharpoonright n) \geq h(n)$ for almost all $n$.*

**Definition 8.11** (Nies [28])**.** *$A \in 2^\omega$ is* facile *if $K(A \upharpoonright n \mid n) \leq h(n)$ for all order functions $h$ and almost all $n$. If $A$ is not facile then $A$ is* difficult*. $A$ is weakly c.e. traceable if for each order function $p$, for all computably bounded functions $f \leq_T A$, there is a c.e. trace for $f$ of size bounded by $p$.*

## 9. Sep 2010: Higher randomness

9.1. **The collection of $\Delta^1_1$-random reals is not hyperarithmetically upward closed.** Input by Yu.

Define

$$\mathbf{F} = \{x \mid \omega_1^x = \omega_1^{\mathrm{CK}} \wedge \text{every hyperarithmetic real is recursive in } x\}.$$

Then $\mathbf{F}$ is a $\Sigma^1_1$ set.

**Lemma 9.1** (Folklore). *If $\omega_1^x = \omega_1^{\mathrm{CK}}$, then there is a real $z \geq_T x$ so that $z \in \mathbf{F}$.*

*Proof.* Suppose that $\omega_1^x = \omega_1^{\mathrm{CK}}$. The set

$$F_x = \{z \geq_T x \mid \text{every hyperarithemtic real is recursive in } z\}$$

is a $\Sigma_1^1(x)$ set. Moreover $F_x$ is not empty since $\mathcal{O}^x \in F_x$. By Gandy basis theorem, there must be some $z \in F_x$ so that $\omega_1^z = \omega_1^x = \omega_1^{\mathrm{CK}}$. Then $z \in \mathbf{F}$. □

**Lemma 9.2.** (1) *For any $x \in \mathbf{F}$ and $\Pi_1^1$-random real $z$, $x \not\leq_h z$;*
(2) *Every $\Pi_1^1$-random real is hyperaithmetic reducible to some real in $\mathbf{F}$;*
(3) *If $x \in \mathbf{F}$, then there is a $\Pi_1^1$-random real hyperaithmetically reducible to $x$.*

*Proof.* (1). Suppose that $x \in \mathbf{F}$, $z$ is a $\Pi_1^1$-random real and $x \leq_h z$. Since $\omega_1^x = \omega_1^{\mathrm{CK}}$, there exists some recursive ordinal $\alpha$ such that $x \leq_h z^{(\alpha)}$. Since $x \geq_T \emptyset^{\alpha+1}$, $\emptyset^{\alpha+1} \leq_T z^{(\alpha)}$. But the set $\{y \mid \emptyset^{(\alpha+1)} \leq_T y^{(\alpha)}\}$ is a $\Delta_1^1$-null set (see [**?**]). $z$ cannot be $\Delta_1^1$-random, a contradiction.

(2). Immediately from Lemma 9.1.

(3). If $x \in \mathbf{F}$, then every $x$-Schnorr random is $\Delta_1^1$-random. Pick up an $x$-Schnorr random real $z \leq_T x'$. So $z$ is $\Delta_1^1$-random. Since $\omega_1^x = \omega_1^{\mathrm{CK}}$, $\omega_1^z = \omega_1^{\mathrm{CK}}$. So $z$ is $\Pi_1^1$-random. □

**Corollary 9.3.** *The collection of the hyperdegrees of $\Delta_1^1$-random reals is not upward closed within the hyperdegrees.*

*Proof.* Any $\Delta_1^1$-random real $x$ with $\omega_1^x = \omega_1^{\mathrm{CK}}$ is $\Pi_1^1$-random. By (2) in Lemma 9.2, there is a $y \in \mathbf{F}$ so that $x \leq_h y$. By (1) in Lemma 9.2, $y$ cannot be hyperarithmetical equivalent to $\Delta_1^1$-random real. □

9.2. **Some trivial observations about $NCR_{\Pi_1^1}$.** Input by Yu.

$$NCR_{\Pi_1^1} = \{x \mid x \text{ is not } \Pi_1^1\text{-random respec to any continuous measure}\}.$$

**Proposition 9.4.** $NCR_{\Pi_1^1} = \{x \mid x \in L_{\omega_1^x}\}$.

Sketch of the proof.

**Lemma 9.5.** *$NCR_{\Pi_1^1}$ is a thin $\Pi_1^1$-set. So $NCR_{\Pi_1^1} \subseteq \{x \mid x \in L_{\omega_1^x}\}$.*

*Proof.* Just same as the proof in Reimann and Slaman [30], $NCR_{\Pi_1^1}$ does not contain a perfect subset.

Just same as the proof in Hjorth and Nies [11], there is a $\Pi_1^1$ set $\mathcal{Q} \subseteq (2^\omega)^3$ so that for each real $x$ and continuous measure $\mu$, $\mathcal{Q}_{\mu,x} = \{y \mid (\mu, x, y) \in \mathcal{Q}\}$ is the largest $\Pi_1^1(x)$ $\mu$-null set. The same as in Reimann and Slaman [31], $NCR_{\Pi_1^1}$ is a $\Pi_1^1$ set. □

**Lemma 9.6.** *If $x \in L_{\omega_1^x}$ and $z \not\geq_h x$, then $z \oplus x \geq_h \mathcal{O}^z$.*

*Proof.* Suppose that $x \in L_{\omega_1^x}$ and $z \not\geq_h x$. Then $\omega_1^z < \omega_1^x$. So $\omega_1^{x\oplus z} > \omega_1^z$. Thus $z \oplus x \geq_h \mathcal{O}^z$. □

**Lemma 9.7.** *If $x \in L_{\omega_1^x}$, then $x \in NCR_{\Pi_1^1}$.*

*Proof.* Given any continuous measure $\mu$. If $x \leq_h \mu$, then $x$ obviously is not $\mu$-random. By Lemma 9.6, $x \oplus \mu \geq_h \mathcal{O}^\mu$. But $\{z \mid z \oplus \mu \geq \mathcal{O}^\mu\}$ is a $\Pi^1_1$-null set. So $x$ cannot be $\Pi^1_1$-$\mu$-random. □

The proposition follows by the Lemmas above.

**Remark**: By Reimann and Slaman [31], $NCR_n$ is countable for any $n \in \omega$. This should be true for any recursive ordinal. Then $NCR_{\Pi^1_1}$ puts a limit for their results. By a more involved argument, one can show that every master code belongs to $NCR_{\Delta^1_1}$. So the uncountability of $NCR_{\Delta^1_1}$ is unprovable under $ZFC$.

## 10. Nov 2010: Results announcement: Characterizing $\emptyset'$-Schnorr randomness via Martin-Löf randomness.

Input by Yu.

### 10.1. Characterizing $\emptyset'$-Schnorr randomness via Martin-Löf randomness.

**Definition 10.1.** *A real $x$ is* **L***-random if for all real $z$ with $z' \leq_T \emptyset'$, $x$ is $z$-random.*

This notion was introduced by Mr. Peng. The following result was proved.

**Theorem 10.2.** *Every* **L***-random is $\emptyset'$-Schnorr-random.*

The method of the proof is a finite injury argument.
Sketch of the proof:

*Proof.* We prove that for every $\emptyset'$-Schnorr test $\{U_n^{\emptyset'}\}_{n\in\omega}$, there is a real $z$ with $z' \leq_T \emptyset'$ such that there is $z$-Martin-Löf-test $\{V_n^z\}_{n\in\omega}$ so that $\bigcap_{n\in\omega} V_n^z \supseteq \bigcap_{n\in\omega} U_n^{\emptyset'}$.

Since $\{U_n^{\emptyset'}\}_{n\in\omega}$ is a $\emptyset'$-Schnorr test, there is a recursive function $f : \omega \times 2^{<\omega} \times \omega \to 2$ so that for every $n$ and $\sigma$,

(1) $\lim_s f(n, \sigma, s) = 0$ or 1;
(2) $\lim_s f(n, \sigma, s) = 1$ if and only if $\sigma \in U_n^{\emptyset'}$.

We build a low real $z$ and $z$-Martin-Löf test $\{V_n^z\}_{n\in\omega}$ by a full approximation priority argument. We need to satisfy two kinds of requirements:

$N_e : \exists^\infty s \Phi_e^{z_s}(e)[s] \downarrow \implies \Phi_e^z(e) \downarrow$;
$P_e : U_{2^e}^{\emptyset'} \subseteq V_e^z$.

To satisfy $P_e$, we need to decompose $P_e$ into infinitely many subrequirements $P_{e,n}$. For every $n, m$, let

$$U_n^{\emptyset'} \upharpoonright m = U_n^{\emptyset'} \cap 2^{\leq l_m^n} = \{\sigma \mid |\sigma| \leq l_m^n \wedge \sigma \in U_n^{\emptyset'}\}$$

where $l_m^n$ is the least number $l$ such that $\mu(U_n^{\emptyset'} \cap 2^l) > 2^{-n}(1-2^{-m})$. Notice that since $\{U_n^{\emptyset'}\}_{n\in\omega}$ is a $\emptyset'$-Schnorr test, we may $\emptyset'$-recursive find $l_m^n$ for every $m$ and $n$.

Set

$$P_{\langle e,n\rangle} : U_{2^e}^{\emptyset'} \upharpoonright n \subseteq V_e^z.$$

It suffices to satisfy those $P_{\langle i,j\rangle}$'s so that $i \leq j$. Then we may set the priority list as $N_i < P_{\langle 0,i\rangle} < P_{\langle 1,i\rangle} < ...P_{\langle i,i\rangle} < N_{i+1}$, $i \in \omega$.

As in the usual finitary injury argument, we build a restriction function $r(e,s) > \phi_e^{z_s}(e)$ for every negative requirement $N_e$ at every stage $e$ where $\phi_e^{z_s}(e)$ is the use function of $\Phi_e^{z_s}(e)[s]$. Set

$$R(e,s) = \sum_{i \leq e} r(i,s).$$

At stage $s$, $N_e$ requires attention if $\Phi_e^{z_s}(e)[s] \downarrow$ but $N_e$ has not received attention (after initialized).

At every stage $s$, for every $n, m$, let

$$U_n^{\emptyset'_s}[s] \upharpoonright m = U_n^{\emptyset'_s}[s] \cap 2^{\leq l_m^n[s]} = \{\sigma \in 2^{<\omega} \mid |\sigma| \leq l_m^n[s] \wedge \sigma \in U_n^{\emptyset'_s}[s]\}$$

where $l_m^n[s]$ is the least number $l$ such that $\mu(U_n^{\emptyset'_s}[s] \cap 2^l) > 2^{-n}(1-2^{-m})$. Obviously $\lim_s l_m^n[s] = l_m^n$.

The basic strategy for $P_{\langle e,n\rangle}$ is: At any stage $s$, for each $\sigma$, there is a follower $\langle e, \sigma, t_s\rangle$ attached to $\sigma$. If $\sigma$ enters $U_e^{\emptyset'_s}[s] \upharpoonright n$ (i.e. $f(e,\sigma,s) = 1$), then we set $z_s(\langle e,\sigma,t_s\rangle) = 1$. If $\sigma$ exit $U_e^{\emptyset'_s}[s] \upharpoonright n$ (i.e. $f(e,\sigma,s) = 0$), then we set $z_s(\langle e,\sigma,t_s\rangle) = 0$. So we may define $V_e^{z_s}[s] = \{\sigma \mid (z_s(\langle e,\sigma,t_s\rangle) = 1)\}$ and $V_e^z = \{\sigma \mid \exists s(z(\langle e,\sigma,t_s\rangle) = 1)\}$.

The rule attributing a follower to $P_{\langle i,j\rangle}$ at stage $s$ is: For any $\sigma$ with $l_j^i[s] \geq |\sigma| > l_{j-1}^i[s]$, we attribute a follower $\langle i,\sigma,t_s\rangle$ to $\sigma$ such that $t_s$ greater than all the parameters mentioned in the higher priority requirements no later than stage $s$.

$P_{\langle i,j\rangle}$ requires attention at stage $s$ if $\sigma$ enters $U_e^{\emptyset'_s}[s] \upharpoonright j$ but $z_s(\langle e,\sigma,t_s\rangle) = 0$. Then we intend set $z_{s+1}(\langle e,\sigma,t_s\rangle) = 1$.

To avoid the confliction between $P_{\langle i_0,j_0\rangle}$ and $P_{\langle i_1,j_1\rangle}$ say $P_{\langle i_0,j_0\rangle} < P_{\langle i_1,j_1\rangle}$, we initialize all the parameters for $P_{\langle i_1,j_1\rangle}$ and set $z_{s+1}(\langle i_1,\sigma,t_s\rangle) = 0$ for any parameter $\langle i_1,\sigma,t_s\rangle$ for $P_{\langle i_1,j_1\rangle}$ once upon $P_{\langle i_0,j_0\rangle}$ receives attention. This cannot happen infinitely often by the definition of $f$.

Notice that there are at most $2^{-2^i-(j-1)}$ measure of clopen sets put in $V_i^z$ by $P_{\langle i,j\rangle}$ for any pair $\langle i,j\rangle$.

Since $\{U_n^{\emptyset'}\}_{n\in\omega}$ is a $\emptyset'$-Schnorr test, a usual finite injury argument will show that $N_e$ will be injured at most finitely many times for every $e$. So $z$ must be low.

For each $P_{\langle i,j\rangle}$ with $j \geq i$, there are $j$ many negative requirements $\{N_e\}_{e\leq j}$ having higher priority than $P_{\langle i,j\rangle}$. For each $e \leq j$, once $N_e$ set up a restriction $r(e,s)$, then $P_{\langle i,j\rangle}$ cannot change its parameters less than $R(e,s)$ anymore until some $P_{\langle i',j'\rangle}$ higher than $N_e$ receives attention. So $P_{\langle i,j\rangle}$ may $j$-times wrongly put clopen sets into $U_i^z$. The measure of the sum of these mistakes is no more than $j \cdot 2^{-2^i-j+1}$. Thus

$$\mu(V_i^z) \leq \sum_{j\in\omega}(j+1)\cdot 2^{-2^i-j+1} \leq 2^{-i}.$$

So $\{V_i^z\}_{i\in\omega}$ is a $z$-Martin-Löf test. By the definition of $V_i^z$, for every $i$, $U_{2^e}^{\emptyset'} \subseteq V_e^z$ for every $e$. So $\bigcap_{e\in\omega} U_e^{\emptyset'} \subseteq \bigcap_{e\in\omega} V_e^z$.

□

**Corollary 10.3.** *For any real $x \geq_T \emptyset'$ and $z$, the followings are equivalent:*

(1) *$z$ is $x$-Schnorr random;*

(2) *For any real $y$ with $y' \leq_T x$, $z$ is weakly-2-random relativized to $y$;*
(3) *For any real $y$ with $y' \leq_T x$, $z$ is Martin-Löf-random relativized to $y$.*

*Proof.* Both (1) $\implies$ (2) and (2) $\implies$ (3) are obvious.

We show that (3) $\implies$ (1). Since $x \geq_T \emptyset'$, there is a real $z_0 \leq_T x$ so that $z_0' \equiv_T x$. Relativizing the proof of Theorem 10.2 to $z_0$, every $z$-Schnorr random real is Martin-Löf-random relativized to $y$ for some $y$ with $z_0 \leq y$ and $y' \leq_T x$.

□

Obviously another direction of Theorem 10.2 is true. So **L**-randomness is the same as $\emptyset'$-Schnorr-randomness.

10.2. **Lowness properties.**

**Theorem 10.4.** *If $x$ is not low, then there is a $\emptyset'$-Schnorr random real which is not $x$-random.*

The method of the proof is a forcing argument which was based on a couple of results due to Diamondstone, Nies and others. They are:

**Theorem 10.5** (Diamondstone [5] )**.** *For any pair of low reals $x$ and $y$, there is a c.e. low real $z$ so that every $z$-random real is both $x$- and $y$-random.*

**Theorem 10.6** (Nies [28])**.** *If $y \leq_T x'$ and every $x$-random is $y$-random, then $y' \leq_T x'$.*

And Theorem 5.6.9 in [28].

The forcing is: $\mathbb{P} = (\mathbf{P}, \leq)$ where $\mathbf{P}$ is the collection of $\Pi^0_1(y)$ set of reals having positive measure for some low real $y$. For $P_1, P_2 \in \mathbf{P}$, $P_1 \subseteq P_2$ if and only if $P_1 \leq P_2$.

So

**Corollary 10.7.** $\mathrm{Low}(\mathrm{Sch}(\emptyset'), \mathrm{W2R}) = \mathrm{Low}(\mathrm{Sch}(\emptyset'), \mathrm{ML}) = \mathrm{Low}(= \{x \mid x' \equiv_T \emptyset'\})$.

## 11. Uniformly $\Sigma^0_3$ index sets

By Frank Stephan. Also see 2012 version of Nies' book.

[28, Problem 5.3.33] Let $C$ be an index set for a class of c.e. sets, namely $e \in C \wedge W_e = W_i \rightarrow i \in C$. We say that $C$ is uniformly $\Sigma^0_3$ if there is a $\Pi^0_2$ predicate $P$ and a effective (= recursive) sequence $(e_0, b_0), (e_1, b_1), \ldots$ such that the following three conditions hold:

- $e \in C \Leftrightarrow \exists b\, [P(e, b)]$;
- $P(e_n, b_n)$ for all $n$;
- For all $e \in C$ there is an $n$ with $W_{e_n} = W_e$.

In other words, $C$ is the closure, under having the same index, of a projection of a c.e. relation contained in $P$. For instance, let $P(e, b)$ be

$$\forall n \forall s \exists t > s\, [K_t(W_{e,t} \upharpoonright_n) \leq K_s(n) + b].$$

This shows that the $K$-trivials are uniformly $\Sigma^0_3$ by a Theorem of Downey, Hirschfeldt, Nies and Stephan (see [28, Thm 5.3.28] for a simpler proof of that theorem). Also the computables - $C$-trivials are u'ly $\Sigma^0_3$ by a similar argument.

The problem was whether each $\Sigma^0_3$ class is already uniformly $\Sigma^0_3$.

First an easy counterexample:

(1) $\{e : |W_e| = \infty\}$ has a $\Sigma^0_3$ Index set; in fact $\Pi^0_2$.

(2) If this set were uniform $\Sigma^0_3$, there would be an effective sequence $(e_n, b_n)$ and a predicate $P$ such that the following holds:

- $P(e_n, b_n)$ for all $n$;
- $W_e$ is infinite iff $P(e, b)$ for some $b$;
- For each infinite $W_e$ there is $n$ such that $W_e = W_{e_n}$.

Thus $n \mapsto W_{e_n}$ is an effective enumeration of all infinite r.e. sets, contradiction.

Next we give a full characterization:

**Theorem.** An index set $C$ is uniformly $\Sigma^0_3$ iff there is a recursive enumeration $e_0, e_1, \ldots$ such that $e \in C \Leftrightarrow \exists n\, [W_{e_n} = W_e]$.

**Proof.**

The definition directly gives that $e \in C \Leftrightarrow W_e = W_{e_n}$ for some $n$. So every uniformly $\Sigma^0_3$ index set belongs to an r.e. class of r.e. sets. For the converse direction, assume that $C = \{e : \exists n\, [W_e = W_{e_n}]\}$ where $e_0, e_1, \ldots$ is an effective sequence of indices. Now let $b_n = n$ for all $n$ and define

$$P(e, b) \Leftrightarrow \forall x \forall s \exists t\, [t > s \wedge W_{e,t}(x) = W_{e_b,t}(x)].$$

In other words, $P(e, b)$ is the $\Pi^0_2$ predicate which holds iff $W_e = W_{e_b}$. Now it follows that $e \in C \Leftrightarrow \exists b\, [P(e, b)]$ and $(e_0, 0), (e_1, 1), \ldots$ is the effective sequence which witnesses together with $P$ that $C$ is uniformly $\Sigma^0_3$.

**Remark.** It is known that there are $\Sigma^0_3$ index sets which do not belong to a uniformly r.e. family of sets. Here some examples:

- The set $\{e : |W_e| = \infty\}$;
- The set $\{e : W_e \subseteq A\}$ where $A$ is a non-r.e. $\Pi^0_2$ set;
- The set $\{e : \exists a \in A\, [W_e = \{a\}]\}$ where $A$ is a non-r.e. $\Sigma^0_3$ set.

If $C$ is a $\Sigma^0_3$ index set of a class containing all finite sets then this class is a uniformly r.e. family and $C$ is uniformly $\Sigma^0_3$ as can be seen as follows: Given a formula such that

$$e \in C \Leftrightarrow \exists b \forall c \exists d\, [Cond(e, b, c, d)]$$

where $Cond$ is a recursive predicate, let now

$$W_{f(e,b)} = \{x \in W_e : \forall c \leq x \exists d\, [Cond(e, b, c, d)]\}.$$

The set $W_{f(e,b)}$ is equal to $W_e$ in the case that $e, b$ satisfy $\forall c \exists d\, [Cond(e, b, c, d)]$; otherwise $W_{f(e,b)}$ is finite. So every index $f(e, b)$ is in $C$ and the class of sets indexed by $C$ is equal to the family $\{W_{f(e,b)} : e, b \in \mathbb{N}\}$.

## 12. December 2010: Characteristic traceability, and weakly DNC sets

Freer, Kjos-Hanssen, and Nies worked at the University of Hawai'i.

**Definition 12.1.** *A trace $(T_n)_{n\in\mathbb{N}}$ is a sequence of finite sets. We say that a function $h$ is a bound for the trace if $\#T_n \le h(n)$ for each $n$. We say $(T_n)_{n\in\mathbb{N}}$ is a trace for function $f$ if $f(n) \in T_n$ for each $n$.*

Recall computable traceability [28, 8.2.15]: a trace $(T_n)_{n\in\mathbb{N}}$ is called computable if there is a computable function $g$ such that $T_n = D_{g(n)}$ (strong index) for each $n$. We say that $A$ is computably traceable if there is an order function $h$ such that each function $f \le_T A$ has a computable trace with bound $h$.

A $\Delta^0_1$-*index* for a set $B \subseteq \mathbb{N}$ is given by a pair of c.e. indices, one for the set, and one for its complement $\mathbb{N} - B$. This has also been called characteristic index [34], because it is equivalent to having an index for the characteristic function of $B$.

**Definition 12.2.** *A trace $(T_n)_{n\in\mathbb{N}}$ is called characteristic if there is a computable function $g$ such that for each $n$, $g(n)$ is a characteristic index for $T_n$.*

The following example shows these traces are more general than computable traces.

**Example 12.3.** *Let $A$ be c.e. via the computable enumeration $(A_s)_{s\in\mathbb{N}}$. Then the modulus function*

$$f_A(n) = \mu s. [A\restriction_n = A_s\restriction_n]$$

*has a characteristic trace with bound $n+1$. It has no computable trace unless $A$ is computable.*

*Proof.* Let $f_0(n) = n$ for each $n$. For each stage $s$, if we have $y \in A_s - A_{s-1}$ where $y$ is least, we redefine $f_s(n) = s$ for all $n$ such that $y < n \le s$. Let $T_n = \{f_s(n)\colon s \in \mathbb{N}\}$. Then $(T_n)_{n\in\mathbb{N}}$ is a characteristic trace for $f$ with bound $n + 1$.

Any function with a computable trace is dominated by a computable function. This is not possible for the modulus function unless $A$ is computable. □

12.1. **Computably dominated sets and weakly d.n.c. sets.** Recall that $A$ is computably dominated (or of HIF degree) if

$$\forall f \le_T A\, \exists g \,\text{computable}\, \forall n\, [f(n) < g(n)].$$

In the definition following apparently weaker property, we only have an inequality.

**Definition 12.4.** *We say $A$ is* weakly d.n.c. *if*

$$\forall f \le_T A\, \exists g \,\textit{computable}\, \forall n\, [f(n) \ne g(n)].$$

However, it's actually the same!

**Proposition 12.5.** *Let $A$ be weakly d.n.c. Then $A$ is computably dominated.*

*Proof.* If $A$ is not computably dominated, there is an increasing function $r \le_T A$ such that for any computable $h$, there are infinitely many $n$ with $h(n) \le r(n)$.

We define inductively a function $f \le_T A$ which agrees somewhere with each (total) computable function $\phi_e$. On input $x$, with oracle $A$ compute the least $e \le x$ such that

$$\forall y \leq x\, \phi_{e,r(x)}(y)\downarrow, \text{ and } \forall y < x\, \phi_{e,r(x)}(y) \neq f(x).$$

Let $f(x) = \phi_{e,r(x)}(x)$. (If there is no $e$, let $f(x) = 0$.)

If $\phi_e$ is total then the function $h(x) = \mu t.\forall y \leq x\, \phi_{e,t}(y)\downarrow$ is total. So for infinitely many $x$, $r(x) \geq h(x)$. So eventually for some $x$ we choose $e$ and ensure $f(x) = \phi_e(x)$. □

### 12.2. **Characteristic traceability equals computable traceability.**

**Definition 12.6.** *Let us say $A$ is* characteristically traceable *if the same definition as above holds for characteristic traces: there is an order function $h$ such that each function $f \leq_T A$ has a characteristic trace with bound $h$.*

Clearly, each characteristically traceable set is weakly d.n.c.: given $f \leq_T A$, pick a characteristic trace $(T_n)_{n\in\mathbb{N}}$, and let $g$ be a computable function such that $g(n) \in \mathbb{N} - T_n$ for each $n$.

**Theorem 12.7.** *Each characteristically traceable set $A$ is already computably traceable.*

*Proof.* $A$ is computably dominated by Proposition 12.5. Also $A$ is c.e. traceable. Hence $A$ is computably traceable by a simple argument due to [13]. □

## 13. Questions

### 13.1. **August 2010: Distribution of a real obtained by tossing a biased coin.** By André.

Let $0 < \delta < 1$. Imagine we repeatedly toss a biased coin where the probability of tails (0) is $\delta$, and thus of heads (1) is $1 - \delta$. Let $r$ be the real in $[0, 1]$ with binary expansion given by this sequence of coin tosses. Describe the distribution function of $r$ depending on $\delta$, i.e., the function

$$f_\delta(x) = P[r \leq x].$$

For instance, $f_{0.5}(x) = x$.

Here is a plot, in a different scale, for the case $\delta = 1/3$. Looks like it is nondifferentiable at the dyadic rationals –Bjørn

Interesting- something like the Cantor function then. Thanks! Andre

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